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SENSITIVITY CONSIDERATIONS IN ACTIVE NETWORK SYNTHESIS

by

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PREFACE

This report is a Scientific Report representing one phase of the work done under AFCRC Contract No. AF 19(628)-1649.

Appreciation is hereby expressed to Mr. Donald J. Rohrbacher with whose assistance some of the ideas in Chapter 3 were developed. Thanks are also due to Mr. Donald Melvin for aid in developing examples and for proof reading.

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CHAPTER 1

INTRODUCTION

Several methods of synthesis of RC active networks have been reported in the literature. These fall in three main categories.

a) Negative impedance converter (NIC) methods

b) High gain ideal amplifier methods

c) Non-ideal active device methods

All methods require the selection of certain parameters (the roots of an auxiliary polynomial or residues at poles). There will be restrictions on these parameters introduced by realizability requirements. In addition, other restrictions may arise if the parameters are to be chosen to minimize the sensitivity of a property of the realization to changes in component values, in particular to variations in an active device parameter like the conversion ratio of the NIC. The selected property of the realization whose sensitivity is to be minimized may be the poles or zeros of the realized transfer function or the coefficients of the corresponding polynomials, or the transfer function itself as a function of frequency.

Horowitz[1] has given a method for the decomposition of the denominator of a transfer impedance in order to yield minimum sensitivity of the cascade NIC realization of Linville to variations in the NIC conversion ratio. In this case the sensitivity of the poles, the coefficients of the denominator and the frequency response are all three simultaneously minimized.

More recently Callahan[2] has given the minimum sensitivity conditions for a realization technique due to Horowitz[3,4] to variations of a parameter similar to the conversion ratio of an NIC. However, the realization technique under discussion does not use an NIC and the introduction of an artificial parameter with respect to whose variations the sensitivity is minimized does not insure optimization with respect to the actual device parameter which is varying.

In this report two previously reported[5,4] realization techniques will be considered. In both of these the active device is characterized by all (four) of its two-port parameters rather than a single one. Thus, the realizations utilize actual devices which are either single stage or multi-stage transistor amplifiers. The questions to be considered, in addition to realizability, are

1. How should arbitrary parameters in the realization procedure be chosen to minimize sensitivity of the transfer function or the poles to variations in the most variable parameter of the active device?
2. What limitations exist for realization by means of a single transistor?

The active devices that are used in the realizations consist of one and two stage common emitter and one stage common base transistor amplifiers. The low frequency equivalent circuit that will be used in all three cases is shown in Fig. 1(a). (For the single stage common emitter connection the reference of the controlled source will be reversed.) The parameters of this circuit will be related to the transistor parameters shown in Fig. 1(b). Typical values of the parameters are given in Table 1. (For reduced sensitivity a series resistance is connected to the emitter making $r_e' = 100$.) The design of the two stage amplifier is shown in Fig. 2.

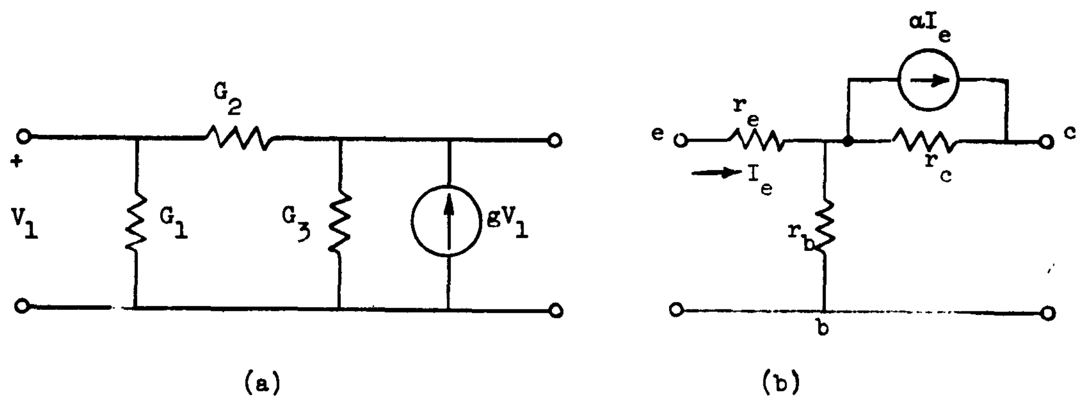


Figure 1.

	G_1	G_2	G_3	g
1 stage common emitter	2×10^{-4}	1.25×10^{-7}	7×10^{-6}	.01
2 stage common emitter	3×10^{-4}	$< 10^{-8}$	5×10^{-4}	.135
normalized to G_1	1	$< 3 \times 10^{-4}$	1.67	450
1 stage common base	.01	7×10^{-6}	3.6×10^{-7}	.01
normalized to G_1	1	7×10^{-4}	3.6×10^{-5}	1

Table 1.

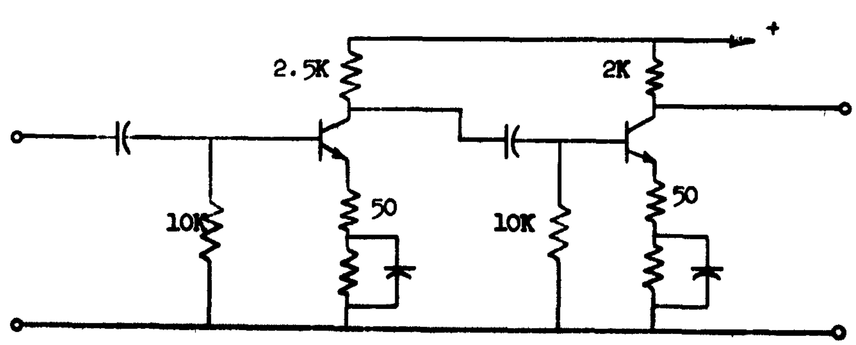


Figure 2.

	1 stage common emitter	1 stage common base	2 stage common emitter
g	$\frac{\alpha}{r_e'}$	$\frac{\alpha}{r_e'}$	$\frac{\alpha^4 r_c^{\,2} - [r_b(1-\alpha) - r_e']^2}{\alpha^2 r_c^{\,2}\left[\frac{r_b r_e'}{r_c} + (1-\alpha)r_e' + 5 \times 10^{-4} r_e'^{\,2}\right]}$
G_1	$\frac{1-\alpha}{r_e'}$	$\frac{1}{r_e'}$	$\frac{\frac{r_e'}{r_c} + (1-\alpha)^2 + 10^{-4}\left[\frac{r_b}{r_c} + 6(1-\alpha)\right] r_e' + 5 \times 10^{-8} r_e'^{\,2} - \left[\frac{r_b(1-\alpha) - r_e'}{\alpha r_c}\right]^2}{\frac{r_b r_e'}{r_c} + (1-\alpha)\, r_e' + 5 \times 10^{-4}\, r_e'^{\,2}}$
G_2	$\frac{r_e}{r_e' r_c}$	$\frac{r_b}{r_c r_e'}$	$\frac{\left[\frac{r_b(1-\alpha) - r_e'}{\alpha r_c}\right]^2}{\frac{r_b r_e'}{r_c} + (1-\alpha)r_e' + 5 \times 10^{-4} r_e'^{\,2}}$
G_3	$\frac{r_b}{r_c r_e'}$	$\frac{r_e}{r_c r_e'}$	$\frac{\frac{r_b^2}{r_c^2} + \frac{r_e'}{r_c} + 10^{-4}\left[\frac{10 r_b}{r_c} + 5(1-\alpha)\right] r_e' + \frac{10^{-10}}{4}\, r_e'^{\,2} - \left[\frac{r_b(1-\alpha) - r_e'}{\alpha r_c}\right]^2}{\frac{r_b r_e'}{r_c} + (1-\alpha)r_e' + 5 \times 10^{-4}\, r_e'^{\,2}}$

$$r_e' = r_e\left(1 + \frac{r_b}{r_c}\right) + (1-\alpha)r_b$$

Table 2.

The parameters of the circuit of Fig. 1(a) are given in terms of those of Fig. 1(b) in Table 2. In the 2 stage case identical transistors are assumed. The resistance r_e includes the emitter resistance and the external resistance in the emitter lead.

CHAPTER 2

REALIZATION ONE

2-1. Summary of the Realization Technique

The realization procedure considered in this Chapter is that due to Kuh.[5] A structure of the form shown in Fig. 3 is assumed.

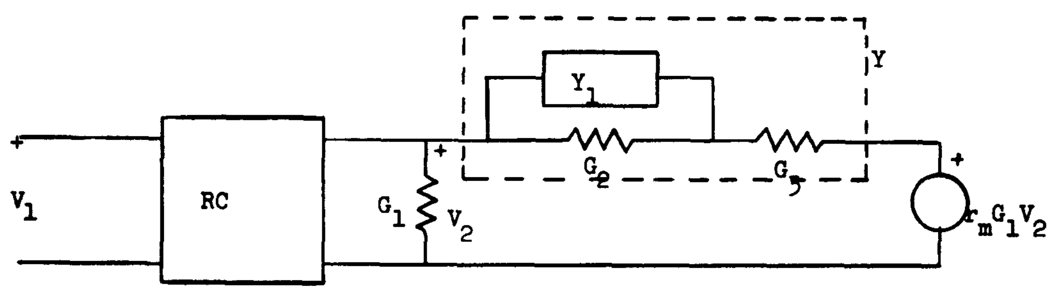


Figure 3.

Within the dashed lines is an RC admittance Y which is to have the partial structure shown, G_3 and the controlled voltage source can be replaced by their current source equivalent and the result takes the form of Fig. 4. Each of the three conductances G_1, G_2 and G_3 has been replaced by two in parallel. The structure within dashed lines is the equivalent circuit of an active device. Because of the reference of the controlled source, the device may be a one stage common base amplifier or a two stage common emitter amplifier. The double primed conductances are external to the active device. G_1'' serves as the load.

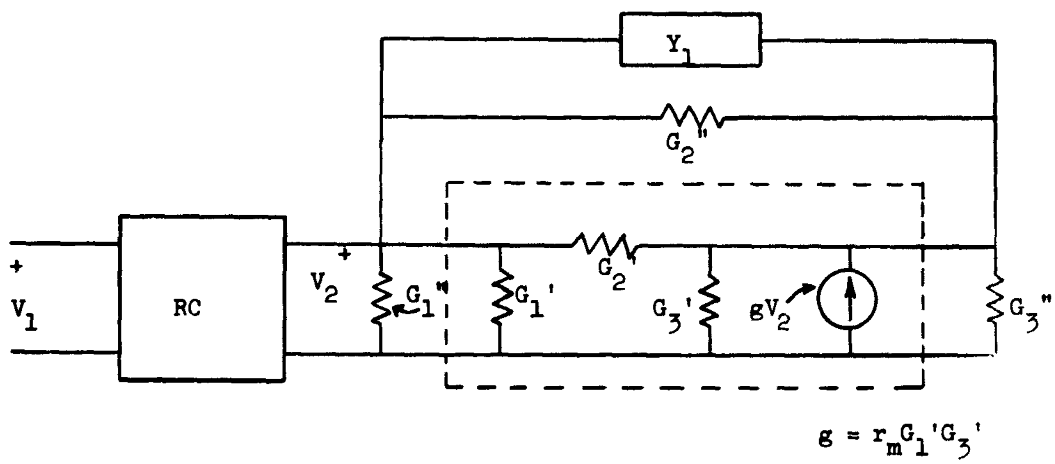


Figure 4.

From Fig. 3 the transfer voltage ratio V_2/V_1 is routinely found to be

$$H(s) = \frac{V_2}{V_1} = \frac{P(s)}{Q(s)} = \frac{-y_{21}}{y_{22} + 1 - (g/G_3-1)Y} \qquad (1)$$

where y_{21} and y_{22} refer to the RC two-port on the left and admittances have been normalized with respect to G_1. Numerator and denominator of the given transfer function P/Q are now divided by an auxiliary polynomial D(s) having only negative real roots and the resulting denominator is expanded in partial fractions. Thus,

$$-y_{21} = \frac{P(s)}{D(s)}$$

$$\frac{Q(s)}{D(s)} = k_\infty s + k_o + \sum \frac{k_i{}'s}{s + \sigma_i{}'} - \sum \frac{k_i{}''s}{s + \sigma_i} \qquad (2)$$

(Actually Q/sD is expanded and the result multiplied by s.)

Note that the denominator of Eq. (1) is the difference of two RC admittances (assuming $g/G_3 > 1$, as evident from Table 1.) Equation (2) also has this form. However, one-to-one identifications cannot yet be made between these two expressions since y_{22} must have as poles all the roots of D(s) because y_{21} does. Hence, a term

$$k + \sum k_i s/(s + \sigma_i)$$

is added and subtracted in Eq. (2) leading to the identifications

$$\left(\frac{g}{G_3} - 1\right) Y = k + \sum \frac{k_i s}{s + \sigma_i} \tag{3}$$

$$y_{22} = (k_o - 1 + k) + k_\infty s + \sum \frac{k_i' s}{s + \sigma_i'} + \sum \frac{(k_i - k_i'')s}{s + \sigma_i} \tag{4}$$

For realizability we require

$$g > G_3 \tag{5}$$

$$k_i > k_i'' \tag{6}$$

$$k > 1 - k_o \quad \text{if } k_o < 1 \quad (k > 0 \text{ otherwise})$$

The constant k is introduced to give Y a nonzero d-c value. The values of G_3 and G_2 are

$$G_3 = Y_{(\infty)} \tag{7}$$

$$G_2 = \frac{Y_{(\infty)}\, Y_{(0)}}{Y_{(\infty)} - Y_{(0)}} \tag{8}$$

from which we find

$$g - G_3 = k + \sum k_i \tag{9}$$

$$G_2 = \frac{k\, G_3}{\sum k_i} \tag{10}$$

2-2. Sensitivity Considerations

The sensitivity of the transfer function to a parameter x is defined as

$$S_x^H = \frac{x}{H}\frac{\partial H}{\partial x} \tag{11}$$

This is a measure of the relative change of H due to a change in x. The parameter x with which we shall concern ourselves is that parameter of the active device which causes the greatest variation in the transfer function. This parameter is the α of the transistor. The emitter resistance is inversely proportional to the base current so that it might be expected that this quantity will be dominant. However, this variation can be reduced by inserting a large enough external resistance in the emitter lead.

By direct application of the definition of sensitivity to the parameters in the last two columns of Table 2 it is found that of the four quantities g, G_1, G_2 and G_3, the one having greatest sensitivity to variations in α is g, by two orders of magnitude. In addition, the variation of the transfer function to changes in the conductances G_1, G_2, G_3 (represented by the primed quantities in Fig. 4) can be reduced if the external conductances shunting these are large enough. Hence, the sensitivity which we shall take as a measure of the performance is $S_g^{H(j\omega,g)}$.

Applying the definition of sensitivity to Eq. (1) leads to

$$S_g^H = \frac{g}{H}\frac{\partial H}{\partial g} = \frac{g\ Y(j\omega)}{G_3 Q(j\omega)} = \frac{g(k + \sum \frac{j\omega k_i}{j\omega + \sigma_i})}{(g - G_3)\ Q(j\omega)} \tag{12}$$

For a given active device g and G_3 are fixed constants (at their nominal values). The polynomial Q(s) is given. Hence, to minimize the magnitude of S_g^H we must minimize the magnitude of the quantity in parentheses. (For convenience we shall deal with its square.) Thus, define the function

$$f = \left| k + \sum \frac{jk_i\omega}{\sigma_i + j\omega} \right|^2 = (k + \sum \frac{\omega^2 k_i}{\sigma_i^2 + \omega^2})^2 + (\sum \frac{\omega\sigma_i k_i}{\sigma_i^2 + \omega^2})^2 \tag{13}$$

It is required to choose the residues in this expression (assuming temporarily that the σ_i poles have been chosen) such that the function f is minimized. However, since g and G_3 are fixed, the choise of the residues is further constrained by Eqs. (6) and (9).

The problem can be handled by the use of Lagrange multipliers. Writing the constraint equation (9) as

$$\phi = k + \sum k_i - g + G_3 = 0 \tag{14}$$

the desired minimum is located at

$$df + \lambda d\phi = 0$$

or

$$(\frac{\partial f}{\partial k} + \lambda)\, dk + \sum (\frac{\partial f}{\partial k_i} + \lambda) dk_i = 0 \tag{15}$$

where λ is as yet undetermined. Since the variations in each of the k's are independent, we require that

$$\frac{\partial f}{\partial k} + \lambda = 0$$

$$\frac{\partial f}{\partial k_j} + \lambda = 0 \qquad j = 1, 2, \ldots, n \tag{16}$$

Using Eq. (13) for f these become

$$k + \sum \frac{\omega^2 k_i}{\sigma_i^2 + \omega^2} = -\frac{\lambda}{2} \tag{17}$$

$$2\left(k + \sum \frac{\omega^2 k_i}{\sigma_i^2 + \omega^2}\right) \frac{\omega^2}{\sigma_j^2 + \omega^2} + \frac{2\omega^2 \sigma_j}{\sigma_j^2 + \omega^2} \sum \frac{\sigma_i k_i}{\sigma_i^2 + \omega^2} + \lambda = 0 \tag{18}$$

Inserting the first of these into the second and also into the constraint equation (14) finally leads to

$$\sum^n \frac{\sigma_i k_i}{\sigma_i^2 + \omega^2} + \frac{\sigma_j}{2\omega^2} \lambda = 0 \qquad j = 1, 2, \ldots, n$$

$$\sum^n \frac{\sigma_i^2 k_i}{\sigma_i^2 + \omega^2} - \frac{\lambda}{2} = g - G_3 \tag{19}$$

These constitute a set of n + 1 linear equations in n + 1 unknowns, λ and the n k_i's. When the k_i's are determined, Eq. (14) will yield k. The determinant of the set is

$$\Delta = \begin{vmatrix} \frac{\sigma_1}{\sigma_1^2+\omega^2} & \frac{\sigma_2}{\sigma_1^2+\omega^2} & \cdots & \frac{\sigma_n}{\sigma_n^2+\omega^2} & \frac{\sigma_1}{2\omega^2} \\ \frac{\sigma_1}{\sigma_1^2+\omega^2} & \frac{\sigma_2}{\sigma_2^2+\omega^2} & \cdots & \frac{\sigma_n}{\sigma_n^2+\omega^2} & \frac{\sigma_2}{2\omega^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\sigma_1}{\sigma_1^2+\omega^2} & \frac{\sigma_2}{\sigma_2^2+\omega^2} & \cdots & \frac{\sigma_n}{\sigma_n^2+\omega^2} & \frac{\sigma_n}{2\omega^2} \\ \frac{\sigma_1^2}{\sigma_1^2+\omega^2} & \frac{\sigma_2^2}{\sigma_2^2+\omega^2} & \cdots & \frac{\sigma_n^2}{\sigma_n^2+\omega^2} & -\frac{1}{2} \end{vmatrix}$$

$$= \frac{\prod^n \sigma_i}{2\omega^2 \prod^n (\sigma_i^2+\omega^2)} \begin{vmatrix} 1 & 1 & \cdots & 1 & \sigma_1 \\ 1 & 1 & \cdots & 1 & \sigma_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 & \sigma_n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n & -\omega^2 \end{vmatrix} \tag{20}$$

The last step is obtained by factoring by columns. Further simplification is possible by subtracting each row from the preceding row up to the nth one, and then repeating with the columns. The result becomes

$$\Delta = \frac{\prod^n \sigma_i}{2\omega^2 \prod^n (\sigma_i^2+\omega^2)} \begin{vmatrix} 0 & 0 & \cdots & 0 & \sigma_1-\sigma_2 \\ 0 & 0 & \cdots & 0 & \sigma_2-\sigma_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & \sigma_n \\ \sigma_1-\sigma_2 & \sigma_2-\sigma_3 & \cdots & \sigma_n & -\omega^2 \end{vmatrix} \tag{21}$$

The value of the determinant is clearly zero for any order greater than three (i.e., $n = 2$). Furthermore, the rank is three, which indicates that of the n residues, $n - 2$ can be arbitrarily chosen.

For the case $n = 2$ the solution for the k_i's becomes

$$k_1 = \frac{(\sigma_1^2 + \omega^2)}{\sigma_1(\sigma_1 - \sigma_2)} (g - G_3)$$

$$k_2 = \frac{-(\sigma_2^2 + \omega^2)}{\sigma_2(\sigma_1 - \sigma_2)} (g - G_3) \tag{22}$$

$$k = \frac{\omega^2}{\sigma_1 \sigma_2} (g - G_3)$$

Thus, either k_1 or k_2 must be negative. Although this is a solution of the sensitivity minimization problem (for $n = 2$), it is of no value since the residues do not fall in the permissible domain restricted by condition (9).

For $n > 2$, let the last $n - 2$ residues be chosen arbitrarily. It is conjectured that removing Eqs. 3 to n and solving for the first two residues will always lead to a negative one. Note that the determinant of the reduced set of equations will be the same for all n. Thus, for $n = 3$ the solutions for k_1 and k_2 are

$$k_1 = \frac{(\sigma_1^2 + \omega^2)}{\sigma_1(\sigma_2 - \sigma_1)} \left[\frac{(\sigma_3 - \sigma_2)\, k_3 \sigma_3}{\sigma_3^2 + \omega^2} - (g - G_3) \right]$$

$$k_2 = \frac{\sigma_2^2 + \omega^2}{\sigma_2(\sigma_2 - \sigma_1)} \left[(g - G_3) - \frac{\sigma_3 k_3 (\sigma_3 - \sigma_1)}{\sigma_3^2 + \omega^2} \right] \tag{23}$$

Assuming $\sigma_2 > \sigma_1$, the factor $(\sigma_2 - \sigma_1)$ will be positive. For the quantities in both square brackets to be positive will then require the condition

$$\frac{\sigma_3 k_3(\sigma_3 - \sigma_1)}{\sigma_3^2 + \omega^2} < (g - G_3) < \frac{\sigma_3 k_3(\sigma_3 - \sigma_2)}{\sigma_3^2 + \omega^2} \tag{24}$$

For this condition to be satisfied requires the term on the extreme right to be greater than the one on the extreme left. Hence,

$$\sigma_3 - \sigma_1 < \sigma_3 - \sigma_2$$

or (25)

$$\sigma_2 < \sigma_1$$

But this is contrary to the assumption $\sigma_2 > \sigma_1$. Hence, one of the two residues will be negative and condition (6) will not be satisfied.

From the results so far obtained it appears that a relative minimum of the function f does not exist within the permissible domain, which is a region in the (n + 1)-dimensional space consisting of the variables k and the k_i's. Hence, the lowest value of f must occur on the boundary of the region. Thinking in terms of three dimensional space, restriction (6) defines the permissible domain as lying outside of the planes defined by $k_i > k_i''$. The boundaries are the planes $k_i = k_i''$. If k_1, say, is held fixed at k_1'' the other two variables can still vary over their permissible range. Hence, the minimum sensitivity will be sought by fixing each of the k_i's in turn to their minimum values and finding the conditions on the other k_i's for minimum sensitivity. This seems like an interminable process; however, the high degree of symmetry of the equations reduces the effort. Furthermore, since the rank of the determinant is three--

which means all but two of the k_i's in Eqs. (19) are to be arbitrarily specified, as already mentioned--and since one of these remaining two is also to be held fixed, there remains a set of two equations to solve in all cases.

Thus, for n = 2 let σ_2 be the larger of the two poles and let k_2 be held fixed. Then, Eqs. (19) reduce to

$$\frac{\sigma_1}{\sigma_1^2 + \omega^2} k_1 + \frac{\sigma_1}{2\omega^2} \lambda = - \frac{\sigma_2 k_2}{\sigma_2^2 + \omega^2}$$

$$\frac{\sigma_1^2}{\sigma_1^2 + \omega^2} k_1 - \frac{1}{2} \lambda = - \frac{\sigma_2^2 k_2}{\sigma_2^2 + \omega^2} + g - G_3 \qquad (26)$$

Solving for k_1 leads to

$$k_1 = (g - G_3) - k_2 \left(\frac{1 + \omega^2/\sigma_1\sigma_2}{1 + \omega^2/\sigma_2^2}\right) \qquad (27)$$

Inserting this into the constraint equation (14) and solving for k leads to

$$k = \frac{\omega^2 k_2}{\sigma_2^2 + \omega^2} \left(\frac{\sigma_2 - \sigma_1}{\sigma_1}\right) \qquad (28)$$

Because σ_2 was chosen as the larger of the two poles k will be positive. There are two cases to consider: (a) if in the original expansion in Eq. (2) $k_o > 1$, then the condition on k is $k > 0$; (b) if, on the other hand $k_o < 1$, then $k > 1 - k_o$. In case (a) Eq. (28) shows that it will always be satisfied. In case (b) the requirement becomes

$$1 - k_o < \frac{\omega^2 k_2}{\sigma_2^2 + \omega^2} \left(\frac{\sigma_2}{\sigma_1} - 1\right) \tag{29}$$

Turning back to Eq. (27), since the minimum value of k_1 is k_1'', we can now write

$$g - G_3 - k_2 \left(\frac{1 + \omega^2/\sigma_1\sigma_2}{1 + \omega^2/\sigma_2^2}\right) > k_1'' \tag{30}$$

This can be solved for k_2; inserting also the lower bound on k_2 leads to

$$k_2'' < k_2 < \left(\frac{1 + \omega^2/\sigma_2^2}{1 + \omega^2/\sigma_1\sigma_2}\right) (g - G_3 - k_1'') \tag{31}$$

It should be kept in mind that the k_i's cannot actually take on their minimum values since Eq. (6) requires the strict inequality. Hence, the absolute minimum sensitivity cannot be achieved. However, it is possible to come within any desired degree of the minimum.

For $n > 2$ all but the first two k_i's are arbitrarily fixed (at values greater than their lower bounds) and the process just carried out repeated. However, now when solving for k_1 there will be additional terms of the form

$$- \sum_{i=3}^{n} \sigma_i k_i / (\sigma_i^2 + \omega^2)$$

on the right side of the first of Eqs. (26) and

$$- \sum_{i=3}^{n} \sigma_i^2 k_i / (\sigma_i^2 + \omega^2)$$

on the right aide of the second. The result will be to subtract from the solution for k_1 additional terms like the second one on the right in Eq. (27). Thus,

$$k_1 = (g - G_3) - k_2 \left(\frac{1 + \omega^2/\sigma_1\sigma_2}{1 + \omega^2/\sigma_2^2}\right) - k_3 \left(\frac{1 + \omega^2/\sigma_1\sigma_3}{1 + \omega^2/\sigma_3^2}\right) - \cdot \cdot \cdot \cdot > k_1'' \qquad (32)$$

Again solving for k_2 and inserting its lower bound leads to

$$k_2'' < k_2 < \left(\frac{1 + \omega^2/\sigma_2^2}{1 + \omega^2/\sigma_1\sigma_2}\right) \left(g - G_3 - k_1'' - k_3 \frac{1 + \omega^2/\sigma_1\sigma_3}{1 + \omega^2/\sigma_3^2} - k_4 \frac{1 + \omega^2/\sigma_1\sigma_4}{1 + \omega^2/\sigma_4^2} \ldots\right) \qquad (33)$$

This condition, of course, is much more difficult to interpret.

Note that the expressions involving the k_i's contain ω as a parameter. When fixing values of the residues a specific value must be assigned to the frequency. An appropriate value might lie in the important frequency range determined by the imaginary parts of the poles of the transfer function.

Up to this point it has been assumed that the poles, σ_i, were fixed as well as the constant $g - G_3$. As for the latter, note that the realizations will consist of either a single common base stage or a two-stage common emitter amplifier. The values for $g - G_3$ given in Table 1 for these two are approximately 1 and 450, respectively. This constitutes a considerable spread. However, these are normalized values with respect to the G_1 of amplifier, that is, with respect to G_1' in Fig. 4. It is possible to get additional values of $g - G_3$ between 1 and 450 by placing an external conductance G_1'' in parallel with G_1' in the two stage case. The normalization will now be with respect to $G_1 = G_1' + G_1''$. Thus, suppose in a given case it is desired to have a normalized value of 20 for $g - G_3$. A G_1'' satisfying the condition

$$\frac{.135 - 5 \times 10^{-4}}{3 \times 10^{-4} + G_1''} = 20$$

or

$$G_1'' = .00643 \qquad (R_1'' = 155 \text{ ohms})$$

will be required.

As for the poles, no conditions for their selection yet have been discussed. Attention is directed back to Eq. (13). For a fixed set of poles and neglecting the constraint equation (9), the smallest value of f will occur at a given frequency if each k_i takes on its lowest possible value k_i''. Hence, to reduce the value of f we should reduce the values of k_i''. But the k_i'''s are those residues of $Q(s)/D(s)$ which are negative. Thus, the poles are to be so chosen that the negative residues will be relatively small.

One objection to the previous paragraph is that the constraint equation (9) cannot be neglected. For a fixed $g - G_3$, if some of the k_i's are small the others must be relatively large in order for their sum to be constant. However, as noted above, it is possible to vary the value of $g - G_3$ between wide limits, so that there will be no objection to the attempt to make the negative residues as small as possible.

Assuming the poles of the transfer function are all complex, the poles of Q/D leading to negative residues alternate with those leading positive residues. Q/D will have the form

$$\frac{Q(s)}{D(s)} = \frac{\prod [(s + a_i)^2 + b_i^2]}{(s + \sigma_1)(s + \sigma_2) \ldots (s + \sigma_m)} \qquad (34)$$

where the σ_i's are ordered in increasing magnitude.

To achieve a residue of small size, it is desired that the numerator factors be small and the denominator factors be large, when evaluated at $s = -\sigma_i$. Nothing precise can be said on how to choose the σ_i's to achieve this result. However, it is noted that a factor of the numerator will have its smallest value when a $-\sigma_i = a_i$; that is, a root of D(s) is equal to the real part of a root of Q(s). This choice will be most effective in reducing the residue if the corresponding imaginary part of the root of Q is relatively small. However, it will be relatively ineffective when the imaginary part is large. The denominator factors will have large values if the σ_i's are far apart.

Thus, in choosing the roots of D(s) the guidlines are:

1. Choose some roots - σ_{in} in the vicinity of the real parts of the zeros of Q(s).

2. Choose additional roots, $-\sigma_{ip}$, lying between these, the one closest to the origin being $-\sigma_{in}$.

3. Arrange the separation between σ_{in} and σ_{pn} such that the difference between adjacent ones is as large as possible.

After the choice of the σ_i's and the calculations of the residues of Q/D, the summation

$$1 - k_o + \sum k_i"$$

is formed. (Assuming $k_o < 1$; otherwise the term $1 - k_o$ can be omitted.) This is the greatest lower bound of the right side of Eq. (9). If this quantity is less than 1, then, according to Table 1, a single stage common base amplifier will be suitable as the active device. If this quantity lies between 1 and 450, a two stage common emitter amplifier can be used.

In the former case smaller values of the k_i's can be used in Eq. (13). Hence, the sensitivity in this case will be smaller than in the case requiring a two stage common emitter amplifier. In this latter contingency, it is possible to improve the sensitivity by choosing an external conductance in parallel with the G_1 of the amplifier, as previously discussed.

However, note that the maximum value of the external conductance shunting G_1' is $y_{22}(0)$. Choosing the k_i's close to the k_i"'s will reduce $y_{22}(0)$, its lower limit being

$$G_1"_{min} = k - (1 - k_o) + \sum \frac{k_i'}{\sigma_i'} \tag{35}$$

If the required external conductance is less than this amount, then $g - G_3$ can be reduced and the k_i's can be brought close to their minimum values. If not, there will be a limit below which $g - G_3$ cannot be reduced and, hence, a limit to the degree in which all the k_i's can be made to approach their minimum values.

2-3. Example

The following transfer function was considered as an example.

$$H(s) = \frac{P(s)}{Q(s)} = \frac{(s+5)(s+7)}{(s^2+s+2)(s^2+2s+3)}$$

The poles are located at $s = -.5 \pm j1.32$ and $s = -1 \pm j1.414$. Preliminary calculations based on the suggested guidelines lead to the choice $D(s) = (s+1)(s+2)(s+3)(s+4)$ so that Eqs. (3) and (4) become

$$Y(s) = \frac{G_3}{g - G_3}\left(k + \frac{k_1}{s+1} + \frac{k_2}{s+3}\right) \tag{36}$$

$$y_{22} = k - .75 + \frac{3s}{s+2} + \frac{6.42s}{s+4} + \frac{(k_1 - .66)s}{s+1} + \frac{(k_2 - 8)s}{s+3}$$

$\omega = 2$ is chosen as the frequency to be used. It is seen that

$$1 - k_o + k_1'' + k_2'' = 9.41 > 1$$

Hence, a 2 stage realization is required. From Eq. (31) with $\sigma_1 = 1$, $\sigma_2 = 3$, $k_2'' = 8$ and $k_1'' = .666$ it follows that

$$8 < k_2 < 278$$

Choosing $k_1 = k_2 = 25$ gives for k from Eq. (9) the value

$$k = 400$$

With these values the sensitivity was calculated to be

$$\left|S_g^H\right| = 36.7 \qquad \text{at } \omega = 2 \tag{37}$$

No attempt was made to reduce the value of $g - G_3$. Let us now make this attempt.

Since $k_2'' = 8$, let us arbitrarily choose $k_2 = 8.2$, which is close to its minimum value. From Eq. (30) it follows that

$$g - G_3 > .666 + 8.2 \frac{(1 + \omega^2/3)}{1 + \omega^2/9} > 25.3$$

at all frequencies. This will require an external conductance G_1'', whose maximum value is $y_{22}(0)$. From Eq. (36) $y_{22} = (0) = k - .75$. Since k is still unknown, it cannot be determined in an unambiguous way how small to choose $g - G_3$ and still be assured that $y_{22}(0)$ will be large enough to permit the required G_1''. Hence, we choose $g - G_3$ to satisfy the requirement that it be greater than 25.3, and then we compute the remaining parameters. If the required external conductance can be supplied by the resulting y_{22}, the task is complete; if not a larger value of $g - G_3$ will be required.

Let us choose $g - G_3 = 26$. This requires an external conductance

$$G_1'' = \frac{.135 - .0005}{26} - 3 \times 10^{-4} = 4.87 \times 10^{-3}$$

From Eq. (31) at $\omega = 2$ we find

$$.666 < k_2 \frac{1 + 4/3}{1 + 4/9} (26 - 8.2) = 12.8$$

Choosing the highest value $k_2 = 12.8$ leads from Eq. (9) to $k = 5$.

Finally, the sensitivity is calculated to be

$$\left| S_g^H \right| = 1.54 \qquad \text{at } \omega = 2 \tag{38}$$

Comparing with Eq. (37) shows an improvement in sensitivity by a factor of 24.

Using the chosen values of the residues leads to the following functions for Y, y_{22} and y_{21}.

$$Y = .00107 + \frac{.00274s}{s + 1} + \frac{.00175s}{s + 3}$$

$$y_{22} = 4.25 + \frac{3s}{s + 2} + \frac{6.42s}{s + 4} + \frac{11.14s}{s + 1} + \frac{.2s}{s + 3} \qquad (39)$$

$$-y_{21} = \frac{(s + 5)\,(s + 7)}{(s + 1)(s + 2)(s + 3)(s + 4)}$$

Since all the transmission zeros are negative real, or at infinity, a ladder network will realize y_{22} and y_{21}. The realization is shown in Fig. 5.

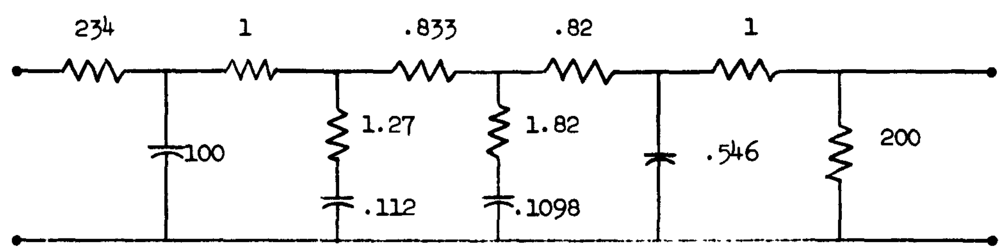
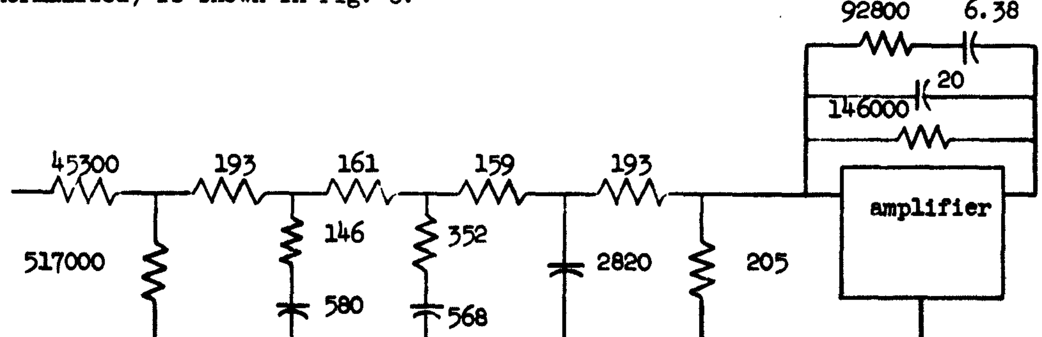


Fig. 5.

Recall that an external conductance $G_1'' = 4.87 \times 10^{-3}$ is required. The 200 ohm resistor (conductance = 5×10^{-3}) at the right can supply this conductance. The normalizing conductance will be

$$G_1 = G_1' + G_1'' = .3 \times 10^{-3} + 4.87 \times 10^{-3} = 5.17 \times 10^{-3} .$$

To denormalize, all capacitances should be multiplied by this value and all resistances should be divided by this value. The complete structure (denormalized) is shown in Fig. 6.



values in ohms and picofarads

Fig. 6.

CHAPTER 3

REALIZATION TWO

3-1. Summary of the Realization Technique

The second realization procedure to be considered is due to Horowitz[3,4]. The desired structure is shown in Fig. 7. The specified function is a

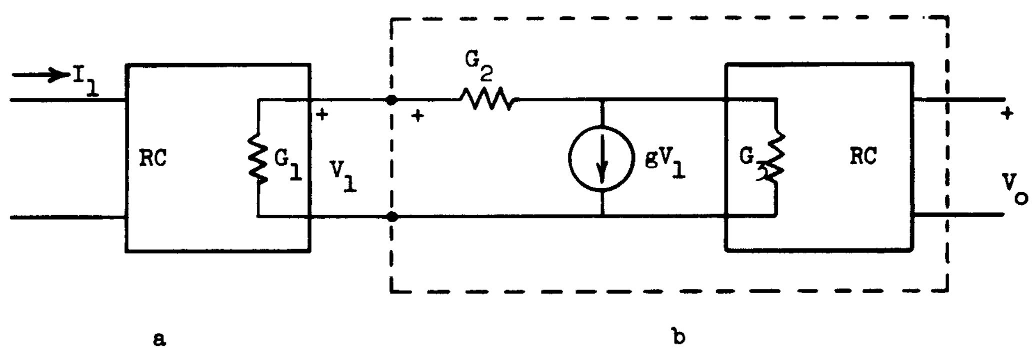


Figure 7.

transfer impedance $z_{21} = V_o/I_1$. This is to be realized as a cascade of two networks, the left hand one being passive RC, the right hand one (inside the dashed lines) being active RC with the specific structure shown. If this structure can be achieved, the result can be redrawn as in Fig. 8. Each of the three conductances has been replaced by two in parallel. The structure within the dashed lines in this figure is the equivalent circuit of an active device. Because of the reference of the controlled source the device may be a single common emitter stage transistor amplifier, or a more extensive amplifier. The double primed conductors are external to the amplifier.

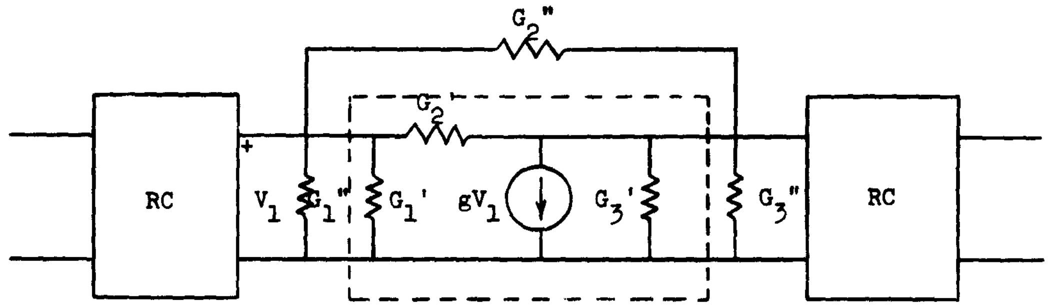


Figure 8.

The transfer impedance of two cascaded networks can be written as

$$z_{21} = \frac{P(s)}{Q(s)} = \frac{z_{21a}\ z_{21b}}{z_{22a} + z_{11b}} \tag{40}$$

where the subscripts refer to the a and b networks in Fig. 7. For a given rational function P/Q, numerator and denominator are again divided by a polynomial D(s) of degree equal to that of Q(s) having negative real zeros only, and the resulting denominator is expanded in partial fractions. Then,

$$\frac{Q(s)}{D(s)} = 1 - K + \sum \frac{k_i'}{s + \sigma_i'} + \left(K - \sum \frac{k_i}{s + \sigma_i}\right) \tag{41}$$

where a constant $K < 1$ has been added and subtracted. The quantity within the parentheses will be positive real (realizable as an RL impedance) only if K is greater than the zero-frequency value of the remaining terms.

$$\left(K > \sum k_i/\sigma_i\right)$$

Since the maximum value of K is 1, a fundamental limitation on the procedure is the fact that the zero frequency value of the sum of the terms having negative residues in Q/D must not exceed unity.

The following identifications can now be made.

$$z_{22a} = 1 - K + \sum \frac{k_i'}{s + \sigma_i'} , \quad z_{21a} = \frac{P_1(s)}{D_1(s)} \tag{42}$$

$$z_{11b} = K - \sum \frac{k_i}{s + \sigma_i} , \quad z_{21b} = \frac{P_2(s)}{D_2(s)} \tag{43}$$

where $P_1P_2 = P$ and $D_1D_2 = D$ with

$$D_1(s) = \sum (s + \sigma_i')$$

$$D_2(s) = \sum (s + \sigma_i) \tag{44}$$

Thus z_{21a} has the same poles as z_{22a} and z_{21b} has the same poles as z_{11b}. The zeros are assigned to z_{11a} and z_{21b} consistent with realizability.

Horowitz[3] considered transfer functions with zeros at infinity only. Balabanian[4] disucssed the incorporation of finite transmission zeros in the a network. The only restriction on the realizability of finite transmission zeros in either the a or the b network is the requirement that a transfer impedance (z_{21a} and z_{21b}) have no pole at infinite, since the corresponding driving point impedance cannot. Since a complex pair of zeros cannot be separated, the assignment of such a pair of zeros to either network requires that the impedances of that network have at least two poles. For a biquadratic transfer function for example, D(s) will have two roots and Q/D will have one positive residue and one negative one. Thus, both z_{11b} and z_{22a} will have but one pole.

$$\frac{P}{Q} = \frac{s^2 + cs + d}{s^2 + as + b}$$

$$\frac{Q}{D} = \frac{s^2 + as + b}{(s + \sigma_1')(s + \sigma_1)} = (1 - K + \frac{k_1'}{s + \sigma_1'}) + (K - \frac{k_1}{s + \sigma_1}) \tag{45}$$

Assigning the complex pair of zeros to either of the networks will cause the corresponding transfer impedance to have a pole at infinity. It is still possible to overcome this difficulty by adding and subtracting a term $A/(s + \sigma_1)$ to Eq. (45), after which the following identifications can be made.

$$z_{22a} = 1 - K + \frac{k_1'}{s + \sigma_1'} + \frac{A}{s + \sigma_1} \tag{46}$$

$$z_{11b} = K - \frac{k_1 + A}{s + \sigma_1} \tag{47}$$

now

$$z_{21a} = \frac{s^2 + cs + d}{(s + \sigma_1')(s + \sigma_1)} \quad ; \quad z_{21b} = \frac{1}{s + \sigma_1} \tag{48}$$

The realizability condition on K simply to make z_{11b} realizable is $K > k_1/\sigma_1$. However, to make the overall transfer function realizable now requires $K > (k_1 + A)/\sigma_1$.

A contemplation of higher order transfer functions quickly shows that the biquadratic is the only case in which the difficulty under discussion can arise. Thus, a biquadratic (with complex zeros and poles) will lead to a Q/D function having two positive and two negative residues. So each complex pair of zeros can be assigned to one of the two subnetworks.

In the discussion following Eq. (41) it was indicated that the realizabion procedure under discussion is not general. Callahan,[2] in considering the sensitivity minimization problem, has stated the following compact realizability condition. Let the transfer function have complex poles only and let $-s_i$ be the ith pole in the second quadrant. Then, the condition for realizability is

$$\arg s_i \leq \frac{\pi}{2} \tag{49}$$

That is, the sum of the angles made by the second quadrant poles with the negative real axis should not exceed $\pi/2$ radians. Turning now to the detailed realization, routine analysis of the b network, shown again in Fig. 9, leads to

$$z_{11}' = \frac{R_2 - z_{11b}}{gz_{11b} - 1} \tag{50}$$

$$z_{21}' = \frac{z_{21b}}{gz_{11b} - 1} \tag{51}$$

where the primed parameters refer to the part of the network within the box. Since z_{11b} and z_{21b} have the same poles, we see that z_{11}' and z_{21}' also have the same poles, and the transmission zeros of the part of the network in the box is the same as the zeros of the overall b network. Thus, once z_{11}' is found, z_{21}' is formed by giving it the same poles as z_{11}' and the zeros of z_{21b}.

Since z_{11b} is an RL impedance function a sketch of $z_{11b}(\sigma)$ will have the form shown in Fig. 10. From Eq. (50) it is seen that the zeros of z_{11}' occur when $R_2 = z_{11b}$ and the poles of z_{11}' occur when $1/g = z_{11b}$. If

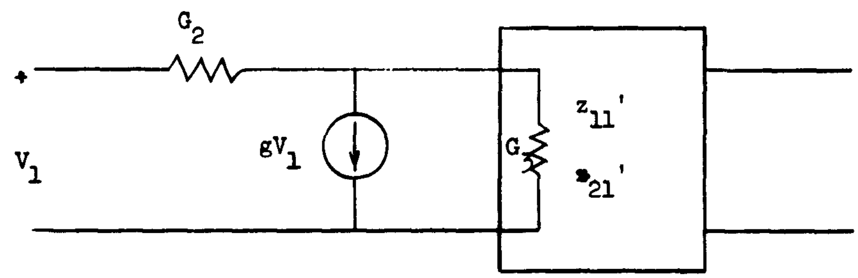


Figure 9.

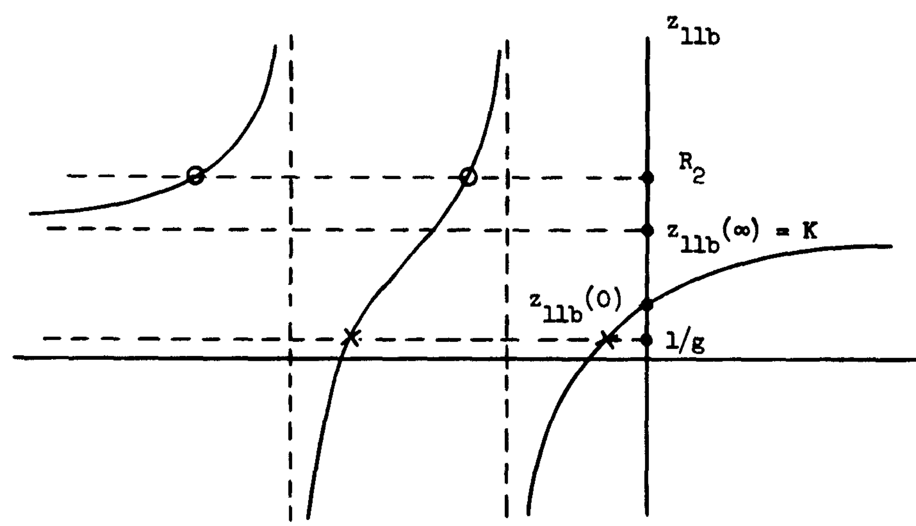


Figure 10.

$$R_2 \geq K = z_{11b}(\infty) \qquad \text{or} \qquad G_2 \leq \frac{1}{z_{11b}(\infty)} \tag{52}$$

$$\frac{1}{g} \leq z_{11b}(0) \qquad \text{or} \qquad g \geq \frac{1}{z_{11b}(0)}$$

then the poles and zeros of z_{11}' will be negative real, they will alternate with each other, and the one nearest the origin will be a pole. Hence, z_{11}' will be an RC impedance.

The remainder of the realization consists of realizing the RC a and b primed networks, given z_{22a}, z_{21a} and z_{11}', z_{21}', which is a straightforward task. Note from Eq. (50) that if z_{11b} has a pole at infinity (which is permissible for an RL impedance), then z_{11}' will be negative at infinity and hence will not be realizable. This is the reason for the original stipulation that D(s) be of the same degree as Q(s).

3-2. Limitations Due to Transistor Parameters

It remains to consider the limitations imposed by realizable values of transistor parameters. Note that for a given z_{11b}, a g can be chosen to satisfy the restriction in Eq. (52). If the active device is to be a single common emitter stage, this value, together with the value in Table 1, will fix the admittance level. The normalizing conductance will be $G_o = .01/g = 1/100g$. Since the G_1, G_2 and G_3 conductances are to be the parallel combinations of the corresponding transistor conductances together with external conductances, they will be greater than the corresponding normalized transistor conductances. Thus,

$$G_1 \geq \frac{2 \times 10^{-4}}{G_o} = .02g \tag{53}$$

$$G_2 \geq \frac{1.25 \times 10^{-7}}{G_o} = 1.25 \times 10^{-5}g \tag{54}$$

$$G_3 \geq \frac{7 \times 10^{-6}}{G_o} = 7 \times 10^{-4}g \tag{55}$$

By combining the above condition on G_2 with the conditions in Eq. (52) there results

$$\frac{1.25 \times 10^{-5}}{z_{11b}(0)} \leq 1.25 \times 10^{-5} g \leq G_2 \leq \frac{1}{z_{11b}(\infty)}$$

or

$$z_{11b}(0) > 1.25 \times 10^{-5}\ z_{11b}(\infty) \tag{56}$$

This appears to be an extremely loose restriction. In fact using Eq. (43) for z_{11b}, the condition becomes

$$K > (1 + 1.25 \times 10^{-5}) \sum \frac{k_i}{\sigma_i} \tag{57}$$

Comparing this with the previous realizability conditions on K

$$(K > \sum k_i/\sigma_i)$$

it is clear that the additional restriction for realizability in a single common emitter stage is negligible. The conclusion is that if the given function can be realized in the comtemplated structure at all, it can be realized with a single common emitter stage as far as the required g and G_2 are concerned.

It still remains to discuss the restrictions on G_1 and G_3, the shunt branches in the equivalent circuit. Note from Fig. 7 that these conductances are within the primed b and the a networks. Thus,

$$G_3 = \frac{1}{z_{11}'(0)} = \frac{g z_{11b}(0) - 1}{R_2 - z_{11b}(0)} \tag{58}$$

$$G_1 = \frac{1}{z_{22a}(0)} \tag{59}$$

Combining the first of these with Eq. (55) leads to

$$\frac{g z_{11b}(0) - 1}{R_2 - z_{11b}(0)} \geq 7 \times 10^{-4} g \tag{60}$$

For a fixed g the left side will take on its largest value if R_2 takes on its smallest value, which from Eq. (52) is $z_{11b}(\infty)$. Using this value in (60) and rearranging leads to

$$(1 + 7 \times 10^{-4}) z_{11b}(0) - 7 \times 10^{-4} \, z_{11b}(\infty) > \frac{1}{g} \tag{61}$$

It is always possible to satisfy this expression by adjusting g, provided the left hand side is positive. Thus,

$$\frac{z_{11b}(\infty)}{z_{11b}(0)} \leq 1430 \tag{62}$$

Finally, combining Eq. (59) with (53) and (52) leads to the condition

$$\frac{z_{22a}(0)}{z_{11b}(0)} \leq 50 \tag{63}$$

Using Eqs. (42) and (43) in this condition leads to

$$K \geq \sum \frac{k_i}{\sigma_i} + \frac{1}{51} \left(1 + \sum \frac{k_i'}{\sigma_i'} - \sum \frac{k_i}{\sigma_i}\right)$$

or

$$K \geq \sum \frac{k_i}{\sigma_i} + \frac{1}{51} \frac{Q(0)}{D(0)} \tag{64}$$

This is a stronger condition than the previous condition on K

$$(K \geq \sum k_i/\sigma_i)$$

Thus conditions (62) and (63) (or (64)) consititute the restrictions that might prevent realizability with a single common emitter stage, assuming the given function is realizable in the contemplated structure.

3-3. Effect of Reversing Transistor

Consider the configuration shown in Fig. 11, which differs from that in Fig. 9 by the position of R_2 relative to the controlled source. Routine

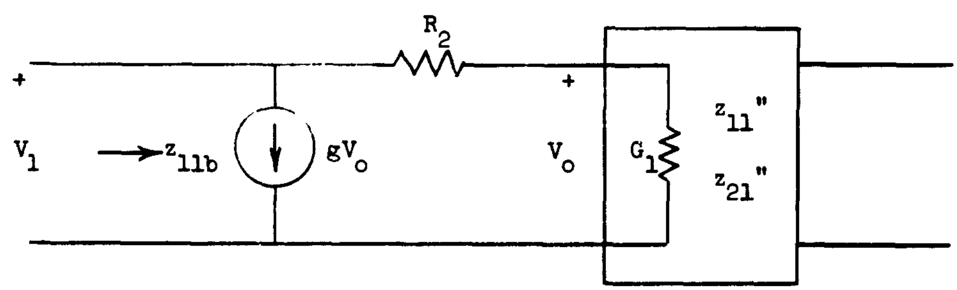


Figure 11.

analysis yields for the double primed parameters of the network on the right

$$z_{11}'' = \frac{R_2 - z_{11b}}{gz_{11b} - 1} \tag{65}$$

$$z_{21}'' = (gR_2 - 1)\frac{z_{21b}}{gz_{11b} - 1} \tag{66}$$

It is seen that z_{11}'' is the same as z_{11}' in Eq. (50) and that z_{21}'' differs z_{21}' in Eq. (51) by a multiplicative constant $(gR_2 - 1)$. As noted in Eq. (52), $(gR_2 - 1)$ is a positive number. In the usual methods of realizing the passive RC networks, the realization is achieved to within a multiplicative gain constant anyway. Hence, the only difference in the two configurations will be the gain levels.

Equations (58) and (59) can still be written but with G_1 and G_3 interchanged. The base and collector terminals of the transistor equivalent circuit will be interchanged. Thus, from Eqs. (58) and (53) for the realizability of the new G_1

$$G_1 = \frac{g z_{11b}(0) - 1}{R_2 - z_{11b}(0)} \geq .02g \tag{67}$$

Again inserting the minimum value of R_2 leads to

$$51 z_{11b}(0) - z_{11b}(\infty) \geq \frac{50}{g} \tag{68}$$

This condition can be satisfied by varying g, provided the left side is positive. Thus, we require

$$\frac{z_{11b}(\infty)}{z_{11b}(0)} \leq 51 \tag{69}$$

This is to be compared with Eq. (62). The present condition on the realizability of G_1 is more stringent than with transistor in the original position.

Similarly, from Eqs. (59) and (55) for the realizability of the new G_3 we find

$$\frac{z_{22a}(0)}{z_{11b}(0)} \leq \frac{10^4}{7} \tag{70}$$

This is to be compared with Eq. (63). Using Eqs. (42) and (43) for z_{11b} and z_{22a}, this condition becomes

$$K \geq \sum \frac{k_i}{\sigma_i} + 7 \times 10^{-4} \frac{Q(0)}{D(0)} \tag{71}$$

This is to be compared with Eq. (64). The comparison indicates that the realizability condition on G_3 is less stringent with the transistor reversed. Thus, it appears that reversing the transistor causes the realizability condition on one of the shunt conductances to be tightened while that on the other conductance is released. If it is found that one or the other of the two conductances cannot be realized with the transistor in one configuration, a reversal might permit realization.

The realizability conditions and values of the parameters for the two single stage common emitter configurations are tabulated in Table 3.

3-4. Sensitivity Considerations

In the last chapter we concerned ourselves with the sensitivity of the transfer function to changes of the parameter g in the equivalent circuit of the active device. In the configuration under discussion here, again there is the possibility of placing external conductances in parallel with the G_1, G_2 and G_3 conductances in the transistor equivalent circuit. Hence, the active device parameter whose variation will cause the greatest change in the response will be g. However, rather than discussing the sensitivity to g of the transfer function, we shall consider the sensitivity of the

	Configuration 1 Base Terminal on Left	Configuration 2 Base Terminal on Right
For realization of G_1 (base to emitter conductance)	$\frac{z_{22a}(0)}{z_{11b}(0)} \leq 50$	$\frac{z_{11b}(\infty)}{z_{11b}(0)} \leq 51$
For realization of G_3 (collector to emitter conductance)	$\frac{z_{11b}(\infty)}{z_{11b}(0)} \leq 1430$	$\frac{z_{22a}(0)}{z_{11b}(0)} \leq 1429$
g	$\geq \frac{1}{z_{11b}(0)}$	$\geq \frac{1}{z_{11b}(0)}$
R_2	$\geq z_{11b}(\infty)$	$\geq z_{11b}(\infty)$
G_1	$\frac{1}{z_{22a}(0)} \geq .02g$	$\frac{gz_{11b}(0) - 1}{R_2 - z_{11b}(0)} \geq .02g$
G_3	$\frac{gz_{11b}(0) - 1}{R_2 - z_{11b}(0)} \geq 7 \times 10^{-4}g$	$\frac{1}{z_{22a}(0)} \geq 7 \times 10^{-4}g$

Table 3.

poles, $S_g^{s_1}$. In the case of the cascade negative impedance converter method of synthesis it has been shown by Horowitz[1] and Callahan[6] that minimizing the one also minimizes the other.

In treating the problem of sensitivity minimization Callahan[2] has written the Q/D in Eq. (41) as follows.

$$\frac{Q(s)}{D(s)} = \frac{AQ_1}{D_1} + \frac{kBQ_2}{D_2} = z_{22a} + kz_{11b} \qquad (72)$$

where the positive and negative residue terms have been grouped together and the coefficients of the highest power terms have been explicitly shown as A and B. k is a fictitious parameter whose value is +1; it is analogous to the NIC conversion ratio in those methods that use a negative impedance converter. (Actually Callahan associated the k with the other term but the result is the same.) By regrouping the terms Q(s) can be written

$$Q(s) = AQ_1D_2 + kBQ_2D_1 = AN_1(s) + kBN_2(s) \qquad (73)$$

where N_1 and N_2 are clearly polynomials with negative real zeros only. Letting $-s_1$ be one of the zeros of Q(s) and $(s + s_1)$ one of its factors, write

$$Q(s) = (s + s_1)\, Q'(s) \qquad (74)$$

Equating the right sides of the last two expressions and solving for $-s_1$ leads to

$$-s_1 = -\frac{AN_1(s) + kBN_2(s)}{Q'(s)} + s \qquad (75)$$

The sensitivity of $-s_1$ to k can now be found as

$$S_k^{-s_i} = \frac{-ds_i}{d(\ln k)} = \frac{-kBN_2(s)}{Q'(s)}\Bigg|_{s=-s_i} \tag{76}$$

Note that if k (whose nominal value is +1) is not introduced at all, and the pole sensitivity to the multiplier B of the polynomial $N_2(s)$ is determined, the result will be the same as Eq. (76) (after setting k = 1). That is,

$$S_B^{-s_i} = S_k^{-s_i} \tag{77}$$

Callahan has shown that the minimum value of the magnitude of $S_k^{-s_i}$ is obtained if all the zeros of both $N_1(s)$ and $N_2(s)$ are double and they alternate, the one closest to the origin being a zero of N_2. Thus, for minimum pole sensitivity Q(s) has the form

$$Q(s) = A(s+a_1)^2\ (s+a_2)^2\ \ldots\ (s+a_m)^2 + B(s+b_1)^2\ \ldots\ (s+b_m)^2 \tag{78}$$

and there are an infinite number of such decompositions. Since Q/D is to be the sum of an RC and an RL impedance, the polynomial D(s) must be

$$D(s) = (s+a_1)(s+a_2)\ \ldots\ (s+a_m)(s+b_1)(s+b_2)\ \ldots\ (s+b_m) \tag{79}$$

and the two impedances will be

$$z_{22a} = \frac{A(s+a_1)(s+a_2)\ \ldots\ (s+a_m)}{(s+b_1)(s+b_2)\ \ldots\ (s+b_m)} \tag{80}$$

$$z_{11b} = \frac{B(s+b_1)(s+b_2)\ \ldots\ (s+b_m)}{(s+a_1)(s+a_2)\ \ldots\ (s+a_m)} = \frac{AB}{z_{22a}} \tag{81}$$

It should be noted that $B = z_{11b}(\infty)$, which is the constant K used before. From here on K will be inserted for B.

In Eq. (78) the two polynomials N_1 and N_2 have the same number of double zeros. Callahan has also shown that there is a unique decomposition, having the same minimum sensitivity, which has the same form as Eq. (76) but in which N_1 has a double zero less than N_2. He refers to this as the "optimum decomposition." However, it can easily be seen that, although a polynomial Q(s) can be decomposed in the stated form, the contemplated realization cannot be carried out. Thus, if the last factor in $N_1(s)$ is missing, z_{11b} in Eq. (81) will have a pole at infinity. Whereas this is permissible for an RL impedance, it will lead to an unrealizable z_{11}', as mentioned earlier. Hence, this optimum decomposition cannot be realized in the structure under discussion.

But what is of greater significance is the observation that sensitivity minimization with respect to a parameter k, which does not correspond to anything specific actually varying, does not give an adequate measure of the performance of the realization. Since g is the varying parameter, the significant sensitivity is $S_g^{-s_i}$.

$$S_g^{-s_i} = -g\frac{\partial s_i}{\partial g} = -\left(K\frac{\partial s_i}{\partial K}\right)\left(\frac{\partial K}{\partial g}\ \frac{g}{K}\right) = S_K^{-s_i}\ S_g^k \tag{82}$$

To find the dependence of K($= z_{11b}(\infty)$) on g, note from Eq. (50) that

$$z_{11}'(\infty) = \frac{R_2 - K}{gK - 1} \tag{83}$$

or

$$K = \frac{R_2 + z_{11}'(\infty)}{1 + gz_{11}'(\infty)} \tag{84}$$

From the last expression we find

$$S_g^K = \frac{-gz_{11}'(\infty)}{1 + gz_{11}'(\infty)} = - g \frac{R_2 - K}{gR_2 - 1} \tag{85}$$

The last step follows from Eq. (83).

Finally, inserting Eqs. (76), (77) and (85) into (82) we find

$$S_g^{-s_i} = gK \left(\frac{R_2 - K}{gR_2 - 1}\right) \left. \frac{N_2(s)}{Q'(s)} \right|_{s=-s_i} \tag{86}$$

Recall that gR_2 is always greater than 1 and that $R_2 \geq K$. Thus, it is possible to make the sensitivity of the poles to variations in g as small as desired by choosing R_2 sufficiently close to K. Furthermore, this can be done regardless of previous minimization of $-s_i$ with respect to changes in K.

Hence, it appears that any effort expended on obtaining a decomposition that minimizes the sensitivity to K can be saved, since the minimization of the sensitivity to g can be achieved by proper choice of R_2. However, the closer that R_2 is chosen to K to minimize sensitivity to g, the smaller will be the external conductance shunting it. The benefits of reduced sensitivity to variations in G_2 will be diminished. Hence, it may often be of value to obtain the Callahan decomposition first so that a greater margin in the choice of R_2 will be available for the same sensitivity. (However, the Callahan decomposition so obtained may not be realizable in a single stage, as illustrated below.)

3-5. Examples

It should be noted at the outset that for a biquadratic transfer function with complex zeros the Callahan decomposition is not realizable. This is clear from Eqs. (80) and (81) which show that both z_{11b} and z_{22a} will be bilinear functions and there will be no way to assign the complex zeros to either z_{21a} or z_{21b}.

Consider the transfer impedance

$$z_{21} = \frac{P(s)}{Q(s)} = \frac{s^2 + s + 4}{(s^2 + 2s + 2)(s^2 + \sqrt{3}s + 1)} \tag{87}$$

Q(s) satisfies the angle condition in Eq. (49) and so this function is realizable. Callahan[2] has given the optimum decomposition of Q(s) and a decomposition in the form of Eq. (78) as follows

$$Q(s) = 0.026(s+.857)^2(s+10.1)^2+.974(s+.1)^2(s+1.51)^2 \tag{88}$$

Both lead to the minimum pole sensitivity as defined by Callahan.

$$\left|S_k^{-1+j1}\right| = 1.44 \qquad S_k^{-.866+j.5} = .714 \tag{89}$$

As already pointed out, the optimum decomposition cannot be realized by the structure under consideration.

As for the decomposition in Eq. (88) it leads to the functions

$$z_{11b} = \frac{.974\ (s + .1)\ (s + 1.51)}{(s + .857)(s + 10.1)} \tag{90}$$

$$z_{22a} = \frac{.026\ (s + .857)\ (s + 10.1)}{(s + .1)\ (s + 1.51)} \tag{91}$$

To see if these functions are realizable in a single stage we apply the conditions in Table 3. The important values are

$$z_{11b}(\infty) = .974 \qquad z_{22a}(\infty) = .026$$

$$z_{11b}(0) = .017 \qquad z_{22a}(0) = 1.48$$

For the transistor in configuration 1 we find that, whereas G_3 can be realized, G_1 cannot. With the transistor reversed we find that again G_3 can be realized but not G_1. Hence, this decomposition is not realizable in a single common emitter stage.

Note that although (69) is not satisfied, the violation is not very great since $z_{11b}(\infty)/z_{11b}(0) = 57$. If it is possible to increase $z_{11b}(0)$ without at the same time modifying the remaining values significantly, it may be possible to achieve a realization. With this thought as guide, the following was obtained.

$$\frac{Q(s)}{D(s)} = \frac{(s^2+2s+2)(s^2+\sqrt{3}s+1)}{(s+.05)(s+.85)(s+1.6)(s+11)}$$

$$= .015 + \frac{.1177}{s+.05} + \frac{.0982}{s+1.6} + \left(.985 - \frac{.042}{s+.85} - \frac{9.95}{s + 11}\right) \tag{92}$$

From which we find

$$z_{11b} = .985 - \frac{.042}{s + .85} - \frac{9.95}{s + 11} = \frac{.985(s + .194)(s + 1.512)}{(s + .85)(s + 11)} \tag{93}$$

$$z_{22a} = .015 + \frac{.1177}{s+.05} + \frac{.0982}{s+1.6} = \frac{.015\ (s + .86)(s + 15.16)}{(s + .05)(s + 1.6)} \tag{94}$$

The infinite and zero frequency values are found to be

$$z_{11b}(0) = .0305 \qquad z_{22a}(\infty) = .015$$

$$z_{11b}(\infty) = .985 \qquad z_{22a}(0) = 2.43$$

From these values it is seen from Table 3 that conditions on both G_1 and G_3 in configuration 2 are satisfied. Hence, a single stage realization is possible.

Turning to the sensitivity, from Eqs. (76) and (77) it is found that

$$\left| S_k^{-1+j1} \right| = 1.03 \times 1.44 \tag{95}$$

which is only three percent greater than the Callahan minimum. However, from Eq. (86) the sensitivity of the same pole to g is

$$\left| S_g^{-1+j1} \right| = (1.03)(1.44)\ g\ \frac{(R_2 - .985)}{gR_2 - 1} \tag{96}$$

From Table 3 we find that g must be chosen greater than 33. However, the condition on G_1 from Table 3 is seen to be

$$G_1 = \frac{.0305g - 1}{R_2 - .0305} > .02g$$

Since the minimum value of R_2 is .985, the denominator of this expression can be in the neighborhood of 1. Hence, we find $g > 100$ is required. Choose $g = 150$, $R_2 = 1.0305$. With these choices, the pole sensitivity becomes

$$\left| S_g^{-1+j1} \right| = (.046)\ (1.44) \tag{97}$$

which is less than 5 percent of the Callahan sensitivity.

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