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ON SLENDER BODIES OF MINIMUM DRAG IN NEWTONIAN FLOW

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ON SLENDER BODIES OF MINIMUM DRAG IN NEWTONIAN FLOW

by

DAVID G. HULL(*)

SUMMARY

In recent papers by Miele (Refs. 1 through 3), the problem of minimizing the pressure drag of slender bodies in Newtonian flow was considered in general and, then, solved for those particular cases in which, of the four geometric properties being considered (thickness, length, enclosed area, and moment of inertia of the contour for two-dimensional shapes and diameter, length, wetted area, and volume for axisymmetric shapes), two are prescribed and the remaining two are free. In this paper, the analysis of Refs. 1 through 3 is extended to the class of problems in which three quantities are prescribed and the remaining is free. After the variational problem is reformulated in order to account for the fact that the pressure coefficient must be nonnegative everywhere, special attention is devoted to those particular cases in which two of the three prescribed quantities are the thickness and the length. In each case, a one-parameter family of extremal solutions is obtained, the parameter being related to the three prescribed quantities. Furthermore, each family of extremal solutions contains three classes of body shapes: (I) an infinitely thin plate or a spike followed by a regular shape, (II) a regular shape only, and (III) a regular shape followed by a constant thickness contour or a cylinder. In all of the cases considered,

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analytical expressions are obtained for the geometry of the optimum shapes and the associated drag coefficients.

1. INTRODUCTION

In the previous reports by Miele (Refs. 1 and 2), the problem of minimizing the pressure drag of slender bodies in Newtonian flow was formulated in general and solved for those particular cases in which, of the four geometric properties being considered (thickness d, length ℓ, enclosed area A, and moment of inertia of the contour M for two-dimensional shapes and thickness d, length ℓ, wetted area S, and volume V for axisymmetric shapes), two are prescribed and the remaining two are free. In this report, the analysis of Refs. 1 and 2 is extended in order to cover the case in which three of the geometric properties are specified and only one is free. However, in order to do so, it is necessary to reformulate the variational problem and include the condition that the pressure coefficient must be nonnegative everywhere. In the interest of brevity, the approach of Ref. 3 is employed, that is, the two-dimensional problem and the axisymmetric problem are considered simultaneously. In particular, attention is devoted to those cases in which two of the three prescribed quantities are the thickness and the length.

2. MINIMUM DRAG PROBLEM

According to Refs. 1 through 3, the Newtonian pressure coefficient for a slender body in an inviscid hypersonic flow at zero angle of attack is given by

$$C_p = 2\dot{y}^2 \tag{1}$$

where x denotes a coordinate in the flow direction, y a coordinate normal to the flow direction, and y the derivative dy/dx. With reference to the portion of the body between stations 0 and x, the drag per unit span, the enclosed area, and the moment of inertia of the contour of a two-dimensional shape and the drag, the wetted area, and the volume of an axisymmetric shape can be written as

Two-dimensional case	Axisymmetric case	
$D(x) = 4q \int_0^x \dot{y}^3 \, dx$	$D(x) = 4\pi q \int_0^x y\dot{y}^3 \, dx$	
$A(x) = 2 \int_0^x y \, dx$	$S(x) = 2\pi \int_0^x y \, dx$	(2)
$M(x) = 2 \int_0^x y^2 \, dx$	$V(x) = \pi \int_0^x y^2 \, dx$	

After the definitions

Two-dimensional case	Axisymmetric case	
$\alpha = D(x)/4q$	$\alpha = D(x)/4\pi q$	
$\beta = A(x)/2$	$\beta = S(x)/2\pi$	(3)
$\gamma = M(x)/2$	$\gamma = V(x)/\pi$	

are introduced and after both sides of Eqs. (2) are differentiated with respect to the independent variable, the following differential constraints are obtained (Ref. 3):

$$\dot{\alpha} - y^n \dot{y}^3 = 0$$
$$\dot{\beta} - y = 0 \qquad (4)$$
$$\dot{\gamma} - y^2 = 0$$

where n = 0 in the two-dimensional case and n = 1 in the axisymmetric case. Since the condition that the pressure coefficient must be nonnegative can be expressed in the form

$$\dot{y} - p^2 = 0 \qquad (5)$$

where p denotes a real variable, the system of differential equations (4) and (5) has one independent variable (x), five dependent variables (α, β, γ, y, p), and one degree of freedom. Consequently, if it is assumed that

$$x_i = y_i = \alpha_i = \beta_i = \gamma_i = 0 \qquad (6)$$

and that some, but not all, of the remaining state variables are specified at the final point, the minimum drag problem is stated as follows: <u>In the class of functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $y(x)$, $p(x)$ which are consistent with the differential constraints (4) and (5) and the initial conditions (6), find that special set which minimizes the difference $\Delta G = G_f - G_i$, where $G = \alpha$.</u>

3. NECESSARY CONDITIONS

This problem is one of the Mayer type with separated end-conditions, so that, after the Lagrange multipliers λ_1 through λ_4 are introduced, the extremal arc is governed by the following Euler-Lagrange equations (Refs. 8 and 9):

$$\dot{\lambda}_1 = 0$$

$$\dot{\lambda}_2 = 0$$

$$\dot{\lambda}_3 = 0 \tag{7}$$

$$\frac{d}{dx}(-3\lambda_1 y^n \dot{y}^2 + \lambda_4) = -\lambda_1 n y^{n-1} \dot{y}^3 - \lambda_2 - 2\lambda_3 y$$

$$0 = -2\lambda_4 p$$

Integration of the first three equations leads to the results

$$\lambda_1 = C_1 , \quad \lambda_2 = C_2 , \quad \lambda_3 = C_3 \tag{8}$$

where C_1, C_2, C_3 are constants. Furthermore, since the independent variable does not appear explicitly in the constraining equations, the additional first integral

$$-2C_1 y^n \dot{y}^3 + C_2 y + C_3 y^2 = C \tag{9}$$

is valid, where C is a constant.

As the fifth Euler equation indicates, the extremal arc is generally discontinuous and is composed of the subarcs

$$\lambda_4 = 0 \quad \text{and/or} \quad p = 0 \tag{10}$$

Along the former subarcs, called regular shapes, the pressure coefficient is always positive or, at most, zero at a single point. Along the latter subarcs, the slope of the body is always zero; hence, they are called zero-slope shapes.

When a discontinuity occurs, the Erdmann-Weierstrass corner conditions must be applied. They require that the constants C_1, C_2, C_3, C have the same value for all of the subarcs included in the extremal arc and that

$$\Delta(-3\lambda_1 y^n \dot{y}^2 + \lambda_4) = \Delta(y^n \dot{y}^3) = 0 \tag{11}$$

Consequently, Eqs. (11) imply that $\lambda_4 = 0$ at a corner point. Furthermore, a discontinuity in the slope is not possible in the two-dimensional case ($n = 0$) but is possible in the axisymmetric case ($n = 1$) if $y = 0$.

The end-conditions are partly of a fixed end-point type and partly of the natural type. The latter must be derived from the transversality condition

$$\left[-C dx + (C_1 + 1)\, d\alpha + C_2 d\beta + C_3 d\gamma + (-3C_1 y^n \dot{y}^2 + \lambda_4)\, dy\right]_f = 0 \tag{12}$$

which is to be satisfied for all systems of differentials consistent with the prescribed end-conditions and implies that $C_1 = -1$.

The application of the Legendre-Clebsch condition indicates that the drag is a minimum if the following inequalities are satisfied:

$$\dot{y} \geq 0 \text{ , along the regular shape}$$
$$\lambda_4 \leq 0 \text{ , along the zero-slope shape} \tag{13}$$

From the previous discussion, it appears that the multiplier λ_4 plays an important role in determining the composition of the extremal arc and, hence, is called the <u>switching function</u>. Its properties are summarized as follows:

$$\lambda_4 = 0 \text{ , along a regular shape}$$
$$\lambda_4 \leq 0 \text{ , along a zero-slope shape} \tag{14}$$
$$\lambda_4 = 0 \text{ , at a corner point}$$

4. GENERAL SOLUTION

After introducing the nondimensional variables

$$\xi = \frac{x}{\ell} , \quad \eta = \frac{y}{d/2} \tag{15}$$

and the definitions

$$K_1 = C(d/2)^{-n} , \quad K_2 = C_2(d/2)^{1-n} , \quad K_3 = C_3(d/2)^{2-n} \tag{16}$$

one can rewrite the first integral (9) in the form

$$\frac{\tau^3}{4}\eta^n\dot{\eta}^3 + K_2\eta + K_3\eta^2 = K_1 \tag{17}$$

in which $\tau = d/\ell$ denotes the thickness ratio. Due to the fact that the pressure coefficient must be nonnegative everywhere, the application of this first integral at both end points of the extremal arc leads to the <u>basic inequalities</u>

$$K_1 \geq 0 , \quad K_1 - K_2 - K_3 \geq 0 \tag{18}$$

Furthermore, the general equations for the regular shape and the zero-slope shape can be written as

$$\xi = \frac{\tau}{\sqrt[3]{4}}\int \sqrt[3]{\frac{\eta^n}{K_1 - K_2\eta - K_3\eta^2}}\, d\eta + \text{Const} \tag{19}$$

$$\eta = \text{Const}$$

respectively.

If η_c denotes the ordinate of a corner point between a zero-slope shape and a regular shape, the <u>transition equation</u>

$$K_2\eta_c + K_3\eta_c^2 = K_1 \tag{20}$$

holds. Furthermore, after the nondimensional switching function is defined as

$$\sigma = \lambda_4 \frac{(d/2)^{1-n}}{\ell} \tag{21}$$

the integration of the fourth Euler-Lagrange equation along a zero-slope shape yields the relationship

$$\sigma = K(\xi_c - \xi) \;, \quad K = K_2 + 2K_3\eta_c \tag{22}$$

This equation supplies the proper sequence of subarcs as follows: (a) If $K > 0$, a regular shape precedes the corner point, and a zero-slope shape follows; furthermore, owing to the monotonic nature of the switching function, no subarc may follow the zero-slope shape. (b) If $K < 0$, a zero-slope shape precedes the corner point, and a regular shape follows; also, no subarc may precede the zero-slope shape.

In view of these statements and the transition equation, the maximum possible number of subarcs composing the extremal arc is three (one regular shape and two zero-slope shapes) with one of the two possible corner points lying along $\eta = 0$ and the other, along $\eta = 1$. If the coordinates of these two corner points are denoted by ξ_o, 0 and ξ_1, 1, the general equations for the shape of the optimum body can be expressed as

$$0 \le \xi \le \xi_o \, , \quad \eta = 0$$

$$\xi_o \le \xi \le \xi_1 \, , \quad \frac{\xi - \xi_o}{\xi_1 - \xi_o} = \frac{\displaystyle\int_0^{\eta} \sqrt[3]{\frac{\eta^n}{K_1 - K_2\eta - K_3\eta^2}}\, d\eta}{\displaystyle\int_0^{1} \sqrt[3]{\frac{\eta^n}{K_1 - K_2\eta - K_3\eta^2}}\, d\eta} \tag{23}$$

$$\xi_1 \le \xi \le 1 \, , \quad \eta = 1$$

For extremal arcs involving only one corner point or no corner points at all, the shape of the optimum body can be formally obtained from Eqs. (23) by setting $\xi_o = 0$ and/or $\xi_1 = 1$. It must be understood, however, that the corner conditions need not be satisfied at these points.

Once the shape is known, the following nondimensional integrals can be evaluated:

$$I_D = \int_0^1 \eta^n \dot{\eta}^3 d\xi \, , \; I_A = I_S = \int_0^1 \eta d\xi \, , \quad I_M = I_V = \int_0^1 \eta^2 d\xi \tag{24}$$

so that, after the corresponding dimensional quantities are written as

<u>Two-dimensional case</u>	<u>Axisymmetric case</u>
$D = \frac{qd^3}{2\ell^2} I_D$	$D = \frac{\pi q d^4}{4\ell^2} I_D$
$A = d\ell \, I_A$	$S = \pi d\ell \, I_S$
$M = \frac{d^2\ell}{2} I_M$	$V = \frac{\pi d^2 \ell}{4} I_V$

(25)

it is possible to express every unknown quantity in terms of the known quantities and the nondimensional integrals (24). Finally, if the reference area for the drag coefficient is chosen to be the frontal area evaluated at $x = \ell$, the following relationship can be established between the drag coefficient and the thickness ratio:

$$\frac{C_D}{\tau^2} = \frac{I_D}{2 - n} \qquad (26)$$

5. OPTIMUM TWO-DIMENSIONAL BODIES

In the previous sections, the minimum drag problem was solved in general for arbitrary boundary conditions. Here, two particular cases associated with the two-dimensional problem ($n = 0$) are analyzed in detail, and the results are summarized in Figs. 1 through 7.

5.1. Given Thickness, Length, and Enclosed Area

If the thickness, the length, and the enclosed area are prescribed while the moment of inertia of the contour is free, the transversality condition leads to $C_3 = 0$ which implies that $K_3 = 0$, so that the basic inequalities (18), the transition equation (20), and the switching function (22) reduce to

$$K_1 \geq 0 \,, \quad K_1 - K_2 \geq 0$$
$$\eta_c = K_1/K_2 \,, \quad \sigma = K_2(\xi_c - \xi) \tag{27}$$

These results indicate that the corner point ξ_0, 0 exists when $K_1 = 0$, which implies that $K_2 < 0$. On the other hand, the corner point ξ_1, 1 exists when $K_1 = K_2$, which implies that $K_2 > 0$. Consequently, these two corner points cannot coexist. Furthermore, for $K_1 > 0$ and $K_1 - K_2 > 0$, no corner points exist.

In conclusion, three classes of bodies enter into the one-parameter family of extremal solutions (Fig. 1): (I) an infinitely thin flat plate followed by a regular shape, (II) a regular shape only, and (III) a regular shape followed by a constant thickness contour. The parameter which governs these solutions, called the <u>shape parameter</u>, can be defined as

$$\varphi = \frac{A}{d\ell} = I_A \tag{28}$$

<u>Bodies of Class I</u>. This class of bodies, which is composed of infinitely thin flat plates followed by regular shapes, is characterized by the conditions

$$K_1 = 0\,, \quad K_2 < 0\,, \quad 0 \le \xi_o \le 1\,, \quad \xi_1 = 1 \tag{29}$$

which are valid for

$$0 \le \varphi \le \frac{2}{5} \tag{30}$$

Consequently, use of Eq. (23) yields the following expression for the shape of the optimum body (Fig. 2):

$$\begin{aligned} &0 \le \xi \le \xi_o\,, \quad \eta = 0 \\ &\xi_o \le \xi \le 1\,, \quad \eta = \left(\frac{\xi - \xi_o}{1 - \xi_o}\right)^{3/2} \end{aligned} \tag{31}$$

where, from Eq. (28), it is seen that the abscissa of the transition point is related to the shape parameter by (Fig. 3)

$$\xi_o = 1 - \frac{5\varphi}{2} \tag{32}$$

Finally, from Eq. (26), the minimum drag coefficient can be written in the form (Fig. 4)

$$\frac{C_D}{\tau^2} = \frac{27}{250\,\varphi^2} \tag{33}$$

<u>Bodies of Class II</u>. The next class of bodies is composed of regular shapes only and is characterized by the conditions

$$K_1 > 0\,, \quad K_1 - K_2 > 0\,, \quad \xi_o = 0\,, \quad \xi_1 = 1 \tag{34}$$

which hold for

$$\frac{2}{5} \le \varphi \le \frac{3}{5} \tag{35}$$

Thus, the shape of the optimum body is given by

$$0 \le \xi \le 1\,, \quad \xi = \frac{1 - (1 - \phi\eta)^{2/3}}{1 - (1 - \phi)^{2/3}} \tag{36}$$

where

$$\phi = K_2/K_1\,, \quad -\infty \le \phi \le 1 \tag{37}$$

is a constant which is related to the shape parameter through the expression

$$\varphi = \frac{1}{5\phi}\,\frac{3 - (3 + 2\phi)\,(1 - \phi)^{2/3}}{1 - (1 - \phi)^{2/3}} \tag{38}$$

The drag coefficient can then be written as follows:

$$\frac{C_D}{\tau^2} = \frac{27}{40\varphi^3}\left[1 - (1 - \varphi)^{5/3}\right]\left[1 - (1 - \varphi)^{2/3}\right]^2 \tag{39}$$

<u>Bodies of Class III</u>. The last class of bodies is composed of regular shapes followed by constant thickness contours and satisfies the conditions

$$K_1 > 0 , \quad K_2 = K_1 , \quad \xi_0 = 0 , \quad 0 \le \xi_1 \le 1 \tag{40}$$

which are valid for

$$\frac{3}{5} \le \varphi \le 1 \tag{41}$$

The equation for the shape of the optimum body is then given by

$$\begin{aligned} &0 \le \xi \le \xi_1 , \quad \eta = 1 - \left(1 - \frac{\xi}{\xi_1}\right)^{3/2} \\ &\xi_1 \le \xi \le 1 , \quad \eta = 1 \end{aligned} \tag{42}$$

where the abscissa of the transition point and the shape parameter are related as follows:

$$\xi_1 = \frac{5}{2}(1 - \varphi) \tag{43}$$

The associated drag coefficient can be expressed in the form

$$\frac{C_D}{\tau^2} = \frac{27}{250(1 - \varphi)^2} \tag{44}$$

The above results can be employed in order to understand the effect of changing any one of the given quantities on the shape of the optimum body and its drag while the other two are kept constant. As an example, if the thickness and the length are constant and the enclosed area is varied, then the pressure drag (Fig. 4) is an absolute minimum when $\varphi = 1/2$. The corresponding optimum shape is a wedge, which is precisely the result obtained by Miele in Ref. 1 for problems where the thickness and the length are given while the enclosed area and the moment of inertia of the contour are free.

5.2. Given Thickness, Length, and Moment of Inertia of the Contour

If the thickness, the length, and the moment of inertia of the contour are prescribed while the enclosed area is free, the transversality condition leads to $C_2 = 0$ which means that $K_2 = 0$. Thus, the basic inequalities, the transition equation, and the switching function become

$$\begin{gathered} K_1 \geq 0 , \quad K_1 - K_3 \geq 0 \\ \eta_c = \sqrt{K_1/K_3} , \quad \sigma = 2K_3\eta_c(\xi_c - \xi) \end{gathered} \tag{45}$$

and imply the following: (a) The corner point ξ_0, 0 occurs when $K_1 = 0$; (b) The corner point ξ_1, 1 occurs when $K_1 = K_3$; and (c) These corner points cannot coexist.

Again, one obtains a one-parameter family of extremal solutions which is composed of the three classes of shapes mentioned in the previous section (Fig. 1). The shape parameter, which governs these solutions, is defined as

$$\varphi = \frac{2M}{d^2 \ell} = I_M \tag{46}$$

<u>Bodies of Class I</u>. The conditions to be satisfied for this class of bodies are

$$K_1 = 0\,, \quad K_3 < 0\,, \quad 0 \le \xi_o \le 1\,, \quad \xi_1 = 1 \tag{47}$$

and are valid for

$$0 \le \varphi \le \frac{1}{7} \tag{48}$$

Consequently, Eq. (23) yields the following expression for the optimum shape (Fig. 5):

$$\begin{aligned} &0 \le \xi \le \xi_o\,, \quad \eta = 0 \\ &\xi_o \le \xi \le 1\,, \quad \eta = \left(\frac{\xi - \xi_o}{1 - \xi_o}\right)^3 \end{aligned} \tag{49}$$

where the abscissa of the transition point is related to the shape parameter by (Fig. 6)

$$\xi_o = 1 - 7\varphi \tag{50}$$

Furthermore, the minimum drag coefficient is given by (Fig. 7)

$$\frac{C_D}{\tau^2} = \frac{27}{686\varphi^2} \tag{51}$$

__Bodies of Class II__. This class of bodies is characterized by the conditions

$$K_1 > 0 \,, \quad K_1 - K_3 > 0 \,, \quad \xi_0 = 0 \,, \quad \xi_1 = 1 \tag{52}$$

which are valid for

$$\frac{1}{7} \le \varphi \le \frac{3}{7} \tag{53}$$

Thus, the expression for the optimum shape is as follows:

$$0 \le \xi \le 1 \,, \quad \xi = \frac{A(\eta, \phi)}{A(1, \phi)} \tag{54}$$

where

$$\phi = \mp \frac{K_3}{K_1} \,, \quad \begin{cases} K_3 \le 0 & 0 \le \phi \le \infty \\ K_3 \ge 0 & 0 \le \phi \le 1 \end{cases} \tag{55}$$

and where

$$A(\eta, \phi) = \pm \left[\frac{\sqrt{\phi}\, \eta}{\sqrt{3} \mp 1 \pm \sqrt[3]{1 \pm \phi \eta^2}} + \frac{\sqrt{3} \pm 1}{2 \sqrt[4]{3}} F(\theta, k) - \sqrt[4]{3}\, E(\theta, k) \right] \tag{56}$$

In these and the subsequent formulas of this section, the upper signs are to be employed when $K_3 \le 0$ which means that $1/7 \le \varphi \le 1/3$, and the lower signs, when $K_3 \ge 0$ which means that $1/3 \le \varphi \le 3/7$. Furthermore, the symbols F and E denote the incomplete elliptic integrals of the first and second kind, respectively, whose argument θ and parameter k are defined as

$$\theta = \text{arc cos}\,\frac{\sqrt{5} \pm 1 \mp \sqrt[3]{1 \pm \phi\eta^2}}{\sqrt{5} \mp 1 \pm \sqrt[3]{1 \pm \phi\eta^2}} \tag{57}$$

$$k = \sqrt{\frac{2 \mp \sqrt{3}}{4}}$$

Next, the constant ϕ is related to the shape parameter through the expression

$$\varphi = \pm \frac{1}{7\phi}\left[\frac{B(1,\ \phi)}{A(1,\ \phi)} - 3\right] \tag{58}$$

where

$$B(\eta,\ \phi) = \sqrt{\phi}\ \eta(1 \pm \phi\eta^2)^{2/3} \tag{59}$$

Finally, the minimum drag coefficient is given by

$$\frac{C_D}{\tau^2} = \frac{27A^3(1,\ \phi)}{14\phi\sqrt{\phi}}\left[4 + \frac{B(1,\ \phi)}{A(1,\ \phi)}\right] \tag{60}$$

<u>Bodies of Class III</u>. The bodies of this class must satisfy the following conditions:

$$K_1 > 0\ ,\quad K_3 = K_1\ ,\quad \xi_o = 0\ ,\quad 0 \leq \xi_1 \leq 1 \tag{61}$$

which are valid for

$$\frac{3}{7} \leq \varphi \leq 1 \tag{62}$$

Consequently, the shape of the optimum body can be expressed as

$$0 \le \xi \le \xi_1 \,, \quad \frac{\xi}{\xi_1} = \frac{A(\eta, 1)}{A(1, 1)}$$
$$\xi_1 \le \xi \le 1 \,, \quad \eta = 1 \tag{63}$$

where the abscissa of the transition point is given by

$$\xi_1 = \frac{7}{4} (1 - \varphi) \tag{64}$$

The drag coefficient for the optimum body can then be written as

$$\frac{C_D}{\tau^2} = \frac{864}{343} \frac{A^3(1, 1)}{(1 - \varphi)^2} \tag{65}$$

6. OPTIMUM AXISYMMETRIC BODIES

In this section, two particular cases associated with the three-dimensional problem ($n = 1$) are considered, and the results are summarized in Figs. 8 through 13.

6.1. Given Thickness, Length, and Wetted Area

If the thickness, the length, and the wetted area are given while the volume is free, the transversality condition leads to $C_3 = 0$ and implies that $K_3 = 0$, so that the basic inequalities, the transition equation, and the switching function reduce to expressions (27). Consequently, the corner point ξ_0, 0 exists for $K_1 = 0$, and the corner point ξ_1, 1 exists when $K_1 = K_2$. Incidentally, these two corner points cannot coexist. Furthermore, for $K_1 > 0$ and $K_1 - K_2 > 0$, no corner points exist (Fig. 1).

It is evident from the previous discussion that three classes of bodies enter into the one-parameter family of extremal solutions: (I) a spike followed by a regular shape, (II) a regular shape only, and (III) a regular shape followed by a cylinder. The shape parameter governing the solutions is defined as

$$\varphi = \frac{S}{\pi d \ell} = I_S \tag{66}$$

Bodies of Class I. For this class of bodies, which is composed of spikes followed by regular shapes, conditions (29) must be satisfied and are valid for

$$0 \le \varphi \le \frac{1}{2} \tag{67}$$

Consequently, use of Eq. (23) yields the following result for the optimum shape (Fig. 8):

$$0 \le \xi \le \xi_o \,, \quad \eta = 0$$
$$\xi_o \le \xi \le 1 \,, \quad \eta = \frac{\xi - \xi_o}{1 - \xi_o} \tag{68}$$

The abscissa of the transition point can be obtained from Eq. (66) and is given by (Fig. 9)

$$\xi_o = 1 - 2\varphi \tag{69}$$

Finally, the minimum drag coefficient can be expressed as (Fig. 10)

$$\frac{C_D}{\tau^2} = \frac{1}{8\varphi^2} \tag{70}$$

<u>Bodies of Class II</u>. This class of bodies, which is composed of regular shapes only, is characterized by the set of conditions (34) which are valid for

$$\frac{1}{2} \le \varphi \le \frac{2}{3} \tag{71}$$

The expression for the shape of the optimum body is given by

$$0 \le \xi \le 1 \,, \quad \xi = \frac{A(\eta, \phi)}{A(1, \phi)} \tag{72}$$

where

$$\phi = K_2/K_1 , \quad -\infty \leq \phi \leq 1$$

and where

$$A(\eta, \phi) = -\sqrt[3]{\phi\eta(1 - \phi\eta)^2} + \frac{1}{2}\log\left[\sqrt[3]{\phi\eta} + \sqrt[3]{1 - \phi\eta}\right]$$
$$+ \frac{1}{\sqrt{3}}\arc\tan\frac{\sqrt{3}\,\sqrt[3]{\phi\eta}}{2\sqrt[3]{1 - \phi\eta} - \sqrt[3]{\phi\eta}} \tag{73}$$

The constant ϕ is related to the shape parameter through the expression

$$\varphi = \frac{1}{\phi}\left[\frac{2}{3} - \frac{1}{2}\frac{B(1, \phi)}{A(1, \phi)}\right] \tag{74}$$

where

$$B(\eta, \phi) = (\phi\eta)^{4/3}(1 - \phi\eta)^{2/3} \tag{75}$$

The associated drag coefficient can be expressed as

$$\frac{C_D}{\tau^2} = \frac{A^3(1, \phi)}{\phi^4}\left[\frac{1}{3} + \frac{1}{2}\frac{B(1, \phi)}{A(1, \phi)}\right] \tag{76}$$

<u>Bodies of Class III</u>. The set of conditions to be satisfied by this class of bodies, which is composed of regular shapes followed by cylinders, is represented by Eqs. (40) and hold for

$$\frac{2}{3} \leq \varphi \leq 1 \tag{77}$$

The equation for the optimum shape can then be written as follows:

$$\begin{aligned} &0 \leq \xi \leq \xi_1 , \quad \frac{\xi}{\xi_1} = \frac{A(\eta, 1)}{A(1, 1)} \\ &\xi_1 \leq \xi \leq 1 , \quad \eta = 1 \end{aligned} \tag{78}$$

where the abscissa of the transition point is given by

$$\xi_1 = 3(1 - \varphi) \tag{79}$$

Finally, the drag coefficient can be expressed as

$$\frac{C_D}{\tau^2} = \left(\frac{2\pi}{9\sqrt{3}}\right)^3 \left[\frac{1}{1 - \varphi}\right]^2 \tag{80}$$

6.2. Given Thickness, Length, and Volume

If the thickness, the length, and the volume are given while the wetted area is free, the transversality condition leads to $C_2 = 0$ and implies that $K_2 = 0$, so that the basic inequalities, the transition equation, and the switching function reduce to expressions (45). These results indicate that the corner point ξ_o, 0 exists when $K_1 = 0$ and the corner point ξ_1, 1 exists when $K_3 = K_1$. As in the problem of Section 5.2, the two corner points cannot occur simultaneously. Furthermore, the three classes of bodies which were discussed in the previous section enter into the one-parameter family of extremal arcs, the shape parameter being defined as

$$\varphi = \frac{4V}{\pi d^2 \ell} = I_V \tag{81}$$

<u>Bodies of Class I</u>. For this class of bodies, the conditions (47) must be satisfied and are valid for

$$0 \leq \varphi \leq \frac{1}{4} \tag{82}$$

Thus, the equation of the optimum shape reduces to (Fig. 11)

$$\begin{aligned} &0 \leq \xi \leq \xi_o\,, \quad \eta = 0 \\ &\xi_o \leq \xi \leq 1\,, \quad \eta = \left(\frac{\xi - \xi_o}{1 - \xi_o}\right)^{3/2} \end{aligned} \tag{83}$$

where the abscissa of the transition point is given by (Fig. 12)

$$\xi_o = 1 - 4\varphi \tag{84}$$

Finally, the drag coefficient can be written as (Fig. 13)

$$\frac{C_D}{\tau^2} = \frac{27}{512\,\varphi^2} \tag{85}$$

<u>Bodies of Class II</u>. The bodies of this class are characterized by the conditions (52) which are valid for

$$\frac{1}{4} \le \varphi \le \frac{1}{2} \tag{86}$$

The geometry of the optimum shape can be obtained from Eq. (23) and is given by

$$0 \le \xi \le 1 \,, \qquad \xi = \frac{A(0,\ \psi) \mp A(\eta,\ \psi)}{A(0,\ \psi) \mp A(1,\ \psi)} \tag{87}$$

where

$$\psi = K_3/K_1 \,, \qquad -\infty \le \psi \le 1 \tag{88}$$

and where

$$A(\eta,\ \psi) = \pm \frac{2\psi\eta^2 - 1}{\sqrt{3} + 1 - \sqrt[3]{1 - (1 - 2\psi\eta^2)^2}} - \frac{\sqrt{3} - 1}{2\sqrt[4]{3}} F(\theta,\ k) + \sqrt[4]{3}\, E(\theta,\ k) \tag{89}$$

The symbols F and E denote the incomplete elliptic integrals of the first and second kind whose argument θ and parameter k are defined as

$$\theta = \text{arc cos}\,\frac{\sqrt{3} - 1 + \sqrt[3]{1 - (1 - 2\phi\eta^2)^2}}{\sqrt{3} + 1 - \sqrt[3]{1 - (1 - 2\phi\eta^2)^2}}$$

$$k = \sqrt{\frac{2 + \sqrt{3}}{4}} \tag{90}$$

Furthermore, the constant ϕ is related to the shape parameter by the relationship

$$\varphi = \frac{1}{2\phi}\left[1 - \frac{B(1, \phi)}{A(0, \phi) \mp A(1, \phi)}\right] \tag{91}$$

where

$$B(\eta, \phi) = \frac{1}{4}\left[1 - (1 - 2\phi\eta^2)^2\right]^{2/3} \tag{92}$$

The drag coefficient is given by

$$\frac{C_D}{\tau^2} = \frac{27}{32\phi^2}\left[A(0, \phi) \mp A(1, \phi)\right]^2\left[A(0, \phi) \mp A(1, \phi) + B(1, \phi)\right] \tag{93}$$

In the above expressions, the upper signs are valid for all values of η providing that $1/4 \leq \varphi \leq \varphi_*$, where

$$\varphi_* = 1 - \frac{1}{4A(0, 1/2)} \tag{94}$$

Otherwise, if $\varphi_* \leq \varphi \leq 1/2$, the upper signs are valid for $\eta \leq 1/\sqrt{2}\phi$, and the lower signs, for $\eta \geq 1/\sqrt{2}$. In closing, it is worth noting that some previous work on bodies of Class II was done by Strand using partly analytical and partly numerical procedures (Ref. 4).

<u>Bodies of Class III</u>. For this class of bodies, the conditions (61) must be satisfied and are valid for

$$1/2 \leq \varphi \leq 1 \tag{95}$$

Hence, the equation of the optimum shape is given by

$$\begin{aligned} &0 \leq \xi \leq \xi_1 , \quad \frac{\xi}{\xi_1} = \frac{1}{2}\left[1 \mp \frac{A(\eta, 1)}{A(0, 1)}\right] \\ &\xi_1 \leq \xi \leq 1 , \quad \eta = 1 \end{aligned} \tag{96}$$

where the upper sign is valid for $\eta \leq 1/2$ and the lower sign, for $\eta \geq 1/2$. Finally, the abscissa of the transition point and the shape parameter are related by the expression

$$\xi_1 = 2(1 - \varphi) \tag{97}$$

The drag coefficient is given by

$$\frac{C_D}{\tau^2} = \frac{27}{16} \frac{A^3(0, 1)}{(1 - \varphi)^2} \tag{98}$$

CONCLUSIONS

In this report, the problem of minimizing the pressure drag of slender bodies in Newtonian flow is considered. Attention is focused on the class of problems which involve, in addition to the thickness and the length, the enclosed area and the moment of inertia of the contour for two-dimensional shapes and the wetted area and the volume for axisymmetric shapes.

When three of the four quantities under consideration are specified, one obtains a one-parameter family of optimum shapes, the parameter being a function of the three prescribed quantities. Generally speaking, this family of solutions is composed of three classes of shapes: (I) a flat plate or a spike followed by a regular shape, (II) a regular shape only, and (III) a regular shape followed by a constant thickness contour or a cylinder.

The above results can be employed in order to understand the effect of changing any one of the three given quantities on the minimum drag while the other two are kept constant. In each case, the drag has an absolute minimum, and the corresponding optimum shapes are those derived by Miele in Refs. 1 through 3. For example, if the thickness, the length, and the enclosed area of a two-dimensional shape are prescribed, the drag is an absolute minimum when $A/d\ell = 1/2$, which corresponds to the wedge solution. Miele found the wedge to be the optimum body for the case where the thickness and the length are specified, while the enclosed area and the moment of inertia of the contour are free.

In closing, three remarks are important:

(a) Some of the optimum shapes obtained in this analysis are concave;

consequently, these bodies should be restudied using the Newton-Busemann pressure coefficient law in accordance with the method developed in Ref. 5.

(b) When the thickness ratio becomes very large, the average value of the slope may become such that the slender body approximation is violated; consequently, this case should be reinvestigated using the exact Newtonian expression for the pressure coefficient, that is, the sine square law (Ref. 9).

(c) When the thickness ratio becomes very small, the friction drag (which was neglected here) may have the same importance as the pressure drag. Consequently, it is of interest to formulate a new minimal problem in which the total drag (the sum of the pressure drag and the friction drag) is extremized (Refs. 6 and 7).

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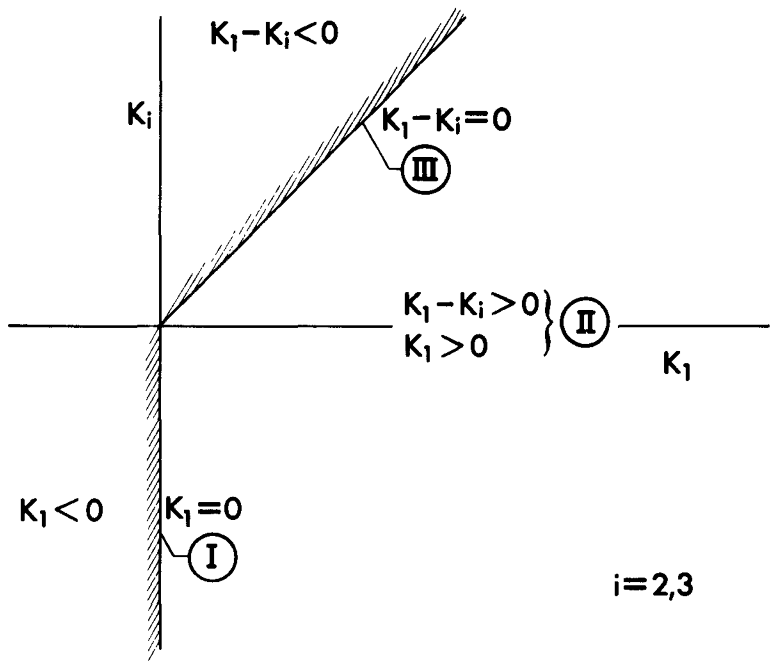

Fig. 1. Relation between characteristic constants and various classes of solutions.

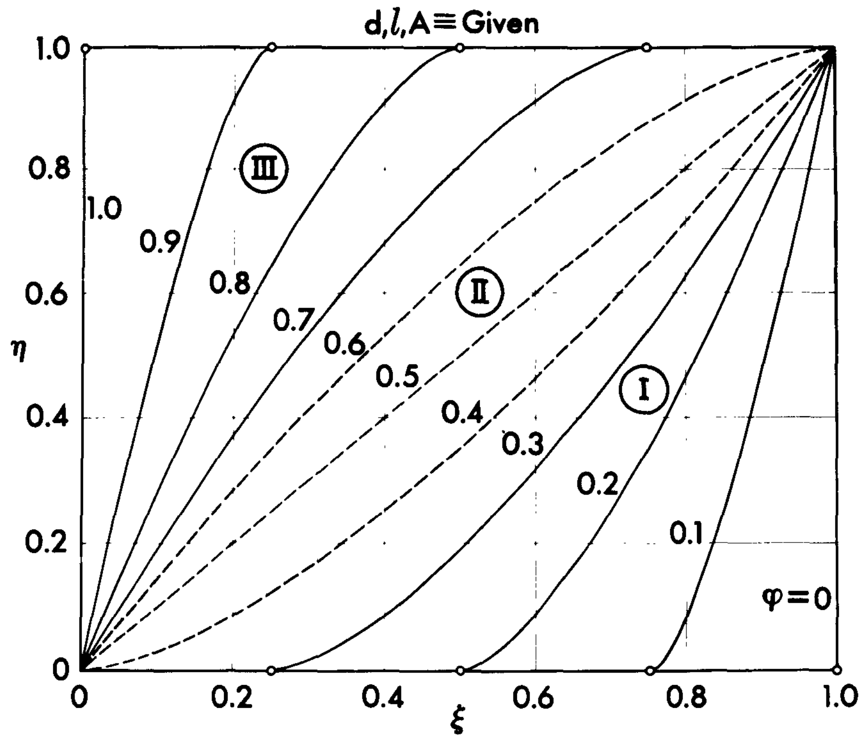

Fig. 2. Optimum shapes for given thickness, length, and enclosed area.

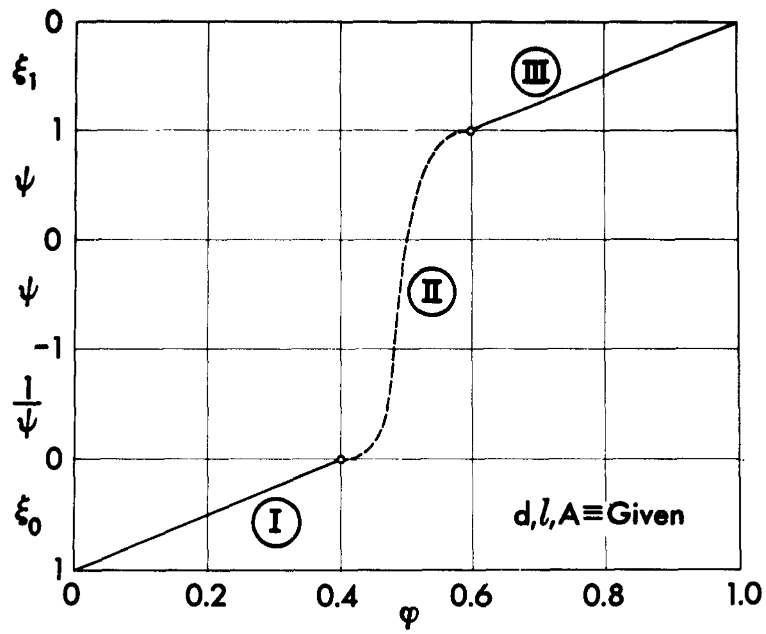

Fig. 3. Shape parameter for given thickness, length, and enclosed area.

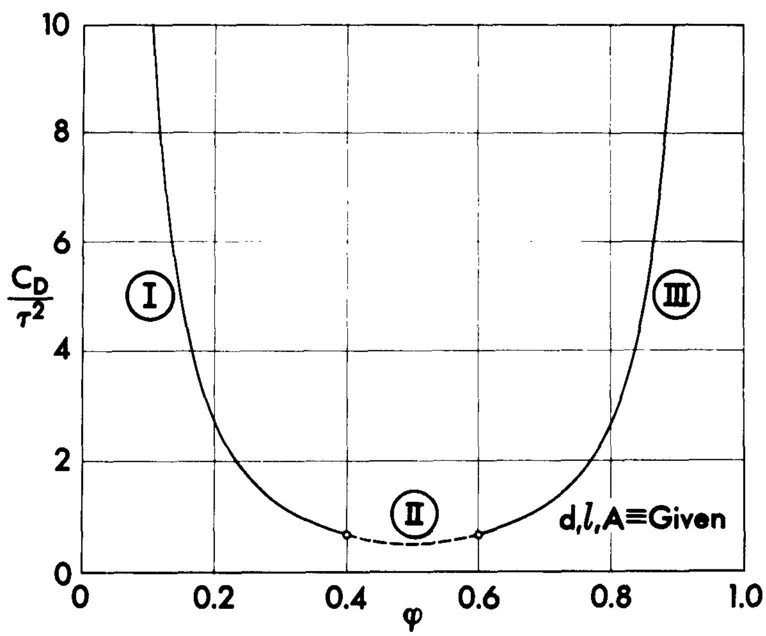

Fig. 4. Minimum drag coefficient for given thickness, length and enclosed area.

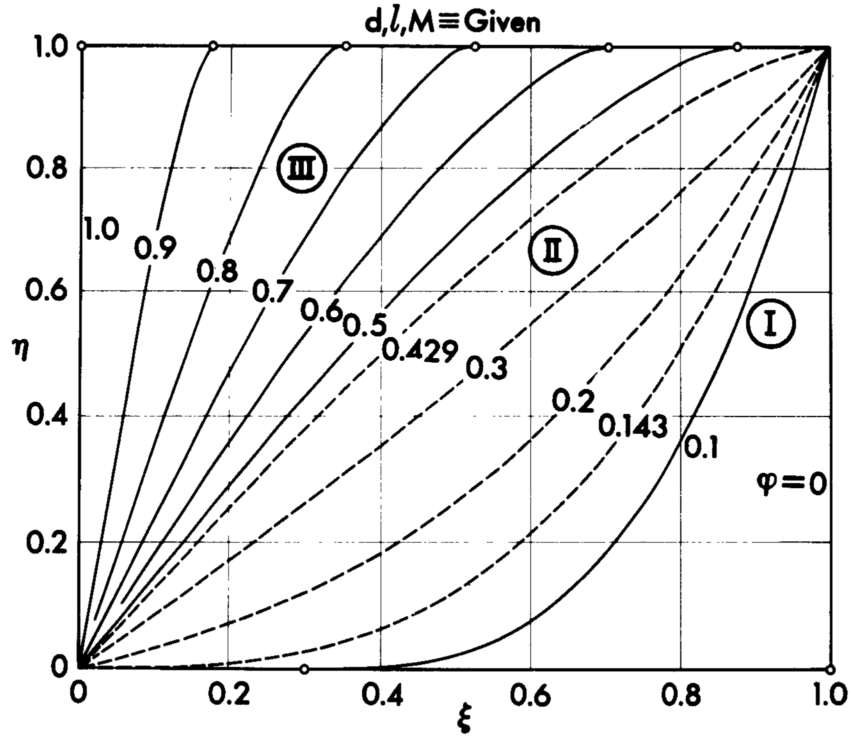

Fig. 5. Optimum shapes for given thickness, length, and moment of inertia of the contour.

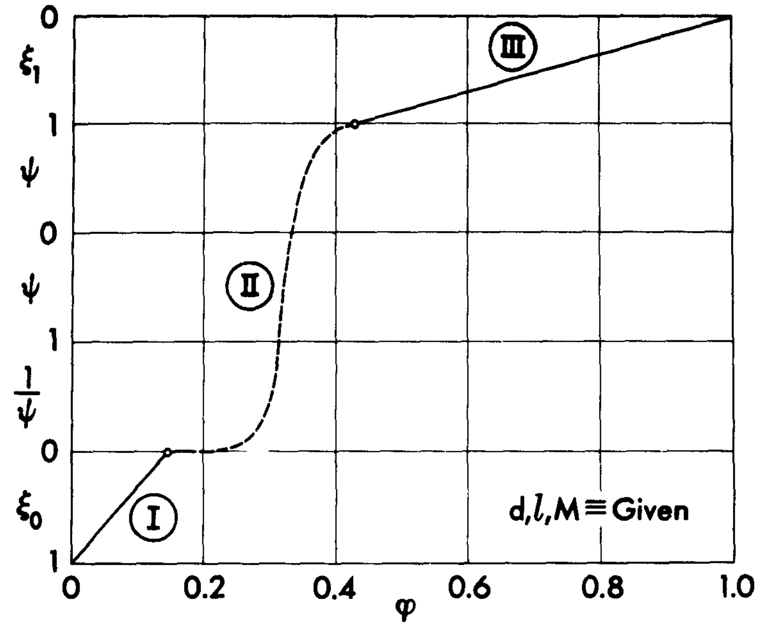

Fig. 6. Shape parameter for given thickness, length, and moment of inertia of the contour.

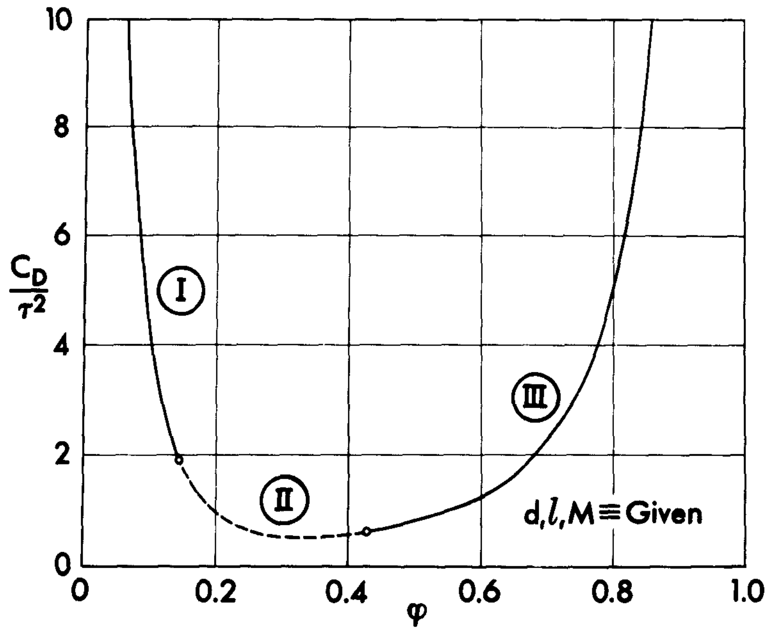

Fig. 7. Minumum drag coefficient for given thickness, length, and moment of inertia of the contour.

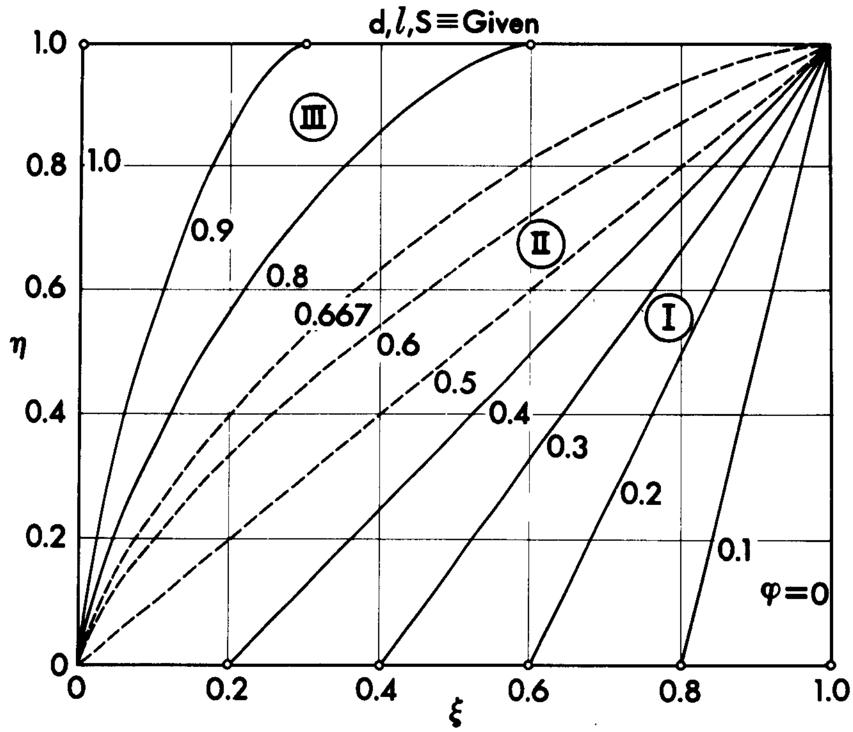

Fig. 8. Optimum shapes for given thickness, length, and wetted area.

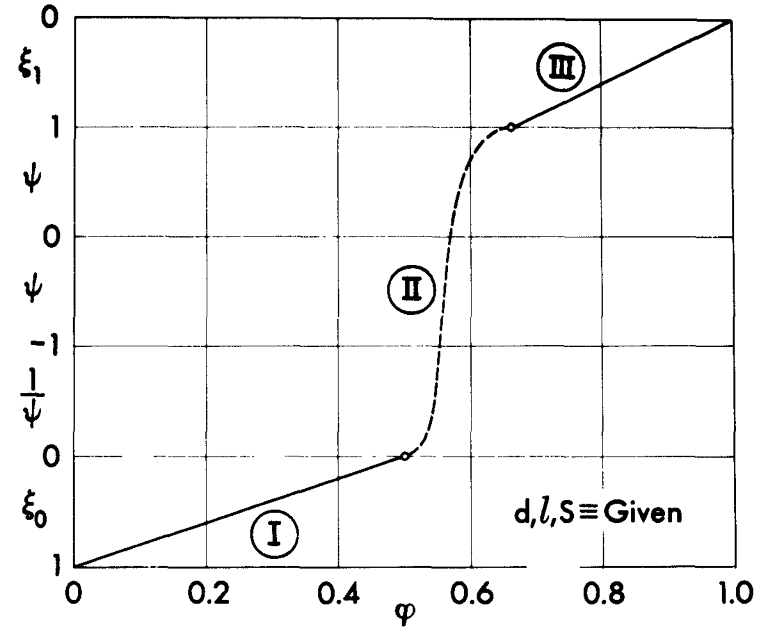

Fig. 9. Shape parameter for given thickness, length, and wetted area.

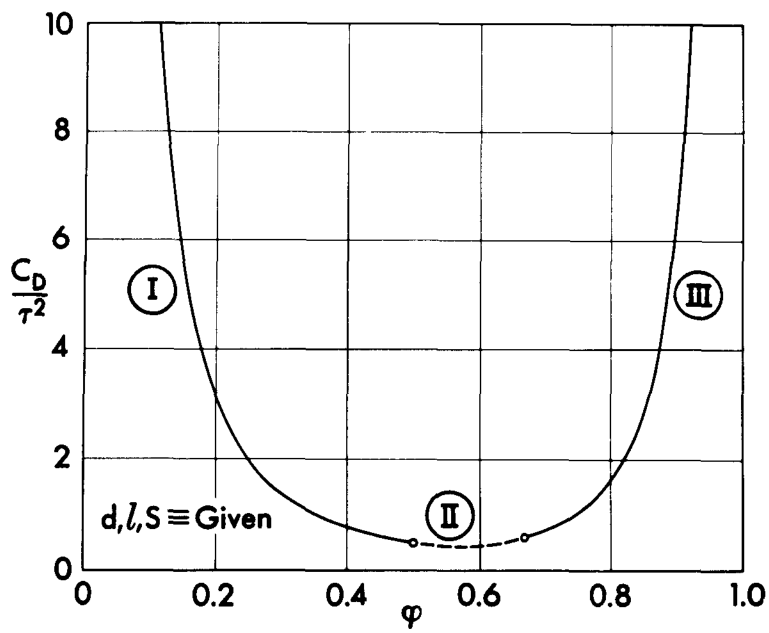

Fig. 10. Minimum drag coefficient for given thickness, length, and wetted area.

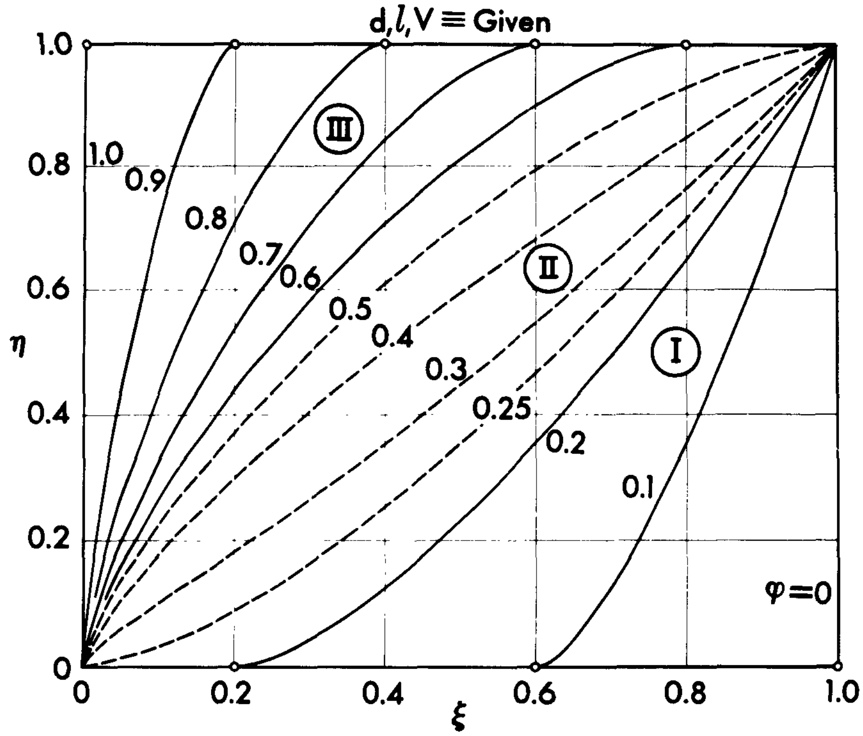

Fig. 11. Optimum shapes for given thickness, length, and volume.

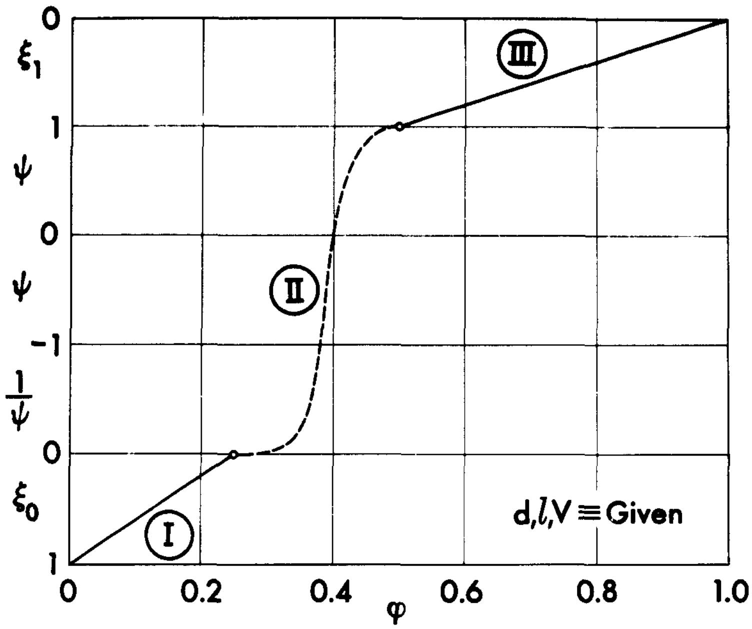

Fig. 12. Shape parameter for given thickness, length, and volume.

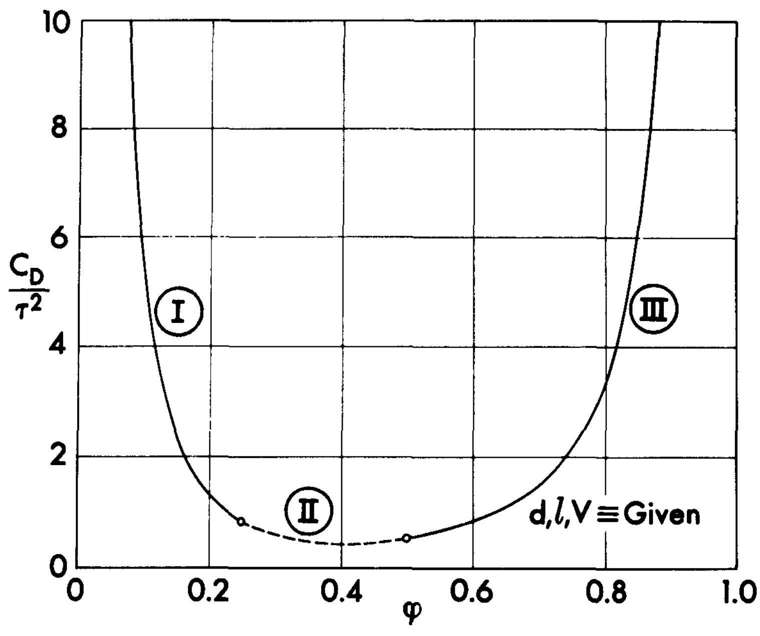

Fig. 13. Minimum drag coefficient for given thickness, length, and volume.