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In [1] Robert Hermann introduced the concept of tangent vector fields on the space of maps of one manifold into another. A special type of these are the "k-vector fields" which were studied in [3], where this author defined their bracket and exponential. This paper explores further the analogy with classical continuous groups. Specifically, we study invariance of systems of partial differential equations under k-vector fields.

1. Introduction

Every map and manifold is C^∞ unless otherwise noted. $J^k = J^k(N,M)$ is the manifold of k-jets $j_x^k(f)$ of order k of maps $f: N \to M$ from the manifold N to the manifold M. α and β are the source and target projections, ρ_i^{k-i}: $J^{k+i} \to J^i$ the usual projection. $T(M)$ denotes the tangent bundle to M, M_y the tangent space at $y \in M$, π the tangent bundle projection. $C^\infty(Q)$ is the algebra (over the reals R) of C^∞ real-valued functions on the manifold Q.

A k-<u>vector</u> <u>field</u> is a map $\Theta: C^\infty(M) \to C^\infty(J^k)$ which is linear over R and satisfies

$$\Theta(FG) = (F \circ \beta)\Theta(G) + (G \circ \beta)\Theta(F) .$$

In [3] the i<u>th</u> prolongation $P^i\Theta: C^\infty(J^i) \to C^\infty(J^{i+k})$

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was defined. This satisfies $P^i\Theta(FG) = (F\circ\rho_i^{k+i})P^i\Theta(G) + (G\circ\rho_i^{k+i})P^i\Theta(F)$; and when $H \in C^\infty(M)$, $P^i\Theta(H\circ\mu) = \Theta(H)\circ\rho_i^{k+i}$. Using these facts one sees that if Θ and ψ are k- and i- vector fields, respectively, then $[\Theta, \psi] = P^i\Theta\circ\psi - P^k\psi\circ\Theta$ is a k+i- vector field.

In local coordinates (x^i) on N, (y^λ) on M, $(x^i, y^\lambda, p^\lambda_{j_1}, \dots, p^\lambda_{j_1\dots j_k})$ on J^k, where $i, j_1, \dots, j_k = 1, \dots, n$; $\lambda = 1, \dots, m$, we follow Kuranishi in defining for each $F \in C^\infty(J^k)$, $\partial^\#_j F \in C^\infty(J^{k+1})$ by

$$\partial^\#_j F = \frac{\partial F}{\partial x^j} + \frac{\partial F}{\partial y^\lambda}p^\lambda_j + \dots + \frac{\partial F}{\partial p^\lambda_{j_1\dots j_k}}p^\lambda_{j_1\dots j_k j}.$$

Then if $\Theta = a^\lambda(\partial/\partial y^\lambda)$ is a k-vector field,

$$P^i\Theta = a^\lambda\frac{\partial}{\partial y^\lambda} + \partial^\#_j a^\lambda\frac{\partial}{\partial p^\lambda_j} + \dots + \partial^\#_{j_1}\dots\partial^\#_{j_i}a^\lambda\frac{\partial}{\partial p^\lambda_{j_1\dots j_i}}.$$

[See 3, Lemma 1.] We shall also need the following Lemma whose proof we omit.

Lemma 1. *Let* Θ *be a k-vector field,* $F_1, F_2 \in C^\infty(J^i)$, $G \in C^\infty(M)$ *and* $F \in C^\infty(J^{i-j})$, *where* $0<j<i$. *Then*

(A) $P^i\Theta(F\circ\rho^i_{i-j}) = (P^{i-j}\Theta(F))\circ\rho^{i+k}_{i-j+k}$,

(B) $P^i\Theta(G\circ\mu) = \Theta(G)\circ\rho^{i+k}_i$,

(C) $P^i\Theta(F_1F_2) = (F_1\circ\rho^{i+k}_i)P^i\Theta(F_2) + (F_2\circ\rho^{i+k}_i)P^i\Theta(F_1)$,

(D) $P^i\Theta(\partial^\#_{j_1}\dots\partial^\#_{j_r}G\circ\rho^i_r) = \partial^\#_{j_1}\dots\partial^\#_{j_r}\Theta(G)\circ\rho^{i+k}_{r+i}$, $r<k$,

(E) $P^i\Theta(\partial^\#_{j_1}\dots\partial^\#_{j_r}F\circ\rho^i_{i-j+r}) = (\partial^\#_{j_1}\dots\partial^\#_{j_r}P^{i-j}\Theta(F))\circ{}^{i+k}_{i-j+k+r}$, $r<j$.

Conversely, if $\phi: C^\infty(J^i) \to C^\infty(J^{i+k})$ *satisfies* (A),...,

(E) <u>when</u> $P^1\Theta$ <u>is replaced by</u> ϕ , <u>then</u> $\phi = P^1\Theta$.

Another important property for us is that if $F \in C^\infty(J^i)$, $f: N \to M$, then $(\partial/\partial x^i)G(j^i(f)) = (\partial_i^\# G)(j^{i+1}(f))$. [2, Prop. 1.10]

Let $I = (-\epsilon, \epsilon)$. An <u>integral curve</u> of Θ starting at $f_0 : N \to M$ is a 1-parameter family $f: N \times I \to M$ with $f_0(x) = f(x,0)$ and

$$\Theta(j_x^k(f)) = \frac{\partial f}{\partial t}(x,t) \quad .$$

Here $(\partial f/\partial t)(x,t) \in M_{f(x,t)}$ is defined to act on any real-valued function F defined in a neighborhood of $f(x,t)$ by $dF(f(x,t))/dt$.

2. Differential Systems

A system Σ of partial differential equations (s.p.d.e.) of order h with N as independent and M as dependent variables is a finitely generated ideal in $C^\infty(J^h)$. A solution of Σ is a map $f: N \to M$ such that $F(j_x^h(f)) = 0$ for all $x \in N$, $F \in \Sigma$. $P^k\Sigma$ denotes the s.p.d.e. of order $h+k$ generated by the functions $F \circ \rho_h^{h+k}$, $\partial_j^\# F \circ \rho_{h+1}^{h+k}$, ..., $\rho_{j_1}^\# \dots \rho_{j_k}^\# F$, $1 \leq j$, $j_t \leq n$, $F \in \Sigma$.

Definition. A k-vector field Θ leaves Σ <u>invariant</u> if for each $F \in \Sigma$, $P^h\Theta(F) \in P^k\Sigma$.

Compare with [2] for the older theory. The intuitive meaning of invariance under a transformation group was that the transformations permute the solutions. We shall show that if f_0 is a solution of Σ which belongs to an integral curve of Θ , then Σ evaluated at this integral curve has zero derivatives at f_0 of all orders.

Lemma 2. *If* Θ *is an invariant vector field of* Σ, *then* Θ *is an invariant vector field for* $P^i\Sigma$, *all* i.

This follows from (D) and (E) in Lemma 1. Using local coordinates, a calculation proves

Lemma 3. *If* $F \in C^\infty(J^i)$, $f: N \times I \to M$, *and* $(\partial f/\partial t) = \Theta(j_x^k(f))$, *then*

$$\frac{\partial}{\partial t}F(j_x^i(f)) = P^i\Theta(F)\Big|_{j_x^{k+i}(f)} .$$

Lemma 4. *If* $f: N \to M$ *is a solution of* Σ, *it is a solution of* $P^i\Sigma$, *all* i.

Theorem 1. *Suppose that*

(A) Θ *is an invariant k-vector field of* Σ,
(B) $f: N \times I \to M$ *satisfies* $(\partial f/\partial t) = \Theta(j_x^k(f))$, *and*
(C) $f(\ ,0): N \to M$ *is a solution of* Σ.

Then

$$\frac{\partial^n}{\partial t^n}F(j_x^h(f))\Big|_{t=0} = 0$$

for all $x \in N$, $F \in \Sigma$, *and* $n = 1,2,\ldots.$

Proof: From Lemma 3,

$$\frac{\partial}{\partial t}F(j_x^h(f)) = P^h\Theta(F)\Big|_{j_x^{k+h}(f)} .$$

However, $P^h\Theta(F) \in P^h\Sigma$, and f is a solution of $P^h\Sigma$ by Lemma 4. Hence $P^h\Theta(F)(j_x^{k+h}(f))\big|_{t=0} = 0$, all $x \in N$.

Let $F^1 = P^h\theta(F) \in P^k\Sigma$. By Lemma 3,

$$P^{h+k}\theta(F)\Big|_{j_x^{2k+h}(f)} = \frac{\partial}{\partial t}F^1(j_x^{h+k}(f)) = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}F(j_x^h(f))\right] .$$

Using Lemma 4 as before, $(\partial^2/\partial t^2)F(j_x^h(f))\big|_{t=0} = 0$.
Continuing in this way, the result follows. Q.E.D.

When the manifolds and functions are real analytic, Theorem 1 implies that integral curves of an invariant vector field which pass through one solution yield solutions for all parameter values.

3. Lie Algebra Structure

Proposition. Let θ and ψ be k-and h-vector fields, respectively. Then

$$P^i[\theta,\psi] = P^{i+h}\theta o P^i\psi - P^{i+k}\psi o P^i\theta .$$

Proof. By induction on i . A local coordinate calculation shows the result for $i = 1$. Call ϕ: $C^\infty(J^i) \to C^\infty(J^{i+h+k})$ the operator on the right-hand side. We shall use Lemma 1. Let $F_1, F_2 \in C^\infty(J^i)$, $G \in C^\infty(M)$, and $F \in C^\infty(J^{i-j})$.

$$P^{i+h}\theta o P^i\psi(F o \rho^i_{i-j}) = P^{i+h}\theta(P^{i-j}\psi(F) o \rho^{i+h}_{i+h-j})$$

$$= (P^{i+h-j}\theta(P^{i-j}\psi(F)) o \rho^{i+h+k}_{i+h+k-j}$$

$$= (P^{i+h-j}\theta P^{i-j}\psi)(F) o \rho^{i+h+k}_{i+h+k-j} ,$$

applying Lemma 1(A) to ψ and θ. Interchanging θ and ψ, we find

$$P^{i}\phi(F\circ\rho^{i}_{i-j}) = (P^{i-j}\phi(F))\circ\rho^{i+h+k}_{i+h+k-j} .$$

Now, by induction, $P^{i-j}\phi(F) = P^{i-j}[\theta, \psi]$. Hence (A) holds for ϕ. The same technique works for (B),...,(E). Q.E.D.

Theorem 2. <u>If θ and ψ are k- and h-vector fields, respectively, which leave Σ invariant, then</u> $[\theta, \psi]$ <u>leaves Σ invariant</u>.

Proof. If $F \in \Sigma$ and Σ is of order i, then $P^{i}[\theta, \psi](F) = P^{i+h}\theta \circ P^{i}\psi(F) - P^{i+k}\psi\circ P^{i}\theta(F)$. However, $P^{i}\psi(F) \in P^{h}\Sigma$. By Lemma 2 θ is an invariant vector field of $P^{h}\Sigma$, so $P^{i+h}\theta\circ P^{i}\psi(F) \in P^{h+k}\Sigma$. Similarly $P^{i+k}\psi\circ P^{i}\theta(F) \in P^{h+k}\Sigma$. Q.E.D.

We conclude that the set of all k-vector fields, $k = 1, 2, \ldots$, leaving Σ invariant forms a Lie algebra under the bracket.

4. An Example

Let $N = E^{n}$, $M = E^{m}$. Consider a s.p.d.e. of the type

$$\frac{\partial y^{\lambda}}{\partial x^{n}} = \phi^{\lambda}(x^{1},\ldots,x^{n-1},y^{\mu},\frac{\partial y^{\mu}}{\partial x^{1}},\ldots,\frac{\partial y^{\mu}}{\partial x^{n-1}}),$$

$\lambda, \mu = 1,\ldots,m$. On J^{1} let $F^{\lambda} = p^{\lambda}_{n} - \phi^{\lambda}(x^{i},y^{\mu},p^{\mu}_{i})$, and let $\sum$ be generated by $F^1, \ldots, F^m$. Then by a calculation one may check that $\Theta = \phi^\lambda(\partial/\partial y^\lambda)$ turns out to be an invariant vector field of $\sum$.

We can see that Θ generates solutions of the Cauchy problem associated with $\sum$. Since Θ is independent of x^n and p_n^λ, it can be considered a 1-vector field on E^{n-1}. Suppose $f_0 : E^{n-1} \to E^m$ is the initial data at $x^n = 0$. Suppose $I = \{ x^n \mid -\epsilon < x^n < \epsilon \}$ and $f : E^{n-1} \times I \to E^m$ is an integral curve of Θ through f_0. But that is merely another way of saying that f is a solution of $\sum$.

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