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SOLUTION OF A STOCKING PROBLEM BY
A SHORTEST ROUTE ALGORITHM

by

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1. Introduction

Suppose some quality of a product varies according to some monotonic function of \( n \) numbers \( L_i \), which we shall call for convenience lengths. The problem is to determine which of these \( n \) lengths to stock (henceforward called standards), given that we can stock only \( m \leq n \) different lengths, such that the loss incurred in reducing or cutting or supplying standards to the demanded lengths is minimized. It is assumed that:

A. The part of the product that is left after reducing, i.e., the scrap, cannot be used again.

B. The loss is a function of the scrap only.

C. All the lengths are demanded equally.

Assumptions B and C are assumed to hold only in Sections 1 and 2.

For instance if we are given three lengths of steel beams of 4 ft., 3 ft., and 1 ft. at a cost of $1 a ft. so that the loss in reducing \( L_i \) to \( L_j \) is just the cost of \( L_i - L_j \), and told to choose two standards, we will pick 4 ft. and 1 ft. as our lengths. To see this, note that 4 ft. must be chosen as a standard, since it is the longest, must be supplied, and anything longer is just wasted (if other than demanded lengths were permitted as standards). For the other standard either 3 ft. or 1 ft. must be chosen, since again, not choosing a demanded length as a standard just wastes the difference between its length and the greatest demanded length less than it, at least. Now it can be seen that 1 ft. must be chosen as a standard since its waste is only $1 (= $4 - $3) and the waste with 3 ft. is $2 (= $3 - $1). As another example suppose we had square box tops with sides of lengths 4 ft., 3 ft., and 1 ft. demanded and the cost is $1 a square foot, and we could stock but two standard lengths. The loss in
not stocking the 1 ft. top would be $3^2 - 1^2 = 8$, and the loss in not
stocking the 3 ft. top would be $4^2 - 3^2 = 7$. Therefore we would stock
the 4 ft. and 1 ft. box tops.

2. The Solution

With the given lengths $L_i$, we associate the transitive, complete,
asyymmetric graph $G$, to be defined. The graph $G$ will also be irreflexive
in this section but not in Sections 3 and 4. Each vertex in $G$ corresponds
to one of the $L_i$, and one more vertex is added to serve as a sink. The
vertex corresponding to the longest length, $L_n$, is the source. Assume the
demanded lengths are ordered, i.e., $L_i \leq L_{i+1}$ for all $i \leq n$. Call the
added vertex (the sink) $L_0$. Now draw an arc from $L_i$ to $L_j$ if $i > j$ and
associate with this arc the loss incurred in assuming that $L_i$ and $L_j$ are
standards and that there are no standards between them, and counting only
that part of the loss incurred in supplying the lengths intermediate be-
 tween $L_i$ and $L_j$. Arcs from $L_i$ to $L_0$ count the loss incurred in sup-
plying $L_i$ in assuming that $L_i$ is the smallest standard. Thus the arc
from $L_i$ to $L_j$,

$$[L_i;L_j] = \frac{1}{n-j_i} f(L_i;L_n)$$

where $f(L_i;L_n)$ is the loss from $L_i$ in supplying $L_n$. Using the steel
beam example the graph $G$ is:
Now let $u_m$ be a set of $m$ of the $L_j$, and $C(u_m)$ be the cost incurred in assuming that each of the $u_m$ are all standards and $A$, $B$, and $C$. Also let $u^i_m$ be the $L_j$ obtained from $G$, defined as above, which correspond to the shortest $m$ step path from $L_n$ to $L_0$. Let $L(u_m)$ be the length of the path in $G$ corresponding to $u_m$. Then,

Theorem: Given $L_n < L_2 < ... < L_m$ and $w/n$, and assuming $A$, $B$, and $C$, then

a. $C(u_m) = L(u_m)$ and hence

b. $\min_{u_m} C(u_m) = C(u^1_m)$

where the minimum is taken over all $u_m$.

Proof: It is clear from the nature of $G$ that each $u_m$ corresponds to a unique $m$ step path from $L_n$ to $L_0$, and that $C(u_m)$ is the length of that path since the length of each arc is the loss incurred from each standard assuming $A$, $B$, and $C$. Therefore $u^1_m$ incurs the least cost and $\min_{u_m} C(u_m) = C(u^1_m)$.

There are several ways of finding the shortest $m$ step path from $L_n$ to $L_0$. In [1] the shortest path is found using matrices, and it can be seen to be extendable to the shortest $m$ step path. Another more efficient method is to label $L_n = 0$ and all the other $L_j = \infty$, then relabel each $L_j$ $m$ times starting with $L_0$ and taking them in order each time. The label on $L_j$ each time is $\min \{L_j - [L_j, L_1]\}$ where $[L_j, L_1]$ as before is the length of the arc from $L_j$ to $L_1$. This is equivalent to remembering only the $n$th row or column from $A^2$ to $A^m$ where $A$ is the matrix associated with $G$ as described [1]. Now in order to find the path we at the $k$th step associate with each vertex $L_1$ the $k$ step path used to obtain the labels of $L_1$. When $L_1$ is being relabeled $j$ and the path used to get to $L_j$ is the new path associated with $L_1$, where $j$ is determined by the minimum function. It can also be seen that we need only use $n-m$ locations at
a time for remembering paths since the path associated with \( L_k \) on the kth step is just \( L_n^{k}, \ldots, L_k \) and since the path associated with \( L_j \) for \( j \neq m-k \) need not be remembered since it will never be used or needed. Similarly only \( n-m \) locations need be remembered for the labels of the \( L_i \) since as before we calculate the label for \( L_{m-1}, \ldots, L_n^{2} \) then \( L_{m-2}, \ldots, L_n \) etc., until \( L_m \) is labeled.

3. Generalization

Now let us relax assumptions B and C. In their stead we will say that there are given probabilities of demand, \( p_i \), for each \( L_i \). We also have a function \( c(L_i, L_j) \) of the endpoints that correspond to inventory or space costs of keeping a supply of standard lengths, \( L_i \), of the product in order to meet the expected demand for lengths greater than \( L_j \) and less than or equal to \( L_i \) during a period of time, where \( L_i \) and \( L_j \) are assumed to be standards with no standards between them. It could also correspond to the initial cost of building a container for the standard length \( L_i \). Thus in G let \( [L_i, L_j] = c(L_i, L_j) + \sum_{n=j+1}^{i-1} p_n f(L_n, L_n) \).

Now using this graph we generalize a number of ways. First, if the problem, as before, requires that we use exactly \( m \) standards, we let \( [L_i, L_j] = \infty \) and \( C(u_m) = L(u_m) \) and \( \min_{u_m} C(u_k) = C(u_m') \) using the notation of the Theorem. Second, if we require the use of \( m \) or less standards, i.e., \( \min_{u_m} C(u_k) \) we then let \( [L_i, L_j] = 0 \) in G and use the same algorithm as before for the \( m \) step path only permitting these loops to be used.

Then if \( u_m' \) denotes the shortest \( m \) or less step path, \( \min_{u_k} C(u_k) = C(u_m') \).

Third, if we rid ourselves entirely of \( m \) and just require that the cost be minimized regardless of the number of standards, \( [L_i, L_j] = 0 \) and the
last equation becomes \( \min_u C(u) = C(u') \) where the minimum is taken over all \( u \) and \( u' \) is the shortest path and may be found by any of the standard methods. Fourth, if space or inventory costs are due to just the number of standards regardless of which of the \( L_i \) are standards, we let \([L_i, L_j] = \infty\) again in \( G \) and calculate \( \min_{u_m} C(u_m) \) for each \( m \). Then \( C(u) = \min_{m} C(u_m') + I(m) \) where \( u \) is the set of standards incurring the least total cost.

4. Example

Let \( L_1 = 1, L_2 = 2, L_3 = 3, L_4 = 6 \)
\[ p_1 = 1/4, k_1 = 2 \quad \text{and} \quad \lambda(L_i, L_j) = 1 - j - l \cdot k_i \]
\[ p_2 = 1/8, k_2 = 1 \]
\[ p_3 = 1/8, k_3 = 1 \]
\[ p_4 = 1/2, k_4 = 2 \]

So the graph is

Therefore the shortest path is \( L_4, L_3, L_2, L_0 \) and the cost is 5 1/2.
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