TECHNICAL REPORT: MATHEMATICS

THE DUAL SIMPLEX METHOD AND ITS APPLICATION
TO THE SYNTHESIS OF MINIMAL WEIGHTS
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THE DUAL SIMPLEX METHOD AND ITS APPLICATION
TO THE SYNTHESIS OF MINIMAL WEIGHTS

by

SZE-TSEN HU

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Lockheed

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FOREWORD

This study was performed by Dr. Sze-Tsen Hu, Professor of Mathematics, University of California at Los Angeles, while acting as consultant to the Electronic Sciences Laboratory, Lockheed Missiles & Space Company, Sunnyvale, California.
ABSTRACT

This report describes an algorithm for applying the dual simplex method in linear programming to the problem of determining whether or not a given switching function is linearly separable. Further, where separability is possible, a description is given of the application of the dual simplex method to finding the most economic system of weights and threshold.

A detailed elementary exposition of the dual simplex method is given in Sections 2 through 4. A completely worked out illustrative numerical example is presented in Section 5. In Sections 6 through 8, the dual simplex method is applied to the problem of determining minimal weights and threshold; this is illustrated by a numerical example which reduces to the example given in Section 5.
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Section 1
INTRODUCTION

Linear programming was first applied to the general synthesis problem of linear separability of switching functions by R. C. Minnick (Ref. 1). Using Minnick's method for a switching function of \( n \) variables, it becomes necessary to solve \( 2^n \) equations in \( 2(n + 1 + 2^n) \) nonnegative variables in such a way that the cost function will be minimized (Ref. 1, p. 11). For \( n = 7 \), which is the lowest dimension of the problem not yet solved, it is necessary to solve 128 equations in 272 nonnegative variables for each of the \( 2^{128} \) switching functions of seven variables. This is by far too large a task for even the best electronic computers to accomplish.

In a joint paper of S. Muroga, I. Toda, and S. Takasu (Ref. 2), which appeared two months following Ref. 1, linear programming was also applied to the general synthesis problem. Unlike Minnick, the authors of Ref. 2 introduced a reduction process which reduced the number of inequalities by applying the deletion law in Boolean algebra. This method is considerably simpler than that of Minnick but not simple enough for actual computation.

In an earlier report by the author (Ref. 3), linear programming was applied to the Willis synthesis method (Ref. 4) for finding canonical separating systems; the canonical partial order was fully used and, hence, the number of inequalities reduced to a minimum. The solution of the resulting system of inequalities for any given regular switching function of \( n \) variables depends completely on that of a system of \( n + 2 \) linear equations in accordance with a well-known theorem on linear inequalities. The problem of solving this system of \( n + 2 \) linear equations is made into a canonical minimum problem of linear programming by introducing one more variable. Then, the usual simplex method is applied to this canonical minimum problem.
In another of the author's reports (Ref. 5), the problem of finding the most economic separating systems was formulated in the form of a standard minimum problem of linear programming. It was proved in Ref. 5 that this standard minimum problem has optimal solutions if and only if it is feasible. However, the usual method in linear programming for solving a standard minimum problem consists of two major steps: first, one must determine whether or not the problem is feasible and, if so, find a feasible solution of the problem; second, using the feasible solution obtained in the first step, one can find an optimal solution of the problem by means of the usual simplex method. The first step can be accomplished by the method given in Ref. 3 which involves another application of the simplex method. It appears that one has to use the simplex method twice to get a set of minimal weights and threshold. Fortunately, this apparent complication is by no means inevitable. In fact, in a recent report by the author (Ref. 6), an algorithm was described for solving this problem by applying the usual simplex method on its dual maximum problem.

The objective of the present report is to give a simpler direct method of solving the minimization problem by making use of the dual simplex method of C. E. Lemke (Ref. 7). The minimization problem is always dually feasible no matter whether the given regular switching function is linearly separable or not. Hence, the problem can be solved by applying the dual simplex once, not twice.

For the convenience of the reader, a detailed exposition of the dual simplex method in the form suitable for our application is given in Sections 2 through 4. An illustrative numerical example is presented in Section 5. In Sections 6 and 7, we will review briefly the preliminaries and formulate the synthesis and minimization problem. The solution of the problem by means of the dual simplex method is given and illustrated in Section 8.
Section 2

STANDARD PROBLEM IN LINEAR PROGRAMMING

A standard problem in linear programming is to find nonnegative real numbers which minimize (or maximize) a given linear function subject to a given system of linear inequalities. Since a maximum problem is reduced to a minimum problem simply by multiplying the given linear function with -1, we will describe and study the standard minimum problem only.

For this purpose, let us consider a given linear function

$$a_0, o + \sum_{j=1}^{q} a_{o,j} t_j$$

(2.1)

of q variables $t_1, \ldots, t_q$, where $a_0, o, a_{o,1}, \ldots, a_{o,q}$ are given real numbers. On the other hand, let

$$a_1, o + \sum_{j=1}^{q} a_{i,j} t_j \geq 0$$

(2.2)

where $i = 1, 2, \ldots, p$, be a given system of p inequalities in the same variables $t_1, \ldots, t_q$ with given real coefficients $a_{i,j}$ and constant terms $a_{i,o}$. Then, the standard minimum problem in linear programming is the problem of finding nonnegative real numbers $t_1, \ldots, t_q$ which minimize the given linear function (2.1) subject to the system (2.2) of p linear inequalities.

Now, let $y$ denote the given linear function (2.1) and let $x_i, (i = 1, 2, \ldots, p)$, denote the linear function on the left side of the inequality (2.2). Consider $y, x_1, \ldots, x_p$ also as variables. Then, the standard minimum problem described above can be restated as follows:
Problem: Find the minimum value of the variable $y$ subject to the following $p + 1$ linear equations:

$$y = a_{o, o} + \sum_{j=1}^{q} a_{o, j} t_j \quad (2.3)$$

$$x_i = a_{i, o} + \sum_{j=1}^{q} a_{i, j} t_j \quad (2.4)$$

where $i = 1, 2, \ldots, p$, and the condition that

$$x_i \geq 0, \quad (i = 1, 2, \ldots, p), \quad (2.5)$$

$$t_j \geq 0, \quad (j = 1, 2, \ldots, q). \quad (2.6)$$

Let $A$ denote the $p + 1$ by $q + 1$ matrix

$$A = \begin{bmatrix} a_{i, j} \end{bmatrix} \quad (2.7)$$

where $0 \leq i \leq p$ and $0 \leq j \leq q$.

A column $\beta$ in $A$ is said to be (lexicographically) positive,

$$\beta \geq 0$$

provided that the first nonzero entry of $\beta$, counting from top down, is positive. A column $\beta$ in $A$ is said to be (lexicographically) greater than another column $\gamma$ in $A$ provided that

$$\beta - \gamma > 0.$$ 

Hereafter, we will omit the word "lexicographically" when there is no danger of misunderstanding.
The matrix \( A \) is said to be dually feasible in case the \( q \) columns

\[
\alpha_j = (a_{0,j}^0, a_{1,j}^0, \ldots, a_{p,j}^0) \geq 0
\]

where \( j = 1, 2, \ldots, q \). Note that the leading column \( \alpha_0 \) in \( A \) is not required to be positive or zero for the dual feasibility of \( A \). In particular, \( A \) is dually feasible if

\[
a_{0,j}^0 > 0, \quad (j = 1, 2, \ldots, q).
\]

If the matrix \( A \) is dually feasible, then our standard minimum problem is also said to be dually feasible.

A trial solution of our problem is obtained by setting

\[
t_j = 0, \quad (j = 1, 2, \ldots, q)
\]

in the \( p + 1 \) linear equations (2.3) and (2.4). Hence, the trial solution of our problem is given by

\[
y = a_{0,0}^0
\]

\[
x_i = a_{1,0}^0, \quad (i = 1, 2, \ldots, p)
\]

\[
t_j = 0, \quad (j = 1, 2, \ldots, q).
\]

If \( A \) is dually feasible, then \( y = a_{0,0}^0 \) is its minimum value for all nonnegative values of the \( q \) variables \( t_j, (j = 1, 2, \ldots, q) \). However, unless

\[
a_{1,0}^0 \geq 0, \quad (i = 1, 2, \ldots, p),
\]

the trial solution obtained above is not a solution of our problem because the condition (2.5) is not satisfied. In fact, we have the following:
Theorem 2.1. The trial solution

\[ y = a_0, \quad x_i = a_i, \quad t_j = 0 \]

where \( i = 1, 2, \ldots, p \) and \( j = 1, 2, \ldots, q \), is an optimal solution of the standard minimum problem if and only if \( A \) is dually feasible and

\[ a_{i, o} \geq 0, \quad (i = 1, 2, \ldots, p). \]
Section 3
DUAL SIMPLEX METHOD

The dual simplex method for a dually feasible, standard minimum problem with matrix

\[
A = \begin{bmatrix}
  a_{i,j}
\end{bmatrix}
\]  

(3.1) & (2.7)

where \(0 \leq i \leq p\) and \(0 \leq j \leq q\), is a finite process of transforming the given problem into equivalent standard minimum problems with matrices

\[
A_0, A_1, A_2, \ldots, A_r
\]

respectively described so that the trial solution of the last problem with matrix \(A_r\) is an optimal solution of the problem.

To insure that all variables remain nonnegative in the final trial solution, we adjoin to the system (2.3) and (2.4) of linear equations in Section 2 the identical equations

\[
t_j = t_j, \quad (j = 1, 2, \ldots, q).
\]

(3.2)

The coefficients and constant terms of the right members of the system (2.3), (2.4) and (3.2) form the following matrix:
In more compact notation, we have

\[ A_O = \begin{bmatrix} 0 & I \\ \end{bmatrix} \]  

\[ A^0 \]

where A denotes the p + 1 by q + 1 matrix (3.1), 0 denotes the q by 1 matrix which consists of q zeroes, and I denotes the q by q unit matrix. Therefore, \( A^0 \) is a p + q + 1 by q + 1 matrix. Let us label the rows of \( A_O \) by the integers \( i = 0, 1, \ldots, p+q \) and the columns of \( A_O \) by the integers \( j = 0, 1, \ldots, q \).

Denote by

\[ a_{i,j}, \quad (0 \leq i \leq p+q, \ 0 \leq j \leq q) \]

the element of \( A_O \) at the \( i \)-th row and the \( j \)-th column.

On the left member of the identical equation (3.2), let us use the symbol

\[ x_{p+j}, \quad (1 \leq j \leq q) \]

for the variable \( t_j \). Then, our standard minimum problem is equivalent to the standard minimum problem with matrix \( A^0 \) which is to find the minimum value of the variable \( y \) subject to the following \( p + q + 1 \) linear equations:

\[ \text{Minimize } y \text{ subject to } \]

\[ \begin{align*}
0 & \leq y \\
0 & \leq y \\
0 & \leq y \\
0 & \leq y \\
\end{align*} \]
\[ y = a_{o,0} + \sum_{j=1}^{q} a_{o,j} t_j \]  
(3.4)

\[ x_1 = a_{1,0} + \sum_{j=1}^{q} a_{1,j} t_j \]  
(3.5)

where \( i = 1, 2, \ldots, p + q \) and the condition that

\[ x_i \geq 0, \quad (i = 1, 2, \ldots, p + q). \]  
(3.6)

Since the given problem \( A \) is assumed to be dually feasible and since \( A_0 \) is obtained by adjoining \( q \) nonnegative rows to \( A \), it is clear that \( A_0 \) is also dually feasible.

The variables \( t_1, t_2, \ldots, t_q \) on the right members of (3.4) and (3.5) are traditionally called the nonbasic variables of the problem \( A_0 \). By the definition of the new variables \( x_{p+j}, (1 \leq j \leq q) \), introduced above, we know that the nonbasic variables \( t_1, \ldots, t_q \) also appear on the left members of (3.5) as \( q \) of the \( p + q \) variables \( x_1, \ldots, x_{p+q} \). Because of this fact, the condition (3.6) implies that \( t_j \geq 0 \) for every \( j = 1, 2, \ldots, q \).

Now, let \( k \) be a given positive integer and assume that our standard minimum problem \( A \) has been reduced to an equivalent standard minimum problem with matrix

\[ A_{k-1} = \| b_{i,j} \| \]

\( (0 \leq i \leq p + q, \ 0 \leq j \leq q) \), which is to find the minimum value of the variable \( y \) subject to the following \( p + q + 1 \) linear equations:

\[ y = b_{o,0} + \sum_{j=1}^{q} b_{o,j} u_j \]  
(3.7)
where \( i = 1, 2, \ldots, p + q \) and the condition
\[
x_i \geq 0, \quad (i = 1, 2, \ldots, p + q).
\] (3.9)

Furthermore, we assume that \( A_{k-1} \) is dually feasible and that the nonbasic variables \( u_1, \ldots, u_q \) on the right members of (3.7) and (3.8) also appear on the left members of (3.8) as \( q \) of the \( p + q \) variables \( x_1, \ldots, x_{p+q} \). Because of this, the condition (3.9) implies that \( u_j \geq 0 \) for every \( j = 1, 2, \ldots, q \).

If \( b_{i_0, 0} \geq 0 \) for every \( i = 1, 2, \ldots, p + q \), then the trial solution
\[
y = b_{0, o},
\]
\[
x_i = b_{i_0, 0}, \quad (i = 1, 2, \ldots, p + q),
\] (3.10), (3.11)
is an optimal solution of the standard minimum problem with matrix \( A_{k-1} \) according to Theorem 2.1. Since the standard minimum problem with matrix \( A_{k-1} \) is equivalent to our original standard minimum problem with matrix \( A \), it follows that the trial solution given by (3.10) and (3.11) is also an optimal solution of our original problem. In this case, we have obtained an optimal solution of our original problem and hence the problem is solved affirmatively.

Now let us assume that \( A_{k-1} \) is dually feasible but not all of the \( b_{i_0, 0} \)'s, \( (1 \leq i \leq p + q) \), are nonnegative. Choose some \( i = i_0 \), \( (1 \leq i_0 \leq p + q) \), with
\[
b_{i_0, 0} \leq 0.
\] (3.12)

Consider the other elements \( b_{i_0, j} \) of the row in \( A_{k-1} \) labelled by \( i_0 \). If
\[
b_{i_0, j} \leq 0, \quad (j = 1, 2, \ldots, q)
\] (3.13)
then the negative value \( x_{i_0} = b_{i_o,0} \) is the maximal possible value of \( x_{i_0} \), subject to the condition

\[
u_j \geq 0, \quad (j = 1, 2, \ldots, q).
\]

(3.14)

In this case, the standard minimum problem with matrix \( A_{k-1} \) has no optimal solution; in fact, the system (3.8) of \( p + q \) linear equations has no nonnegative solution in \( x_1, \ldots, x_p+q \). Since the standard minimum problem with matrix \( A_{k-1} \) is equivalent to our original standard minimum problem with matrix \( A \), it follows that our original problem has no optimal solution if there exists an integer \( i_0 \) such that

(3.12) and (3.13) both hold. In this case, the system (2.4) of \( p \) linear equations fails to have nonnegative solution in \( x_1, \ldots, x_p \) and \( t_1, \ldots, t_j \). Hence, the problem is solved negatively in this case.

Next, let us assume that (3.13) is false. Consider the columns \( \beta_j \) of \( A_{k-1} \) with

\[
b_{i_o,j} > 0
\]

(3.15)

and select from these a column \( \beta_{j_0} \) of \( A_{k-1} \) such that

\[
\frac{\beta_{j_0}}{b_{i_o,j_0}} \leq \frac{\beta_j}{b_{i_o,j}}
\]

(3.16)

in the lexicographical order defined in Section 2 for all \( j \) satisfying (3.15). In particular, we have

\[
b_{i_o,j_0} > 0.
\]

(3.17)

The fundamental operation used to derive \( A_k \) from \( A_{k-1} \) is the Gaussian elimination, traditionally called pivoting (on rows) in the theory of linear programming. Once the integers \( i_0 \) and \( j_0 \) have been chosen as above, the element \( b_{i_o,j_o} \) is called the pivot element.
The Garrett elimination with \( b_{i_0}, j_0 \) as the pivot element can be described as follows. We first solve the \( i_0 \)-th equation of the system (3.8) for \( u_{j_0} \) in terms of \( x_{i_0} \) and the remaining \( u_j \)'s; in other words, we write the \( i_0 \)-th equation of (3.8) in the following form:

\[
 u_{j_0} = \frac{1}{b_{i_0}, j_0} \left( -b_{i_0}, j_0 \sum_{j \neq j_0} b_{i_0}, j \ u_j + x_{i_0} \right). \tag{3.18}
\]

Then substitute \( u_{j_0} \) in each equation of (3.7) and (3.8) by the right member of (3.18). Thus, the variables \( y \) and \( x_1, \ldots, x_{p+q} \) are expressed as linear functions of the \( q \) variables

\[
 u_1, \ldots, u_{j_0-1}, x_{i_0}, u_{j_0+1}, \ldots, u_q. 
\]

For the sake of neatness, let us denote these \( q \) variables by \( v_1, \ldots, v_q \); in other words, let

\[
 v_j = \begin{cases} 
 u_j, & \text{(if } j \neq j_0) \\
 x_{i_0}, & \text{(if } j = j_0) 
\end{cases}. 
\]

The result of the Garrett elimination with \( b_{i_0}, j_0 \) as pivot element is that the equations (3.7) and (3.8) are transformed into an equivalent system consisting of \( p + q + 1 \) linear equations of the following form:

\[
 y = c_{0, o} + \sum_{j=1}^{q} c_{0, j} \ v_j \tag{3.19}
\]

\[
 x_i = c_{1, o} + \sum_{j=1}^{q} c_{1, j} \ v_j \tag{3.20}
\]

where \( i = 1, 2, \ldots, p + q \).
Since the new nonbasic variables $v_1, \ldots, v_q$ are the variables $u_1, \ldots, u_{j_0-1}, x_{i_0},\ldots, u_{j_0+1}, \ldots, u_q$, they also appear on the left members of (3.20) as $p+q$ of the $x_1, \ldots, x_{p+q}$. Let

$$A_k = \begin{bmatrix} c_{i,j}\end{bmatrix}$$

$(0 \leq i \leq p+q, 0 \leq j \leq q)$ denote the matrix of constant terms and coefficients on the right members of (3.19) and (3.20).

Since (3.19) and (3.20) are derived by substituting (3.18) into (3.7) and (3.8), the elements $c_{i,j}$ of $A_k$ can be easily computed from those of $A_{k-1}$ as follows: For each $j = 0, 1, \ldots, q$, let $\beta_j$ and $\gamma_j$ denote the $j$-th column in $A_{k-1}$ and $A_k$ respectively, that is to say,

$$\beta_j = \left( b_{o,j}, b_{1,j}, \ldots, b_{p+q,j} \right)$$

$$\gamma_j = \left( c_{o,j}, c_{1,j}, \ldots, c_{p+q,j} \right) .$$

Then, one can easily verify that

$$\gamma_j = \begin{cases} \beta_j \cdot \frac{b_{i_0,j}}{b_{i_0,j}^2}, & \text{(if } j \neq j_0) \\ \beta_j \cdot \frac{1}{b_{i_0,j}}, & \text{(if } j = j_0) \end{cases} .$$

In words, the matrix $A_k$ can be obtained from the matrix $A_{k-1}$ as follows: First, divide the $j_0$-th column $\beta_{j_0}$ of $A_{k-1}$ by the pivot element $b_{i_0,j_0}$; second, subtract from every other column $\beta_j$, $j \neq j_0$, this new $j_0$-th column multiplied by $b_{i_0,j}$. The result is the matrix $A_k$.

Note that, in the resulting matrix $A_k$, we have $c_{i_0,j} = 1$ and $c_{i_0,j} = 0$ for every $j \neq j_0$. Therefore, the second step in the preceding paragraph can be interpreted.
as to subtract from every column $\beta_{j'}$, $j \neq j_0$, a multiple of the new $j_0$-th column so that its element $c_{i_0,j}$ in the $i_0$-th row vanishes.

Having determined the columns of the new matrix $A_k$, we will prove that this matrix $A_k$ is still dually feasible. For this purpose, let $j$ be an arbitrary integer satisfying $1 \leq j \leq q$. We will prove that $\gamma_j \geq 0$ lexicographically.

To do this, let us first consider the case $j = j_0$. In this case, since

$$\beta_{j_0} > 0, \quad b_{i_0,j_0} > 0$$

by the dual feasibility of $A_{k-1}$ and (3.17), we have

$$\gamma_{j_0} = \frac{1}{b_{i_0,j_0}} (\beta_{j_0}) > 0.$$ 

Next, assume that $j \neq j_0$. Then we have

$$\gamma_j = \beta_{j_0} - \frac{b_{i_0,j_0}}{b_{i_0,j_0}} \beta_{j_0}$$

(3.21)

$$= b_{i_0,j} \left( \frac{\beta_{j_0} - \beta_{j_0}}{b_{i_0,j}} \right).$$

(3.22)

If $b_{i_0,j} \leq 0$, then it follows from (3.21) that

$$\gamma_j \geq \beta_{j_0} \geq 0.$$ 

If $b_{i_0,j} > 0$, then (3.16) holds. Hence, by (3.22), we have

$$\gamma_j \geq 0.$$ 

This completes the proof that $A_k$ is dually feasible.
Thus, our given dually feasible standard minimum problem with matrix $A$ has been reduced to an equivalent dually feasible minimum problem with matrix $A_k = ||c_{i,j}||$, $(0 \leq i \leq p + q, 0 \leq j \leq q)$, which is to find the minimum value of the variable $y$ subject to the $p + q + 1$ linear equations (3.19) and (3.20) and the condition (3.9). Since the new nonbasic variables $v_1, \cdots, v_q$ also appear on the left members of (3.20) as $q$ of the $p + q$ variables $x_1, \cdots, x_{p+q}$, we have

$$v_j \geq 0, \quad (j = 1, 2, \cdots, q). \quad (3.23)$$

This completes the inductive construction of the sequence of equivalent dually feasible minimum problems with matrices

$$A_0, A_1, \cdots, A_{k-1}, A_k, \cdots \quad (3.24)$$

It remains to prove the finiteness of the sequence (3.24). For this purpose, let us consider the leading column $\gamma_o$ in the matrix $A_k$. By the dual feasibility of $A_{k-1}$ and (3.17), we have

$$\beta_{j_0} > 0. \quad (3.25)$$

On the other hand, it follows from (3.12) and (3.17) that

$$-\frac{b_{i_0}^o}{b_{i_0}^o, j_0} > 0. \quad (3.26)$$

By (3.25) and (3.26), we have

$$\gamma_o = \beta_o - \frac{b_{i_0}^o}{b_{i_0}^o, j_0} \beta_{j_0} > \beta_o. \quad (3.27)$$

Hence, as $k$ increases, the leading column in $A_k$ increases strictly in the lexicographical order. Since there is only a finite number of possible sets of $q$ nonbasic variables chosen from the $p + q$ variables $x_1, \cdots, x_{p+q}$ and any choice uniquely determines the leading column, the process must stop at some $A_k$. 3-9
This can happen either if there is no negative element in the leading column $\gamma_o$ of $A_k$, or if there is a negative element

$$c_{i_0,0} < 0$$

in the leading column $\gamma_o$ of $A_k$ but no positive element in the $i_0$-th row of $A_k$; i.e.,

$$c_{i_0,j} \leq 0, \quad (0 \leq j \leq q).$$

In the first case, the trial solution

$$y = c_{0,0}$$

$$x_i = c_{i,0}, \quad (1 \leq i \leq p + q)$$

of the problem with matrix $A_k$ is an optimal solution of our given standard minimum problem with matrix $A$. Hence, in this case, the problem is solved affirmatively.

In the second case, the negative value $c_{i_0,0}$ is the largest value of the variable $x_{i_0}$ and consequently our standard minimum problem with matrix $A$ has no nonnegative solution. Hence, in this case, the problem is solved negatively.
For computational purpose, the dual simplex method, which is described and proved in Section 3, can be worked exclusively on the matrices involved.

The given standard minimum problem is represented by the matrix

\[ A = \begin{bmatrix} a_{1,j} \end{bmatrix}, \quad (0 \leq i \leq p, \ 0 \leq j \leq q) \]

The dual simplex method is essentially a finite process of deriving from \( A \) a sequence of \( p + q + 1 \) by \( q + 1 \) matrices:

\[ A_0, A_1, \ldots, A_r \]

The initial matrix \( A_0 \) is given by

\[ A_0 = \begin{bmatrix} A \end{bmatrix} \]

as described in Section 3. In the actual computation, we will write \( A_0 \) in the form of the following tableau \( T_0 \) which will be called the initial tableau:
For each \( k = 1, 2, \ldots, r \), the matrix \( A_k \) is represented by a similar tableau \( T_k \) in which the columns are headed by const., \( v_1, v_2, \ldots, v_q \), the new nonbasic variables. The operation of deriving \( T_k \) from \( T_{k-1} \) is described as follows:

Consider the tableau \( T_{k-1} \) which has been constructed by the preceding step:

<table>
<thead>
<tr>
<th>Const.</th>
<th>( t_1 )</th>
<th>( t_2 )</th>
<th>( \ldots )</th>
<th>( t_q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( a_{o,0} )</td>
<td>( a_{o,1} )</td>
<td>( a_{o,2} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( a_{1,0} )</td>
<td>( a_{1,1} )</td>
<td>( a_{1,2} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>( a_{2,0} )</td>
<td>( a_{2,1} )</td>
<td>( a_{2,2} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( x_p )</td>
<td>( a_{p,0} )</td>
<td>( a_{p,1} )</td>
<td>( a_{p,2} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( x_{p+1} )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( x_{p+2} )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( x_{p+q} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

\( (T_k) \)

\( (T_{k-1}) \)

<table>
<thead>
<tr>
<th>Const.</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( \ldots )</th>
<th>( u_q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( b_{o,0} )</td>
<td>( b_{o,1} )</td>
<td>( b_{o,2} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( b_{1,0} )</td>
<td>( b_{1,1} )</td>
<td>( b_{1,2} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>( b_{2,0} )</td>
<td>( b_{2,1} )</td>
<td>( b_{2,2} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( x_p )</td>
<td>( b_{p,0} )</td>
<td>( b_{p,1} )</td>
<td>( b_{p,2} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( x_{p+1} )</td>
<td>( b_{p+1,0} )</td>
<td>( b_{p+1,1} )</td>
<td>( b_{p+1,2} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( x_{p+2} )</td>
<td>( b_{p+2,0} )</td>
<td>( b_{p+2,1} )</td>
<td>( b_{p+2,2} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( x_{p+q} )</td>
<td>( b_{p+q,0} )</td>
<td>( b_{p+q,1} )</td>
<td>( b_{p+q,2} )</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

4-2
If the constant column in $T_{k-1}$ contains no negative entry, then it gives an optimal solution of the given standard minimum problem. Precisely, the minimum value of the linear function

$$y = a_{0,0} + \sum_{j=1}^{q} a_{0,j} t_j$$

is $b_{0,0}$ reached when

$$t_j = x_{p+j} = b_{p+j,0}$$

for every $j = 1, 2, \ldots, q$.

Otherwise, there is at least one entry in the constant column of $T_{k-1}$ which is negative. Choose an $i_0$ with

$$b_{i_0,0} < 0.$$ 

Then consider the row of $T_{k-1}$ which is headed by $x_{i_0}$. If this row contains no positive entry, then our given problem has no solution. Otherwise, consider the columns $\beta_j$ of $T_{k-1}$ such that

$$b_{i_0,j} > 0$$

and select from these a column $\beta_{i_0}$ of $T_{k-1}$ such that

$$\frac{\beta_{i_0,j}}{b_{i_0,j}} \leq \frac{\beta_j}{b_{i_0,j}}$$

for every $j$ satisfying $b_{i_0,j} > 0$. Thus, we have obtained the pivot element $b_{i_0,j_0}$. In numerical computations, the pivot element is usually marked by a circle around the numerical entry in the tableau $T_{k-1}$.
Having chosen the pivot element $b^i_{o,j^o}$ in the tableau $T_{k-1}$, we can construct the next tableau $T_k$ as follows:

First, change the nonbasic variable $u_{j^o}$ into $x_{i^o}$. Thus, we obtain the nonbasic variables $v_1, \ldots, v_q$ given by

$$v_j = \begin{cases} u_j, & \text{(if } j \neq j^o) \\ x_{i^o}, & \text{(if } j = j^o). \end{cases}$$

Next, the column $\gamma_{j^o}$ in $T_k$ headed by $v_{j^o} = x_{i^o}$ is obtained by dividing $\beta_{j^o}$ by the pivot element $b^i_{o,j^o}$; in symbols,

$$\gamma_{j^o} = \frac{\beta_{j^o}}{b^i_{o,j^o}}.$$

Next, the constant column $\gamma_o$ in $T_k$ is obtained from $\beta_o$ by subtracting $b^i_{o,0} \gamma_{j^o}$; in symbols,

$$\gamma_o = \beta_o - b^i_{o,0} \gamma_{j^o}.$$

Since $b^i_{o,0}$ is negative, we actually have

$$\gamma_o = \beta_o + \left( \frac{b^i_{o,0}}{b^i_{o,0}} \right) \gamma_{j^o}.$$

Finally, the column $\gamma_j$ in $T_k$ headed by $v_j = u_j$, $j \neq j^o$, is obtained from $\beta_j$ by subtracting $b^i_{o,j} \gamma_{j^o}$; in symbols,

$$\gamma_j = \beta_j - b^i_{o,j} \gamma_{j^o}.$$

This completes the construction of $T_k$. 

4-4

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Section 5
AN ILLUSTRATIVE EXAMPLE

In the present section, let us study the standard minimum problem of finding non-negative real numbers

\[ t_1, t_2, t_3, t_4, t_5, t_6 \]

which minimize a given linear function

\[ y = t_1 + t_2 + t_3 + t_4 + t_5 + t_6 \]

subject to a system of 10 linear inequalities

\[
\begin{align*}
    t_6 - t_5 & \geq 0 \\
    t_6 - t_1 - t_4 & \geq 0 \\
    t_6 - t_2 - t_3 & \geq 0 \\
    t_1 + t_2 + t_3 - t_6 - 1 & \geq 0 \\
    t_2 + t_4 - t_6 - 1 & \geq 0 \\
    t_1 + t_5 - t_6 - 1 & \geq 0 \\
    t_2 - t_1 & \geq 0 \\
    t_3 - t_2 & \geq 0 \\
    t_4 - t_3 & \geq 0 \\
    t_5 - t_4 & \geq 0 
\end{align*}
\]

Introduce slack variables \( x_i \) \((1 \leq i \leq 10)\), for the left members of the inequalities. Thus, the given problem is reduced to find the minimum value of the variable \( y \) subject to the following eleven equations:
\[y = t_1 + t_2 + t_3 + t_4 + t_5 + t_6\]
\[x_1 = t_6 - t_5\]
\[x_2 = t_6 - t_1 - t_4\]
\[x_3 = t_6 - t_2 - t_3\]
\[x_4 = t_1 + t_2 + t_3 - t_6 - 1\]
\[x_5 = t_2 + t_4 - t_6 - 1\]
\[x_6 = t_1 + t_5 - t_6 - 1\]
\[x_7 = t_2 - t_1\]
\[x_8 = t_3 - t_2\]
\[x_9 = t_4 - t_3\]
\[x_{10} = t_5 - t_4\]

and the condition that

\[x_i \geq 0, \quad (1 \leq i \leq 10)\]
\[t_j \geq 0, \quad (1 \leq j \leq 6).\]

The matrix \(A\) of the right members of these equations is as follows:

\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & -1 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & -1 & -1 & 0 & 0 & 1 \\
-1 & 1 & 1 & 1 & 0 & 0 & -1 \\
-1 & 0 & 1 & 0 & 1 & 0 & -1 \\
-1 & 1 & 0 & 0 & 0 & 1 & -1 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0
\end{bmatrix}
\]
Since all coefficients of the linear function $y$ are positive, the problem is dually feasible. Let

$$x_{10+j} = t_j, \quad (1 \leq j \leq 6).$$

Then we obtain the initial tableau $T_0$ as follows:

<table>
<thead>
<tr>
<th></th>
<th>Const.</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_3$</th>
<th>$t_4$</th>
<th>$t_5$</th>
<th>$t_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>-1</td>
<td>(1)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$x_5$</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$x_6$</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$x_7$</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_8$</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_9$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_{10}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$t_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$t_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$t_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$t_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$t_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$t_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
The constant column contains 3 negative entries in the rows headed by \( x_4 \), \( x_5 \), and \( x_6 \). Each of these rows contains some positive entries. Let us choose the row headed by \( x_4 \). This row contains 3 positive entries in the columns headed by \( t_1 \), \( t_2 \), and \( t_3 \). Of these 3 columns, the one headed by \( t_1 \) is the least in the lexicographical order and hence (3.16) is satisfied. Therefore, we have to choose the element 1 located in the row headed by \( x_4 \) and in the column headed by \( t_1 \) as the pivot element for the construction of the next tableau \( T_1 \).

Thus, we indicate the pivot element in \( T_0 \) by a circle around it. Then, by following the process of constructing \( T_k \) from \( T_{k-1} \) described in Section 4, we obtain our next tableau \( T_1 \), listed as follows:

<table>
<thead>
<tr>
<th></th>
<th>Const.</th>
<th>( x_4 )</th>
<th>( t_2 )</th>
<th>( t_3 )</th>
<th>( t_4 )</th>
<th>( t_5 )</th>
<th>( t_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>( \Box )</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( x_6 )</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_7 )</td>
<td>-1</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( x_8 )</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_9 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_{10} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( t_1 )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( t_3 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( t_4 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( t_5 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( t_6 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\( T_1 \)
The constant column of $T_1$ contains 3 negative entries in the rows headed by $x_2$, $x_5$ and $x_7$. Let us choose $x_2$. Then, by (3.16), the pivot element is in the column headed by $t_3$. Hence, we obtain the next tableau $T_2$ as follows:

<table>
<thead>
<tr>
<th>Const.</th>
<th>$x_4$</th>
<th>$t_2$</th>
<th>$x_2$</th>
<th>$t_3$</th>
<th>$t_5$</th>
<th>$t_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_5$</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_6$</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$x_7$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_8$</td>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_9$</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_{10}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$t_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$t_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$t_3$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
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<tr>
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<tr>
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</tr>
<tr>
<td>$t_6$</td>
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<td>0</td>
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<td>0</td>
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</tr>
</tbody>
</table>

The constant column of $T_2$ contains 4 negative entries in the rows headed by $x_3$, $x_5$, $x_6$ and $x_9$. Let us choose $x_3$. Then the pivot element is in the column headed by $t_6$. Thus, we obtain:
The constant column of $T_3$ contains 4 negative entries in the rows headed by $x_5$, $x_6$, $x_7$, and $x_9$. Let us choose $x_5$. Then the pivot element is in the column headed by $t_2$. Thus we obtain:

<table>
<thead>
<tr>
<th></th>
<th>Const.</th>
<th>$x_4$</th>
<th>$t_2$</th>
<th>$x_2$</th>
<th>$t_4$</th>
<th>$t_5$</th>
<th>$x_3$</th>
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<tbody>
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</tr>
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<td>$x_2$</td>
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</tr>
<tr>
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</tr>
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<td>-1</td>
</tr>
<tr>
<td>$x_6$</td>
<td>-1</td>
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<td>0</td>
<td>-1</td>
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<td>1</td>
<td>0</td>
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<tr>
<td>$x_7$</td>
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<td>1</td>
<td>0</td>
<td>0</td>
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<td>-1</td>
</tr>
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<td></td>
</tr>
<tr>
<td>$x_9$</td>
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<td>-1</td>
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<td>-1</td>
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</tr>
<tr>
<td>$x_{10}$</td>
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</tbody>
</table>
The constant column of $T_4$ contains 3 negative entries in the rows headed by $x_6$, $x_8$, and $t_3$. Let us choose $x_6$. Then the pivot element is in the column headed by $t_5$. Thus we obtain:

<table>
<thead>
<tr>
<th></th>
<th>Const.</th>
<th>$x_4$</th>
<th>$x_5$</th>
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<td>1</td>
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<td>1</td>
</tr>
</tbody>
</table>

The constant column of $T_5$ contains 2 negative entries in the rows headed by $x_8$ and $t_3$. Let us choose $x_8$. Then the pivot element is in the column headed by $t_4$.

Thus we obtain:
Since the constant column of \( T_6 \) contains no negative entry, we obtain an optimal solution of our problem; namely, the minimum value of the linear function
\[
y = t_1 + t_2 + t_3 + t_4 + t_5 + t_6
\]
subject to the given constraints is
\[
y = 16
\]
reached when

\[ \begin{align*}
  t_1 &= 1 \\
  t_2 &= 2 \\
  t_3 &= 2 \\
  t_4 &= 3 \\
  t_5 &= 4 \\
  t_6 &= 4
\end{align*} \]

This completes the illustrative example.

In the preceding example, we have always chosen the first negative entry, counting from top down next to the \( y \) row, in the constant column of the tableaux. For definiteness, one can make this a rule in practice especially when programming to electronic computers.
Section 6
SWITCHING FUNCTIONS

Let $Q$ denote the set which consists of the two integers 0 and 1. For any given integer $n \geq 1$, consider the Cartesian power

$$Q^n = Q \times \ldots \times Q$$

which is the Cartesian product of $n$ copies of $Q$. Thus, the elements of $Q^n$ are the $2^n$ ordered $n$-tuples

$$\left(x_1, x_2, \ldots, x_n\right)$$

where the $k$-th coordinate $x_k$ is in $Q$ for every $k = 1, 2, \ldots, n$. Hereafter, $Q^n$ will be called the $n$-cube and its $2^n$ elements will be called its points.

By a switching function (or truth function or Boolean function) of $n$ variables, we mean any subset $F$ of the $n$-cube $Q^n$. Since $Q^n$ has $2^n$ points, there are $2^{2^n}$ different switching functions of $n$ variables.

A switching function $F$ of $n$ variables is said to be linearly separable provided that there exist $n+1$ real numbers $w_1, w_2, \ldots, w_n, w_{n+1}$ such that, for every point $x = (x_1, \ldots, x_n)$ in $Q^n$, we have $x \in F$ if and only if

$$w_1 x_1 + w_2 x_2 + \ldots + w_n x_n \leq w_{n+1}.$$ 

The set $w = (w_1, w_2, \ldots, w_n, w_{n+1})$ is called a separating system of $F$; the real numbers $w_1, w_2, \ldots, w_n$ are called the weights, and the real number $w_{n+1}$ is called the threshold. By taking the threshold $w_{n+1}$ as small as possible while the weights $w_1, \ldots, w_n$ are held fixed, we may assume that, in case $F$ is not empty, there exists a point $x = (x_1, \ldots, x_n)$ in $F$ such that

$$w_1 x_1 + w_2 x_2 + \ldots + w_n x_n = w_{n+1}.$$
Consider the complement \( F' = \mathbb{R}^n - F \) of \( F \). For every point \( y = (y_1, \ldots, y_n) \) in \( F' \), we have
\[
 w_1 y_1 + w_2 y_2 + \ldots + w_n y_n > w_{n+1}.
\]

Let \( M \) denote the minimal value of
\[
 w_1 y_1 + w_2 y_2 + \ldots + w_n y_n - w_{n+1}
\]
for all points \( y = (y_1, \ldots, y_n) \) in \( F' \) in case \( F' \) is not empty. This positive real number \( M \) is called the margin of the separating system \( W \) (Ref. 8, p. 6). A separating system \( W = (w_1, \ldots, w_{n+1}) \) of \( F \) is said to be normal provided that \( M = 1 \). Every separating system \( W \) of \( F \) can be normalized by dividing each \( w_i \), \( (i = 1, 2, \ldots, n+1) \), by the margin \( M \) of \( W \). In particular, every linearly separable switching function \( F \) has a normal separating system.

Let \( W = (w_1, \ldots, w_{n+1}) \) be any normal separating system of a given linearly separable switching function \( F \) of \( n \) variables. Then, for every point
\[
 x = \left( x_1, \ldots, x_n \right) \in \mathbb{R}^n
\]
we have:
\[
 w_1 x_1 + \ldots + w_n x_n \leq w_{n+1} \quad \text{(if } x \in F) \]
\[
 w_1 x_1 + \ldots + w_n x_n \geq w_{n+1} + 1 \quad \text{(if } x \in F^c).
\]

By a canonical switching function of \( n \) variables, we mean a linearly separable switching function \( F \) of \( n \) variables which admits a separating system \( W = (w_1, \ldots, w_n, w_{n+1}) \) satisfying
\[
 0 \leq w_1 \leq w_2 \leq \ldots \leq w_n.
\]
In words, the weights \( w_1, w_2, \ldots, w_n \) in \( W \) are nonnegative and nondecreasing.
It is well-known (Refs. 9 and 10) that every linearly separable switching function \( F \) of \( n \) variables can be reduced to a unique canonical switching function by permuting and complementing a number of the variables.

By a weight function of \( n \) variables, we mean a homogenous linear function \( w : \mathbb{R}^n \to \mathbb{R} \) on the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) with real values. Precisely, there are \( n \) real numbers \( w_1, \ldots, w_n \) such that, for an arbitrary point \( x = (x_1, \ldots, x_n) \) of \( \mathbb{R}^n \), we have

\[
w(x) = w_1 x_1 + \ldots + w_n x_n.
\]

The real numbers \( w_1, \ldots, w_n \) are called the coefficients of the weight function \( w \).

A weight function \( w : \mathbb{R}^n \to \mathbb{R} \) with coefficients \( w_1, \ldots, w_n \) is said to be canonical provided

\[
0 \leq w_1 \leq w_2 \leq \ldots \leq w_n.
\]

By means of the canonical weight functions of \( n \) variables, we can define a partial order in the \( n \)-cube \( Q^n \) as follows: Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) be any two points of \( Q^n \); then we define \( x \leq y \) if and only if \( w(x) \leq w(y) \) for every canonical weight function \( w \) of variables. This partial order in \( Q^n \) is called the canonical partial order (Ref. 10).

Using the canonical partial order \( \leq \) in the \( n \)-cube, we can define the regular switching functions as follows: a switching function \( F \) of \( n \) variables is said to be regular if and only if it satisfies the regularity condition:

\[
\text{If } x \in F \text{ and } y \leq x, \text{ then } y \in F.
\]

Obviously, every canonical switching function of \( n \) variables is regular. In Ref. 11, it was proved that every regular switching function of \( n \leq 5 \) variables is canonical; in Ref. 12, an example is given which shows that not every regular switching function is canonical.
Section 7
SYNTHESIS AND MINIMIZATION

Let $F$ be an arbitrary switching function of $n$ variables. The synthesis problem for the linear separability of $F$ is to determine whether or not $F$ is linearly separable and to find a separating system $(w_1, \ldots, w_n, w_{n+1})$ for $F$ in case $F$ is linearly separable.

Among various synthesis methods for linear separability introduced in the literature, the one given by D.G. Willis (Ref. 4), turns out to be the most convenient because it involves as few linear inequalities as possible. In Ref. 4, the synthesis problem for the linear separability of arbitrary switching functions of $n$ variables was reduced to that of the regular switching functions of $n$ variables. Indeed, it remains to determine whether or not a given regular switching function $F$ is linearly separable and to find a canonical separating system $(w_1, \ldots, w_n, w_{n+1})$ for $F$ in case $F$ is linearly separable and hence canonical.

For the convenience of the reader, we describe briefly the Willis synthesis method as follows:

Let $F$ be an arbitrary nonempty regular switching function of $n$ variables as defined in Section 6. Let $L$ denote the set of all maximal points of $F$ with respect to the canonical partial order in the $n$-cube $Q^n$; and let $M$ denote the set of all minimal points of the complement $F' = Q^n - F$ with respect to the same canonical partial order in $Q^n$. Let

$$a_i = (a_{i1}, \ldots, a_{in})$$

be the points of $L$ and

$$a_i' = (a'_{i1}, \ldots, a'_{in})$$

be the points of $M$. Then the points $a_i$ and $a_i'$ can be ordered so that

$$a_1 < a_2 < \ldots < a_{\lambda} < a_{\lambda} < a_{\lambda + 1} < \ldots < a_{\lambda + \mu}$$

where $\lambda$ is the number of points in $L$ and $\mu$ is the number of points in $M$. The separating system is then defined by

$$w_1 = \sum_{i=1}^{\lambda} a_{i1}$$

$$w_n = \sum_{i=\lambda+1}^{\lambda+\mu} a'_{i1}$$

$$w_{n+1} = \sum_{i=\lambda+\mu+1}^{\lambda+2\mu} a_{i1}$$

$$w_{n+2} = \sum_{i=\lambda+2\mu+1}^{\lambda+3\mu} a'_{i1}$$

$$\vdots$$
be the points of $M$. Consider the following system of linear inequalities:

$$\begin{align*}
\sum_{k=1}^{n} a_{ik} w_k &\leq w_{n+1}, \quad (i = 1, 2, \ldots, \lambda) \\
\sum_{k=1}^{n} b_{jk} w_k &\geq w_{n+1} + 1, \quad (j = 1, 2, \ldots, \mu) \\
0 &\leq w_1 \leq w_2 \leq \ldots \leq w_n
\end{align*} \quad (7.1)$$

Then, the Willis synthesis theorem states that the given regular switching function $F$ is linearly separable if and only if the system (7.1) of linear inequalities has a solution (and hence an integral solution) in $w_1, \ldots, w_n$ and $w_{n+1}$.

In Ref. 13, the system (7.1) was solved by Fan's principle of bounding solutions. An elimination process for solving the system (7.1) was formulated in Ref. 14, and methods of successive approximation were applied to solve (7.1) in Ref. 15. Finally, the simplex method in linear programming was used in finding a solution of the system (7.1) in Ref. 3.

The next problem is naturally the minimization problem which is to find the most economical solution of the system (7.1) in case the given regular switching function $F$ is linearly separable. In other words, the minimization problem is to find a solution

$$\left( w_1, \ldots, w_n, w_{n+1} \right)$$

of the system (7.1) which makes some cost function

$$\varphi \left( w_1, \ldots, w_n, w_{n+1} \right)$$

minimal. The precise formulation of the problem is as follows:
First, let us pick the cost function $\phi$. Assume that the cost of realizing the $w_i$, $(i = 1, 2, \ldots, n, n+1)$, is proportional to the magnitude of $w_i$. Under this assumption, the cost function $\phi$ will be a homogeneous linear function

$$\phi(w_1, \ldots, w_{n+1}) = \sum_{i=1}^{n+1} \gamma_i w_i$$ (7.2)

where the coefficients $\gamma_1, \ldots, \gamma_{n+1}$ are nonnegative real numbers. In the literature, (see Ref. 16), two different cost functions have been studied; one of these is defined by $\gamma_i = 1$ for all $i = 1, 2, \ldots, n+1$ and the other is given by $\gamma_i = 1$ for all $i \leq n$ and $\gamma_{n+1} = 0$.

Having fixed the cost function $\phi$ by (7.2), the minimization problem for the given regular switching function $F$ is that of finding a canonical normal separating system which minimizes $\phi$ ; in other words, nonnegative real numbers

$w_1, w_2, \ldots, w_n, w_{n+1}$

are to be determined which minimize the cost function (7.2) and satisfy the system (7.1) of linear inequalities.

Next, let us write the system (7.1) of linear inequalities in the form of (2.2). Thus, the first $\lambda$ linear inequalities become

$$- \sum_{k=1}^{n} a_{ik} w_k + w_{n+1} \geq 0, \quad (i = 1, 2, \ldots, \lambda).$$ (7.3)

The next $\mu$ linear inequalities of (7.1) become

$$-1 + \sum_{k=1}^{n} b_{jk} w_k - w_{n+1} \geq 0, \quad (j = 1, 2, \ldots, \mu).$$ (7.4)
Under the condition that the variables \( w_1, \ldots, w_{n+1} \) are nonnegative, the remaining inequalities in (7.1) are equivalent to following \( n-1 \) linear inequalities:

\[
W_k - W_{k-1} \geq 0, \quad (k = 2, 3, \ldots, n).
\]

(7.5)

Consequently, the minimization problem for the given regular switching function \( F \) is a standard minimization problem in linear programming of finding \( q = n + 1 \) nonnegative real numbers

\[
\left( w_1, w_2, \ldots, w_n, w_{n+1} \right)
\]

which minimize the linear cost function (7.2) and also satisfy the system of \( p = \lambda + \mu + n - 1 \) linear inequalities (7.3), (7.4), and (7.5).
Section 8

SOLUTION OF THE MINIMIZATION PROBLEM

In Section 7, we have seen that the minimization problem for a given regular switching function $F$ is a standard minimization problem in linear programming. In the present section, we are concerned with the methods of solving this problem. For this purpose, we assume that the first $n$ coefficients in the cost function (7.2) are positive; in symbols,

$$\gamma_i > 0, \quad (i = 1, 2, \ldots, n). \quad (8.1)$$

Economically, this means that it costs something to realize the weights $w_1, \ldots, w_n$ physically. For the two different cost functions studied in Ref. 16, $\gamma_i = 1$ for all $i = 1, 2, \ldots, n$. Hence, the condition (8.1) is rather reasonable.

Let us briefly review some standard terminology in linear programming. By a feasible solution of the minimization problem for $F$, we mean a set of $n + 1$ non-negative real numbers

$$\left( w_1, w_2, \ldots, w_n, w_{n+1} \right)$$

which satisfy the linear inequalities (7.3), (7.4), and (7.5). By an optimal solution of the problem, we mean a feasible solution which minimizes the cost function (7.2).

The minimization problem for $F$ is said to be feasible provided that it has a feasible solution. Hence, the problem is feasible if and only if the given regular switching function $F$ is linearly separable.

Furthermore, for the nontrivial case where $F$ is nonempty and different from the whole $n$-cube, it was proved in Ref. 5 that the minimization problem for $F$ has optimal solutions if and only if it is feasible. Hence every linearly separable regular switching function $F$ of $n$ variables has a minimal canonical normal separating system.
As to methods of solving the minimization problem for a given regular switching function \( F \) of \( n \) variables, we certainly think of the usual simplex method in linear programming. This method splits into two parts of work. The first part is the search of a feasible solution of the problem as in Ref. 3; and the second part is the application of the simplex technique to find an optimal solution starting from the feasible solution found in the first part. Because of this, the usual simplex method is rather inefficient for this particular minimization problem.

On the other hand, the dual simplex method described in Section 3 suits especially well for solving our particular standard minimization problem. Indeed, under the reasonable assumption (8.1), the minimization problem for any nonempty regular switching function \( F \) of \( n \) variables is always dually feasible as defined in Section 2. Hence, we can apply the dual simplex method to our problem right away.

As an illustrative example, let us consider the regular switching function

\[
F = 531
\]

of five variables in the notation introduced in Refs. 12 and 17. Precisely, \( F \) consists of the following ten points of \( Q^5 \):

\[
\begin{align*}
(0, 0, 0, 0, 0) & \quad (0, 0, 0, 0, 1) \\
(1, 0, 0, 0, 0) & \quad (1, 1, 0, 0, 0) \\
(0, 1, 0, 0, 0) & \quad (1, 0, 1, 0, 0) \\
(0, 0, 1, 0, 0) & \quad (1, 0, 0, 1, 0) \\
(0, 0, 0, 1, 0) & \quad (0, 1, 1, 0, 0).
\end{align*}
\]

The maximal points of \( F \) can be read from its label 531 in accordance with Ref. 12. In fact, \( F \) has three maximal points, namely,

\[
\begin{align*}
(0, 0, 0, 0, 1) \\
(1, 0, 0, 1, 0) \\
(0, 1, 1, 0, 0).
\end{align*}
\]
To find the minimal points of the complement $F'$ of $F$, we may first find the dual $F^*$ of $F$ defined in Ref. 18. According to the table at the end of Ref. 18, the dual of $F = 531$ is

$$F^* = 54321/32/1$$

in the notation of Ref. 17. By the table given at the end of Ref. 12, the regular switching function $F^*$ has three maximal points, namely,

$$(0, 0, 0, 1, 1)$$
$$(1, 0, 1, 0, 1)$$
$$(0, 1, 1, 1, 0).$$

By Theorem 5.1 in Ref. 18, it follows immediately that the complement $F'$ of $F$ has three minimal points, namely,

$$(1, 1, 1, 0, 0)$$
$$(0, 1, 0, 1, 0)$$
$$(1, 0, 0, 0, 1).$$

Having obtained the maximal points of $F$ and the minimal points of its complement $F'$, we can exhibit the system of inequalities (7.3), (7.4) and (7.5) in the form of the following ten linear inequalities:

\[
\begin{align*}
w_6 - w_5 & \geq 0 \\
w_6 - w_1 - w_4 & \geq 0 \\
w_6 - w_2 - w_5 & \geq 0 \\
w_1 + w_2 + w_3 - w_6 - 1 & \geq 0 \\
w_2 + w_4 - w_6 - 1 & \geq 0 \\
w_1 + w_5 - w_6 - 1 & \geq 0 \\
w_2 - w_1 & \geq 0 \\
w_3 - w_2 & \geq 0 \\
w_4 - w_3 & \geq 0 \\
w_5 - w_4 & \geq 0.
\end{align*}
\]
Next, for the definiteness, let us choose the cost function to be the linear function

\[ y = w_1 + w_2 + w_3 + w_4 + w_5 + w_6. \]

Thus, we obtain exactly the illustrative example in Section 5 as our standard minimum problem. Hence, the given regular switching function \( F = 531 \) has a minimal normal canonical separating system

\[ (w_1, w_2, w_3, w_4, w_5, w_6) \]

with

\[
\begin{align*}
  w_1 &= 1 \\
  w_2 &= 2 \\
  w_3 &= 2 \\
  w_4 &= 3 \\
  w_5 &= 4 \\
  w_6 &= 4.
\end{align*}
\]
Section 9
REFERENCES


9-1

LOCKHEED MISSILES & SPACE COMPANY


