I. Introduction

1.1 If $M_0, M_1, M_2, \ldots$ are positive numbers, we denote by $C\{M_n\}$ the class of all complex functions $f$ on the real line for which there exist constants $\beta = \beta_f$ and $B = B_f$ such that

$$\|D^n f\| \leq \beta^n M_n \quad (n = 0, 1, 2, \ldots),$$

where $D = d/dx$ and $\| \|$ is the supremum norm: $\|f\| = \sup |f(x)|$, $-\infty < x < \infty$.

The class of all members of $C\{M_n\}$ which are periodic, with period $2\pi$, will be denoted by $C_p\{M_n\}$.

The sequence $\{M_n\}$ is said to be logarithmically convex if $\{\log M_n\}$ is convex, i.e., if $M_n^2 \leq M_{n-1}M_{n+1}$ for $n = 1, 2, 3, \ldots$. If $\{\tilde{M}_n\}$ is the largest logarithmically convex minorant of $\{M_n\}$, then $C\{M_n\} = C\{\tilde{M}_n\}$ and $C_p\{M_n\} = C_p\{\tilde{M}_n\}$. This follows from the inequalities

$$\|D^n f\| \leq 2 \|D^p f\|^{r-p} \|D^r f\|^{r-p} \quad (0 \leq p \leq n < r),$$

which are due to Kolmogoroff [6; pp. 211, 216].

Hence we may assume, without loss of generality, that $\{M_n\}$ is logarithmically convex; unless the contrary is stated, this assumption will be made from now on.

Since $C\{M_n\} = C\{\lambda M_n\}$, for every positive constant $\lambda$, we may also assume

Sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin under Contract No.: DA-11-022-ORD-2059.
that $M_0 = 1$. It will be convenient to define $A_0 = 1$ and

$$A_n = \left( \frac{M_n}{n!} \right)^{1/n} \quad (n = 1, 2, 3, \ldots).$$

1.2. Leibnitz' formula

$$D^n(f \cdot g) = \sum_{j=0}^{n} \binom{n}{j} D^j f \cdot D^{n-j} g$$

shows that each $C\{M_n\}$ is an algebra, under pointwise addition and multiplication:

the above assumptions on $\{M_n\}$ show that $M_j M_{n-j} \leq M_n$ if $0 \leq j \leq n$, and

therefore the inequalities $\|D^n f\| \leq \beta_1 B_1^n M_n$ and $\|D^n g\| \leq \beta_2 B_2^n M_n$ imply

$$\|D^n(f \cdot g)\| \leq \sum_{j=0}^{n} \binom{n}{j} \beta_1 B_1^j B_2^{n-j} M_{n-j} \leq \beta_1 \beta_2 M_n \sum_{j=0}^{n} \binom{n}{j} B_1^j B_2^{n-j} = \beta_1 \beta_2 (B_1 + B_2)^n M_n.$$  

1.3. The algebra $C\{M_n\}$ is called quasianalytic if the zero-function is the only member of $C\{M_n\}$ such that $D^n f(x_0) = 0$ for $n = 0, 1, 2, \ldots$, at some point $x_0$. Otherwise, $C\{M_n\}$ is non-quasianalytic. The theorem of Denjoy and Carleman ([1], [6]) states that $C\{M_n\}$ is quasianalytic if and only if

$$\sum_{n=0}^{\infty} \frac{M_n}{M_{n+1}} = \infty.$$  

Since $\{\log M_n\}$ is convex and $M_0 = 1$, we see that $(M_n/M_{n+1})^n \leq M_n^{-1}$, so that the condition (6) implies

$$\sum_{n=1}^{\infty} M_n^{-1/n} = \infty.$$
that $M_0 = 1$. It will be convenient to define $A_0 = 1$ and

$$A_n = \left( \frac{M_n}{n!} \right)^{1/n} \quad (n = 1, 2, 3, \ldots) \quad (3)$$

1.2. Leibnitz' formula

$$D^n(f \cdot g) = \sum_{j=0}^{n} \binom{n}{j} D^j f \cdot D^{n-j} g \quad (4)$$

shows that each $C\{M_n\}$ is an algebra, under pointwise addition and multiplication:

the above assumptions on $\{M_n\}$ show that $M_j M_{n-j} \leq M_n$ if $0 \leq j \leq n$, and

therefore the inequalities $\|D^n f\| \leq \beta_1 B_1^n M_n$ and $\|D^n g\| \leq \beta_2 B_2^n M_n$ imply

$$\|D^n (f \cdot g)\| \leq \sum_{j=0}^{n} \binom{n}{j} \beta_1 B_1^j M_j \beta_2 B_2^{n-j} M_{n-j} \quad \leq \beta_1 \beta_2 M_n \sum_{j=0}^{n} \binom{n}{j} B_1^j B_2^{n-j} = \beta_1 \beta_2 (B_1 + B_2)^n M_n \quad (5)$$

1.3. The algebra $C\{M_n\}$ is called quasianalytic if the zero-function is the only member of $C\{M_n\}$ such that $D^n f(x_0) = 0$ for $n = 0, 1, 2, \ldots$, at some point $x_0$. Otherwise, $C\{M_n\}$ is non-quasianalytic. The theorem of Denjoy and Carleman ([1], [6]) states that $C\{M_n\}$ is quasianalytic if and only if

$$\sum_{0}^{\infty} \frac{M_n}{M_{n+1}} = \infty \quad (6)$$

Since $\{\log M_n\}$ is convex and $M_0 = 1$, we see that $(M_n / M_{n+1})^n \leq M_n^{-1}$, so that the condition (6) implies

$$\sum_{1}^{\infty} M_n^{-1/n} = \infty \quad (7)$$
To prove the converse we appeal to the inequality [7]

$$\sum (a_1 a_2 \ldots a_n)^{1/n} \leq e \sum a_n,$$

valid for $a_i > 0$, and take $a_i = M_{i-1} / M_i$.

Thus (7) is also a necessary and sufficient condition for quasianalyticity.

1.4. If $1/f \in \mathcal{C}(M_n)$ whenever $f \in \mathcal{C}(M_n)$ and $\inf_x |f(x)| > 0$, we call $\mathcal{C}(M_n)$ inverse-closed; a similar definition applies to $\mathcal{C}_p(M_n)$.

The problem with which we are concerned, and which is solved in the present paper, is the description of all inverse-closed non-quasianalytic algebras $\mathcal{C}(M_n)$. It turns out that they are precisely those for which there is a constant $K$ such that the inequalities

(8) $$A_s \leq KA_n$$

hold whenever $s \leq n$; here $\{A_n\}$ is defined by (3).

The condition (8) is satisfied with $K = 1$ precisely when $\{A_n\}$ is an increasing sequence. Accordingly, we shall call $\{A_n\}$ almost increasing if (8) is satisfied for some $K < \infty$.

1.5. Actually, a more striking dichotomy exist than was indicated in the preceding paragraph. Our main results may be summarized as follows:

THEOREM A. Suppose $\{A_n\}$ is almost increasing. Then $\mathcal{C}(M_n)$ is inverse-closed. Furthermore, if $f \in \mathcal{C}(M_n)$ and if $\phi$ is an analytic function in an open set which contains the closure of the range of $f$, then $\phi \circ f \in \mathcal{C}(M_n)$. 
THEOREM B. Suppose \( C\{M_n\} \) is non-quasianalytic and \( \{A_n\} \) is not almost increasing. Then there exists an \( f \in C_p\{M_n\} \) and an entire function \( \phi \) such that

(i) if \( \lambda \) is any complex number, then \( (\lambda - f)^{-1} \) is not in \( C\{M_n\} \);

(ii) \( \phi \cdot f \) is not in \( C\{M_n\} \).

The symbol \( \phi \cdot f \) indicates the function defined by: \( (\phi \cdot f)(x) = \phi(f(x)) \).

Since \( f \) is bounded, (i) shows that \( C\{M_n\} \) is not inverse-closed, by taking \( |\lambda| > \|f\| \). Actually, (i) shows more: for some \( f \in C_p\{M_n\} \) the spectrum of \( f \) (relative to the algebra \( C_p\{M_n\} \)) consists of the whole plane, although the range of \( f \) is compact. We state the result for \( C_p\{M_n\} \) rather than for \( C\{M_n\} \) to emphasize that the phenomenon (i) is not caused by the behavior of \( f \) near infinity, but that it is present in non-quasianalytic algebras on the circle.

It would be interesting to extend Theorem B to quasianalytic classes.

1.6. The problem treated here has the following background. Let \( A \) be the class of all functions on the circle which are sums of absolutely convergent trigonometric series. Katznelson ([4], [2]) proved that if \( \phi \) is defined on the real line and if \( \phi \cdot f \in A \) for all real \( f \in A \), then \( \phi \) must be analytic on the line. Malliavin [5] has proved that corresponding to every inverse-closed non-quasianalytic class \( C\{M_n\} \) there is a real \( f \in A \) such that \( \phi \cdot f \in A \) only if \( \phi \in C\{M_n\} \). It is known that the intersection of all non-quasianalytic classes is precisely the class \( C\{n!\} \), which consists of analytic functions (a proof is included in Part IV).

If it were true that the intersection of all inverse-closed non-quasianalytic classes is also \( C\{n!\} \), then Malliavin's result would imply Katznelson's. But it is not so:
THEOREM C. The intersection of all inverse-closed non-quasianalytic classes is precisely the class $C\{(n \log n)^n\}$.

Since $C\{M_n\}$ is a subclass of $C\{M_n^n\}$ if and only if $\{(M_n^*/M_n)^{1/n}\}$ is bounded above [1; p. 19] and since Stirling's formula implies that
\[\frac{1}{n} \cdot \log n \cdot (n!)^n \to \infty,\]
we see that $C\{n!\}$ is a proper subclass of $C\{(n \log n)^n\}$.

In particular, it follows that there exist non-quasianalytic algebras which are not inverse-closed, a fact which seems to have escaped previous notice.

II. PROOF OF THEOREM A.

2.1. THEOREM. Suppose $A_s \leq KA_n$ whenever $s \leq n$, for some fixed $K$. If $\sigma, \beta, B$ are positive constants, if

\[(1) \quad \|D^n f\| \leq \beta B^n M_n \quad (n = 0, 1, 2, \ldots)\]

and if $|f(x)| \geq \sigma$ $(-\infty < x < \infty)$, then

\[(2) \quad \|D^n (I/f)\| \leq \beta_1 B_1^n M_n \quad (n = 0, 1, 2, \ldots),\]

where $\beta_1 = 2/\sigma$, $B_1 = BK(1 + 2\beta/\sigma)$.

This is due to Malliavin [5]. We include the proof since the quantitative version stated here is needed for Theorem 2.3.

Proof. Choose $\epsilon$ so that $2\beta \epsilon = (1 - \epsilon)\sigma$, then choose $\{r_n\}$ so that $B K A_n r_n = \epsilon$ $(n = 0, 1, 2, \ldots)$. Fix $n$, fix $x_0$, and define

\[Q(z) = f(x_0) + Df(x_0)z + \ldots + \frac{D^n f(x_0)}{n!} z^n.\]
For $1 \leq s \leq n$ we have

$$|D^s f(x_0)|/s! \leq \beta B^s A^s \leq \beta (BKA_n)^s$$

and hence $|z| \leq r_n$ implies

$$|Q(z)| \geq \sigma - \beta \sum_{s=1}^{n} (BKA_n r_n)^s > \sigma - \beta \sum_{1}^{\infty} \epsilon^s$$

(3)

$$= \sigma - \frac{\beta \epsilon}{1 - \epsilon} = \frac{\sigma}{2}.$$

The first $n$ derivatives of $Q$ at $z = 0$ are equal to the first $n$ derivatives of $f$ at $x = x_0$. Hence $D^n (l/f)(x_0) = D^n (l/Q)(0)$, and Cauchy's formula gives

(4)

$$D^n (l/f)(x_0) = \frac{n!}{2\pi i} \int_{|z|=r_n} \frac{dz}{z^{n+1} Q(z)}.$$

We conclude from (3) and (4) that

$$|D^n (l/f)(x_0)| \leq \frac{2}{\sigma} \cdot \frac{n!}{r_n^n} = \frac{2}{\sigma} (\frac{B K}{\epsilon})^n M_n,$$

which completes the proof.

2.2. LEMMA. Suppose $\{f_p\}$ is a sequence of functions on the real line which converges pointwise to a function $f$, and which satisfies the inequalities

(5)

$$\|D^n f_p\| \leq R_n < \infty \quad (n = 0, 1, 2, \ldots; \ p = 1, 2, 3, \ldots)$$

Then we also have $\|D^n f\| \leq R_n$ for all $n \geq 0$.

Proof. Suppose that $D^j f$ exists and that $D^j f_p \to D^j f$ pointwise (for $j = 0$, this is part of the hypothesis). Fix $x$ and $\epsilon > 0$, restrict $y$ so that $0 < |y - x| < \epsilon/R_{j+2}$. Then

(6)

$$\frac{D^j f_p (y) - D^j f_p (x)}{y - x} = D^j f_p (x) = \frac{x-x}{2} D^{j+2} f_p (\xi).$$
for some $\xi$ between $x$ and $y$. Write (6) once more, with $q$ in place of $p$, and subtract the two equations. The right side is less than $\epsilon$; letting $p, q \to \infty$, the quotients on the left converge to the same limit, namely $\{D^j f(y) - D^j f(x)\}/(y - x)$. Hence $\{D^{j+1} f_p(x)\}$ is a Cauchy sequence. Let $L$ be its limit. Then (6) gives

$$|\frac{D^j f(y) - D^j f(x)}{y - x} - L| \leq \epsilon$$

as soon as $0 < |y - x| < \epsilon/R_{j+2}$. Thus $D^{j+1} f$ exists and $D^{j+1} f_p \to D^{j+1} f$ pointwise.

The proof is completed by induction.

2.3. THEOREM. Suppose $f \in C(M_n)$, $\{A_n\}$ is almost increasing, and $\phi$ is analytic in an open set which contains the closure of the range of $f$. Then $\phi \cdot f \in C(M_n)$.

Proof. There exists $\Gamma$, a union of finitely many rectifiable curves in the domain of $\phi$, and there exists $\sigma > 0$, such that

$$|z - f(x)| \geq \sigma$$

for all $z \in \Gamma$ and all real $x$, and such that

$$\phi(f(x)) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(z)}{z - f(x)} \, dz \quad (-\infty < x < \infty).$$

There is a sequence of partitions of $\Gamma$, by points $z_0(p), z_1(p), \ldots, z_N(p)$, such that the functions $g_p$ defined by

$$g_p(x) = \frac{1}{2\pi i} \sum_{j=1}^{N} \frac{\phi(z_j)}{z_j - f(x)} \frac{z_j(p) - z_{j-1}(p)}{z_j - f(x)}$$

converge to $\phi(f(x))$, as $p \to \infty$. 


Choosing $\beta$ and $B$ so that $\|D^nf\| \leq \beta B^pM_n$ and $\|f - z\| \leq \beta$ for all $z \in \Gamma$, Theorem 2.1 shows that

$$\|D^n(\frac{1}{z_j - f})\| \leq \beta_1 B^pM_n \quad (n \geq 0).$$

Since $\phi$ is bounded on $\Gamma$ and since $\sum|z_j^{(p)} - z_{j-1}^{(p)}|$ does not exceed the length of $\Gamma$, we see from (10) and (11) that

$$\|D^n \gamma_p\| \leq \beta_2 B^pM_n \quad (n \geq 0).$$

Lemma 2.2 now implies that

$$\|D^n(\phi \cdot f)\| \leq \beta_2 B^pM_n \quad (n \geq 0),$$

and this completes the proof.

III. PROOF OF THEOREM B.

3.1. LEMMA. Suppose $\{a(n)\}$ is a sequence of positive numbers such that $\{na(n)\}$ is increasing but $\{a(n)\}$ is not almost increasing. Then there exist sequences of integers, $\{s_i\}$ and $\{m_i\}$, both tending to $\infty$, such that

$$\frac{a(s_i)}{a(m_is_i)} \rightarrow \infty \quad (i \rightarrow \infty).$$

Proof. Put

$$\gamma(s) = \sup \{\frac{a(s)}{a(s+1)}, \frac{a(s)}{a(s+2)}, \frac{a(s)}{a(s+3)}, \ldots\}.$$ 

Since $\{a(n)\}$ is not almost increasing, we have $\sup_s \gamma(s) = \infty$. Since $\{a(n)\}$ increases, we have
Also, if $s \leq n$, then $n = ms + t$ with $0 \leq t < s$, and so $a(ms) \leq na(n)/ms \leq 2a(n)$. Thus $a(s)/a(ms) \geq a(s)/2a(n)$, which gives

$$(4) \quad \sup_{m \geq 1} \frac{a(s)}{a(ms)} \geq \frac{1}{2} \gamma(s).$$

Since $\sup \gamma(s) = \infty$, (4) shows that (i) holds for some sequences $\{s_i\}$, $\{m_i\}$; by (3) this is only possible if $m_i \to \infty$.

If $\gamma(s) = \infty$, for all $s$, we can take for $\{s_i\}$ any sequence tending to $\infty$, and then find $\{m_i\}$ so that (i) holds. If $\gamma(s_0) < \infty$ for some $s_0$, then $\inf a(n) > 0$, and (i) implies that $a(s_i) \to \infty$, i.e., that $s_i \to \infty$.

3.2 LEMMA. Suppose $C\{M_n\}$ is non-quasianalytic and $I$ is a closed interval in the interior of a closed interval $J$ on the real line. Then there exists a constant $\beta$ and a function $h$ such that $h(x) = 1$ on $I$, $h(x) = 0$ off $J$, $0 \leq h \leq 1$, and

$$(5) \quad \|D^n h\| \leq \beta M_n \quad (n = 0, 1, 2, \ldots).$$

Proof. Put $a_n = M_{n-1}/M_n$. Then $\{a_n\}$ decreases monotonically, and $\sum a_n < \infty$. There exists a monotonically decreasing sequence $\{b_n\}$ such that $a_n/b_n \to 0$ and $\sum b_n < \infty$. Put $M_1 = (b_1, b_2, \ldots$). Then $\sum M_{n-1}/M_n = \sum b_n < \infty$ and $\{M_n^*\}$ is logarithmically convex. Hence $C\{M_n^*\}$ is non-quasianalytic. Also,

$$(6) \quad \left\{ \frac{M_{1/n}^*}{M_n} \right\}^{1/n} = \left\{ \frac{b_1 \ldots b_{1/n}}{a_1 \ldots a_n} \right\}^{1/n} \to 0 \quad (n \to \infty).$$

Since $C\{M_n\}$ is non-quasianalytic, there is a function $g \in C\{M_n^*\}$ such
that \( g(x) = 0 \) if \( x \leq 0 \), \( g(x) = 1 \) if \( x \geq x_0 \) for some \( x_0 > 0 \). Bang [1; p. 55] (see also Mandelbrojt [6; p. 103]) has indicated a very simple construction which achieves this. Affine changes of variables (which do not affect the class \( C(M_n^*) \)) then give functions \( h_1, h_2 \in C(M_n^*) \) such that \( h_1 = 0 \) to the left of \( J \), \( h_1 = 1 \) on \( I \) and to the right of \( I \), \( h_2 = 0 \) to the right of \( J \), \( h_2 = 1 \) on \( I \) and to the left of \( I \). Put \( h = h_1 h_2 \). Then \( h \) has the required properties, except that \( (5) \) is replaced by

\[
\|D^n h\| \leq B^n M_n^* \quad (n = 0, 1, 2, \ldots)
\]

for some constant \( B \). Setting \( \beta = \max_n B^n M_n^*/M_n \), \( (6) \) shows that \( \beta < \infty \), and \( (7) \) shows that \( (5) \) holds.

3.3. We now turn to the proof of Theorem B. Put

\[
\mu_n = M_n/M_{n+1} \quad (n = 0, 1, 2, \ldots)
\]

By the Denjoy-Carleman Theorem, \( \sum \mu_n < \infty \). Replacing \( M_n \) by \( k^n M_n \), if necessary, we may assume, without loss of generality, that

\[
\sum_0^\infty \mu_n < \frac{1}{2}
\]

We define

\[
f_s(x) = \mu_s^s M_s \exp\{ix/\mu_s\} \quad (s = 0, 1, 2, \ldots)
\]

and note that

\[
D^n (f_s^m) = (im/\mu_s)^n f_s^m \quad (s, n \geq 0, \ m \geq 1)
\]

The convexity of \( \{\log M_n\} \) shows that \( M_n^{s+1-n} \leq M_n^s M_n^{s-n} \) if \( 0 \leq n \leq s \); if \( s + 1 \leq n \), we have similarly \( M_n^{n-s} \leq M_n^{n-s-1} M_n \). Thus the inequality
Applying (12) to (11), with \( m = 1 \), we see that

\[
\| D^nf_s \| = \mu_s^{s-n} M_s \leq M_n \quad (s, n \geq 0)
\]

holds in all cases.

In particular, taking \( n = 0 \),

\[
\| f_s^m \| = \| f_s \| \leq M_0 = 1 \quad (s \geq 0, \ m \geq 1)
\]

By (9), we can place disjoint closed intervals \( J_k \) in \((0, 2\pi)\) which contain intervals \( I_k \) in their interiors, with \( m(I_k) = 2\pi \mu_k \), and Lemma 3.2 shows that there are functions \( h_k \) and constants \( \beta_k > k \) such that \( h_k = 1 \) on \( I_k \), \( h_k = 0 \) off \( J_k \), and

\[
\| D^nh_k \| \leq \beta_k M_n \quad (u, k \geq 0)
\]

Put \( a(0) = 1 \) and define \( a(n) \) by

\[
n a(n) = M_n \quad (n \geq 1)
\]

By hypothesis, \( \{ A_n \} \) is not almost increasing. By Stirling's formula, \( \{ a(n)/A_n \} \) is bounded above and below by positive numbers. Hence \( \{ a(n) \} \) is not almost increasing. Our standing assumptions on \( \{ M_n \} \) (logarithmic convexity, and \( M_0 = 1 \)) imply that \( \{ n a(n) \} \) increases. Thus Lemma 3.1 applies, and there are sequences \( \{ s_k \} \), \( \{ m_k \} \), tending to \( \infty \), such that \( s_k > k \), \( 2s_k > \beta_k \), and

\[
\frac{a(s_k)}{a(m_k s_k)} \to \infty \quad (k \to \infty)
\]
We extend the functions \( h_k \cdot f_{s_k} \), defined in \((0, 2\pi)\), to be periodic, with period \(2\pi\), and define

\[
f(x) = \sum_{k=0}^{\infty} \frac{1}{\beta_k} h_k(x) f_{s_k}(x) .
\]

By (13), (15), and Leibnitz' formula, we have \( \|D^n(h_k f_{s_k})\| \leq 2^n M_n \). The functions \( h_k \) have disjoint supports. Hence if \( g \) is any partial sum of the series (18), we have \( \|D^n g\| \leq 2^n M_n \), and we conclude from Lemma 2.2 that \( \|D^n f\| \leq 2^n M_n \). Thus \( f \in C_p(M_n) \).

Since 0 is in the range of \( f \), it is clear that \( f^{-1} \) is not in \( C(M_n) \). Fix \( \lambda \neq 0 \), put \( F = (1 - f/\lambda)^{-1} \), and assume (this will lead to a contradiction) that \( F \in C(M_n) \). For some \( B < \infty \) we then have

\[
\|D^n F\| \leq B^n M_n \quad (n \geq 1) .
\]

For large enough \( k \), \( |\lambda| \beta_k > 1 \). Since \( h_k = 1 \) on \( I_k \) and \( h_j = 0 \) on \( I_k \) if \( j \neq k \), we have

\[
F(x) = \sum_{m=0}^{\infty} (\lambda \beta_k)^{-m} m^n f_{s_k}(x) \quad (x \in I_k, \ k \geq k_0) .
\]

By (11) and (14), the series (20) may be differentiated term by term any number of times, since the resulting series converge uniformly on \( I_k \). Since \( s_k > k \), we have \( \mu_{s_k} \leq \mu_k \), so that there is a point \( x_k \in I_k \) at which \( \exp\{ix/\mu_{s_k}\} > 0 \).

Differentiating (20) \( n \) times at \( x_k \) therefore gives

\[
D^n F(x_k) = i^n \sum_{m=0}^{\infty} (m/\mu_{s_k})^n f_{s_k}(x_k)^{\lambda \beta_k^{-m}} ,
\]
by (11). By (19), no term in the series (21) exceeds $B^n M_n$. Taking $m = m_k$ and $n = m_k s_k$, (10) shows therefore that

$$\frac{s_k}{m_k M_k s_k} \leq B \frac{m_k s_k}{m_k M_k s_k}$$

Taking $n$th roots in (22) and using (16), we obtain

$$\frac{a(s_k)}{a(m_k s_k)} \leq B |\lambda k|^{1/s_k} \leq 2B |\lambda|^{1/s_k}.$$  

The last term in (23) is bounded, as $k \to \infty$, and this contradicts (17).

Thus $(1 - f/\lambda)^{-1}$ is not in $C \{M_n\}$, and part (i) of Theorem B is proved.

Part (ii) is proved quite similarly. Suppose

$$\phi(z) = \sum_{m=0}^{\infty} \frac{c_m z^m}{\beta_m k} s_k$$

and put $g(x) = \phi(f(x))$. On $I_k$ we have, in place of (20),

$$g(x) = \sum_{m=0}^{\infty} \frac{c_m x^m}{\beta_m k} s_k$$

and we can choose $x_k \in I_k$ so that $f(x_k) > 0$. In place of (23) we obtain

$$\frac{1}{m_k s_k} \frac{a(s_k)}{a(m_k s_k)} \leq 2B.$$  

Since $c_m^{1/m} \leq c_m^{1/m}$, this gives

$$\frac{1}{m_k^{1/m}} \leq 2B \cdot \frac{a(m_k s_k)}{a(s_k)}.$$  

But $\{c_m^{1/m}\}$ can tend to 0 without satisfying (25), since the right side of (25) tends to 0 as $k \to \infty$, by (16).

This completes the proof.
IV. PROOF OF THEOREM C.

4.1. Let us now assume that \(C\{M_n\}\) is non-quasianalytic and inverse-closed.

By Theorem B, \(\{A_n\}\) is then almost increasing, and so is \(\{a_n\}\), if
\[a_n = M_n^{1/n}/n.\]
Choose \(K\) so that \(a_s \leq Ka_n\) if \(s \leq n\).

Since \(\sum M_n^{-1/n} < \infty\) (see § 1.3), \(\sum (a_n)^{-1} < \infty\). But
\[
\sum_{1/2 \leq s \leq n} \frac{1}{s a_s} \geq \frac{1}{Ka_n} \cdot \sum_{s} \frac{1}{s} \sim \frac{1}{Ka_n} \cdot \frac{1}{2} \log n.
\]

The sum on the left tends to 0 as \(n \to \infty\), hence \(a_n/log n \to \infty\), and this means that \(C\{M_n\}\) contains \(C\{(n \log n)^n\}\) and therefore proves one half of Theorem C.

4.2. To prove the other half, we consider a function \(f \not\in C\{(n \log n)^n\}\), and we shall construct a non-quasianalytic class \(C\{M_n\}\), with \(\{a_n\}\) increasing, such that \(f \not\in C\{M_n\}\).

Since \(f \not\in C\{(n \log n)^n\}\), either some derivative of \(f\) fails to be bounded, in which case \(f\) belongs to no \(C\{M_n\}\), or there is a sequence \(\{n_i\}\) such that
\[
(1) \quad \|D_{i+1}^n f\| > (i^3 n_i \log n_i)^{n_i} ;
\]

we can make \(\{n_i\}\) increase so rapidly that
\[
(2) \quad n_{i+1} > n_i \log (i^2 \log n_i) .
\]

Define
\[
(3) \quad \phi(n_i) = n_i \log (i^2 n_i \log n_i)
\]
and
\[
(4) \quad \phi(n) = a_i + b_i n + n \log n \quad (n_i \leq n \leq n_{i+1}) ,
\]

where \(a_i\) and \(b_i\) are so chosen that the definitions of \(\phi(n)\) agree when
n = n_1, \quad n = n_{i+1}. \quad \text{Thus}

(5) \quad a_i + b_i \log n_i = \log (i^2 \log n_i)
\quad a_i + b_i \log n_{i+1} = \log ((i+1)^2 \log n_{i+1}).

From this we deduce that \( a_i < 0 \), and, via (2), that

(6) \quad b_i > \log (i^2 \log n_{i+1}) - 1.

Now put \( M_n = \exp \{ \phi(n) \} \). If \( n_1 \leq n \leq n_{i+1} \), then

(7) \quad \exp \{-b_i\} < e^{i^2 \log n_{i+1}},

and hence, by (6),

(8) \quad \frac{M_n}{M_{n+1}} = \exp \{ \phi(n) - \phi(n+1) \} = \exp \{-b_i\} \cdot \frac{n^n}{(n+1)^{n+1}}
\quad < \frac{e}{i \log n_{i+1}} \cdot \left(1 + \frac{1}{n}\right)^{-n} \cdot \frac{1}{n} < \frac{1}{n^2 \log n_{i+1}}.

It follows that

(9) \quad \sum_{n_1+1}^{n_{i+1}} \frac{M_{n-1}}{M_n} < \frac{1}{i^2 \log n_{i+1}} \quad \sum_{n_1}^{n_{i+1}} \frac{1}{n} < \frac{1}{i},

so that \( C(M_n) \) is non-quasianalytic.

Next,

(10) \quad a_n = \frac{\phi(n)}{n} - \log n = b_i + \frac{a_i}{n} \quad (n_1 \leq n \leq n_{i+1}),

and since \( a_i < 0 \), \( \{a_n\} \) increases. We can also arrange our construction so that \( b_{i+1} > b_i \), and then \( \phi \) will be convex. (This is not really necessary, since
the convergence of \( \frac{\sum M_n}{M_{n+1}} \) assures the non-quasianalyticity of \( C\{M_n\} \)
even without logarithmic convexity of \( \{M_n\} \).

By (1) and (3), \( f \notin C\{M_n\} \), and the proof of Theorem C is thus complete.

4.3. THEOREM. The intersection of all non-quasianalytic classes \( C\{M_n\} \) is the class \( C\{n!\} \). (Our reason for including a proof of this result is stated in § 1.6.)

Proof. If \( A \underset{\mathbf{n_i}}{<} A \) for some sequence \( \{n_i\} \) tending to \( \infty \) and some constant \( A \), if \( f \in C\{M_n\} \), and if \( D^n f(0) = 0 \) for \( n = 0, 1, 2, \ldots \), then for each \( x \neq 0 \) there exists \( \xi = \xi(x, n_i) \) such that

\[
|f(x)| = \left| D^\xi f(x) x^{\frac{n_i}{n_i!}} \right| \leq |\beta| B \frac{n_i}{n_i!} x^{\frac{n_i}{n_i!}} \leq |\beta| \cdot |B x|^{\frac{n_i}{n_i!}},
\]

where \( \beta, B \) depend on \( f \). If \( |B x| < 1 \), it follows that \( f(x) = 0 \). Hence \( C\{M_n\} \) is quasianalytic.

Thus \( C\{n!\} \) is contained in every non-quasianalytic \( C\{M_n\} \).

To prove the converse, suppose \( f \not\in C\{n!\} \). Then there is a sequence \( \{n_i\} \) such that

\[
\|D^{n_i} f\| > (3 n_i)^{n_i}
\]

and

\[
n_{i+1} > n_i \text{ log } (4 n_i).
\]

Put \( \phi(n_i) = n_i \text{ log } (4 n_i) \), \( \phi(n) = a_n + b n \) for \( n_i \leq n \leq n_{i+1} \), where

\[
a_n + b_{n_i} n_i = n_i \text{ log } (4 n_i)
\]

\[
a_n + b_{n_{i+1}} n_{i+1} = n_{i+1} \text{ log } (4 (i + 1)^2 n_{i+1}).
\]
and define \( M_n = \exp \{ \phi(n) \} \). As in § 4.2, we now have \( b_1 > \log (\frac{n^2}{n+1}) - 1 \), hence
\[
\frac{M_n}{M_{n+1}} = e^{-b_1} < \frac{e^{-2}}{n+1} \quad (n_1 \leq n \leq n_1+1),
\]
and
\[
\sum_{n_1+1}^{n_1+n+1} \frac{M_{n+1}}{M_n} < 1^2.
\]

Thus \( C\{M_n\} \) is non-quasianalytic, and since our definition of \( \phi \) shows that \( f \notin C\{M_n\} \), the proof is complete.

V. MISCELLANEOUS RESULTS

5.1. THEOREM. Every non-quasianalytic algebra \( C\{M_n\} \) is contained in an inverse-closed algebra \( C\{M_n^*\} \) which is minimal in the following sense: if \( C\{M_n\} \) contains \( C\{M_n^*\} \) and if \( C\{M_n^*\} \) is inverse-closed, then \( C\{M_n\} \) contains \( C\{M_n^*\} \).

Proof. Put \( A_n^* = \max_{s \leq n} A_s^{} \) and \( M_n^* = n! A_n^* \). Since \( M_n \leq M_n^* \) we have
\( C\{M_n\} \subseteq C\{M_n^*\} \). Since \( \{A_n^*\} \) increases, \( C\{M_n^*\} \) is inverse-closed.

(Note that the proof of Theorem A made no use of logarithmic convexity.)

Now suppose \( C\{M_n\} \subseteq C\{M_n^*\} \) and \( C\{M_n^*\} \) is inverse-closed. Since \( C\{M_n^*\} \) is non-quasianalytic, Theorem B shows that \( \{A_n^*\} \) is almost increasing, where \( A_n^* = (M_n^*/n!)^{1/n} \). Hence there are constants \( \lambda, k \), such that \( M_n \leq \lambda^n M_n^* \) and \( A_s \leq kA_n^* \) if \( s \leq n \). This implies \( A_s \leq \lambda kA_n^* \), hence \( A_n^* \leq \lambda kA_n^* \), hence \( M_n^* \leq (\lambda k)^n M_n^* \), and thus \( C\{M_n^*\} \subseteq C\{M_n^*\} \).
5.2. THEOREM. There exist non-quasianalytic algebras $C(M_n)$ which contain no inverse-closed non-quasianalytic $C(M'_n)$.

Proof. Theorem 4.3 shows that there is a non-quasianalytic $C(M_n)$ such that
\[
\left\{ \frac{M_{n_1}}{n_1 \log n_1} \right\}^{1/n_1} \to 0
\]
for some sequence $\{n_1\}$. If $C(M'_n) \subset C(M_n)$, it follows that $C(M'_n)$ does not contain $C((n \log n)^n)$, and hence Theorem C shows that $C(M'_n)$ cannot be both inverse-closed and non-quasianalytic.

5.3. COMPLEX HOMOMORPHISMS OF $C_p(M_n)$.

Since we are investigating certain function algebras, it is appropriate to study their maximal ideals and the complex homomorphisms which exist on them. We restrict ourselves to the algebras $C_p(M_n)$, for simplicity, for then we are dealing with functions on the circle $T$, i.e., on a compact space.

If $C_p(M_n)$ is inverse-closed, there are no problems. For each $x \in T$, let $I_x$ be the set of all $f \in C_p(M_n)$ which vanish at $x$. Then $I_x$ is clearly a maximal ideal in $C_p(M_n)$. Conversely, assume $I$ is a maximal ideal different from every $I_x$. For each $x$, there is a function $f_x \in I$ such that $f_x(x) \neq 0$, and the compactness of $T$ shows that there are points $x_1, \ldots, x_n$ such that
\[
g = \sum_{i=1}^{n} f_{x_i} > 0.
\]
But $g \in I$, and since $C_p(M_n)$ is inverse-closed (by assumption), we have $1 \in I$, hence $I = C_p(M_n)$. We summarize:

If $C_p(M_n)$ is inverse-closed, then every maximal ideal $I$ in $C_p(M_n)$ is of the form $I = I_x$, and every complex homomorphism $\psi$ of $C_p(M_n)$ is of the form $\psi(f) = f(x)$, for some $x \in T$. 
(By a complex homomorphism of $C_p\{M_n\}$ we mean a multiplicative linear functional which maps $C_p\{M_n\}$ onto the complex field. We make no continuity assumptions. Indeed, we have not introduced a topology in $C_p\{M_n\}$.)

If $C_p\{M_n\}$ is not inverse-closed, then, on the other hand, there do also exist other maximal ideals. For if $f \in C_p\{M_n\}$, if $f$ has no zero on $T$, and if $1/f \not\in C_p\{M_n\}$, then $f$ generates a proper ideal in $C_p\{M_n\}$ which, by Zorn's lemma, is contained in a maximal ideal $I$; since $f \not\in I$, $I$ is different from $I_x$ for all $x \in T$.

It is nevertheless conceivable that all complex homomorphisms are of the form $\psi(f) = f(x)$ for some $x \in T$, so that the quotient algebras $C_p\{M_n\}/I$ are different from the complex field, whenever $I$ is not one of the ideals $I_x$.

We shall now prove that this conjecture is true, under the additional assumption that $C_p\{M_n\}$ is non-quasianalytic and that $\log M_n = O(n^2)$. We divide the proof into several steps. Our growth condition will only be used at the end.

We consider a fixed $C_p\{M_n\}$, and a fixed complex homomorphism $\psi$ of $C_p\{M_n\}$.

(i) There is a point $x_0 \in T$ such that $\psi(f) = 0$ for all $f \in C_p\{M_n\}$ which vanish near $x_0$, (i.e., in a neighborhood of $x_0$).

For if there is no such point, the compactness of $T$ shows that there are segments $V_1, \ldots, V_m$ and functions $f_1, \ldots, f_m$ such that $f_i = 0$ on $V_i$ but $\psi(f_i) = 1$. Putting $f = f_1 \ldots f_m$, we have $f = 0$, $\psi(f) = \psi(f_1) \ldots \psi(f_m) = 1$, and hence $\psi(0) = 1$, a contradiction.

For simplicity, we assume from now on that $x_0 = 0$. 

(ii) Suppose \( f \in C_p(M_n) \) and \( f(x) = x \) near 0. Then \( \psi(f) = 0 \).

Proof. Put \( \psi(f) = a \). If \( a \neq 0 \), then there exists \( g \in C_p(M_n) \) such that \( g(x) = (x - a)^{-1} \) near 0; this is so since \( (x - a)^{-1} \) is analytic near 0, and we can multiply by one of functions \( h \) constructed in Lemma 3.2.

Then \( (f - a) \cdot g = 1 \) near 0, and (i) shows that \( \psi(f - a) \psi(g) = 1 \). But \( \psi(f - a) = \psi(f) - a = 0 \), a contradiction.

(iii) If \( f \in C(M_n) \), \( f(0) = 0 \), and \( g(x) = f(x)/x \), then \( g \in C(M_{n+1}) \).

Proof. Repeated differentiation of the equation \( f(x) = xg(x) \) yields

\[
D^{n+1}f(x) = xD^{n+1}g(x) + (n + 1)D^ng(x) \quad (n \geq 0).
\]

As \( |x| \to \infty \), \( D^n g(x) \to 0 \), and \( D^{n+1}g(x) = 0 \) at every local maximum of \( |D^n g| \). Hence \( \|D^n g\| \leq \|D^{n+1}f\| \).

(iv) If \( f \in C_p(M_n) \) and \( f(0) = 0 \), then \( \psi(f) = 0 \).

Proof. There are functions \( g, h \in C_p(M_n) \) such that \( g \equiv 1 \) near 0, the support of \( g \) lies in \( [-\pi + \delta, \pi - \delta] \) for some \( \delta > 0 \), and \( h(x) = x \) on the support of \( g \).

Put \( F = fg/h \). Since \( h = x \) where \( fg \neq 0 \), \( F = fg/x \). Since \( fg \in C_p(M_n) \), (iii) shows that \( F \in C_p(M_{n+1}) \). But if \( \log M_n = 0(n^2) \), then \( C(M_{n+1}) = C(M_n) \) \([1; p. 22]\). Thus \( F \in C_p(M_n) \).

By (i), \( \psi(g) = 1 \); by (ii), \( \psi(h) = 0 \). Hence \( \psi(f) = \psi(f)\psi(g) = \psi(fg) = \psi(Fh) = \psi(F)\psi(h) = 0 \).

We now summarize the result:

**Theorem.** If \( C_p(M_n) \) is non-quasianalytic, if \( \log M_n = 0(n^2) \), and if \( \psi \) is
a complex homomorphism of $C_p(M_n)$, then $\psi(f) = f(x)$ for some $x \in T$.

We conclude with the remark that there exist non-quasianalytic algebras $C(M_n)$ which are not inverse-closed and which fail to satisfy the condition $\log M_n = O(n^2)$. (In fact, if $\omega_n \to \infty$ and if $\lambda_n / n! \to \infty$, the technique used in the proof of Theorem 4.3 allows us to construct non-quasianalytic $C(M_n)$ such that $M_n > \omega_n$ for infinitely many $n$, and also $M_n < \lambda_n$ for infinitely many $n$.) For these algebras we do not yet know all complex homomorphisms.
REFERENCES

1. T. Bang, Om Quasi-Analytiske Funktioner, Nyt Nordisk Forlag, Copenhagen, 1946.


