NWL REPORT NO. 1833

TABLE OF EXPECTATIONS OF MEAN SQUARES IN THE ANALYSIS OF VARIANCE FOR CROSSED CLASSIFICATIONS

BY
KLAUS ABT
COMPUTATION AND ANALYSIS LABORATORY

U. S. NAVAL WEAPONS LABORATORY
DAHLGREN, VIRGINIA

DATE: 2 APRIL 1963

U. S. Naval Weapons Laboratory
Dahlgren, Virginia

Table of Expectations of Mean Squares in the Analysis of Variance for Crossed Classifications

by

Klaus Abt
Computation and Analysis Laboratory

NWL REPORT NO. 1833
Foundational Research Project K12007/29C
18 October 1962

TABLE OF CONTENTS

ABSTRACT

This report contains a table of the expectations of mean squares in the analysis of variance for crossed classifications and the description of the table. In an appendix the method is outlined which was used to obtain the mean square expectations.

The table covers the two-way, the three-way and the four-way classification with unequal but proportional as well as with equal cell numbers. The mean square expectations (and the appropriate variance ratios for testing all nullhypotheses) are listed for all possible combinations of fixed and random effects classifications. Also, rules are given for generating the mean square expectations for all situations in the general case of an n-way crossed classification.

FOREWORD

The work covered by this report was performed as part of Foundational Research Project No. K12007/29C, "Table of Expectations of Mean Squares in the Analysis of Variance (General)." The date of completion was 18 October 1962.

APPROVED FOR RELEASE:

/S/ R. H. LYDDANE
Technical Director

I. Introduction

In the application of analysis of variance techniques, the expectations of mean squares serve two essential purposes: (1) They determine the appropriate F-value for exactly or approximately testing a stated nullhypothesis and (2) they indicate how to estimate a given variance component. For the conceptually simplest and, at the same time, most frequently occurring classification of statistical data, the crossed classification (i.e., a classification into rows, columns, slices, etc., by logical reasoning), the expectations of mean squares in the analysis of variance are commonly known provided that there are equal numbers of replicated observations in all "cells" formed by the classification and that the underlying model is either of the "fixed effects" type ("Model I") or of the "random effects" type ("Model II"). Here the terms "fixed effects" and "random effects" relate to the classical definitions by Eisenhart (1947), i.e., in the linear model, the effects of a given way of classification (the row effects, for example) upon the variable under investigation are conceived to be fixed (non-random) or to have been randomly sampled from parent infinite populations, respectively.

If one deals, however, with a three- or more-way classification with an underlying model of the "mixed effects" type, say, (i.e., with a model containing both fixed and random effects classifications) it requires, even in the case of "equal cell numbers", some time to find in the literature -- if it is at all possible -- ready formulas of mean square expectations. Some of these formulas are given, for example, by Anderson and Bancroft (1952), Bennett and Franklin (1954), Cornfield and Tukey (1956) and Brownlee (1960). One reason for the said rarity seems to be the fact that various models and methods of different degrees of complexity have been proposed and applied to this case of mixed effects. This necessarily led to differing mean square expectations and thereby, understandably, to some confusion. These facts are reviewed in the very lucid summary paper by Plackett (1960). In the discussion of Plackett's paper voices also arose that the theoreticians of analysis of variance should keep closer to the needs of the practice rather than construct highly complicated models which are theoretically right but practically of not too much use.

In the most general orthogonal case of unequal but proportional cell numbers (in crossed classifications, see Appendix B) few attempts have been made to develop general formulas which would be applicable to all models; see the papers of Wilk and Kempthorne (1956) and of Bankier and Walpole (1957). These authors use the method of sampling from finite populations which seemed to be the only one capable of handling the derivations of mean square expectations in this case of proportional cell numbers. The method, which also in case of equal cell numbers is applied by various authors, see for example Bennett and Franklin (1954) and Cornfield and Tukey (1956), is extremely cumbersome, however. Moreover, in case of proportional cell numbers, it seems as if this method leads to questionable expectations of mean squares for the fixed and the mixed effects models (see paragraph II. 3, below).

Stimulated by these facts the author of this report felt that for the practitioner of analysis of variance a unified and simplified underlying model and method for the derivation of mean square expectations should be employed and the expectations themselves be tabulated. In particular, it was thought that the model, the method and the table should also cover the most general orthogonal cases of unequal but proportional cell numbers.

This aim was accomplished, it is believed, for n-way crossed classifications in cases where the underlying model is of the fixed, random or mixed effects type. The results are presented in this report. Section IV contains the formulas for the expectations of mean squares (and the appropriate variance ratios for testing all nullhypotheses) in the analysis of variance for the two-way, three-way and four-way classification, for all three types of models and for equal as well as unequal but proportional cell numbers. The method used for the derivations is described in Appendix A. This method is general, however, and can be applied also to other than crossed classifications. Otherwise its application is restricted to the "classical" cases of fixed, random or mixed effects. The expectations of mean squares when the sampling of classification effects is from finite populations (non-exhaustive) cannot be obtained by the method.

II. Description and use of the table

II.1. Notation and linear model

The notation used in this report for the expectations of mean squares and for the method of their derivation was chosen such that on one hand the table would be as easily readable as possible and on the other hand that the notation would not deviate too much from that widely used in the literature.

Each criterion of classification (into "rows" or "columns" or "slices," etc.) is called a "factor," and the classes within each classification or factor are called "levels" of that factor. This terminology is used for simplicity only, and it may well be noted that, for example, the blocks in a randomized block design also will fall under this definition of factor levels, the "factor" being that of replications.

The factors are symbolized by capital script letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, etc., where these letters never represent any numerical value. They are used only to identify the classifications, or as arguments, for example in the term MS ($\mathcal{A}$) = "mean square (of estimated level effects) for factor $\mathcal{A}$."

All terms related to a given factor will show this by a symbol or a subscript which is the corresponding letter or letter type in the Latin or Greek alphabet: The numbers of levels of the factors are A, B, C, etc.; the general subscripts indicating the factor levels are α, β, γ, etc. Thus, one has

$$\text{for factor } \mathcal{A}\text{:} \quad \alpha = 1, \ldots, A\ ;$$
$$\text{for factor } \mathcal{B}\text{:} \quad \beta = 1, \ldots, B\ ;$$
$$\text{for factor } \mathcal{C}\text{:} \quad \gamma = 1, \ldots, C\ ; \text{ etc.}$$

In the underlying linear model for the analysis of variance "true" factor level (or "main") effects are denoted by small Latin letters and their respective interaction effects by combinations of the corresponding small Latin letters. The general (true) mean is always called μ. For example, a_α is the (true) effect of level α of factor $\mathcal{A}$ over the general (true) mean μ; $ab_{\alpha\beta}$ is the interaction effect of levels α and β of factors $\mathcal{A}$ and $\mathcal{B}$, respectively, etc. These factorial effects are defined in terms of expectations of the cell responses as is shown in Appendix A. In a two-way classification, for example, such a "true" cell response is denoted by $X_{\alpha\beta}$.

In case of random sampling from infinite populations one has furthermore the notation for variances:

$$V\left[a_\alpha\right] = \sigma_a^2 ,$$

$$V\left[ab_{\alpha\beta}\right] = \sigma_{ab}^2, \text{ etc.}$$

For the classification "Replication within cells" the script letter $\mathcal{R}$ is used, with the corresponding letters R, ρ, and r in the relating terms. By that one has, for example, $r_{\alpha\beta\rho}$ as the "residual" or error term in the model for a two-way classification. Here, then, ρ runs from 1 to $R_{\alpha\beta}$, where $R_{\alpha\beta}$ is the number of replicated observations in the cell which is determined by level α of factor $\mathcal{A}$ and by level β of factor $\mathcal{B}$. The residual or error term always represents the combination of unit error, unit-"treatment" interaction and technical error. σ_r^2 is the residual or error variance, with the three above-mentioned error sources.

The general subscript of an actual observation x is composed of the factor level subscripts determining the cell in which x is observed and of ρ, the index of the particular replication in that cell. Thus, for example, $x_{\alpha\beta\rho}$ denotes "ρth observation in cell '$\alpha\beta$' of a two-way crossed classification."

Using the terms explained before, the observation $x_{\alpha\beta\rho}$ in a two-way crossed classification is then expressed by the linear model

$$\begin{aligned} x_{\alpha\beta\rho} &= X_{\alpha\beta} + r_{\alpha\beta\rho} \\ &= \mu + a_\alpha + b_\beta + ab_{\alpha\beta} + r_{\alpha\beta\rho} \,. \end{aligned}$$

Correspondingly, one has for the analysis of variance of a three-way crossed classification the following underlying model:

$$\begin{aligned} x_{\alpha\beta\gamma\rho} &= X_{\alpha\beta\gamma} + r_{\alpha\beta\gamma\rho} \\ &= \mu + a_\alpha + b_\beta + c_\gamma + ab_{\alpha\beta} + ac_{\alpha\gamma} + bc_{\beta\gamma} + abc_{\alpha\beta\gamma} + r_{\alpha\beta\gamma\rho} \,. \end{aligned}$$

For further discussion of the model see paragraph II.3 below and Appendix A.

Whenever summation (of first order terms) takes place over any one of the subscripts, this subscript is replaced by a dot. Thus, for example, $x_{a.}$ means the sum of all x-values observed at level a of factor $\mathfrak{A}$ in a two-way classification. Correspondingly, $R_{a.}$ means the number of all observations made at level a of factor $\mathfrak{A}$. Average values are denoted, as usual, by bars, for example, $\bar{x}_{a.} = x_{a.}/R_{a.}$ means the average of all observations made at level a of factor $\mathfrak{A}$.

In the formulas for expectations of mean squares there also appear the symbols k_a'', k_a''', . . . , k_b'', k_b''', etc. They are defined in paragraph II.3 below. The test values F' denote the approximate variance ratios as suggested by Cochran (1951). They are explained more fully together with the numerator and denominator mean squares (for example, $MS_1(\mathfrak{A})$ and $MS_2(\mathfrak{A})$, respectively, for factor $\mathfrak{A}$) in paragraph II.4 below. An asterisk ("*") attached to an F-value (sometimes additionally to the prime) indicates that this variance ratio only approximately tests the stated nullhypothesis by distributional reasons. This also is further discussed in paragraph II.4 below.

All cases for which the mean square expectations and/or the variance ratios are given in the table are marked by indicative symbols, which are explained in paragraph II.2 below. For example, "[2.$\mathfrak{B}$.R]" means a two-way classification, with factor $\mathfrak{B}$ of the random effects type (factor $\mathfrak{A}$ being "fixed") and with $R_{\alpha\beta} \equiv R$ replicated observations in each cell. Finally, the abbreviation EMS is used for "expectation(s) of mean square(s)."

II. 2. Scope of the table

The table given in section IV contains the EMS (expectations of mean squares) for two-way, three-way and four-way crossed classifications when the factor levels are either randomly sampled from infinite parent populations ("random effects") or when they are fixed ("fixed effects") or when the factor levels are of both types together in any possible combination ("mixed effects"). In the last case "any possible combination" merely means that either all factors are "random," or the first factor is "fixed" and all others are "random," or the first two are "fixed" etc., or, finally, that all factors are "fixed" and none are "random."

The formulas are given for the case of equal cell numbers R as well as for the most general orthogonal case of unequal but proportional cell numbers. As Wilk and Kempthorne (1956) already put it, "a case of 'proportional numbers' can arise quite naturally when there are unequal numbers of observations corresponding to only one factor of classification." Actually, the formulas for the EMS are simplified considerably if one goes from the most general cases to those where corresponding to one, two or more factors of classification the numbers are equal. Because of the fact that they can easily be derived from the EMS for the most general cases the EMS for these "partial proportional" cases, as they may be called, are not given. Naturally, the cases of "equal cell numbers R" could equally as easily be derived from the most general ones.

These cases are, however, extensively listed because of their frequent occurrence in practice. Again the EMS are not given for the case $R \equiv 1$ ("one observation per cell"), because they can most simply be obtained from the cases with equal numbers R per cell.

In many situations the testing of nullhypotheses is only possible by construction of linear combinations of mean squares for the numerator and the denominator quantities of the variance ratio. Both quantities have to be constructed such that they will have equal expectations if the stated nullhypothesis is true. This test procedure is only approximate, however, and sometimes requires cumbersome derivations for the approximate variance ratio F' and its effective degrees of freedom, f_1 and f_2. In order to save the user of the table the burden of deriving these formulas, the appropriate test values for all nullhypotheses, along with the necessary formulas for the degrees of freedom, have been included in the table. By a matter of consistency, however, this also led to the inclusion of well known ordinary F-values, and the author wishes to take excuse for this from the reader.

Throughout the table the approximate test procedure of Cochran (1951) was applied. The formulas given are more fully described in paragraph II. 4 below. The author is aware of the fact that for four- and more-way classifications the approximations may not be good. For the "partial proportional" cases (as they were called above and for which the EMS are not given) only the F'- and F-values are listed. These test values show how the situation gradually simplifies when the numbers of observations per level become equal for one, two and more factors.

In this connection, it is worthwhile to mention that, in testing nullhypotheses concerning main effects of or interactions between fixed factors, it makes no difference whether or not the numbers of observations at the levels of the fixed factors are equal, the numbers of observations having no influence upon the proper F-values.

The estimation of variance components is easily achieved by using the formulas for the F'- and F-values. This procedure is more fully described in paragraph II. 4 below.

Because of space limitations and practical considerations it was decided to go only up to the four-way classification in the present table. However, the structures of the formulas for the two-way, three-way and four-way classifications indicate sufficiently the rules under which the EMS have to be formed for the general n-way classification. These rules are given in paragraph II.5.

In order to facilitate the use of the table tabular summaries for the three classifications dealt with are given. In each of these summaries the different cases are arranged in rows and columns according to the various models and characteristics of the cell numbers, respectively. The column headings give the cell numbers expressed according to the proportionality conditions imposed upon them. These conditions are:

$$R_{\alpha\beta} = \frac{R_{\alpha.}R_{.\beta}}{R_{..}} \qquad \text{for a two-way classification,}$$

$$R_{\alpha\beta\gamma} = \frac{R_{\alpha..}R_{.\beta.}R_{..\gamma}}{R^2_{...}} \qquad \text{for a three-way classification, and}$$

$$R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}R_{..\gamma.}R_{...\delta}}{R^3_{....}} \qquad \text{for a four-way classification.}$$

For a derivation of these conditions see Appendix B.

Thus, if the numbers of observations at the levels of factor $\mathcal{B}$ in a two-way classification are equal, say, it implies that $R_{.\beta} = \text{const.} = \frac{R_{..}}{B}$ and one gets for this case $R_{\alpha\beta} = \frac{R_{\alpha.}R_{.\beta}}{R_{..}} = \frac{R_{\alpha.}}{B}$. According to the procedure thus exemplified, the column headings in the tabular summaries are formed.

Each case for which formulas are given is marked by an indicative symbol, which again is shown on the page where the EMS and/or the variance ratios are listed. A double line frame indicates that the EMS and the variance ratios are given; a single line frame shows that only the variance ratios are listed. The symbol is composed of three parts which are separated by points. The first is the number 2, 3 or 4, showing the respective number of ways of classification. The next part indicates which of the factors are "random" by showing the respective script letters. Thus the script letters which do not appear are those relating to fixed effects factors. The third and last part of the symbol either consists of the subscripts of those factors for which the numbers of replicated observations per level are unequal, or it is simply "R" or "1", indicating "R observations in each cell" or "1 observation in each cell," respectively. In the first case, therefore, the small Greek letters which do not appear are those relating to factors at the levels of which the numbers of observations are equal. For example, "[3.$\mathcal{BC}$.α]" means: "Three-way (crossed) classification. Factor $\mathcal{A}$ fixed, factors $\mathcal{B}$ and $\mathcal{C}$ random. Unequal numbers of observations at the levels of factor $\mathcal{A}$, equal numbers of observations at the levels of factors $\mathcal{B}$ and $\mathcal{C}$." Another example, namely [2.$\mathcal{B}$.R], was explained at the end of paragraph II.1. It may be noted that only the typical cases received symbols. Also given for each one of the three classifications is a table of the corresponding mean squares in the analysis of variance in their general form as well as in their computational form.

Again because of space limitations, the case of orthogonal contrasts in fixed factors and their interactions with other factors is not treated in this report. However, some remarks are made in paragraph II.6 with reference to a later report which will cover this subject.

II. 3. Discussion of mean square expectation formulas

The derivation of the formulas for the expectations of mean squares (EMS) given in this table is exemplified and shown in Appendix A for the two-way crossed classification with unequal but proportional cell numbers. The method of derivation is based upon defining the components of the linear model in terms of expectations of the "true" cell responses. This is done by a generalized expectation operation which takes care of both "random" and "fixed" effects. Once the terms of the linear model have thus been defined, and after some distributional assumptions concerning them are stated, no further assumptions about them whatsoever are made for the sake of arriving at the final mean square expectation formulas, and the latter then follow in a straightforward process. For the special case of equal cell numbers in the two-way crossed classification with one factor fixed, the other random, the method used for the present table resembles in several aspects that used by Scheffé (1956a) in a critical paper about the mixed model. It will be noted that in the model used in the present report (see Appendix A) the true interaction effects are tied to the main effects by definition. This is a more realistic situation than that of "independent" interactions, as has been pointed out in another paper by Scheffé (1956b).

The relatively simple structure of the formulas, even in the most general cases, may surprise the reader who is familiar with the formulas for the case of proportional cell numbers in the paper of Wilk and Kempthorne (1956). In their "Table 3" ("EMS for special cases of a two-factor experiment") these authors have - besides other deviations - σ_{ab}^2 appearing in the mean square expectations of both factors even if these factors are "fixed."

A sufficient explanation for the discrepancies between the mean square expectations obtained by these authors and the expectations in the present table may be given in considering the definition of the true factorial effects. Namely, in the case of unequal but proportional cell numbers the usual least squares estimates of main effect- and interaction effect-contrasts would be biased if related to Wilk and Kempthorne's definition of the corresponding true contrasts. If Wilk and Kempthorne's true effects are marked by asterisks but otherwise the notation of the present report is retained, one gets, for example, for the contrast of the estimates of two $\mathcal{B}$ factor effects, b^*_β and $b^*_{\beta'}$ with $\beta \neq \beta'$, in a two-way classification with $\mathcal{A}$ "fixed" and $\mathcal{B}$ "random":

$$E\left[\hat{b}^*_\beta - \hat{b}^*_{\beta'}\right] = E\left[\bar{x}_{..\beta.} - \bar{x}_{..\beta'.}\right]$$

$$= \frac{\sum_a R_{a.} X_{a\beta}}{R_{..}} - \frac{\sum_a R_{a.} X_{a\beta'}}{R_{..}}$$

$$\neq b^*_\beta - b^*_{\beta'} = \frac{\sum_a X_{a\beta}}{A} - \frac{\sum_a X_{a\beta'}}{A}$$

(Here the term on the right hand side of the inequality sign is the definition of $b_\beta^* - b_{\beta'}^*$ in Wilk and Kempthorne's paper.)

In the method used in the present report the estimates of the true contrasts are unbiased, see paragraph 2c of Appendix A.

When the cell numbers become equal the formulas for the most general cases as given in the present table reduce - with one exception - to the familiar ones found here and there in textbooks and papers. The only exception is that, in case of mixed models, the interaction variance components carrying the subscript of a fixed factor are multiplied by the ratio of the number of levels of that factor over that same number minus one. For example in case $[3.\mathcal{BC}.R]$, $\sigma_{ab}{}^2$ is multiplied by $\frac{A}{A\text{-}1}$ in all EMS. Because the multiplication by this coefficient is consistent, it does not influence the familiar procedure for testing nullhypotheses. It has, however, an influence upon the estimation of the variance components concerned (in the above example upon that of $\sigma_{ab}{}^2$). The explanation for the appearance of these multipliers is as follows. In the method of sampling from finite populations, the interaction variance in a two-way classification, say, is defined as

$$\left(\sigma_{ab}{}^2\right)_{\text{F.P.}} = \frac{1}{(A^* - 1)(B^* - 1)} \sum_{\alpha=1}^{A^*} \sum_{\beta=1}^{B^*} ab_{\alpha\beta}^2$$

Here "F.P." stands for "finite populations" and A^* and B^* are the sizes of the respective populations of factor levels from which A and B levels, respectively, are randomly sampled. The subtraction of 1 from both A^* and B^* is done in analogy to the degrees of freedom $(A-1)$ and $(B-1)$ in the samples. In fact, when it comes to exhaustive sampling in factor $\mathcal{A}$ and to sampling from an infinite population in factor $\mathcal{B}$, i.e., when $A \to A^*$ and $B^* \to \infty$, one deals with the classical mixed model. If, however, $A = A^*$, it does not make much sense to keep to the analogy of providing one degree of freedom for the estimation of the mean and of subtracting it from A^*, because, actually, this population average then is "known" and does not have to be estimated. Therefore, if one has in mind to make the transition $A \to A^*$ it would be more logical to define

$$\left(\sigma_{ab}{}^2\right)'_{\text{F.P.}} = \frac{1}{A^* \, (B^*-1)} \sum_{\alpha=1}^{A^*} \sum_{\beta=1}^{B^*} ab_{\alpha\beta}^2$$

Then, when $A \to A^*$, independently from B as long as $B < B^*$, one has the relation, leaving the subscript "F.P." off:

$$\left(\sigma_{ab}{}^2\right) = \frac{A}{A-1} \left(\sigma_{ab}{}^2\right)'$$

This, in fact, reflects the relation between the interaction variances (mixed models) usually found in the literature and those given in the present table, where then $(\sigma_{ab}{}^2)$ stands for the commonly used variance and $(\sigma_{ab}{}^2)'$ for the one used in this table. (For the above-discussed example the definition of $(\sigma_{ab}{}^2)'$ or simply of $\sigma_{ab}{}^2$ in this report is $\sigma_{ab}{}^2 = E\,[ab^2_{\alpha\beta}]$. This definition appears to be straightforward if a random sampling process from an infinite population is involved as is the case here for the levels of factor $\mathcal{B}$. See also Appendix A.) Comments emphasizing the adequacy of the coefficient $\frac{1}{A}$ rather than that of $\frac{1}{A-1}$ were also made by D. R. Cox and H. E. Daniels in the discussion of the paper by Plackett (1960).

So far, except for the coefficients k''_a, k''_b, . . ., k'''_a, k'''_b . . ., etc., the formulas for the EMS in the present table are self-explanatory after the notation has been explained in paragraph II.1. The k-coefficients are introduced only for the simplicity of writing. The number of their apostrophies indicates whether they relate to the two-way, three-way or four-way classification; their subscript indicates to which factor they belong. Thus, for example, the k-coefficient relating to factor $\mathcal{B}$ in a three-way classification is defined as

$$k'''_b = \frac{1}{R^2_{\cdots}} \sum_{\beta=1}^{B} R^2_{\cdot\beta\cdot}$$

Here $R_{\cdot\beta\cdot}$ and $R_{\cdots}$ are the numbers of observations at level β of factor $\mathcal{B}$ and the total number of observations in the whole analysis, respectively. These k-coefficients are explicitly defined in the table for all three classifications.

The EMS for a two-way classification with proportional cell numbers when both factors are random (case $[2.\mathcal{A}\mathcal{B}.\alpha\beta]$ in the present table) seem to have first been given by H. Fairfield Smith (1951). The expectations of mean squares of H. F. Smith take on exactly the form of those presented in this table if one replaces his terms by the symbols of the present table as follows:

a by $\dfrac{R_{\alpha\cdot}}{\sqrt{R_{\cdot\cdot}}}$

b by $\dfrac{R_{\cdot\beta}}{\sqrt{R_{\cdot\cdot}}}$

$$S'aa' = (Sa)^2 - Sa^2 \text{ by } R_{\cdot\cdot}\left(1 - \frac{1}{R^2_{\cdot\cdot}} \sum_{\alpha=1}^{A} R^2_{\alpha\cdot}\right) = R_{\cdot\cdot}\,(1-k''_a)$$

$S'bb'$ by $R_{\cdot\cdot}\,(1-k''_b)$

N by $R_{\cdot\cdot}$

p by A and V_a by $\sigma_a{}^2$

q by B and V_β by $\sigma_b{}^2$

$V_{\alpha\beta}$ by $\sigma_{ab}{}^2$ and V_o by $\sigma_r{}^2$.

A final remark concerns the correctness of the formulas presented. After the mean square expectation formulas had been derived they were checked by application of the rules for obtaining EMS's as given in paragraph II.5 below. Complete conformity was observed in all cases.

II. 4. Approximate F-tests and the estimation of variance components

The appropriate variance ratios for testing all nullhypotheses are listed in the present table in order to save the user the burden of deriving them, especially in the complex cases, as mentioned earlier. This listing of variance ratios may, however, also be of help for the planning of experiments or sample surveys in that it shows under which conditions exact F-tests will be available and under which they will not. This also is the main reason why the F- and F'-values are listed for the "partial proportional" cases (where the EMS are not explicitly given). By that the reader may see at a glance also in these cases in which situation with respect to the testing of nullhypotheses he is or will be.

The numerator and denominator quantities and their effective degrees of freedom f_1 and f_2, respectively, for the approximate F'-tests are given following Cochran (1951) and, for the degrees of freedom, Satterthwaite (1946). These two quantities, which are actually linear combinations of mean squares from the analysis of variance, are denoted (for testing a nullhypothesis $\sigma_a{}^2 = 0$, say) by $MS_1(\mathfrak{A})$ and $MS_2(\mathfrak{A})$, in analogy to the numerator mean square, $MS(\mathfrak{A})$, in the ordinary case. The expectations of the two quantities are equal if the nullhypothesis is true. Thus, in the above example, $E[MS_1(\mathfrak{A})] = E[MS_2(\mathfrak{A})]$ if $\sigma_a{}^2 = 0$ is true. The subscripts 1 and 2 always refer to the numerator and the denominator in F', respectively.

Cochran proposes to have all coefficients positive in both the numerator and the denominator linear combination of mean squares because such linear forms are better represented by a Type III approximation than those where some coefficients are negative. The two quantities have always been constructed according to this suggestion. In fact, the coefficients as given in the table for the cases of proportional cell numbers (where the absolute value of the coefficients is not simply unity) will never be negative. This is true because the k-coefficients, as can easily be seen, are always smaller than one, and products like $Ak_a^{n\cdot\cdot}$, $Bk_b^{\cdot m\cdot}$ etc., as appearing in the coefficients for the residual mean square, $MS(\mathfrak{R})$, are always greater than or equal to one. (The last statement can easily be proven geometrically.) The case in which these products are equal to one (whereby the residual mean square drops out of the corresponding linear combination because of $Ak_a'' - 1 = 0$, for example) arises when the numbers of observations at the levels of the corresponding factor become equal. More explicitly, for the above example of a two-way classification: $R_{a.} = \text{const.} = \frac{R_{..}}{A}$ will make $k_a'' = \frac{1}{R_{..}^2} \sum_a R_{a.}^2 = \frac{1}{A}$ and thereby $Ak_a'' - 1 = 0$.

The numerator quantity in F' is always such that its expectation is larger than that of the denominator quantity if the alternative hypothesis is true.

In his above-mentioned paper Cochran discusses the accuracy of the F' approximation only for the case when 3 variances (mean squares) are involved in testing a nullhypothesis. This accuracy is rather good as shown in his Table III ("True significance probability of F' at the apparent 5% level"). However, there can be no doubt that the accuracy decreases considerably if many mean squares are involved in F' as is the case in some F'-values in the four-way classification, for example. Therefore, it is recommended to use the F'-values in this table, in which more than three mean squares are involved, with caution. There will be no sense in keeping exactly to the 5% significance level, say, of the F-distribution. Only a definitely significant F'-value or a definitely non-significant F'-value will lead to a statement whether the nullhypothesis should be rejected or not rejected, with a no-decision region of $0.01 \leq \alpha \leq 0.10$, say.

The approximate degrees of freedom f_1 and f_2 of F' will always have to be rounded to the nearest integer in order to compare the computed F'-value with the tabled percentage point of F.

Another type of approximation in testing nullhypotheses by variance ratios in the analysis of variance is present if the underlying model is of the mixed effects type. In this case all those variance ratios which test for fixed main effects and fixed or mixed interaction effects and whose denominator quantities have expectations larger than σ_r^2, the error variance, are not distributed as F under the stated nullhypothesis. This and the fact that the nullhypothesis on the fixed factor effects in a two-way classification with underlying mixed model can be tested by Hotelling's T^2 has been discussed by Scheffé (1956a). This author, however, doubts whether the exact test procedure is worth the extra computational labor. Later, Scheffé (1959) almost ruled out the application of Hotelling's T^2 for cases of mixed models in three- and more-way classifications in saying that in these cases its use is "numerically so complicated that it is unlikely ever to be applied in practice." Following Scheffé, however, Imhof (1960) has given exact test procedures for the three-way classification with one factor fixed and the other two random.

Up to now it seems to be unknown how good the approximation of the "classical" F-values is in these situations. In the present table they are marked by asterisks ("*"). The user of this table should keep in mind the approximate character of these F-values when applying them to test a stated nullhypothesis. This especially will apply when both the prime and the asterisk are attached to an F-symbol, thus indicating "double" approximation.

The estimation of variance components is easily achieved with the help of the listed F- and F'-values. As a general rule the following can be stated:

$$\begin{bmatrix}\text{Estimate of}\\ \text{Variance}\\ \text{Component}\end{bmatrix} = \frac{\left[\begin{pmatrix}\text{Numerator quantity of the } F\text{-}\\ \text{or } F'\text{-value for testing the}\\ \text{corresponding nullhypothesis}\end{pmatrix} \text{ minus } \begin{pmatrix}\text{Denominator}\\ \text{quantity of this}\\ F\text{- or } F'\text{-value}\end{pmatrix}\right]}{\begin{bmatrix}\text{Coefficient of variance component to}\\ \text{be estimated in numerator quantity}\end{bmatrix}}$$

Thus, for example, in case $[3.\mathcal{ABC}.\alpha\beta\gamma]$, one has as the best estimate of σ_a^2:

$$\hat{\sigma}_a^2 = \frac{\mathrm{MS}_1(\mathcal{A}) - \mathrm{MS}_2(\mathcal{A})}{R\ldots(1 - k_a'')/(A-1)} \quad .$$

For the sampling variances of variance component estimates the reader is referred to the papers by Crump (1951), Tukey (1956), Welch (1956) and Searle (1958).

II. 5. Generalization for the n-way crossed classification

Mean square expectations for the n-way crossed classification can be induced from those given in this table for the two-way, three-way and four-way classification. The rules of doing this are given below. They could be separately listed for all models and for proportional as well as for equal cell numbers. For the sake of simplicity, however, the rules are given for the case of "proportional cell numbers, all factors random" only, with additional rules of how to change these formulas if one, two, or more factors are fixed. Any case of equal numbers can then easily be obtained by equating the k-coefficients to the reciprocals of the numbers of levels of the corresponding factors. (For example, in the two-way classification, if there are equal numbers of observations at the levels of factor $\mathcal{A}$, one substitutes $k_a'' = \frac{1}{A}$.)

1. Rules for obtaining expectations of mean squares in an n-way crossed classification, case of unequal but proportional cell numbers, all factors random.

1a. Each mean square expectation contains the residual variance component σ_r^2 (with coefficient 1); further it contains all variance components (with coefficients as described in 1b. below) with subscript combinations containing those small Latin letters which correspond to the script letters in the designating symbol of the respective mean square. Thus, for example, the expectation of the mean square for the first order interaction $\mathcal{AB}$ in the five-way classification will contain σ_r^2 plus the following components (with coefficients as indicated in 1b. below): σ^2_{abcde}, σ^2_{abcd}, σ^2_{abce}, σ^2_{abde}, σ^2_{abc}, σ^2_{abd}, σ^2_{abe}, and σ^2_{ab}.

In general, in an n-way classification with all factors random, the expectation of a mean square designated by, say, f script letters thus will contain $\sigma_r{}^2$ plus 2^{n-f} additional variance components related to main effects and interactions. Main effects, therefore, are characterized by $f = 1$ and their expectations will contain $\sigma_r{}^2$ plus 2^{n-1} additional variance components.

1b. The coefficients of the 2^{n-f} variance components are obtained as follows: Each component is multiplied by the total number of observations in the n-way classification. (In a five-way classification, say, $R_{\cdot\cdot\cdot\cdot\cdot}$ is the symbol for this number.) For each Latin letter in the subscript of the variance component which corresponds to a designating script letter in the respective mean square, there will be a coefficient (1-k) divided by the number of degrees of freedom corresponding to that particular script letter. Thus, in the above example, all 8 variance components (not including $\sigma_r{}^2$) in E[MS ($\mathcal{AB}$)] will have

$$\frac{R_{\cdot\cdot\cdot\cdot\cdot}\,(1-k_a^{''''})(1-k_b^{''''})}{(A-1)(B-1)}$$

as a common coefficient. Corresponding to all other Latin letters in the subscript the variance component will be multiplied by a k-coefficient with that very subscript. Thus, finally, in the above example, the coefficient of σ^2_{abcde} in E[MS ($\mathcal{AB}$)], say, will be

$$\frac{R_{\cdot\cdot\cdot\cdot\cdot}\,(1-k_a^{''''})(1-k_b^{''''})k_c^{''''}k_d^{''''}k_e^{''''}}{(A-1)(B-1)}$$

2. <u>Rules for deriving mean square expectations from the formulas obtained under 1a. and 1b. above for the case of unequal but proportional cell numbers, one factor or more fixed, the others random</u>.

2a. If a factor is fixed all variance components containing the corresponding small Latin letter in their subscripts are deleted in the expectations of those mean squares among whose designating script letters is not the script letter of the fixed factor. Thus, in the before-mentioned example of a five-way classification, if factor $\mathcal{C}$ is fixed, say, the components σ^2_{abcde}, σ^2_{abcd}, σ^2_{abce} and σ^2_{abc} are deleted in E[MS ($\mathcal{AB}$)]. They are, however, not deleted in the expectation of MS ($\mathcal{AC}$), for example.

2b. In the expectations of those mean squares among whose designating script letters is that of the fixed factor, the coefficient (1-k) corresponding to the fixed factor is deleted. In the above example of a five-way classification with factor $\mathcal{C}$ fixed,

$(1 - k_c^{**})$ is thus deleted in the common multiplier of all 8 variance components (not including σ_r^2) in $E[\mathrm{MS}(\mathcal{AC})]$.

2c. In the expectation of the mean square due to the fixed factor the term containing the corresponding variance component is replaced by the weighted sum of the squared (true) level effects divided by the corresponding degrees of freedom. Thus, in the example, if again factor $\mathcal{C}$ is assumed to be fixed, $\frac{R_{.....}}{C-1}\ \sigma_c^2$ in $E[\mathrm{MS}(\mathcal{C})]$ (after application of rule 2b.) is replaced by

$$\frac{1}{C-1} \sum_{\gamma=1}^{C} R_{..\gamma..}\, c_\gamma^2 \quad .$$

2d. If two or more factors are fixed, rules 2a. - 2c. simultaneously apply with respect to all fixed factors. Moreover, the terms including interaction variance components due to fixed factors only are replaced by the corresponding weighted sums of squared (true) interaction effects, divided by the respective degrees of freedom. Thus in the five-way classification with factors $\mathcal{A}$ and $\mathcal{B}$ fixed, say, the term $\frac{R_{.....}}{(A-1)(B-1)}\ \sigma_{ab}^2$ in $E[\mathrm{MS}(\mathcal{AB})]$ (after application of rule 2b.) is replaced by

$$\frac{1}{(A-1)(B-1)} \sum_{\alpha=1}^{A} \sum_{\beta=1}^{B} R_{\alpha\beta...}\, ab_{\alpha\beta}^2 \quad .$$

II. 6. <u>The case of orthogonal contrasts in fixed factors.</u>

In case of orthogonal contrasts in fixed factors, i.e., when the (overall) sum of squares for main effects of a fixed factor or for interaction effects involving one or more fixed factors is split into independent components with one degree of freedom each, the situation with respect to expected values and testing of nullhypotheses is not, in general, a priori obvious.

However, if one deals with a fixed model it is not difficult to see that such a single degree of freedom component will have expectation σ_r^2 if the corresponding nullhypothesis is true. This applies both for unequal but proportional and for equal cell numbers. Therefore, in this case of a fixed model, one will have as denominator quantity in the respective variance ratio the mean square for replications or the mean square for the highest order interaction (if the latter can be assumed not existent), respectively, depending upon whether or not one has replicated observations in the cells.

In the case of a mixed model with unequal but proportional cell numbers the expectations of the said orthogonal components are such that the nullhypotheses concerned cannot be tested following the pattern given for the (overall) mean squares in this table. The expectations of the components and adequate procedures for testing the corresponding nullhypotheses will be given in a later report.

The situation with respect to orthogonal contrasts simplifies to a given extent when in case of mixed models the cell numbers are equal. This also will be discussed in the above-mentioned later report.

III. References

1. Anderson, R. L. and Bancroft, T. A. (1952), Statistical Theory in Research, McGraw Hill Book Co., Inc., New York.

2. Bankier, J. D. and Walpole, R. E. (1957), Components of Variance Analysis for Proportional Frequencies, Ann. Math. Stat., 28, 742-743.

3. Bennett, C. A. and Franklin, N. L. (1954), Statistical Analysis in Chemistry and the Chemical Industry, John Wiley & Sons, Inc., New York.

4. Brownlee, K. A. (1960), Statistical Theory and Methodology in Science and Engineering, John Wiley & Sons, Inc., New York.

5. Cochran, W. G. (1951), Testing a Linear Relation Among Variances, Biometrics, 7, 17-32.

6. Cornfield, J. and Tukey, J. W. (1956), Average Values of Mean Squares in Factorials, Ann. Math. Stat., 27, 907-949.

7. Crump, S. L. (1951), The Present Status of Variance Component Analysis, Biometrics, 7, 1-16.

8. Eisenhart, C. (1947), The Assumptions Underlying the Analysis of Variance, Biometrics, 3, 1-21.

9. Imhof, J. P. (1960), A Mixed Model for the Complete Three-way Layout with Two Random-Effects Factors, Ann. Math. Stat., 31, 906-928.

10. Plackett, R. L. (1960), Models in the Analysis of Variance, Journal Roy. Stat. Soc., B, 22, 195-217.

11. Satterthwaite, F. E. (1946), An Approximate Distribution of Estimates of Variance Components, Biometrics 2, 110-114.

12. Scheffé, H. (1956a) A "Mixed Model" for the Analysis of Variance, Ann. Math. Stat., 27, 23-36.

13. Scheffé, H. (1956b), Alternative Models for the Analysis of Variance, Ann. Math. Stat., 27, 251-271.

14. Scheffé, H. (1959), The Analysis of Variance, John Wiley & Sons, Inc., New York.

15. Searle, S. R. (1958), Sampling Variances of Estimates of Components of Variance, Ann. Math. Stat., 29, 167-178.

III. References (continued)

16. Smith, H. Fairfield (1951), Analysis of Variance with Unequal but Proportionate Numbers of Observations in the Sub-Classes of a Two-Way Classification, Biometrics, 7, 70-74.

17. Tukey, J. W. (1956), Variances of Variance Components: I. Balanced Designs, Ann. Math. Stat., 27, 722-736.

18. Welch, B. L. (1956), On Linear Combinations of Several Variances, Journal American Statistical Association, 51, 132-148.

19. Wilk, M. B. and Kempthorne, O. (1956), Some Aspects of the Analysis of Factorial Experiments in a Completely Randomized Design, Ann. Math. Stat., 27, 950-985.

IV. 1 TABLE OF MEAN SQUARE EXPECTATIONS AND VARIANCE RATIOS IN THE ANALYSIS OF VARIANCE FOR THE TWO-WAY CROSSED CLASSIFICATION

Two-way crossed classification. Table I: Definition of symbols.

Model for analysis of variance:

$x_{\alpha\beta\rho} = \mu + a_\alpha + b_\beta + ab_{\alpha\beta} + r_{\alpha\beta\rho}$, with $r_{\alpha\beta\rho} \sim \mathrm{NID}\,(o, \sigma_r^2)$.

$\alpha = 1, \ldots, A$; $\beta = 1, \ldots, B$; $\rho = 1, \ldots, R_{\alpha\beta}$

A and B: numbers of levels in factors $\mathfrak{A}$ and $\mathfrak{B}$, respectively.

$R_{\alpha\beta}$ = number of replicated observations in cell "$\alpha\beta$"

$R_{\alpha.} = \sum_{\beta=1}^{B} R_{\alpha\beta}$, $\quad R_{.\beta} = \sum_{\alpha=1}^{A} R_{\alpha\beta}$

$R_{..} = \sum_{\alpha}\sum_{\beta} R_{\alpha\beta}$ = total number of observations

$x_{\alpha\beta\rho}$ = ρth observation in cell "$\alpha\beta$"

$x_{\alpha\beta.} = \sum_{\rho=1}^{R_{\alpha\beta}} x_{\alpha\beta\rho}$, $\quad x_{\alpha..} = \sum_{\beta=1}^{B} x_{\alpha\beta.}$, etc.

$\bar{x}_{\alpha\beta.} = \frac{x_{\alpha\beta.}}{R_{\alpha\beta}}$, $\quad \bar{x}_{\alpha..} = \frac{x_{\alpha..}}{R_{\alpha.}}$, etc.

$k''_a = \frac{1}{R^2_{..}} \sum_{\alpha=1}^{A} R^2_{\alpha.}$, $\quad k''_b = \frac{1}{R^2_{..}} \sum_{\beta=1}^{B} R^2_{.\beta}$

F', F^*, F'^*: Prime and/or asterisk attached to F-value:
Variance ratio only approximately distributed as F under stated nullhypothesis H_o. See paragraph II.4.

Two-way crossed classification. Table II: Mean squares. $R_{\alpha\beta} = \frac{R_{\alpha.} R_{.\beta}}{R_{..}}$

For definition of symbols used see [2-way classification. Table I], page 19.

	DF	MS (Mean square)
$\mathcal{A}$	$A-1$	$\mathrm{MS}(\mathcal{A}) = \frac{1}{A-1} \sum_{\alpha} R_{\alpha.} (\bar{x}_{\alpha..} - \bar{x}_{...})^2 = \frac{1}{A-1} \left[\sum_{\alpha} \frac{x^2_{\alpha..}}{R_{\alpha.}} - \frac{x^2_{...}}{R_{..}} \right]$
$\mathcal{B}$	$B-1$	$\mathrm{MS}(\mathcal{B}) = \frac{1}{B-1} \sum_{\beta} R_{.\beta} (\bar{x}_{.\beta.} - \bar{x}_{...})^2 = \frac{1}{B-1} \left[\sum_{\beta} \frac{x^2_{.\beta.}}{R_{.\beta}} - \frac{x^2_{...}}{R_{..}} \right]$
$\mathcal{AB}$	$(A-1)(B-1)$	$\mathrm{MS}(\mathcal{AB}) = \frac{1}{(A-1)(B-1)} \sum_{\alpha} \sum_{\beta} R_{\alpha\beta} (\bar{x}_{\alpha\beta.} - \bar{x}_{\alpha..} - \bar{x}_{.\beta.} + \bar{x}_{...})^2$
		$= \frac{1}{(A-1)(B-1)} \left[\sum_{\alpha} \sum_{\beta} \frac{x^2_{\alpha\beta.}}{R_{\alpha\beta}} - \sum_{\alpha} \frac{x^2_{\alpha..}}{R_{\alpha.}} - \sum_{\beta} \frac{x^2_{.\beta.}}{R_{.\beta}} + \frac{x^2_{...}}{R_{..}} \right]$
$\mathcal{R}$	$R_{..}-AB$	$\mathrm{MS}(\mathcal{R}) = \frac{1}{R_{..}-AB} \sum_{\alpha} \sum_{\beta} \sum_{\rho} (x_{\alpha\beta\rho} - \bar{x}_{\alpha\beta.})^2$
		$= \frac{1}{R_{..}-AB} \left[\sum_{\alpha} \sum_{\beta} \sum_{\rho} x^2_{\alpha\beta\rho} - \sum_{\alpha} \sum_{\beta} \frac{x^2_{\alpha\beta.}}{R_{\alpha\beta}} \right]$

2-way classification
Table II: Mean squares.

Two-way crossed classification. Table III: Tabular summary.

For definition of symbols used see [2-way classification. Table I], page 19.

Cases for which mean square expectations (EMS) and/or variance ratios are tabulated on the subsequent pages are marked by the following indicative symbols. Double line frame: EMS and variance ratios given; single line frame: Only variance ratios given.

	$R_{\alpha\beta} = \dfrac{R_{\alpha.} R_{.\beta}}{R_{..}}$	$R_{.\beta} = \text{const.} = \dfrac{R_{..}}{B}$: $R_{\alpha\beta} = \dfrac{R_{\alpha.}}{B}$	$R_{\alpha\beta} \equiv R$	$R_{\alpha\beta} \equiv 1$
$\mathfrak{A}$ and $\mathfrak{B}$ random	$2.\mathfrak{AB}..\alpha\beta$ Page 22	$2.\mathfrak{AB}..\alpha$ Page 22	$2.\mathfrak{AB}..R$ Page 23	$2.\mathfrak{AB}.1$ Page 23
$\mathfrak{A}$ fixed, $\mathfrak{B}$ random	$2.\mathfrak{B}..\alpha\beta$ Page 24	$2.\mathfrak{B}..\alpha$ Page 24	$2.\mathfrak{B}..R$ Page 25	$2.\mathfrak{B}..1$ Page 25
$\mathfrak{A}$ and $\mathfrak{B}$ fixed	$2..\alpha\beta$ Page 26	Put $R_{\alpha\beta} = \dfrac{R_{\alpha.}}{B}$ in [2..$\alpha\beta$], page 26.	Put $R_{\alpha\beta} \equiv R$ in [2..$\alpha\beta$], page 26.	$2..1$ Page 26

Note. For the case of orthogonal contrasts in fixed factors see paragraph II.6.

2-way classification
Table III: Tabular summary

[2.$\mathcal{AB}$.$a\beta$] and [2.$\mathcal{AB}$.a]: $\mathcal{A}$ and $\mathcal{B}$ random; $R_{a\beta} = \dfrac{R_{a.}R_{.\beta}}{R_{..}}$ and $R_{a\beta} = \dfrac{R_{a.}}{B}$, respectively.

For definition of symbols used see [2-way classification. Table I, II and III], pages 19-21.

	EMS ([2.$\mathcal{AB}$.$a\beta$])†	H_o	Testing H_o: [2.$\mathcal{AB}$.$a\beta$]: $R_{a\beta} = \dfrac{R_{a.}R_{.\beta}}{R_{..}}$	Testing H_o: [2.$\mathcal{AB}$.a]: $R_{a\beta} = \dfrac{R_{a.}}{B}$
$\mathcal{A}$	$\sigma_r^2 + R_{..}\dfrac{1-k_a''}{A-1}\left[k_b''\sigma_{ab}^2 + \sigma_a^2\right]$	$\sigma_a^2 = 0$	$F'(\mathcal{A})$ 1)	$F = \dfrac{MS(\mathcal{A})}{MS(\mathcal{AB})}$
$\mathcal{B}$	$\sigma_r^2 + R_{..}\dfrac{1-k_b''}{B-1}\left[k_a''\sigma_{ab}^2 + \sigma_b^2\right]$	$\sigma_b^2 = 0$	$F'(\mathcal{B})$ 2)	
$\mathcal{AB}$	$\sigma_r^2 + R_{..}\dfrac{(1-k_a'')(1-k_b'')}{(A-1)\ (B-1)}\sigma_{ab}^2$	$\sigma_{ab}^2 = 0$	$F = MS(\mathcal{AB})/MS(\mathcal{R})$	
$\mathcal{R}$	σ_r^2	—	—	

†) For EMS of [2.$\mathcal{AB}$.a] put $k_b'' = \dfrac{1}{B}$

1) $F'(\mathcal{A}) = MS_1(\mathcal{A})/MS(\mathcal{AB})$, where

$$MS_1(\mathcal{A}) = \frac{(1-k_b'')MS(\mathcal{A})}{(B-1)k_b''} + \frac{(Bk_b'' - 1)MS(\mathcal{R})}{(B-1)k_b''} ,$$

with effective degrees of freedom:

$$f_1 = [MS_1(\mathcal{A})]^2 \Bigg/ \left\{ \frac{1}{A-1}\left[\frac{(1-k_b'')MS(\mathcal{A})}{(B-1)k_b''}\right]^2 + \frac{1}{R_{..}-AB}\left[\frac{(Bk_b''-1)MS(\mathcal{R})}{(B-1)k_b''}\right]^2 \right\}$$

2) $F'(\mathcal{B}) = MS_1(\mathcal{B})/MS(\mathcal{AB})$ analogous to $F'(\mathcal{A})$: Interchange respective symbols.

2.$\mathcal{AB}$.$a\beta$ + 2.$\mathcal{AB}$.a

[2.$\mathfrak{AB}$.R] and [2.$\mathfrak{AB}$.1]: $\mathfrak{A}$ and $\mathfrak{B}$ random; $R_{\alpha\beta} \equiv R$ and $R_{\alpha\beta} \equiv 1$, respectively.

For definition of symbols used see [2-way classification. Tables I, II and III], pages 19-21.

	EMS ([2.$\mathfrak{AB}$.R])†	H_o	Testing H_o	
			[2.$\mathfrak{AB}$.R]: $R_{\alpha\beta} \equiv R$	[2.$\mathfrak{AB}$.1]: $R_{\alpha\beta} \equiv 1$
$\mathfrak{A}$	$\sigma_r^2 + R\,\sigma_{ab}^2 + RB\,\sigma_a^2$	$\sigma_a^2 = 0$	$F = \mathrm{MS}(\mathfrak{A})/\mathrm{MS}(\mathfrak{AB})$	
$\mathfrak{B}$	$\sigma_r^2 + R\,\sigma_{ab}^2 + RA\,\sigma_b^2$	$\sigma_b^2 = 0$	$F = \mathrm{MS}(\mathfrak{B})/\mathrm{MS}(\mathfrak{AB})$	
$\mathfrak{AB}$	$\sigma_r^2 + R\,\sigma_{ab}^2$	$\sigma_{ab}^2 = 0$	$F = \dfrac{\mathrm{MS}(\mathfrak{AB})}{\mathrm{MS}(\mathfrak{R})}$	Not possible
$\mathfrak{R}$	σ_r^2	——	——	——

†) For EMS of [2.$\mathfrak{AB}$.1] put $R = 1$ and delete line "$\mathfrak{R}$".

2.$\mathfrak{AB}$.R + 2.$\mathfrak{AB}$.1

$[2.\mathcal{B}.\alpha\beta]$ and $[2.\mathcal{B}.\alpha]$: $\mathcal{A}$ fixed, $\mathcal{B}$ random; $R_{\alpha\beta} = \frac{R_{\alpha.}R_{.\beta}}{R_{..}}$ and $R_{\alpha\beta} = \frac{R_{\alpha.}}{B}$, respectively.

For definition of symbols used see [2-way classification. Tables I, II and III], pages 19-21.

	EMS $([2.\mathcal{B}.\alpha\beta])^{\dagger}$	H_o	Testing H_o: $[2.\mathcal{B}.\alpha\beta]$: $R_{\alpha\beta} = \frac{R_{\alpha.}R_{.\beta}}{R_{..}}$	Testing H_o: $[2.\mathcal{B}.\alpha]$: $R_{\alpha\beta} = \frac{R_{\alpha.}}{B}$
$\mathcal{A}$	$\sigma_r^2 + \frac{R_{..}k_b''}{A-1}\sigma_{ab}^2 + \frac{1}{A-1}\sum_{\alpha} R_{\alpha.}a_{\alpha}^2$	$a_{\alpha} \equiv 0$	$F'^{*}(\mathcal{A})$ 1)	$F^{*} = \frac{MS(\mathcal{A})}{MS(\mathcal{AB})}$
$\mathcal{B}$	$\sigma_r^2 + R_{..}\frac{1-k_b''}{B-1}\sigma_b^2$	$\sigma_b^2 = 0$	$F = MS(\mathcal{B})/MS(\mathcal{R})$	
$\mathcal{AB}$	$\sigma_r^2 + \frac{R_{..}(1-k_b'')}{(A-1)(B-1)}\sigma_{ab}^2$	$\sigma_{ab}^2 = 0$	$F = MS(\mathcal{AB})/MS(\mathcal{R})$	
$\mathcal{R}$	σ_r^2	———	———	

†) For EMS of $[2.\mathcal{B}.\alpha]$ put $k_b'' = \frac{1}{B}$

1) $F'^{*}(\mathcal{A}) = MS_1(\mathcal{A})/MS(\mathcal{AB})$ as in $[2.\mathcal{AB}.\alpha\beta]$, page 22.

$\boxed{2.\mathcal{B}.\alpha\beta}$ + $\boxed{2.\mathcal{B}.\alpha}$

[2.$\mathfrak{B}$.R] and [2.$\mathfrak{B}$.1]: $\mathfrak{A}$ fixed, $\mathfrak{B}$ random; $R_{a\beta} \equiv R$ and $R_{a\beta} \equiv 1$, respectively.

For definition of symbols used see [2-way classification. Tables I, II and III], pages 19-21.

	EMS ([2.$\mathfrak{B}$.R])†	H_o	Testing H_o	
			[2.$\mathfrak{B}$.R]: $R_{a\beta} \equiv R$	[2.$\mathfrak{B}$.1]: $R_{a\beta} \equiv 1$
$\mathfrak{A}$	$\sigma_r^2 + \frac{RA}{A-1}\sigma_{ab}^2 + \frac{RB}{A-1}\sum_a \alpha_a^2$	$\alpha_a \equiv 0$	$F^* = MS(\mathfrak{A})/MS(\mathfrak{AB})$	
$\mathfrak{B}$	$\sigma_r^2 + RA\,\sigma_b^2$	$\sigma_b^2 = 0$	$F = \frac{MS(\mathfrak{B})}{MS(\mathfrak{R})}$	If $\sigma_{ab}^2 = 0$ can be assumed: $F = MS(\mathfrak{B})/MS(\mathfrak{AB})$
$\mathfrak{AB}$	$\sigma_r^2 + \frac{RA}{A-1}\sigma_{ab}^2$	$\sigma_{ab}^2 = 0$	$F = \frac{MS(\mathfrak{AB})}{MS(\mathfrak{R})}$	Not possible
$\mathfrak{R}$	σ_r^2	——	——	——

†) For EMS of [2.$\mathfrak{B}$.1] put $R = 1$ and delete line "$\mathfrak{R}$".

$\boxed{2.\mathfrak{B}.R}$ + $\boxed{2.\mathfrak{B}.1}$

[2~.$\alpha\beta$] and [2~.1]: $\mathfrak{A}$ and $\mathfrak{B}$ fixed; $R_{\alpha\beta} = \frac{R_{\alpha.}R_{.\beta}}{R_{..}}$ and $R_{\alpha\beta} \equiv 1$, respectively.

For definition of symbols used see [2-way classification. Tables I, II and III], pages 19-21.

	EMS ([2~.$\alpha\beta$])†	H_o	Testing H_o: [2~.$\alpha\beta$]: $R_{\alpha\beta} = \frac{R_{\alpha.}R_{.\beta}}{R_{..}}$	Testing H_o: [2~.1]: $R_{\alpha\beta} \equiv 1$
$\mathfrak{A}$	$\sigma_r^2 + \frac{1}{A-1}\sum_\alpha R_{\alpha.} a_\alpha^2$	$a_\alpha \equiv 0$	$F = \frac{MS(\mathfrak{A})}{MS(\mathfrak{R})}$	If $ab_{\alpha\beta} \equiv 0$ can be assumed: $F = MS(\mathfrak{A})/MS(\mathfrak{AB})$
$\mathfrak{B}$	$\sigma_r^2 + \frac{1}{B-1}\sum_\beta R_{.\beta} b_\beta^2$	$b_\beta \equiv 0$	$F = \frac{MS(\mathfrak{B})}{MS(\mathfrak{R})}$	If $ab_{\alpha\beta} \equiv 0$ can be assumed: $F = MS(\mathfrak{B})/MS(\mathfrak{AB})$
$\mathfrak{AB}$	$\sigma_r^2 + \frac{1}{(A-1)(B-1)}\sum_\alpha\sum_\beta R_{\alpha\beta} ab_{\alpha\beta}^2$	$ab_{\alpha\beta} \equiv 0$	$F = \frac{MS(\mathfrak{AB})}{MS(\mathfrak{R})}$	Not possible
$\mathfrak{R}$	σ_r^2	———	———	———

†) For EMS of [2~.1] put $R_{\alpha\beta} \equiv 1$ and delete line "$\mathfrak{R}$".

2~.$\alpha\beta$ + 2~.1

IV. 2 TABLE OF MEAN SQUARE EXPECTATIONS AND VARIANCE RATIOS IN THE ANALYSIS OF VARIANCE FOR THE THREE-WAY CROSSED CLASSIFICATION

Three-way crossed classification. Table I: Definition of symbols.

Model for analysis of variance:

$x_{\alpha\beta\gamma\rho} = \mu + a_\alpha + b_\beta + c_\gamma + ab_{\alpha\beta} + ac_{\alpha\gamma} + bc_{\beta\gamma} + abc_{\alpha\beta\gamma} + r_{\alpha\beta\gamma\rho}$, with $r_{\alpha\beta\gamma\rho} \sim \mathrm{NID}(0, \sigma_r^2)$.

$\alpha = 1, \ldots, A;\ \beta = 1, \ldots, B;\ \gamma = 1, \ldots, C;\ \rho = 1, \ldots, R_{\alpha\beta\gamma}$

A, B and C: numbers of levels in factors $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$, respectively.

$R_{\alpha\beta\gamma}$ = number of replicated observations in cell "$\alpha\beta\gamma$".

$R_{\alpha\beta.} = \sum_{\gamma=1}^{C} R_{\alpha\beta\gamma}$, $R_{\alpha..} = \sum_{\beta=1}^{B} R_{\alpha\beta.}$, etc.

$R_{...} = \sum_\alpha \sum_\beta \sum_\gamma R_{\alpha\beta\gamma}$ = total number of observations.

$x_{\alpha\beta\gamma\rho}$ = ρth observation in cell "$\alpha\beta\gamma$".

$x_{\alpha\beta\gamma.} = \sum_{\rho=1}^{R_{\alpha\beta\gamma}} x_{\alpha\beta\gamma\rho}$, $x_{\alpha\beta..} = \sum_{\gamma=1}^{C} x_{\alpha\beta\gamma.}$, etc.

$\bar{x}_{\alpha\beta\gamma.} = \dfrac{x_{\alpha\beta\gamma.}}{R_{\alpha\beta\gamma}}$, $\bar{x}_{\alpha\beta..} = \dfrac{x_{\alpha\beta..}}{R_{\alpha\beta.}}$, etc.

$k_a''' = \dfrac{1}{R_{...}^2} \sum_{\alpha=1}^{A} R_{\alpha..}^2$, $k_b''' = \dfrac{1}{R_{...}^2} \sum_{\beta=1}^{B} R_{.\beta.}^2$, $k_c''' = \dfrac{1}{R_{...}^2} \sum_{\gamma=1}^{C} R_{..\gamma}^2$

F'; F^*, F'^*: Prime and/or asterisk attached to F-value:

Variance ratio only approximately distributed as F under stated nullhypothesis H_o.

See paragraph II.4.

Three-way crossed classification. Table II: Mean Squares. $R_{\alpha\beta\gamma} = \dfrac{R_{\alpha\cdot\cdot}R_{\cdot\beta\cdot}R_{\cdot\cdot\gamma}}{R^2_{\cdot\cdot\cdot}}$

For definition of symbols used see [3-way classification. Table I], page 28.

	DF	MS (Mean square)
$\mathfrak{A}$	$A-1$	$\mathrm{MS}(\mathfrak{A}) = \dfrac{1}{A-1}\sum_{\alpha} R_{\alpha\cdot\cdot}(\bar{x}_{\alpha\cdot\cdot\cdot} - \bar{x}_{\cdot\cdot\cdot\cdot})^2 = \dfrac{1}{A-1}\left[\sum_{\alpha}\dfrac{x^2_{\alpha\cdot\cdot\cdot}}{R_{\alpha\cdot\cdot}} - \dfrac{x^2_{\cdot\cdot\cdot}}{R_{\cdot\cdot\cdot}}\right]$
Lines "$\mathfrak{B}$" and "$\mathfrak{C}$" analogous to line "$\mathfrak{A}$": Interchange respective symbols.		
$\mathfrak{A}\mathfrak{B}$	$(A-1)(B-1)$	$\mathrm{MS}(\mathfrak{A}\mathfrak{B}) = \dfrac{1}{(A-1)(B-1)}\sum_{\alpha}\sum_{\beta} R_{\alpha\beta\cdot}(\bar{x}_{\alpha\beta\cdot\cdot} - \bar{x}_{\alpha\cdot\cdot\cdot} - \bar{x}_{\cdot\beta\cdot\cdot} + \bar{x}_{\cdot\cdot\cdot\cdot})^2$ $= \dfrac{1}{(A-1)(B-1)}\left[\sum_{\alpha}\sum_{\beta}\dfrac{x^2_{\alpha\beta\cdot\cdot}}{R_{\alpha\beta\cdot}} - \sum_{\alpha}\dfrac{x^2_{\alpha\cdot\cdot\cdot}}{R_{\alpha\cdot\cdot}} - \sum_{\beta}\dfrac{x^2_{\cdot\beta\cdot\cdot}}{R_{\cdot\beta\cdot}} + \dfrac{x^2_{\cdot\cdot\cdot\cdot}}{R_{\cdot\cdot\cdot}}\right]$
Lines "$\mathfrak{A}\mathfrak{C}$" and "$\mathfrak{B}\mathfrak{C}$" analogous to line "$\mathfrak{A}\mathfrak{B}$": Interchange respective symbols.		
$\mathfrak{A}\mathfrak{B}\mathfrak{C}$	$(A-1)(B-1)(C-1)$	$\mathrm{MS}(\mathfrak{A}\mathfrak{B}\mathfrak{C}) = \dfrac{1}{(A-1)(B-1)(C-1)}\sum_{\alpha}\sum_{\beta}\sum_{\gamma}\left[(\bar{x}_{\alpha\beta\gamma\cdot} - \bar{x}_{\alpha\cdot\gamma\cdot} - \bar{x}_{\cdot\beta\gamma\cdot} + \bar{x}_{\cdot\cdot\gamma\cdot}) - (\bar{x}_{\alpha\beta\cdot\cdot} - \bar{x}_{\alpha\cdot\cdot\cdot} - \bar{x}_{\cdot\beta\cdot\cdot} + \bar{x}_{\cdot\cdot\cdot\cdot})\right]^2$ $= \dfrac{1}{(A-1)(B-1)(C-1)}\left[\sum_{\alpha}\sum_{\beta}\sum_{\gamma}\dfrac{x^2_{\alpha\beta\gamma\cdot}}{R_{\alpha\beta\gamma}} - \sum_{\alpha}\sum_{\beta}\dfrac{x^2_{\alpha\beta\cdot\cdot}}{R_{\alpha\beta\cdot}} - \sum_{\alpha}\sum_{\gamma}\dfrac{x^2_{\alpha\cdot\gamma\cdot}}{R_{\alpha\cdot\gamma}} - \sum_{\beta}\sum_{\gamma}\dfrac{x^2_{\cdot\beta\gamma\cdot}}{R_{\cdot\beta\gamma}} + \right.$ $\left. + \sum_{\alpha}\dfrac{x^2_{\alpha\cdot\cdot\cdot}}{R_{\alpha\cdot\cdot}} + \sum_{\beta}\dfrac{x^2_{\cdot\beta\cdot\cdot}}{R_{\cdot\beta\cdot}} + \sum_{\gamma}\dfrac{x^2_{\cdot\cdot\gamma\cdot}}{R_{\cdot\cdot\gamma}} - \dfrac{x^2_{\cdot\cdot\cdot\cdot}}{R_{\cdot\cdot\cdot}}\right]$
$\mathfrak{R}$	$R_{\cdot\cdot\cdot} - ABC$	$\mathrm{MS}(\mathfrak{R}) = \dfrac{1}{R_{\cdot\cdot\cdot}-ABC}\sum_{\alpha}\sum_{\beta}\sum_{\gamma}\sum_{\rho}(x_{\alpha\beta\gamma\rho} - \bar{x}_{\alpha\beta\gamma\cdot})^2 = \dfrac{1}{R_{\cdot\cdot\cdot}-ABC}\left[\sum_{\alpha}\sum_{\beta}\sum_{\gamma}\sum_{\rho} x^2_{\alpha\beta\gamma\rho} - \sum_{\alpha}\sum_{\beta}\sum_{\gamma}\dfrac{x^2_{\alpha\beta\gamma\cdot}}{R_{\alpha\beta\gamma}}\right]$

3-way classification
Table II: Mean squares

Three-way crossed classification. Table III: Tabular summary.
For definition of symbols used see [3-way classification. Table I], page 28.

Cases for which mean square expectations (EMS) and/or variance ratios are tabulated on the subsequent pages are marked by the following indicative symbols. Double line frame: EMS and variance ratios given; single line frame: Only variance ratios given.

	$R_{\alpha\beta\gamma} = \frac{R_{\alpha..}R_{.\beta.}R_{..\gamma}}{R^2_{...}}$	$R_{..\gamma} = \text{const.} = \frac{R_{...}}{C}$: $R_{\alpha\beta\gamma} = \frac{R_{\alpha..}R_{.\beta.}}{CR_{...}}$	$R_{.\beta\gamma} = \text{const.} = \frac{R_{...}}{BC}$: $R_{\alpha\beta\gamma} = \frac{R_{\alpha..}}{BC}$	$R_{\alpha\beta\gamma} \equiv R$	$R_{\alpha\beta\gamma} \equiv 1$
$\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ random	$\boxed{3.\mathcal{ABC}.\alpha\beta\gamma}$ Page 31	$\boxed{3.\mathcal{ABC}.\alpha\beta}$ Page 33	$\boxed{3.\mathcal{ABC}.\alpha}$ Page 33	$\boxed{3.\mathcal{ABC}.R}$ Page 34	$\boxed{3.\mathcal{ABC}.1}$ Page 34
$\mathcal{A}$ fixed, $\mathcal{B}$ and $\mathcal{C}$ random	$\boxed{3.\mathcal{BC}.\alpha\beta\gamma}$ Page 36	$\boxed{3.\mathcal{BC}.\alpha\beta}$ Page 38	$\boxed{3.\mathcal{BC}.\alpha}$ Page 38	$\boxed{3.\mathcal{BC}.R}$ Page 39	$\boxed{3.\mathcal{BC}.1}$ Page 39
$\mathcal{A}$ and $\mathcal{B}$ fixed, $\mathcal{C}$ random	$\boxed{3.\mathcal{C}.\alpha\beta\gamma}$ Page 40	$\boxed{3.\mathcal{C}.\alpha\beta}$ Page 40	Put $R_{\alpha\beta\gamma} = \frac{R_{\alpha..}}{BC}$ in "$\alpha\beta\gamma$" - cases, pages 40 and 43.	$\boxed{3.\mathcal{C}.R}$ Page 42	$\boxed{3.\mathcal{C}.1}$ Page 42
$\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ fixed	$\boxed{3..\alpha\beta\gamma}$ Page 43	Put $R_{\alpha\beta\gamma} = \frac{R_{\alpha..}R_{.\beta.}}{CR_{...}}$ in [3..$\alpha\beta\gamma$], page 43.		Put $R_{\alpha\beta\gamma} \equiv R$ in [3..$\alpha\beta\gamma$], page 43.	$\boxed{3..1}$ Page 43

Note. For the case of orthogonal contrasts in fixed factors see paragraph II.6.

[3.$\mathcal{ABC}.\alpha\beta\gamma$]: $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ random; $R_{\alpha\beta\gamma} = \dfrac{R_{\alpha..}R_{.\beta.}R_{..\gamma}}{R_{...}^2}$

For definition of symbols used see [3-way classification. Tables I, II and III], pages 28-30.

	EMS	H_o	Testing H_o
$\mathcal{A}$	$\sigma_r^2 + R_{...}\dfrac{1-k_a^{'''}}{A-1}\left[k_b^{'''}k_c^{'''}\sigma_{abc}^2 + k_b^{'''}\sigma_{ab}^2 + k_c^{'''}\sigma_{ac}^2 + \sigma_a^2\right]$	$\sigma_a^2 = 0$	$F'(\mathcal{A})$ 1)
$\mathcal{B}$	$\sigma_r^2 + R_{...}\dfrac{1-k_b^{'''}}{B-1}\left[k_a^{'''}k_c^{'''}\sigma_{abc}^2 + k_a^{'''}\sigma_{ab}^2 + k_c^{'''}\sigma_{bc}^2 + \sigma_b^2\right]$	$\sigma_b^2 = 0$	$F'(\mathcal{B})$ 2)
$\mathcal{C}$	$\sigma_r^2 + R_{...}\dfrac{1-k_c^{'''}}{C-1}\left[k_a^{'''}k_b^{'''}\sigma_{abc}^2 + k_a^{'''}\sigma_{ac}^2 + k_b^{'''}\sigma_{bc}^2 + \sigma_c^2\right]$	$\sigma_c^2 = 0$	$F'(\mathcal{C})$ 3)
$\mathcal{AB}$	$\sigma_r^2 + R_{...}\dfrac{(1-k_a^{'''})(1-k_b^{'''})}{(A-1)(B-1)}\left[k_c^{'''}\sigma_{abc}^2 + \sigma_{ab}^2\right]$	$\sigma_{ab}^2 = 0$	$F'(\mathcal{AB})$ 4)
$\mathcal{AC}$	$\sigma_r^2 + R_{...}\dfrac{(1-k_a^{'''})(1-k_c^{'''})}{(A-1)(C-1)}\left[k_b^{'''}\sigma_{abc}^2 + \sigma_{ac}^2\right]$	$\sigma_{ac}^2 = 0$	$F'(\mathcal{AC})$ 5)
$\mathcal{BC}$	$\sigma_r^2 + R_{...}\dfrac{(1-k_b^{'''})(1-k_c^{'''})}{(B-1)(C-1)}\left[k_a^{'''}\sigma_{abc}^2 + \sigma_{bc}^2\right]$	$\sigma_{bc}^2 = 0$	$F'(\mathcal{BC})$ 6)
$\mathcal{ABC}$	$\sigma_r^2 + R_{...}\dfrac{(1-k_a^{'''})(1-k_b^{'''})(1-k_c^{'''})}{(A-1)(B-1)(C-1)}\sigma_{abc}^2$	$\sigma_{abc}^2 = 0$	$F = \dfrac{MS(\mathcal{ABC})}{MS(\mathcal{R})}$
$\mathcal{R}$	σ_r^2	——	1) - 6): See next page.

3.$\mathcal{ABC}.\alpha\beta\gamma$

[3.$\mathcal{ABC}.\alpha\beta\gamma$], continued

1) $F'(\mathcal{A}) = \mathrm{MS}_1(\mathcal{A})/\mathrm{MS}_2(\mathcal{A})$, where

$$\mathrm{MS}_1(\mathcal{A}) = \mathrm{MS}(\mathcal{A}) + \frac{(B-1)(C-1)k_b'''k_c'''\,\mathrm{MS}(\mathcal{ABC})}{(1-k_b''')(1-k_c''')}$$

$$\mathrm{MS}_2(\mathcal{A}) = \frac{(B-1)k_b'''\,\mathrm{MS}(\mathcal{AB})}{1-k_b'''} + \frac{(C-1)k_c'''\,\mathrm{MS}(\mathcal{AC})}{1-k_c'''} + \frac{(Bk_b'''-1)(Ck_c'''-1)\,\mathrm{MS}(\mathcal{R})}{(1-k_b''')(1-k_c''')}$$

with effective degrees of freedom for $\mathrm{MS}_1(\mathcal{A})$ and $\mathrm{MS}_2(\mathcal{A})$, respectively:

$$f_1 = [\mathrm{MS}_1(\mathcal{A})]^2 \Bigg/ \left\{ \frac{1}{A-1}[\mathrm{MS}(\mathcal{A})]^2 + \frac{1}{(A-1)(B-1)(C-1)}\left[\frac{(B-1)(C-1)k_b'''k_c'''\,\mathrm{MS}(\mathcal{ABC})}{(1-k_b''')(1-k_c''')}\right]^2 \right\}$$

$$f_2 = [\mathrm{MS}_2(\mathcal{A})]^2 \Bigg/ \left\{ \frac{1}{(A-1)(B-1)}\left[\frac{(B-1)k_b'''\,\mathrm{MS}(\mathcal{AB})}{1-k_b'''}\right]^2 + \frac{1}{(A-1)(C-1)}\left[\frac{(C-1)k_c'''\,\mathrm{MS}(\mathcal{AC})}{1-k_c'''}\right]^2 + \frac{1}{R_{\cdots}-ABC}\left[\frac{(Bk_b'''-1)(Ck_c'''-1)\,\mathrm{MS}(\mathcal{R})}{(1-k_b''')(1-k_c''')}\right]^2 \right\}$$

2) $F'(\mathcal{B}) = \mathrm{MS}_1(\mathcal{B})/\mathrm{MS}_2(\mathcal{B})$
3) $F'(\mathcal{C}) = \mathrm{MS}_1(\mathcal{C})/\mathrm{MS}_2(\mathcal{C})$
analogous to $F'(\mathcal{A})$: Interchange respective symbols.

4) $F'(\mathcal{AB}) = \mathrm{MS}_1(\mathcal{AB})/\mathrm{MS}(\mathcal{ABC})$, where

$$\mathrm{MS}_1(\mathcal{AB}) = \frac{(1-k_c''')\,\mathrm{MS}(\mathcal{AB})}{(C-1)k_c'''} + \frac{(Ck_c'''-1)\,\mathrm{MS}(\mathcal{R})}{(C-1)k_c'''}$$

with effective degrees of freedom

$$f_1 = [\mathrm{MS}_1(\mathcal{AB})]^2 \Bigg/ \left\{ \frac{1}{(A-1)(B-1)}\left[\frac{(1-k_c''')\,\mathrm{MS}(\mathcal{AB})}{(C-1)k_c'''}\right]^2 + \frac{1}{R_{\cdots}-ABC}\left[\frac{(Ck_c'''-1)\,\mathrm{MS}(\mathcal{R})}{(C-1)k_c'''}\right]^2 \right\}$$

5) $F'(\mathcal{AC}) = \mathrm{MS}_1(\mathcal{AC})/\mathrm{MS}(\mathcal{ABC})$
6) $F'(\mathcal{BC}) = \mathrm{MS}_1(\mathcal{BC})/\mathrm{MS}(\mathcal{ABC})$
analogous to $F'(\mathcal{AB})$: Interchange respective symbols.

[3.$\mathcal{ABC}.\alpha\beta\gamma$] continued

[3.$\mathcal{ABC}.\alpha\beta$] and [3.$\mathcal{ABC}.\alpha$]: $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ random; $R_{\alpha\beta\gamma} = \frac{R_{\alpha..}R_{.\beta.}}{CR_{...}}$ and $R_{\alpha\beta\gamma} = \frac{R_{\alpha..}}{BC}$, respectively.

For definition of symbols used see [3-way classification. Table I, II and III], pages 28-30.

	H_o	Testing H_o	
		[3.$\mathcal{ABC}.\alpha\beta$], $\left(R_{..\gamma} = \frac{R_{...}}{C}\right)$: $R_{\alpha\beta\gamma} = \frac{R_{\alpha..}R_{.\beta.}}{CR_{...}}$	[3.$\mathcal{ABC}.\alpha$], $\left(R_{.\beta\gamma} = \frac{R_{...}}{BC}\right)$: $R_{\alpha\beta\gamma} = \frac{R_{\alpha..}}{BC}$
$\mathcal{A}$	$\sigma_a^2 = 0$	$F'(\mathcal{A})$ 1)	$F'(\mathcal{A})$ 1)
$\mathcal{B}$	$\sigma_b^2 = 0$	$F'(\mathcal{B})$ 2)	$F'(\mathcal{B})$ 2)
$\mathcal{C}$	$\sigma_c^2 = 0$	$F'(\mathcal{C})$ 3)	$F'(\mathcal{C})$ 3)
$\mathcal{AB}$	$\sigma_{ab}^2 = 0$	$F = MS(\mathcal{AB}) / MS(\mathcal{ABC})$	$F = MS(\mathcal{AB}) / MS(\mathcal{ABC})$
$\mathcal{AC}$	$\sigma_{ac}^2 = 0$	$F'(\mathcal{AC})$ 5)	$F = MS(\mathcal{AC}) / MS(\mathcal{ABC})$
$\mathcal{BC}$	$\sigma_{bc}^2 = 0$	$F'(\mathcal{BC})$ 6)	$F'(\mathcal{BC})$ 6)
$\mathcal{ABC}$	$\sigma_{abc}^2 = 0$	$F = MS(\mathcal{ABC}) / MS(\mathcal{R})$	$F = MS(\mathcal{ABC}) / MS(\mathcal{R})$
$\mathcal{R}$	—	—	—
		1), 2), 3), 5), 6): F' - values as in [3.$\mathcal{ABC}.\alpha\beta\gamma$], page 32, with $k_c''' = \frac{1}{C}$.	1), 2), 3), 6): F'-values as in [3.$\mathcal{ABC}.\alpha\beta\gamma$], page 32, with $k_b''' = \frac{1}{B}$ and $k_c''' = \frac{1}{C}$.
		For mean square expectations put $k_c''' = \frac{1}{C}$ in those of [3.$\mathcal{ABC}.\alpha\beta\gamma$], page 31.	For mean square expectations put $k_b''' = \frac{1}{B}$ and $k_c''' = \frac{1}{C}$ in those of [3.$\mathcal{ABC}.\alpha\beta\gamma$], page 31.

3.$\mathcal{ABC}.\alpha\beta$ + 3.$\mathcal{ABC}.\alpha$

[3.$\mathfrak{ABC}$.R] and [3.$\mathfrak{ABC}$.1]: $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ random; $R_{\alpha\beta\gamma} \equiv R$ and $R_{\alpha\beta\gamma} \equiv 1$, respectively.
For definition of symbols used see [3-way classification. Table I, II and III], pages 28-30.

	EMS ([3.$\mathfrak{ABC}$.R])†	H_o	Testing H_o	
			[3.$\mathfrak{ABC}$.R]: $R_{\alpha\beta\gamma} \equiv R$	[3.$\mathfrak{ABC}$.1]: $R_{\alpha\beta\gamma} \equiv 1$
$\mathfrak{A}$	$\sigma_r^2 + R\sigma_{abc}^2 + RC\sigma_{ab}^2 + RB\sigma_{ac}^2 + RBC\sigma_a^2$	$\sigma_a^2 = 0$	$F'(\mathfrak{A})$ 1)	
$\mathfrak{B}$	$\sigma_r^2 + R\sigma_{abc}^2 + RC\sigma_{ab}^2 + RA\sigma_{bc}^2 + RAC\sigma_b^2$	$\sigma_b^2 = 0$	$F'(\mathfrak{B})$ 2)	
$\mathfrak{C}$	$\sigma_r^2 + R\sigma_{abc}^2 + RB\sigma_{ac}^2 + RA\sigma_{bc}^2 + RAB\sigma_c^2$	$\sigma_c^2 = 0$	$F'(\mathfrak{C})$ 3)	
$\mathfrak{AB}$	$\sigma_r^2 + R\sigma_{abc}^2 + RC\sigma_{ab}^2$	$\sigma_{ab}^2 = 0$	$F = \text{MS}(\mathfrak{AB})/\text{MS}(\mathfrak{ABC})$	
$\mathfrak{AC}$	$\sigma_r^2 + R\sigma_{abc}^2 + RB\sigma_{ac}^2$	$\sigma_{ac}^2 = 0$	$F = \text{MS}(\mathfrak{AC})/\text{MS}(\mathfrak{ABC})$	
$\mathfrak{BC}$	$\sigma_r^2 + R\sigma_{abc}^2 + RA\sigma_{bc}^2$	$\sigma_{bc}^2 = 0$	$F = \text{MS}(\mathfrak{BC})/\text{MS}(\mathfrak{ABC})$	
$\mathfrak{ABC}$	$\sigma_r^2 + R\sigma_{abc}^2$	$\sigma_{abc}^2 = 0$	$F = \dfrac{\text{MS}(\mathfrak{ABC})}{\text{MS}(\mathfrak{R})}$	Not possible
$\mathfrak{R}$	σ_r^2	——	1)_3): See next page.	

†) For EMS of [3.$\mathfrak{ABC}$.1] put $R = 1$ and delete line "R".

[3.$\mathcal{ABC}$.R] and [3.$\mathcal{ABC}$.1], continued.

1) $F'(\mathcal{A}) = MS_1(\mathcal{A})/MS_2(\mathcal{A})$, where

$$MS_1(\mathcal{A}) = MS(\mathcal{A}) + MS(\mathcal{ABC})$$

$$MS_2(\mathcal{A}) = MS(\mathcal{AB}) + MS(\mathcal{AC})$$

with effective degrees of freedom for $MS_1(\mathcal{A})$ and $MS_2(\mathcal{A})$, respectively:

$$f_1 = [MS_1(\mathcal{A})]^2 \Big/ \left\{ \frac{1}{A-1}[MS(\mathcal{A})]^2 + \frac{1}{(A-1)(B-1)(C-1)}[MS(\mathcal{ABC})]^2 \right\}$$

$$f_2 = [MS_2(\mathcal{A})]^2 \Big/ \left\{ \frac{1}{(A-1)(B-1)}[MS(\mathcal{AB})]^2 + \frac{1}{(A-1)(C-1)}[MS(\mathcal{AC})]^2 \right\}$$

2) $F'(\mathcal{B}) = MS_1(\mathcal{B})/MS_2(\mathcal{B})$

3) $F'(\mathcal{C}) = MS_1(\mathcal{C})/MS_2(\mathcal{C})$

(2 and 3) analogous to $F'(\mathcal{A})$: Interchange respective symbols.

[3.$\mathcal{BC}.\alpha\beta\gamma$]: $\mathcal{A}$ fixed, $\mathcal{B}$ and $\mathcal{C}$ random; $R_{\alpha\beta\gamma} = \dfrac{R_{\alpha..}R_{.\beta.}R_{..\gamma}}{R^2_{...}}$

For definition of symbols used see [3-way classification. Table I, II and III], pages 28-30.

	EMS	H_o	Testing H_o
$\mathcal{A}$	$\sigma_r^2 + \frac{R_{...}}{A-1}\left[k_b''' k_c''' \sigma_{abc}^2 + k_b''' \sigma_{ab}^2 + k_c''' \sigma_{ac}^2\right] + \frac{1}{A-1}\sum_a R_{a..} a_a^2$	$a_a \equiv 0$	$F'^{*}(\mathcal{A})$ 1)
$\mathcal{B}$	$\sigma_r^2 + R_{...}\frac{1-k_b'''}{B-1}\left[k_c''' \sigma_{bc}^2 + \sigma_b^2\right]$	$\sigma_b^2 = 0$	$F'(\mathcal{B})$ 2)
$\mathcal{C}$	$\sigma_r^2 + R_{...}\frac{1-k_c'''}{C-1}\left[k_b''' \sigma_{bc}^2 + \sigma_c^2\right]$	$\sigma_c^2 = 0$	$F'(\mathcal{C})$ 3)
$\mathcal{AB}$	$\sigma_r^2 + \frac{R_{...}(1-k_b''')}{(A-1)(B-1)}\left[k_c''' \sigma_{abc}^2 + \sigma_{ab}^2\right]$	$\sigma_{ab}^2 = 0$	$F'^{*}(\mathcal{AB})$ 4)
$\mathcal{AC}$	$\sigma_r^2 + \frac{R_{...}(1-k_c''')}{(A-1)(C-1)}\left[k_b''' \sigma_{abc}^2 + \sigma_{ac}^2\right]$	$\sigma_{ac}^2 = 0$	$F'^{*}(\mathcal{AC})$ 5)
$\mathcal{BC}$	$\sigma_r^2 + R_{...}\frac{(1-k_b''')(1-k_c''')}{(B-1)(C-1)}\sigma_{bc}^2$	$\sigma_{bc}^2 = 0$	$F = \frac{MS(\mathcal{BC})}{MS(\mathcal{R})}$
$\mathcal{ABC}$	$\sigma_r^2 + \frac{R_{...}(1-k_b''')(1-k_c''')}{(A-1)(B-1)(C-1)}\sigma_{abc}^2$	$\sigma_{abc}^2 = 0$	$F = \frac{MS(\mathcal{ABC})}{MS(\mathcal{R})}$
$\mathcal{R}$	σ_r^2	——	1) — 5) See next page.

[3.$\mathcal{BC}.\alpha\beta\gamma$] continued.

1) $F'^{*}(\mathcal{A}) = MS_1(\mathcal{A})/MS_2(\mathcal{A})$ as in [3.$\mathcal{ABC}.\alpha\beta\gamma$], page 32.

2) $F'(\mathcal{B}) = MS_1(\mathcal{B})/MS(\mathcal{BC})$, where

$$MS_1(\mathcal{B}) = \frac{(1-k_c''')MS(\mathcal{B})}{(C-1)k_c'''} + \frac{(Ck_c'''-1)MS(\mathcal{R})}{(C-1)k_c'''}$$

with effective degrees of freedom

$$f_1 = [MS_1(\mathcal{B})]^2 \Bigg/ \left\{ \frac{1}{B-1}\left[\frac{(1-k_c''')MS(\mathcal{B})}{(C-1)k_c'''}\right]^2 + \frac{1}{R_{\cdots}-ABC}\left[\frac{(Ck_c'''-1)MS(\mathcal{R})}{(C-1)k_c'''}\right]^2 \right\}$$

3) $F'(\mathcal{C}) = MS_1(\mathcal{C})/MS(\mathcal{BC})$, analogous to $F'(\mathcal{B})$: Interchange respective symbols.

4) $F'^{*}(\mathcal{AB}) = MS_1(\mathcal{AB})/MS(\mathcal{ABC})$

5) $F'^{*}(\mathcal{AC}) = MS_1(\mathcal{AC})/MS(\mathcal{ABC})$

as in [3.$\mathcal{ABC}.\alpha\beta\gamma$], page 32.

[3.$\mathcal{BC}.a\beta$] and [3.$\mathcal{BC}.a$]: $\mathfrak{A}$ fixed, $\mathcal{B}$ and $\mathcal{C}$ random; $R_{a\beta\gamma} = \frac{R_{a..}R_{.\beta.}}{CR_{...}}$ and $R_{a\beta\gamma} = \frac{R_{a..}}{BC}$, respectively. For definition of symbols used see [3-way classification. Table I, II and III], pages 28-30.			
	H_o	Testing H_o [3.$\mathcal{BC}.a\beta$], $\left(R_{..\gamma} = \frac{R_{...}}{C}\right)$: $R_{a\beta\gamma} = \frac{R_{a..}R_{.\beta.}}{CR_{...}}$	[3.$\mathcal{BC}.a$], $\left(R_{.\beta\gamma} = \frac{R_{...}}{BC}\right)$: $R_{a\beta\gamma} = \frac{R_{a..}}{BC}$
$\mathfrak{A}$	$a_a \equiv 0$	$F'^{*}(\mathfrak{A})$ 1)	$F'^{*}(\mathfrak{A})$ 1)
$\mathcal{B}$	$\sigma_b^2 = 0$	$F = \text{MS}(\mathcal{B})/\text{MS}(\mathcal{BC})$	$F = \text{MS}(\mathcal{B})/\text{MS}(\mathcal{BC})$
$\mathcal{C}$	$\sigma_c^2 = 0$	$F'(\mathcal{C})$ 3)	$F = \text{MS}(\mathcal{C})/\text{MS}(\mathcal{BC})$
$\mathfrak{A}\mathcal{B}$	$\sigma_{ab}^2 = 0$	$F^{*} = \text{MS}(\mathfrak{A}\mathcal{B})/\text{MS}(\mathfrak{A}\mathcal{BC})$	$F^{*} = \text{MS}(\mathfrak{A}\mathcal{B})/\text{MS}(\mathfrak{A}\mathcal{BC})$
$\mathfrak{A}\mathcal{C}$	$\sigma_{ac}^2 = 0$	$F'^{*}(\mathfrak{A}\mathcal{C})$ 5)	$F^{*} = \text{MS}(\mathfrak{A}\mathcal{C})/\text{MS}(\mathfrak{A}\mathcal{BC})$
$\mathcal{BC}$	$\sigma_{bc}^2 = 0$	$F = \text{MS}(\mathcal{BC})/\text{MS}(\mathfrak{R})$	$F = \text{MS}(\mathcal{BC})/\text{MS}(\mathfrak{R})$
$\mathfrak{A}\mathcal{BC}$	$\sigma_{abc}^2 = 0$	$F = \text{MS}(\mathfrak{A}\mathcal{BC})/\text{MS}(\mathfrak{R})$	$F = \text{MS}(\mathfrak{A}\mathcal{BC})/\text{MS}(\mathfrak{R})$
$\mathfrak{R}$	—	—	—
		1), 3), 5): F'^{*} - and F' - values as in [3.$\mathcal{BC}.a\beta\gamma$], page 37, with $k_c''' = \frac{1}{C}$	1): $F'^{*}(\mathfrak{A}) = \text{MS}_1(\mathfrak{A})/\text{MS}_2(\mathfrak{A})$ as in [3.$\mathfrak{A}\mathcal{BC}.R$], page 35.
		For mean square expectations put $k_c''' = \frac{1}{C}$ in those of [3.$\mathcal{BC}.a\beta\gamma$], page 36.	For mean square expectations put $k_b''' = \frac{1}{B}$ and $k_c''' = \frac{1}{C}$ in those of [3.$\mathcal{BC}.a\beta\gamma$], page 36.

3.$\mathcal{BC}.a\beta$ + 3.$\mathcal{BC}.a$

[3.$\mathcal{BC}$.R] and [3.$\mathcal{BC}$.1]: $\mathcal{A}$ fixed, $\mathcal{B}$ and $\mathcal{C}$ random; $R_{\alpha\beta\gamma} = R$ and $R_{\alpha\beta\gamma} = 1$, respectively.
For definition of symbols used see [3-way classification. Table I, II and III], pages 28-30.

	EMS([3.$\mathcal{BC}$.R])†	H_o	Testing H_o: [3.$\mathcal{BC}$.R]: $R_{\alpha\beta\gamma} = R$	Testing H_o: [3.$\mathcal{BC}$.1]: $R_{\alpha\beta\gamma} = 1$
$\mathcal{A}$	$\sigma_r^2 + \frac{RA}{A-1}\sigma_{abc}^2 + \frac{RAC}{A-1}\sigma_{ab}^2 + \frac{RAB}{A-1}\sigma_{ac}^2 + \frac{RBC}{A-1}\sum_a a_a^2$	$a_a = 0$	$F'^*(\mathcal{A})$ 1)	
$\mathcal{B}$	$\sigma_r^2 + RA\sigma_{bc}^2 + RAC\sigma_b^2$	$\sigma_b^2 = 0$	$F = \text{MS}(\mathcal{B})/\text{MS}(\mathcal{BC})$	
$\mathcal{C}$	$\sigma_r^2 + RA\sigma_{bc}^2 + RAB\sigma_c^2$	$\sigma_c^2 = 0$	$F = \text{MS}(\mathcal{C})/\text{MS}(\mathcal{BC})$	
$\mathcal{AB}$	$\sigma_r^2 + \frac{RA}{A-1}\sigma_{abc}^2 + \frac{RAC}{A-1}\sigma_{ab}^2$	$\sigma_{ab}^2 = 0$	$F^* = \text{MS}(\mathcal{AB})/\text{MS}(\mathcal{ABC})$	
$\mathcal{AC}$	$\sigma_r^2 + \frac{RA}{A-1}\sigma_{abc}^2 + \frac{RAB}{A-1}\sigma_{ac}^2$	$\sigma_{ac}^2 = 0$	$F^* = \text{MS}(\mathcal{AC})/\text{MS}(\mathcal{ABC})$	
$\mathcal{BC}$	$\sigma_r^2 + RA\sigma_{bc}^2$	$\sigma_{bc}^2 = 0$	$F = \frac{\text{MS}(\mathcal{BC})}{\text{MS}(\mathcal{R})}$	If $\sigma_{abc}^2 = 0$ can be assumed: $F = \text{MS}(\mathcal{BC})/\text{MS}(\mathcal{ABC})$
$\mathcal{ABC}$	$\sigma_r^2 + \frac{RA}{A-1}\sigma_{abc}^2$	$\sigma_{abc}^2 = 0$	$F = \frac{\text{MS}(\mathcal{ABC})}{\text{MS}(\mathcal{R})}$	Not possible
$\mathcal{R}$	σ_r^2	—	1) $F'^*(\mathcal{A}) = \text{MS}_1(\mathcal{A})/\text{MS}_2(\mathcal{A})$ as in [3.$\mathcal{ABC}$.R], page 35.	

†) For EMS of [3.$\mathcal{BC}$.1] put $R = 1$ and delete line "$\mathcal{R}$".

3.$\mathcal{BC}$.R + 3.$\mathcal{BC}$.1

$[3.\mathcal{C}.\alpha\beta\gamma]$ and $[3.\mathcal{C}.\alpha\beta]$: $\mathcal{A}$ and $\mathcal{B}$ fixed, $\mathcal{C}$ random; $R_{\alpha\beta\gamma} = \frac{R_{\alpha..}R_{.\beta.}R_{..\gamma}}{R^2_{...}}$ and $R_{\alpha\beta\gamma} = \frac{R_{\alpha..}R_{.\beta.}}{CR_{...}}$, respectively.

For definition of symbols used see [3-way classification, Table I, II and III], pages 28-30.

	EMS($[3.\mathcal{C}.\alpha\beta\gamma]$)†	H_o	Testing H_o: $[3.\mathcal{C}.\alpha\beta\gamma]$: $R_{\alpha\beta\gamma} = \frac{R_{\alpha..}R_{.\beta.}R_{..\gamma}}{R^2_{...}}$	Testing H_o: $[3.\mathcal{C}.\alpha\beta]$, $\left(R_{..\gamma} = \frac{R_{...}}{C}\right)$: $R_{\alpha\beta\gamma} = \frac{R_{\alpha..}R_{.\beta.}}{CR_{...}}$
$\mathcal{A}$	$\sigma_r^2 + \frac{R_{...}k_c^{'''}}{A-1}\sigma_{ac}^2 + \frac{1}{A-1}\sum_\alpha R_{\alpha..}a_\alpha^2$	$a_\alpha = 0$	$F'^*(\mathcal{A})$ 1)	$F^* = \frac{MS(\mathcal{A})}{MS(\mathcal{AC})}$
$\mathcal{B}$	$\sigma_r^2 + \frac{R_{...}k_c^{'''}}{B-1}\sigma_{bc}^2 + \frac{1}{B-1}\sum_\beta R_{.\beta.}b_\beta^2$	$b_\beta = 0$	$F'^*(\mathcal{B})$ 2)	$F^* = \frac{MS(\mathcal{B})}{MS(\mathcal{BC})}$
$\mathcal{C}$	$\sigma_r^2 + R_{...}\frac{1-k_c^{'''}}{C-1}\sigma_c^2$	$\sigma_c^2 = 0$	$F = MS(\mathcal{C})/MS(\mathcal{R})$	
$\mathcal{AB}$	$\sigma_r^2 + \frac{R_{...}k_c^{'''}}{(A-1)(B-1)}\sigma_{abc}^2 + \frac{1}{(A-1)(B-1)}\sum_\alpha\sum_\beta R_{\alpha\beta.}ab_{\alpha\beta}^2$	$ab_{\alpha\beta} = 0$	$F'^*(\mathcal{AB})$ 3)	$F^* = \frac{MS(\mathcal{AB})}{MS(\mathcal{ABC})}$
$\mathcal{AC}$	$\sigma_r^2 + \frac{R_{...}(1-k_c^{'''})}{(A-1)(C-1)}\sigma_{ac}^2$	$\sigma_{ac}^2 = 0$	$F = MS(\mathcal{AC})/MS(\mathcal{R})$	
$\mathcal{BC}$	$\sigma_r^2 + \frac{R_{...}(1-k_c^{'''})}{(B-1)(C-1)}\sigma_{bc}^2$	$\sigma_{bc}^2 = 0$	$F = MS(\mathcal{BC})/MS(\mathcal{R})$	
$\mathcal{ABC}$	$\sigma_r^2 + \frac{R_{...}(1-k_c^{'''})}{(A-1)(B-1)(C-1)}\sigma_{abc}^2$	$\sigma_{abc}^2 = 0$	$F = MS(\mathcal{ABC})/MS(\mathcal{R})$	
$\mathcal{R}$	σ_r^2	—	1) - 3): See next page.	

†) For EMS of $[3.\mathcal{C}.\alpha\beta]$ put $k_c^{'''} = \frac{1}{C}$

$\boxed{3.\mathcal{C}.\alpha\beta\gamma}$ + $\boxed{3.\mathcal{C}.\alpha\beta}$

[3.$\mathcal{C}.\alpha\beta\gamma$] continued.

1) $F'^{*}(\mathcal{A}) = \mathrm{MS}_1(\mathcal{A})/\mathrm{MS}(\mathcal{A}\mathcal{C})$, where

$$\mathrm{MS}_1(\mathcal{A}) = \frac{(1-k_c''')\mathrm{MS}(\mathcal{A})}{(C-1)k_c'''} + \frac{(Ck_c'''-1)\mathrm{MS}(\mathcal{R})}{(C-1)k_c'''}$$

with effective degrees of freedom

$$f_1 = [\mathrm{MS}_1(\mathcal{A})]^2 \bigg/ \left\{ \frac{1}{A-1}\left[\frac{(1-k_c''')\mathrm{MS}(\mathcal{A})}{(C-1)k_c'''}\right]^2 + \frac{1}{R_{\ldots}-ABC}\left[\frac{(Ck_c'''-1)\mathrm{MS}(\mathcal{R})}{(C-1)k_c'''}\right]^2 \right\}$$

2) $F'^{*}(\mathcal{B}) = \mathrm{MS}_1(\mathcal{B})/\mathrm{MS}(\mathcal{B}\mathcal{C})$ analogous to $F'^{*}(\mathcal{A})$: Interchange respective symbols.

3) $F'^{*}(\mathcal{A}\mathcal{B}) = \mathrm{MS}_1(\mathcal{A}\mathcal{B})/\mathrm{MS}(\mathcal{A}\mathcal{B}\mathcal{C})$ as in [3.$\mathcal{A}\mathcal{B}\mathcal{C}.\alpha\beta\gamma$], page 32.

[3.$\mathcal{C}.\alpha\beta\gamma$] continued

[3.$\mathcal{C}$.R] and [3.$\mathcal{C}$.1]: $\mathcal{A}$ and $\mathcal{B}$ fixed, $\mathcal{C}$ random; $R_{\alpha\beta\gamma} = R$ and $R_{\alpha\beta\gamma} = 1$, respectively.

For definition of symbols used see [3-way classification. Table I, II and III], pages 28-30.

	EMS([3.$\mathcal{C}$.R])†	H_o	Testing H_o: [3.$\mathcal{C}$.R]: $R_{\alpha\beta\gamma} = R$	Testing H_o: [3.$\mathcal{C}$.1]: $R_{\alpha\beta\gamma} = 1$
$\mathcal{A}$	$\sigma_r^2 + \frac{RAB}{A-1}\sigma_{ac}^2 + \frac{RBC}{A-1}\sum_a a_\alpha^2$	$a_\alpha = 0$	$F^* = MS(\mathcal{A})/MS(\mathcal{AC})$	
$\mathcal{B}$	$\sigma_r^2 + \frac{RAB}{B-1}\sigma_{bc}^2 + \frac{RAC}{B-1}\sum_\beta b_\beta^2$	$b_\beta = 0$	$F^* = MS(\mathcal{B})/MS(\mathcal{BC})$	
$\mathcal{C}$	$\sigma_r^2 + RAB\sigma_c^2$	$\sigma_c^2 = 0$	$F = \frac{MS(\mathcal{C})}{MS(\mathcal{R})}$	If $\sigma_{abc}^2 = 0$ can be assumed: $F = MS(\mathcal{C})/MS(\mathcal{ABC})$
$\mathcal{AB}$	$\sigma_r^2 + \frac{RAB}{(A-1)(B-1)}\sigma_{abc}^2 + \frac{RC}{(A-1)(B-1)}\sum_\alpha\sum_\beta ab_{\alpha\beta}^2$	$ab_{\alpha\beta} = 0$	$F^* = MS(\mathcal{AB})/MS(\mathcal{ABC})$	
$\mathcal{AC}$	$\sigma_r^2 + \frac{RAB}{A-1}\sigma_{ac}^2$	$\sigma_{ac}^2 = 0$	$F = \frac{MS(\mathcal{AC})}{MS(\mathcal{R})}$	If $\sigma_{abc}^2 = 0$ can be assumed: $F = MS(\mathcal{AC})/MS(\mathcal{ABC})$
$\mathcal{BC}$	$\sigma_r^2 + \frac{RAB}{B-1}\sigma_{bc}^2$	$\sigma_{bc}^2 = 0$	$F = \frac{MS(\mathcal{BC})}{MS(\mathcal{R})}$	If $\sigma_{abc}^2 = 0$ can be assumed: $F = MS(\mathcal{BC})/MS(\mathcal{ABC})$
$\mathcal{ABC}$	$\sigma_r^2 + \frac{RAB}{(A-1)(B-1)}\sigma_{abc}^2$	$\sigma_{abc}^2 = 0$	$F = \frac{MS(\mathcal{ABC})}{MS(\mathcal{R})}$	Not possible
$\mathcal{R}$	σ_r^2	—	—	—

†) For EMS of [3.$\mathcal{C}$.1] put $R = 1$ and delete line "$\mathcal{R}$".

3.$\mathcal{C}$.R + 3.$\mathcal{C}$.1

[3..$\alpha\beta\gamma$] and [3..1]: $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ fixed; $R_{\alpha\beta\gamma} = \frac{R_{\alpha..}R_{.\beta.}R_{..\gamma}}{R_{...}^2}$ and $R_{\alpha\beta\gamma} \equiv 1$, respectively.

For definition of symbols used see [3-way classification. Table I, II and III], pages 28-30.

	EMS ([3..$\alpha\beta\gamma$])†	H_o	Testing H_o: [3..$\alpha\beta\gamma$]: $R_{\alpha\beta\gamma} = \frac{R_{\alpha..}R_{.\beta.}R_{..\gamma}}{R_{...}^2}$	Testing H_o: [3..1]: $R_{\alpha\beta\gamma} \equiv 1$
$\mathcal{A}$	$\sigma_r^2 + \frac{1}{A-1}\sum_{\alpha} R_{\alpha..}a_\alpha^2$	$a_\alpha \equiv 0$	$F = \frac{MS(\mathcal{A})}{MS(\mathcal{R})}$	If $abc_{\alpha\beta\gamma} \equiv 0$ can be assumed: $F = \frac{MS(\mathcal{A})}{MS(\mathcal{ABC})}$
$\mathcal{B}$	$\sigma_r^2 + \frac{1}{B-1}\sum_{\beta} R_{.\beta.}b_\beta^2$	$b_\beta \equiv 0$	$F = \frac{MS(\mathcal{B})}{MS(\mathcal{R})}$	$F = \frac{MS(\mathcal{B})}{MS(\mathcal{ABC})}$
$\mathcal{C}$	$\sigma_r^2 + \frac{1}{C-1}\sum_{\gamma} R_{..\gamma}c_\gamma^2$	$c_\gamma \equiv 0$	$F = \frac{MS(\mathcal{C})}{MS(\mathcal{R})}$	$F = \frac{MS(\mathcal{C})}{MS(\mathcal{ABC})}$
$\mathcal{AB}$	$\sigma_r^2 + \frac{1}{(A-1)(B-1)}\sum\sum_{\alpha\beta} R_{\alpha\beta.}ab_{\alpha\beta}^2$	$ab_{\alpha\beta} \equiv 0$	$F = \frac{MS(\mathcal{AB})}{MS(\mathcal{R})}$	$F = \frac{MS(\mathcal{AB})}{MS(\mathcal{ABC})}$
$\mathcal{AC}$	$\sigma_r^2 + \frac{1}{(A-1)(C-1)}\sum\sum_{\alpha\gamma} R_{\alpha.\gamma}ac_{\alpha\gamma}^2$	$ac_{\alpha\gamma} \equiv 0$	$F = \frac{MS(\mathcal{AC})}{MS(\mathcal{R})}$	$F = \frac{MS(\mathcal{AC})}{MS(\mathcal{ABC})}$
$\mathcal{BC}$	$\sigma_r^2 + \frac{1}{(B-1)(C-1)}\sum\sum_{\beta\gamma} R_{.\beta\gamma}bc_{\beta\gamma}^2$	$bc_{\beta\gamma} \equiv 0$	$F = \frac{MS(\mathcal{BC})}{MS(\mathcal{R})}$	$F = \frac{MS(\mathcal{BC})}{MS(\mathcal{ABC})}$
$\mathcal{ABC}$	$\sigma_r^2 + \frac{1}{(A-1)(B-1)(C-1)}\sum\sum\sum_{\alpha\beta\gamma} R_{\alpha\beta\gamma}abc_{\alpha\beta\gamma}^2$	$abc_{\alpha\beta\gamma} \equiv 0$	$F = \frac{MS(\mathcal{ABC})}{MS(\mathcal{R})}$	Not possible
$\mathcal{R}$	σ_r^2	—	—	—

†) For EMS of [3..1] put $R_{\alpha\beta\gamma} \equiv 1$ and delete line "$\mathcal{R}$".

3..$\alpha\beta\gamma$ + 3..1

IV. 3 TABLE OF MEAN SQUARE EXPECTATIONS AND VARIANCE RATIOS IN THE ANALYSIS OF VARIANCE FOR THE FOUR-WAY CROSSED CLASSIFICATION

Four-way crossed classification. Table I: Definition of symbols.

Model for analysis of variance:

$$x_{\alpha\beta\gamma\delta\rho} = \mu + a_\alpha + b_\beta + c_\gamma + d_\delta + ab_{\alpha\beta} + ac_{\alpha\gamma} + ad_{\alpha\delta} + bc_{\beta\gamma} + bd_{\beta\delta} + cd_{\gamma\delta} + abc_{\alpha\beta\gamma} + abd_{\alpha\beta\delta} + acd_{\alpha\gamma\delta} + bcd_{\beta\gamma\delta}$$

$$+ abcd_{\alpha\beta\gamma\delta} + r_{\alpha\beta\gamma\delta\rho} \quad , \text{ with } r_{\alpha\beta\gamma\delta\rho} \sim NID(O,\sigma_r^2).$$

$\alpha = 1,\ldots,A;\ \beta = 1,\ldots,B;\ \gamma = 1,\ldots,C;\ \delta = 1,\ldots,D;\ \rho = 1,\ldots,R_{\alpha\beta\gamma\delta}$

A, B, C and D: **numbers of levels in factors $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$, respectively.**

$R_{\alpha\beta\gamma\delta}$ = **number of replicated observations in cell "$\alpha\beta\gamma\delta$".**

$$R_{\alpha\beta\gamma.} = \sum_{\delta=1}^{D} R_{\alpha\beta\gamma\delta} \quad , \quad R_{\alpha\beta..} = \sum_{\gamma=1}^{C} R_{\alpha\beta\gamma.} \quad , \text{ etc.}$$

$$R_{....} = \sum_\alpha\sum_\beta\sum_\gamma\sum_\delta R_{\alpha\beta\gamma\delta} = \text{total number of observations.}$$

$x_{\alpha\beta\gamma\delta\rho}$ = **ρth observation in cell "$\alpha\beta\gamma\delta$".**

$$x_{\alpha\beta\gamma\delta.} = \sum_{\rho=1}^{R_{\alpha\beta\gamma\delta}} x_{\alpha\beta\gamma\delta\rho} \quad , \quad x_{\alpha\beta\gamma..} = \sum_{\delta=1}^{D} x_{\alpha\beta\gamma\delta.} \quad , \text{ etc.}$$

$$\bar{x}_{\alpha\beta\gamma\delta.} = \frac{x_{\alpha\beta\gamma\delta.}}{R_{\alpha\beta\gamma\delta}} \quad , \quad \bar{x}_{\alpha\beta\gamma..} = \frac{x_{\alpha\beta\gamma..}}{R_{\alpha\beta\gamma.}} \quad , \text{ etc.}$$

$$k_a^{''''} = \frac{1}{R_{....}^2}\sum_{\alpha=1}^{A} R_{\alpha...}^2 \quad , \quad k_b^{''''} = \frac{1}{R_{....}^2}\sum_{\beta=1}^{B} R_{.\beta..}^2 \quad , \quad k_c^{''''} = \frac{1}{R_{....}^2}\sum_{\gamma=1}^{C} R_{..\gamma.}^2 \quad , \quad k_d^{''''} = \frac{1}{R_{....}^2}\sum_{\delta=1}^{D} R_{...\delta}^2$$

F', F^*, F'^*: **Prime and/or asterisk attached to F-value:**

Variance ratio only approximately distributed as F under stated nullhypothesis H_o. See paragraph II.4

4-way classification
Table I : Definition of symbols

Four-way crossed classification. Table II: Mean squares. $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}R_{..\gamma.}R_{...\delta}}{R^3_{....}}$

For definition of symbols used see [4-way classification. Table I], page 45.

	DF	MS (Mean Square)
$\mathcal{A}$	$A-1$	MS($\mathcal{A}$) $= \frac{1}{A-1}\sum_\alpha R_{\alpha...}(\bar{x}_{\alpha....} - \bar{x}_{.....})^2 = \frac{1}{A-1}\left[\sum_\alpha \frac{x^2_{\alpha....}}{R_{\alpha...}} - \frac{x^2_{.....}}{R_{....}}\right]$
Lines "$\mathcal{B}$", "$\mathcal{C}$" and "$\mathcal{D}$" analogous to line "$\mathcal{A}$": Interchange respective symbols.		
$\mathcal{AB}$	$(A-1)(B-1)$	MS($\mathcal{AB}$) $= \frac{1}{(A-1)(B-1)}\sum_\alpha\sum_\beta R_{\alpha\beta..}(\bar{x}_{\alpha\beta...} - \bar{x}_{\alpha....} - \bar{x}_{.\beta...} + \bar{x}_{.....})^2 = \frac{1}{(A-1)(B-1)}\left[\sum_\alpha\sum_\beta \frac{x^2_{\alpha\beta...}}{R_{\alpha\beta..}} - \sum_\alpha \frac{x^2_{\alpha....}}{R_{\alpha...}} - \sum_\beta \frac{x^2_{.\beta...}}{R_{.\beta..}} + \frac{x^2_{.....}}{R_{....}}\right]$
Lines "$\mathcal{AC}$", "$\mathcal{AD}$", "$\mathcal{BC}$", "$\mathcal{BD}$" and "$\mathcal{CD}$" analogous to line "$\mathcal{AB}$": Interchange respective symbols.		
$\mathcal{ABC}$	$(A-1)(B-1)(C-1)$	MS($\mathcal{ABC}$) $= \frac{1}{(A-1)(B-1)(C-1)}\sum_\alpha\sum_\beta\sum_\gamma R_{\alpha\beta\gamma.}\left[(\bar{x}_{\alpha\beta\gamma..} - \bar{x}_{\alpha.\gamma..} - \bar{x}_{.\beta\gamma..} + \bar{x}_{..\gamma..}) - (\bar{x}_{\alpha\beta...} - \bar{x}_{\alpha....} - \bar{x}_{.\beta...} + \bar{x}_{.....})\right]^2$ $= \frac{1}{(A-1)(B-1)(C-1)}\left[\sum_\alpha\sum_\beta\sum_\gamma \frac{x^2_{\alpha\beta\gamma..}}{R_{\alpha\beta\gamma.}} - \sum_\alpha\sum_\beta \frac{x^2_{\alpha\beta...}}{R_{\alpha\beta..}} - \sum_\alpha\sum_\gamma \frac{x^2_{\alpha.\gamma..}}{R_{\alpha.\gamma.}} - \sum_\beta\sum_\gamma \frac{x^2_{.\beta\gamma..}}{R_{.\beta\gamma.}} + \sum_\alpha \frac{x^2_{\alpha....}}{R_{\alpha...}} + \sum_\beta \frac{x^2_{.\beta...}}{R_{.\beta..}} + \sum_\gamma \frac{x^2_{..\gamma..}}{R_{..\gamma.}} - \frac{x^2_{.....}}{R_{....}}\right]$
Lines "$\mathcal{ABD}$", "$\mathcal{ACD}$" and "$\mathcal{BCD}$" analogous to line "$\mathcal{ABC}$": Interchange respective symbols.		
$\mathcal{ABCD}$	$(A-1)(B-1)(C-1)(D-1)$	MS($\mathcal{ABCD}$) $= \frac{1}{(A-1)(B-1)(C-1)(D-1)}\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta R_{\alpha\beta\gamma\delta}\big[\{(\bar{x}_{\alpha\beta\gamma\delta.} - \bar{x}_{\alpha.\gamma\delta.} - \bar{x}_{.\beta\gamma\delta.} + \bar{x}_{..\gamma\delta.}) - (\bar{x}_{\alpha\beta.\delta.} - \bar{x}_{\alpha..\delta.} - \bar{x}_{.\beta.\delta.} + \bar{x}_{...\delta.})\}$ $-\{(\bar{x}_{\alpha\beta\gamma..} - \bar{x}_{\alpha.\gamma..} - \bar{x}_{.\beta\gamma..} + \bar{x}_{..\gamma..}) - (\bar{x}_{\alpha\beta...} - \bar{x}_{\alpha....} - \bar{x}_{.\beta...} + \bar{x}_{.....})\}\big]^2$ $= \frac{1}{(A-1)(B-1)(C-1)(D-1)}\Big[\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta \frac{x^2_{\alpha\beta\gamma\delta.}}{R_{\alpha\beta\gamma\delta}} - \sum_\alpha\sum_\beta\sum_\gamma \frac{x^2_{\alpha\beta\gamma..}}{R_{\alpha\beta\gamma.}} - \sum_\alpha\sum_\beta\sum_\delta \frac{x^2_{\alpha\beta.\delta.}}{R_{\alpha\beta.\delta}} - \sum_\alpha\sum_\gamma\sum_\delta \frac{x^2_{\alpha.\gamma\delta.}}{R_{\alpha.\gamma\delta}} - \sum_\beta\sum_\gamma\sum_\delta \frac{x^2_{.\beta\gamma\delta.}}{R_{.\beta\gamma\delta}}$ $+ \sum_\alpha\sum_\beta \frac{x^2_{\alpha\beta...}}{R_{\alpha\beta..}} + \sum_\alpha\sum_\gamma \frac{x^2_{\alpha.\gamma..}}{R_{\alpha.\gamma.}} + \sum_\alpha\sum_\delta \frac{x^2_{\alpha..\delta.}}{R_{\alpha..\delta}} + \sum_\beta\sum_\gamma \frac{x^2_{.\beta\gamma..}}{R_{.\beta\gamma.}} + \sum_\beta\sum_\delta \frac{x^2_{.\beta.\delta.}}{R_{.\beta.\delta}} + \sum_\gamma\sum_\delta \frac{x^2_{..\gamma\delta.}}{R_{..\gamma\delta}}$ $- \sum_\alpha \frac{x^2_{\alpha....}}{R_{\alpha...}} - \sum_\beta \frac{x^2_{.\beta...}}{R_{.\beta..}} - \sum_\gamma \frac{x^2_{..\gamma..}}{R_{..\gamma.}} - \sum_\delta \frac{x^2_{...\delta.}}{R_{...\delta}} + \frac{x^2_{.....}}{R_{....}}\Big]$
$\mathcal{R}$	$R_{....} - ABCD$	MS($\mathcal{R}$) $= \frac{1}{R_{....} - ABCD}\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta\sum_\rho (x_{\alpha\beta\gamma\delta\rho} - \bar{x}_{\alpha\beta\gamma\delta.})^2 = \frac{1}{R_{....} - ABCD}\left[\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta\sum_\rho x^2_{\alpha\beta\gamma\delta\rho} - \sum_\alpha\sum_\beta\sum_\gamma\sum_\delta \frac{x^2_{\alpha\beta\gamma\delta.}}{R_{\alpha\beta\gamma\delta}}\right]$

4-way classification

Table II: Mean squares

Four-way crossed classification. Table III: Tabular summary.
For definition of symbols used see [4-way classification. Table I], page 45.

Cases for which mean square expectations (EMS) and variance ratios are tabulated on the subsequent pages are marked by the following indicative symbols.

Double line frame: EMS and variance ratios given; Single line frame: only variance ratios given.

	$R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}R_{..\gamma.}R_{...\delta}}{R^3_{....}}$	$R_{...\delta} = \text{const.} = \frac{R_{....}}{D}$; $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}R_{..\gamma.}}{DR^2_{....}}$	$R_{..\gamma\delta} = \text{const.} = \frac{R_{....}}{CD}$; $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}}{CDR_{....}}$	$R_{.\beta\gamma\delta} = \text{const.} = \frac{R_{....}}{BCD}$; $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}}{BCD}$	$R_{\alpha\beta\gamma\delta} = R$	$R_{\alpha\beta\gamma\delta} = 1$
$\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ random	4.$\mathfrak{ABCD}$.$\alpha\beta\gamma\delta$ Page 48	4.$\mathfrak{ABCD}$.$\alpha\beta\gamma$ Page 51	4.$\mathfrak{ABCD}$.$\alpha\beta$ Page 51	4.$\mathfrak{ABCD}$.α Page 51	4.$\mathfrak{ABCD}$.R Page 52	4.$\mathfrak{ABCD}$.1 Page 52
$\mathfrak{A}$ fixed, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ random	4.$\mathfrak{BCD}$.$\alpha\beta\gamma\delta$ Page 54	4.$\mathfrak{BCD}$.$\alpha\beta\gamma$ Page 56	4.$\mathfrak{BCD}$.$\alpha\beta$ Page 56	4.$\mathfrak{BCD}$.α Page 56	4.$\mathfrak{BCD}$.R Page 57	4.$\mathfrak{BCD}$.1 Page 57
$\mathfrak{A}$ and $\mathfrak{B}$ fixed, $\mathfrak{C}$ and $\mathfrak{D}$ random	4.$\mathfrak{CD}$.$\alpha\beta\gamma\delta$ Page 59	4.$\mathfrak{CD}$.$\alpha\beta\gamma$ Page 61	4.$\mathfrak{CD}$.$\alpha\beta$ Page 61	Put $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}}{BCD}$ in "$\alpha\beta\gamma\delta$" – cases, pages 59, 64 and 67.	4.$\mathfrak{CD}$.R Page 62	4.$\mathfrak{CD}$.1 Page 62
$\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ fixed, $\mathfrak{D}$ random	4.$\mathfrak{D}$.$\alpha\beta\gamma\delta$ Page 64	4.$\mathfrak{D}$.$\alpha\beta\gamma$ Page 64	Put $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}}{CDR_{....}}$ in "$\alpha\beta\gamma\delta$" – cases, pages 64 and 67.		4.$\mathfrak{D}$.R Page 66	4.$\mathfrak{D}$.1 Page 66
$\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ fixed	4..$\alpha\beta\gamma\delta$ Page 67	Put $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}R_{..\gamma.}}{DR^2_{....}}$ in [4..$\alpha\beta\gamma\delta$], page 67.			Put $R_{\alpha\beta\gamma\delta} = R$ in [4..$\alpha\beta\gamma\delta$], page 67.	4..1 Page 67

Note: For the case of orthogonal contrasts in fixed factors see paragraph II.6

4-way classification
Table III: Tabular summary

[4.$\mathcal{ABCD}.\alpha\beta\gamma\delta$]: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ random; $R_{\alpha\beta\gamma\delta} = \dfrac{R_{\alpha\ldots} R_{.\beta..} R_{..\gamma.} R_{\ldots\delta}}{R^3_{\ldots\ldots}}$

For definition of symbols used see [4-way classification. Table I, II and III], pages 45-47.

	EMS	H_o	Testing H_o
$\mathcal{A}$	$\sigma_r^2 + R_{\ldots\ldots} \dfrac{1-k_a^{''''}}{A-1}\left[k_b^{''''} k_c^{''''} k_d^{''''} \sigma_{abcd}^2 + k_b^{''''} k_c^{''''} \sigma_{abc}^2 + k_b^{''''} k_d^{''''} \sigma_{abd}^2 + k_c^{''''} k_d^{''''} \sigma_{acd}^2 + k_b^{''''} \sigma_{ab}^2 + k_c^{''''} \sigma_{ac}^2 + k_d^{''''} \sigma_{ad}^2 + \sigma_a^2\right]$	$\sigma_a^2 = 0$	$F'(\mathcal{A})$ 1)
	Lines "$\mathcal{B}$", "$\mathcal{C}$" and "$\mathcal{D}$" analogous to line "$\mathcal{A}$" : Interchange respective symbols.		
$\mathcal{AB}$	$\sigma_r^2 + R_{\ldots\ldots} \dfrac{(1-k_a^{''''})(1-k_b^{''''})}{(A-1)(B-1)}\left[k_c^{''''} k_d^{''''} \sigma_{abcd}^2 + k_c^{''''} \sigma_{abc}^2 + k_d^{''''} \sigma_{abd}^2 + \sigma_{ab}^2\right]$	$\sigma_{ab}^2 = 0$	$F'(\mathcal{AB})$ 2)
	Lines "$\mathcal{AC}$", "$\mathcal{AD}$", "$\mathcal{BC}$", "$\mathcal{BD}$" and "$\mathcal{CD}$" analogous to line "$\mathcal{AB}$" : Interchange respective symbols.		
$\mathcal{ABC}$	$\sigma_r^2 + R_{\ldots\ldots} \dfrac{(1-k_a^{''''})(1-k_b^{''''})(1-k_c^{''''})}{(A-1)(B-1)(C-1)}\left[k_d^{''''} \sigma_{abcd}^2 + \sigma_{abc}^2\right]$	$\sigma_{abc}^2 = 0$	$F'(\mathcal{ABC})$ 3)
	Lines "$\mathcal{ABD}$", "$\mathcal{ACD}$" and "$\mathcal{BCD}$" analogous to line "$\mathcal{ABC}$" : Interchange respective symbols.		
$\mathcal{ABCD}$	$\sigma_r^2 + R_{\ldots\ldots} \dfrac{(1-k_a^{''''})(1-k_b^{''''})(1-k_c^{''''})(1-k_d^{''''})}{(A-1)(B-1)(C-1)(D-1)} \sigma_{abcd}^2$	$\sigma_{abcd}^2 = 0$	$F = \dfrac{MS(\mathcal{ABCD})}{MS(\mathcal{R})}$
$\mathcal{R}$	σ_r^2	—	1)–3): See next 2 pages.

4.$\mathcal{ABCD}.\alpha\beta\gamma\delta$

[4.$\mathcal{ABCD}.\alpha\beta\gamma\delta$] continued.

1) $F'(\mathcal{A}) = \mathrm{MS}_1(\mathcal{A})/\mathrm{MS}_2(\mathcal{A})$, where (definition of L-terms below):

$$\mathrm{MS}_1(\mathcal{A}) = \mathrm{MS}(\mathcal{A}) + L(\mathcal{ABC}) + L(\mathcal{ABD}) + L(\mathcal{ACD}) + L(\mathcal{R})$$

$$\mathrm{MS}_2(\mathcal{A}) = L(\mathcal{AB}) + L(\mathcal{AC}) + L(\mathcal{AD}) + L(\mathcal{ABCD}),$$

with effective degrees of freedom for $\mathrm{MS}_1(\mathcal{A})$ and $\mathrm{MS}_2(\mathcal{A})$, respectively:

$$f_1 = [\mathrm{MS}_1(\mathcal{A})]^2 \Big/ \left\{ \frac{[\mathrm{MS}(\mathcal{A})]^2}{A-1} + \frac{[L(\mathcal{ABC})]^2}{(A-1)(B-1)(C-1)} + \frac{[L(\mathcal{ABD})]^2}{(A-1)(B-1)(D-1)} + \frac{[L(\mathcal{ACD})]^2}{(A-1)(C-1)(D-1)} + \frac{[L(\mathcal{R})]^2}{R_{....} - ABCD} \right\}$$

$$f_2 = [\mathrm{MS}_2(\mathcal{A})]^2 \Big/ \left\{ \frac{[L(\mathcal{AB})]^2}{(A-1)(B-1)} + \frac{[L(\mathcal{AC})]^2}{(A-1)(C-1)} + \frac{[L(\mathcal{AD})]^2}{(A-1)(D-1)} + \frac{[L(\mathcal{ABCD})]^2}{(A-1)(B-1)(C-1)(D-1)} \right\}$$

Definition of L-terms :

$$L(\mathcal{ABC}) = \frac{(B-1)(C-1)\,\bar{k}_b\bar{k}_c}{(1-\bar{k}_b)(1-\bar{k}_c)}\,\mathrm{MS}(\mathcal{ABC})$$

$$L(\mathcal{ABD}) = \frac{(B-1)(D-1)\,\bar{k}_b\bar{k}_d}{(1-\bar{k}_b)(1-\bar{k}_d)}\,\mathrm{MS}(\mathcal{ABD})$$

$$L(\mathcal{ACD}) = \frac{(C-1)(D-1)\,\bar{k}_c\bar{k}_d}{(1-\bar{k}_c)(1-\bar{k}_d)}\,\mathrm{MS}(\mathcal{ACD})$$

$$L(\mathcal{R}) = \frac{(B\bar{k}_b-1)(C\bar{k}_c-1)(D\bar{k}_d-1)}{(1-\bar{k}_b)(1-\bar{k}_c)(1-\bar{k}_d)}\,\mathrm{MS}(\mathcal{R})$$

$$L(\mathcal{AB}) = \frac{(B-1)\,\bar{k}_b}{1-\bar{k}_b}\,\mathrm{MS}(\mathcal{AB})$$

$$L(\mathcal{AC}) = \frac{(C-1)\,\bar{k}_c}{1-\bar{k}_c}\,\mathrm{MS}(\mathcal{AC})$$

$$L(\mathcal{AD}) = \frac{(D-1)\,\bar{k}_d}{1-\bar{k}_d}\,\mathrm{MS}(\mathcal{AD})$$

$$L(\mathcal{ABCD}) = \frac{(B-1)(C-1)(D-1)\,\bar{k}_b\bar{k}_c\bar{k}_d}{(1-\bar{k}_b)(1-\bar{k}_c)(1-\bar{k}_d)}\,\mathrm{MS}(\mathcal{ABCD})$$

[4.$\mathcal{ABCD}.\alpha\beta\gamma\delta$] continued

[4.$\mathcal{ABCD}.\alpha\beta\gamma\delta$] continued.

2) $F'(\mathcal{AB}) = \mathrm{MS}_1(\mathcal{AB})/\mathrm{MS}_2(\mathcal{AB})$, where

$$\mathrm{MS}_1(\mathcal{AB}) = \mathrm{MS}(\mathcal{AB}) + \frac{(C-1)(D-1)k_c^{''''}k_d^{''''}\,\mathrm{MS}(\mathcal{ABCD})}{(1-k_c^{''''})(1-k_d^{''''})}$$

$$\mathrm{MS}_2(\mathcal{AB}) = \frac{(C-1)k_c^{''''}\mathrm{MS}(\mathcal{ABC})}{1-k_c^{''''}} + \frac{(D-1)k_d^{''''}\mathrm{MS}(\mathcal{ABD})}{1-k_d^{''''}} + \frac{(Ck_c^{''''}-1)(Dk_d^{''''}-1)(\mathrm{MS}(\mathcal{R})}{(1-k_c^{''''})(1-k_d^{''''})},$$

with effective degrees of freedom for $\mathrm{MS}_1(\mathcal{AB})$ and $\mathrm{MS}_2(\mathcal{AB})$, respectively:

$$f_1 = [\mathrm{MS}_1(\mathcal{AB})]^2 \Bigg/ \left\{ \frac{[\mathrm{MS}(\mathcal{AB})]^2}{(A-1)(B-1)} + \frac{\left[\dfrac{(C-1)(D-1)k_c^{''''}k_d^{''''}\,\mathrm{MS}(\mathcal{ABCD})}{(1-k_c^{''''})(1-k_d^{''''})}\right]^2}{(A-1)(B-1)(C-1)(D-1)} \right\}$$

$$f_2 = [\mathrm{MS}_2(\mathcal{AB})]^2 \Bigg/ \left\{ \frac{\left[\dfrac{(C-1)k_c^{''''}\mathrm{MS}(\mathcal{ABC})}{1-k_c^{''''}}\right]^2}{(A-1)(B-1)(C-1)} + \frac{\left[\dfrac{(D-1)k_d^{''''}\mathrm{MS}(\mathcal{ABD})}{1-k_d^{''''}}\right]^2}{(A-1)(B-1)(D-1)} + \frac{\left[\dfrac{(Ck_c^{''''}-1)(Dk_d^{''''}-1)\mathrm{MS}(\mathcal{R})}{(1-k_c^{''''})(1-k_d^{''''})}\right]^2}{R_{....}-ABCD} \right\}$$

3) $F'(\mathcal{ABC}) = \mathrm{MS}_1(\mathcal{ABC})/\mathrm{MS}(\mathcal{ABCD})$, where

$$\mathrm{MS}_1(\mathcal{ABC}) = \frac{(1-k_d^{''''})\,\mathrm{MS}(\mathcal{ABC})}{(D-1)k_d^{''''}} + \frac{(Dk_d^{''''}-1)\mathrm{MS}(\mathcal{R})}{(D-1)k_d^{''''}},$$

with effective degrees of freedom

$$f_1 = [\mathrm{MS}_1(\mathcal{ABC})]^2 \Bigg/ \left\{ \frac{\left[\dfrac{(1-k_d^{''''})\mathrm{MS}(\mathcal{ABC})}{(D-1)k_d^{''''}}\right]^2}{(A-1)(B-1)(C-1)} + \frac{\left[\dfrac{(Dk_d^{''''}-1)\mathrm{MS}(\mathcal{R})}{(D-1)k_d^{''''}}\right]^2}{R_{....}-ABCD} \right\}$$

[4.$\mathcal{ABCD}.\alpha\beta\gamma\delta$] continued

$[4.\mathcal{ABCD}.\alpha\beta\gamma]$, $[4.\mathcal{ABCD}.\alpha\beta]$ and $[4.\mathcal{ABCD}.\alpha]$: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ random; $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}R_{..\gamma.}}{DR^2_{....}}$, $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}}{CDR_{....}}$ and $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}}{BCD}$, respectively.

For definition of symbols used see [4-way classification. Table I, II and III], pages 45-47.

	H_o	Testing H_o: $[4.\mathcal{ABCD}.\alpha\beta\gamma]$, $\left(R_{...\delta} = \frac{R_{....}}{D}\right)$: $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}R_{..\gamma.}}{DR^2_{....}}$	$[4.\mathcal{ABCD}.\alpha\beta]$, $\left(R_{..\gamma\delta} = \frac{R_{....}}{CD}\right)$: $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}}{CDR_{....}}$	$[4.\mathcal{ABCD}.\alpha]$, $\left(R_{.\beta\gamma\delta} = \frac{R_{....}}{BCD}\right)$: $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}}{BCD}$
$\mathcal{A}$	$\sigma_a^2 = 0$	$F'(\mathcal{A})$ [1]	$F'(\mathcal{A})$ [1]	$F'(\mathcal{A})$ [1]
Lines "$\mathcal{B}$", "$\mathcal{C}$" and "$\mathcal{D}$" analogous to line "$\mathcal{A}$": Interchange respective symbols.				
$\mathcal{AB}$	$\sigma_{ab}^2 = 0$	$F'(\mathcal{AB})$ [2]	$F'(\mathcal{AB})$ [2]	$F'(\mathcal{AB})$ [2]
Lines "$\mathcal{AC}$", "$\mathcal{AD}$", "$\mathcal{BC}$", "$\mathcal{BD}$" and "$\mathcal{CD}$" analogous to line "$\mathcal{AB}$": Interchange respective symbols.				
$\mathcal{ABC}$	$\sigma_{abc}^2 = 0$	$F = MS(\mathcal{ABC})/MS(\mathcal{ABCD})$	$F = MS(\mathcal{ABC})/MS(\mathcal{ABCD})$	$F = MS(\mathcal{ABC})/MS(\mathcal{ABCD})$
$\mathcal{ABD}$	$\sigma_{abd}^2 = 0$	$F'(\mathcal{ABD})$ [3]	$F = MS(\mathcal{ABD})/MS(\mathcal{ABCD})$	$F = MS(\mathcal{ABD})/MS(\mathcal{ABCD})$
$\mathcal{ACD}$	$\sigma_{acd}^2 = 0$	$F'(\mathcal{ACD})$ [3]	$F'(\mathcal{ACD})$ [3]	$F = MS(\mathcal{ACD})/MS(\mathcal{ABCD})$
$\mathcal{BCD}$	$\sigma_{bcd}^2 = 0$	$F'(\mathcal{BCD})$ [3]	$F'(\mathcal{BCD})$ [3]	$F'(\mathcal{BCD})$ [3]
$\mathcal{ABCD}$	$\sigma_{abcd}^2 = 0$	$F = MS(\mathcal{ABCD})/MS(\mathcal{R})$	$F = MS(\mathcal{ABCD})/MS(\mathcal{R})$	$F = MS(\mathcal{ABCD})/MS(\mathcal{R})$
$\mathcal{R}$	——	——	——	——
		[1] and [2]: As in $[4.\mathcal{ABCD}.\alpha\beta\gamma\delta]$, pages 49 and 50, with $k_d^{=} = \frac{1}{D}$. [3]: Analogous (interchange respective symbols!) to $F'(\mathcal{ABC})$ in $[4.\mathcal{ABCD}.\alpha\beta\gamma\delta]$, page 50.	[1] and [2]: As in $[4.\mathcal{ABCD}.\alpha\beta\gamma\delta]$, pages 49 and 50, with $k_c^{=} = \frac{1}{C}$ and $k_d^{=} = \frac{1}{D}$. [3]: Analogous (Interchange respective symbols!) to $F'(\mathcal{ABC})$ in $[4.\mathcal{ABCD}.\alpha\beta\gamma\delta]$, page 50.	[1] and [2]: As in $[4.\mathcal{ABCD}.\alpha\beta\gamma\delta]$, pages 49 and 50, with $k_b^{=} = \frac{1}{B}$, $k_c^{=} = \frac{1}{C}$ and $k_d^{=} = \frac{1}{D}$. [3]: Analogous (interchange respective symbols!) to $F'(\mathcal{ABC})$ in $[4.\mathcal{ABCD}.\alpha\beta\gamma\delta]$, page 50.
		For mean square expectations put $k_d^{=} = \frac{1}{D}$ in those of $[4.\mathcal{ABCD}.\alpha\beta\gamma\delta]$, page 48.	For mean square expectations put $k_c^{=} = \frac{1}{C}$ and $k_d^{=} = \frac{1}{D}$ in those of $[4.\mathcal{ABCD}.\alpha\beta\gamma\delta]$, page 48.	For mean square expectations put $k_b^{=} = \frac{1}{B}$, $k_c^{=} = \frac{1}{C}$ and $k_d^{=} = \frac{1}{D}$ in those of $[4.\mathcal{ABCD}.\alpha\beta\gamma\delta]$, page 48.

$4.\mathcal{ABCD}.\alpha\beta\gamma$ + $4.\mathcal{ABCD}.\alpha\beta$ + $4.\mathcal{ABCD}.\alpha$

[4.$\mathfrak{ABCD}$.R] and [4.$\mathfrak{ABCD}$.1] : $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ random; $R_{\alpha\beta\gamma\delta} \equiv R$ and $R_{\alpha\beta\gamma\delta} \equiv 1$, respectively.
For definition of symbols used see [4-way classification. Table I, II and III], pages 45-47.

	EMS ([4.$\mathfrak{ABCD}$.R])†	H_o	Testing H_o: [4.$\mathfrak{ABCD}$.R] : $R_{\alpha\beta\gamma\delta} \equiv R$	Testing H_o: [4.$\mathfrak{ABCD}$.1] : $R_{\alpha\beta\gamma\delta} \equiv 1$
$\mathfrak{A}$	$\sigma_r^2 + R\sigma_{abcd}^2 + RD\sigma_{abc}^2 + RC\sigma_{abd}^2 + RB\sigma_{acd}^2 + RCD\sigma_{ab}^2 + RBD\sigma_{ac}^2 + RBC\sigma_{ad}^2 + RBCD\sigma_a^2$	$\sigma_a^2 = 0$	$F'(\mathfrak{A})$ 1)	
Lines "$\mathfrak{B}$", "$\mathfrak{C}$" and "$\mathfrak{D}$" analogous to line "$\mathfrak{A}$": Interchange respective symbols.				
$\mathfrak{AB}$	$\sigma_r^2 + R\sigma_{abcd}^2 + RD\sigma_{abc}^2 + RC\sigma_{abd}^2 + RCD\sigma_{ab}^2$	$\sigma_{ab}^2 = 0$	$F'(\mathfrak{AB})$ 2)	
Lines "$\mathfrak{AC}$", "$\mathfrak{AD}$", "$\mathfrak{BC}$", "$\mathfrak{BD}$" and "$\mathfrak{CD}$" analogous to line "$\mathfrak{AB}$": Interchange respective symbols.				
$\mathfrak{ABC}$	$\sigma_r^2 + R\sigma_{abcd}^2 + RD\sigma_{abc}^2$	$\sigma_{abc}^2 = 0$	$F = \mathrm{MS}(\mathfrak{ABC})/\mathrm{MS}(\mathfrak{ABCD})$	
Lines "$\mathfrak{ABD}$", "$\mathfrak{ACD}$" and "$\mathfrak{BCD}$" analogous to line "$\mathfrak{ABC}$": Interchange respective symbols.				
$\mathfrak{ABCD}$	$\sigma_r^2 + R\sigma_{abcd}^2$	$\sigma_{abcd}^2 = 0$	$F = \dfrac{\mathrm{MS}(\mathfrak{ABCD})}{\mathrm{MS}(\mathfrak{R})}$	Not possible
$\mathfrak{R}$	σ_r^2	——	1) and 2): See next page.	

†) For EMS of [4.$\mathfrak{ABCD}$.1] put $R = 1$ and delete line "$\mathfrak{R}$".

4.$\mathfrak{ABCD}$.R + 4.$\mathfrak{ABCD}$.1

[4.$\mathcal{ABCD}.R$] and [4.$\mathcal{ABCD}$.1] continued.

1) $F'(\mathcal{A}) = MS_1(\mathcal{A})/MS_2(\mathcal{A})$, where

$$MS_1(\mathcal{A}) = MS(\mathcal{A}) + MS(\mathcal{ABC}) + MS(\mathcal{ABD}) + MS(\mathcal{ACD})$$

$$MS_2(\mathcal{A}) = MS(\mathcal{AB}) + MS(\mathcal{AC}) + MS(\mathcal{AD}) + MS(\mathcal{ABCD}),$$

with effective degrees of freedom for $MS_1(\mathcal{A})$ and $MS_2(\mathcal{A})$, respectively:

$$f_1 = [MS_1(\mathcal{A})]^2 \Big/ \left\{ \frac{[MS(\mathcal{A})]^2}{A-1} + \frac{[MS(\mathcal{ABC})]^2}{(A-1)(B-1)(C-1)} + \frac{[MS(\mathcal{ABD})]^2}{(A-1)(B-1)(D-1)} + \frac{[MS(\mathcal{ACD})]^2}{(A-1)(C-1)(D-1)} \right\}$$

$$f_2 = [MS_2(\mathcal{A})]^2 \Big/ \left\{ \frac{[MS(\mathcal{AB})]^2}{(A-1)(B-1)} + \frac{[MS(\mathcal{AC})]^2}{(A-1)(C-1)} + \frac{[MS(\mathcal{AD})]^2}{(A-1)(D-1)} + \frac{[MS(\mathcal{ABCD})]^2}{(A-1)(B-1)(C-1)(D-1)} \right\}$$

2) $F'(\mathcal{AB}) = MS_1(\mathcal{AB})/MS_2(\mathcal{AB})$, where

$$MS_1(\mathcal{AB}) = MS(\mathcal{AB}) + MS(\mathcal{ABCD})$$

$$MS_2(\mathcal{AB}) = MS(\mathcal{ABC}) + MS(\mathcal{ABD}),$$

with effective degrees of freedom for $MS_1(\mathcal{AB})$ and $MS_2(\mathcal{AB})$, respectively:

$$f_1 = [MS_1(\mathcal{AB})]^2 \Big/ \left\{ \frac{[MS(\mathcal{AB})]^2}{(A-1)(B-1)} + \frac{[MS(\mathcal{ABCD})]^2}{(A-1)(B-1)(C-1)(D-1)} \right\}$$

$$f_2 = [MS_2(\mathcal{AB})]^2 \Big/ \left\{ \frac{[MS(\mathcal{ABC})]^2}{(A-1)(B-1)(C-1)} + \frac{[MS(\mathcal{ABD})]^2}{(A-1)(B-1)(D-1)} \right\}$$

[4.$\mathcal{ABCD}.R$] + [4.$\mathcal{ABCD}$.1] continued

$[4.\mathcal{BCD}.\alpha\beta\gamma\delta]$: $\mathcal{A}$ fixed, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ random; $R_{\alpha\beta\gamma\delta} = \dfrac{R_{\alpha\ldots} R_{.\beta..} R_{..\gamma.} R_{\ldots\delta}}{R^3_{\ldots.}}$

For definition of symbols used see [4-way classification. Table I, II and III], pages 45-47.

	EMS	H_o	Testing H_o
$\mathcal{A}$	$\sigma_r^2 + \frac{R_{\ldots.}}{A-1}\left[k_b'''k_c'''k_d'''\sigma^2_{abcd} + k_b'''k_c'''\sigma^2_{abc} + k_b'''k_d'''\sigma^2_{abd} + k_c'''k_d'''\sigma^2_{acd} + k_b'''\sigma^2_{ab} + k_c'''\sigma^2_{ac} + k_d'''\sigma^2_{ad}\right] + \frac{1}{A-1}\sum_\alpha R_{\alpha\ldots}\alpha_\alpha^2$	$\alpha_\alpha = 0$	$F'^{*}(\mathcal{A})$ 1)
$\mathcal{B}$	$\sigma_r^2 + R_{\ldots.}\frac{1-k_b'''}{B-1}\left[k_c'''k_d'''\sigma^2_{bcd} + k_c'''\sigma^2_{bc} + k_d'''\sigma^2_{bd} + \sigma^2_b\right]$	$\sigma_b^2 = 0$	$F'(\mathcal{B})$ 2)
Lines "$\mathcal{C}$" and "$\mathcal{D}$" analogous to line "$\mathcal{B}$" : Interchange respective symbols.			
$\mathcal{AB}$	$\sigma_r^2 + \frac{R_{\ldots.}(1-k_b''')}{(A-1)(B-1)}\left[k_c'''k_d'''\sigma^2_{abcd} + k_c'''\sigma^2_{abc} + k_d'''\sigma^2_{abd} + \sigma^2_{ab}\right]$	$\sigma^2_{ab} = 0$	$F'^{*}(\mathcal{AB})$ 3)
Lines "$\mathcal{AC}$" and "$\mathcal{AD}$" analogous to line "$\mathcal{AB}$" : Interchange respective symbols.			
$\mathcal{BC}$	$\sigma_r^2 + R_{\ldots.}\frac{(1-k_b''')(1-k_c''')}{(B-1)(C-1)}\left[k_d'''\sigma^2_{bcd} + \sigma^2_{bc}\right]$	$\sigma^2_{bc} = 0$	$F'(\mathcal{BC})$ 4)
Lines "$\mathcal{BD}$" and "$\mathcal{CD}$" analogous to line "$\mathcal{BC}$" : Interchange respective symbols.			
$\mathcal{ABC}$	$\sigma_r^2 + \frac{R_{\ldots.}(1-k_b''')(1-k_c''')}{(A-1)(B-1)(C-1)}\left[k_d'''\sigma^2_{abcd} + \sigma^2_{abc}\right]$	$\sigma^2_{abc} = 0$	$F'^{*}(\mathcal{ABC})$ 5)
Lines "$\mathcal{ABD}$" and "$\mathcal{ACD}$" analogous to line "$\mathcal{ABC}$" : Interchange respective symbols.			
$\mathcal{BCD}$	$\sigma_r^2 + R_{\ldots.}\frac{(1-k_b''')(1-k_c''')(1-k_d''')}{(B-1)(C-1)(D-1)}\sigma^2_{bcd}$	$\sigma^2_{bcd} = 0$	$F = \frac{MS(\mathcal{BCD})}{MS(\mathcal{R})}$
$\mathcal{ABCD}$	$\sigma_r^2 + \frac{R_{\ldots.}(1-k_b''')(1-k_c''')(1-k_d''')}{(A-1)(B-1)(C-1)(D-1)}\sigma^2_{abcd}$	$\sigma^2_{abcd} = 0$	$F = \frac{MS(\mathcal{ABCD})}{MS(\mathcal{R})}$
$\mathcal{R}$	σ_r^2	—	1) – 5): See ne t page.

4.$\mathcal{BCD}.\alpha\beta\gamma\delta$

[4.$\mathcal{BCD}.\alpha\beta\gamma\delta$] continued.

1) $F'^{*}(\mathcal{A}) = MS_1(\mathcal{A})/MS_2(\mathcal{A})$ as in [4.$\mathcal{ABCD}.\alpha\beta\gamma\delta$], page 49.

2) $F'(\mathcal{B}) = MS_1(\mathcal{B})/MS_2(\mathcal{B})$, where

$$MS_1(\mathcal{B}) = MS(\mathcal{B}) + \frac{(C-1)(D-1)k_c^{''''}k_d^{''''}MS(\mathcal{BCD})}{(1-k_c^{''''})(1-k_d^{''''})}$$

$$MS_2(\mathcal{B}) = \frac{(C-1)k_c^{''''}MS(\mathcal{BC})}{1-k_c^{''''}} + \frac{(D-1)k_d^{''''}MS(\mathcal{BD})}{1-k_d^{''''}} + \frac{(Ck_c^{''''}-1)(Dk_d^{''''}-1)MS(\mathcal{R})}{(1-k_c^{''''})(1-k_d^{''''})},$$

with effective degrees of freedom for $MS_1(\mathcal{B})$ and $MS_2(\mathcal{B})$, respectively:

$$f_1 = [MS_1(\mathcal{B})]^2 \Bigg/ \left\{ \frac{[MS(\mathcal{B})]^2}{B-1} + \frac{\left[\dfrac{(C-1)(D-1)k_c^{''''}k_d^{''''}MS(\mathcal{BCD})}{(1-k_c^{''''})(1-k_d^{''''})}\right]^2}{(B-1)(C-1)(D-1)} \right\}$$

$$f_2 = [MS_2(\mathcal{B})]^2 \Bigg/ \left\{ \frac{\left[\dfrac{(C-1)k_c^{''''}MS(\mathcal{BC})}{1-k_c^{''''}}\right]^2}{(B-1)(C-1)} + \frac{\left[\dfrac{(D-1)k_d^{''''}MS(\mathcal{BD})}{1-k_d^{''''}}\right]^2}{(B-1)(D-1)} + \frac{\left[\dfrac{(Ck_c^{''''}-1)(Dk_d^{''''}-1)MS(\mathcal{R})}{(1-k_c^{''''})(1-k_d^{''''})}\right]^2}{R_{\dots\dots}-ABCD} \right\}$$

3) $F'^{*}(\mathcal{AB}) = MS_1(\mathcal{AB})/MS_2(\mathcal{AB})$ as in [4.$\mathcal{ABCD}.\alpha\beta\gamma\delta$], page 50.

4) $F'(\mathcal{BC}) = MS_1(\mathcal{BC})/MS(\mathcal{BCD})$, where

$$MS_1(\mathcal{BC}) = \frac{(1-k_d^{''''})MS(\mathcal{BC})}{(D-1)k_d^{''''}} + \frac{(Dk_d^{''''}-1)MS(\mathcal{R})}{(D-1)k_d^{''''}}$$

with effective degrees of freedom

$$f_1 = [MS_1(\mathcal{BC})]^2 \Bigg/ \left\{ \frac{\left[\dfrac{(1-k_d^{''''})MS(\mathcal{BC})}{(D-1)k_d^{''''}}\right]^2}{(B-1)(C-1)} + \frac{\left[\dfrac{(Dk_d^{''''}-1)MS(\mathcal{R})}{(D-1)k_d^{''''}}\right]^2}{R_{\dots\dots}-ABCD} \right\}$$

5) $F'^{*}(\mathcal{ABC}) = MS_1(\mathcal{ABC})/MS(\mathcal{ABCD})$ as in [4.$\mathcal{ABCD}.\alpha\beta\gamma\delta$], page 50.

[4.$\mathcal{BCD}.\alpha\beta\gamma\delta$] continued

$[4.\mathfrak{BCD}.\alpha\beta\gamma]$, $[4.\mathfrak{BCD}.\alpha\beta]$ and $[4.\mathfrak{BCD}.\alpha]$: $\mathfrak{A}$ fixed, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ random; $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}R_{..\gamma.}}{DR^2_{....}}$, $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}}{CDR_{....}}$ and $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}}{BCD}$, respectively.

For definition of symbols used see [4-way classification. Table I, II and III], pages 45-47.

	H_o	Testing H_o: $[4.\mathfrak{BCD}.\alpha\beta\gamma]$, $\left(R_{...\delta} = \frac{R_{....}}{D}\right)$: $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}R_{..\gamma.}}{DR_{....}}$	Testing H_o: $[4.\mathfrak{BCD}.\alpha\beta]$, $\left(R_{..\gamma\delta} = \frac{R_{....}}{CD}\right)$: $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}}{CDR_{....}}$	Testing H_o: $[4.\mathfrak{BCD}.\alpha]$, $\left(R_{.\beta\gamma\delta} = \frac{R_{....}}{BCD}\right)$: $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}}{BCD}$
$\mathfrak{A}$	$a_a = 0$	$F'^{*}(\mathfrak{A})$ 1)	$F'^{*}(\mathfrak{A})$ 1)	$F'^{*}(\mathfrak{A})$ 1)
$\mathfrak{B}$	$\sigma_b^2 = 0$	$F'(\mathfrak{B})$ 2)	$F'(\mathfrak{B})$ 2)	$F'(\mathfrak{B})$ 2)
Lines "$\mathfrak{C}$" and "$\mathfrak{D}$" analogous to line "$\mathfrak{B}$": Interchange respective symbols.				
$\mathfrak{AB}$	$\sigma_{ab}^2 = 0$	$F'^{*}(\mathfrak{AB})$ 3)	$F'^{*}(\mathfrak{AB})$ 3)	$F'^{*}(\mathfrak{AB})$ 3)
Lines "$\mathfrak{AC}$" and "$\mathfrak{AD}$" analogous to line "$\mathfrak{AB}$": Interchange respective symbols.				
$\mathfrak{BC}$	$\sigma_{bc}^2 = 0$	$F = MS(\mathfrak{BC})/MS(\mathfrak{BCD})$	$F = MS(\mathfrak{BC})/MS(\mathfrak{BCD})$	$F = MS(\mathfrak{BC})/MS(\mathfrak{BCD})$
$\mathfrak{BD}$	$\sigma_{bd}^2 = 0$	$F'(\mathfrak{BD})$ 4)	$F = MS(\mathfrak{BD})/MS(\mathfrak{BCD})$	$F = MS(\mathfrak{BD})/MS(\mathfrak{BCD})$
$\mathfrak{CD}$	$\sigma_{cd}^2 = 0$	$F'(\mathfrak{CD})$ 4)	$F'(\mathfrak{CD})$ 4)	$F = MS(\mathfrak{CD})/MS(\mathfrak{BCD})$
$\mathfrak{ABC}$	$\sigma_{abc}^2 = 0$	$F^{*} = MS(\mathfrak{ABC})/MS(\mathfrak{ABCD})$	$F^{*} = MS(\mathfrak{ABC})/MS(\mathfrak{ABCD})$	$F^{*} = MS(\mathfrak{ABC})/MS(\mathfrak{ABCD})$
$\mathfrak{ABD}$	$\sigma_{abd}^2 = 0$	$F'^{*}(\mathfrak{ABD})$ 5)	$F^{*} = MS(\mathfrak{ABD})/MS(\mathfrak{ABCD})$	$F^{*} = MS(\mathfrak{ABD})/MS(\mathfrak{ABCD})$
$\mathfrak{ACD}$	$\sigma_{acd}^2 = 0$	$F'^{*}(\mathfrak{ACD})$ 5)	$F'^{*}(\mathfrak{ACD})$ 5)	$F^{*} = MS(\mathfrak{ACD})/MS(\mathfrak{ABCD})$
$\mathfrak{BCD}$	$\sigma_{bcd}^2 = 0$	$F = MS(\mathfrak{BCD})/MS(\mathfrak{R})$	$F = MS(\mathfrak{BCD})/MS(\mathfrak{R})$	$F = MS(\mathfrak{BCD})/MS(\mathfrak{R})$
$\mathfrak{ABCD}$	$\sigma_{abcd}^2 = 0$	$F = MS(\mathfrak{ABCD})/MS(\mathfrak{R})$	$F = MS(\mathfrak{ABCD})/MS(\mathfrak{R})$	$F = MS(\mathfrak{ABCD})/MS(\mathfrak{R})$
$\mathfrak{R}$	——	——	——	——
		1), 2), 3): As in $[4.\mathfrak{BCD}.\alpha\beta\gamma\delta]$, page 55, with $k_d^{=} = \frac{1}{D}$. 4) and 5): Analogous (Interchange respective symbols!) to $F'(\mathfrak{BC})$ and $F'^{*}(\mathfrak{ABC})$, respectively, in $[4.\mathfrak{BCD}.\alpha\beta\gamma\delta]$, page 55.	1), 2), 3): As in $[4.\mathfrak{BCD}.\alpha\beta\gamma\delta]$, page 55, with $k_c^{=} = \frac{1}{D}$ and $k_d^{=} = \frac{1}{D}$. 4) and 5): Analogous (Interchange respective symbols!) to $F'(\mathfrak{BC})$ and $F'^{*}(\mathfrak{ABC})$, respectively, in $[4.\mathfrak{BCD}.\alpha\beta\gamma\delta]$, page 55.	1), 2), 3): As in $[4.\mathfrak{BCD}.\alpha\beta\gamma\delta]$, page 55, with $k_b^{=} = \frac{1}{B}$, $k_c^{=} = \frac{1}{C}$ and $k_d^{=} = \frac{1}{D}$.
		For mean square expectations put $k_d^{=} = \frac{1}{D}$ in those of $[4.\mathfrak{BCD}.\alpha\beta\gamma\delta]$, page 54.	For mean square expectations put $k_c^{=} = \frac{1}{C}$ and $k_d^{=} = \frac{1}{D}$ in those of $[4.\mathfrak{BCD}.\alpha\beta\gamma\delta]$, page 54.	For mean square expectations put $k_b^{=} = \frac{1}{B}$, $k_c^{=} = \frac{1}{C}$ and $k_d^{=} = \frac{1}{D}$ in those of $[4.\mathfrak{BCD}.\alpha\beta\gamma\delta]$, page 54.

$\boxed{4.\mathfrak{BCD}.\alpha\beta\gamma}$ + $\boxed{4.\mathfrak{BCD}.\alpha\beta}$ + $\boxed{4.\mathfrak{BCD}.\alpha}$

[4.$\mathcal{BCD}$.R] and [4.$\mathcal{BCD}$.1]: $\mathcal{A}$ fixed, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ random; $R_{\alpha\beta\gamma\delta} \equiv R$ and $R_{\alpha\beta\gamma\delta} \equiv 1$, respectively.
For definition of symbols used see [4-way classification. Table I, II and III], pages 45-47.

	EMS([4.$\mathcal{BCD}$.R])†	H_o	Testing H_o: [4.$\mathcal{BCD}$.R] : $R_{\alpha\beta\gamma\delta} \equiv R$	Testing H_o: [4.$\mathcal{BCD}$.1] : $R_{\alpha\beta\gamma\delta} \equiv 1$
$\mathcal{A}$	$\sigma_r^2 + \frac{RA}{A-1}\sigma_{abcd}^2 + \frac{RAD}{A-1}\sigma_{abc}^2 + \frac{RAC}{A-1}\sigma_{abd}^2 + \frac{RAB}{A-1}\sigma_{acd}^2 + \frac{RACD}{A-1}\sigma_{ab}^2 + \frac{RABD}{A-1}\sigma_{ac}^2 + \frac{RABC}{A-1}\sigma_{ad}^2 + \frac{RBCD}{A-1}\sum_a a_a^2$	$a_a = 0$	$F'^{*}(\mathcal{A})$ 1)	
$\mathcal{B}$	$\sigma_r^2 + RA\sigma_{bcd}^2 + RAD\sigma_{bc}^2 + RAC\sigma_{bd}^2 + RACD\sigma_b^2$	$\sigma_b^2 = 0$	$F'(\mathcal{B})$ 2)	
Lines "$\mathcal{C}$" and "$\mathcal{D}$" analogous to line "$\mathcal{B}$": Interchange respective symbols.				
$\mathcal{AB}$	$\sigma_r^2 + \frac{RA}{A-1}\sigma_{abcd}^2 + \frac{RAD}{A-1}\sigma_{abc}^2 + \frac{RAC}{A-1}\sigma_{abd}^2 + \frac{RACD}{A-1}\sigma_{ab}^2$	$\sigma_{ab}^2 = 0$	$F'^{*}(\mathcal{AB})$ 3)	
Lines "$\mathcal{AC}$" and "$\mathcal{AD}$" analogous to line "$\mathcal{AB}$": Interchange respective symbols.				
$\mathcal{BC}$	$\sigma_r^2 + RA\sigma_{bcd}^2 + RAD\sigma_{bc}^2$	$\sigma_{bc}^2 = 0$	$F = MS(\mathcal{BC})/MS(\mathcal{BCD})$	
Lines "$\mathcal{BD}$" and "$\mathcal{CD}$" analogous to line "$\mathcal{BC}$": Interchange respective symbols.				
$\mathcal{ABC}$	$\sigma_r^2 + \frac{RA}{A-1}\sigma_{abcd}^2 + \frac{RAD}{A-1}\sigma_{abc}^2$	$\sigma_{abc}^2 = 0$	$F^{*} = MS(\mathcal{ABC})/MS(\mathcal{ABCD})$	
Lines "$\mathcal{ABD}$" and "$\mathcal{ACD}$" analogous to line "$\mathcal{ABC}$": Interchange respective symbols.				
$\mathcal{BCD}$	$\sigma_r^2 + RA\sigma_{bcd}^2$	$\sigma_{bcd}^2 = 0$	$F = \frac{MS(\mathcal{BCD})}{MS(\mathcal{R})}$	If $\sigma_{abcd}^2 = 0$ can be assumed: $F = MS(\mathcal{BCD})/MS(\mathcal{ABCD})$
$\mathcal{ABCD}$	$\sigma_r^2 + \frac{RA}{A-1}\sigma_{abcd}^2$	$\sigma_{abcd}^2 = 0$	$F = \frac{MS(\mathcal{ABCD})}{MS(\mathcal{R})}$	Not possible
$\mathcal{R}$	σ_r^2	—	1) – 3): See next page.	

†) For EMS of [4.$\mathcal{BCD}$.1] put $R = 1$ and delete line "$\mathcal{R}$".

4.$\mathcal{BCD}$.R + 4.$\mathcal{BCD}$.1

[4.$\mathcal{BCD}$.R] and [4.$\mathcal{BCD}$.1] continued.

1) $F'^{*}(\mathcal{A}) = \mathrm{MS}_1(\mathcal{A})/\mathrm{MS}_2(\mathcal{A})$ as in [4.$\mathcal{ABCD}$.R] + [4.$\mathcal{ABCD}$.1], page 53.

2) $F'(\mathcal{B}) = \mathrm{MS}_1(\mathcal{B})/\mathrm{MS}_2(\mathcal{B})$, where

$$\mathrm{MS}_1(\mathcal{B}) = \mathrm{MS}(\mathcal{B}) + \mathrm{MS}(\mathcal{BCD})$$

$$\mathrm{MS}_2(\mathcal{B}) = \mathrm{MS}(\mathcal{BC}) + \mathrm{MS}(\mathcal{BD}),$$

with effective degrees of freedom for $\mathrm{MS}_1(\mathcal{B})$ and $\mathrm{MS}_2(\mathcal{B})$, respectively:

$$f_1 = [\mathrm{MS}_1(\mathcal{B})]^2 \Big/ \left\{ \frac{1}{B-1}[\mathrm{MS}(\mathcal{B})]^2 + \frac{1}{(B-1)(C-1)(D-1)}[\mathrm{MS}(\mathcal{BCD})]^2 \right\}$$

$$f_2 = [\mathrm{MS}_2(\mathcal{B})]^2 \Big/ \left\{ \frac{1}{(B-1)(C-1)}[\mathrm{MS}(\mathcal{BC})]^2 + \frac{1}{(B-1)(D-1)}[\mathrm{MS}(\mathcal{BD})]^2 \right\}$$

3) $F'^{*}(\mathcal{AB}) = \mathrm{MS}_1(\mathcal{AB})/\mathrm{MS}_2(\mathcal{AB})$ as in [4.$\mathcal{ABCD}$.R] + [4.$\mathcal{ABCD}$.1], page 53.

[4.$\mathcal{CD}.\alpha\beta\gamma\delta$] : $\mathcal{A}$ and $\mathcal{B}$ fixed, $\mathcal{C}$ and $\mathcal{D}$ random; $R_{\alpha\beta\gamma\delta} = \dfrac{R_{\alpha\ldots}R_{.\beta..}R_{..\gamma.}R_{\ldots\delta}}{R^3_{\ldots.}}$

For definition of symbols used see [4-way classification. Table I, II and III], pages 45-47.

	EMS	H_o	Testing H_o
$\mathcal{A}$	$\sigma_r^2 + \frac{R_{\ldots.}}{A-1}\left[k_c''' k_d''' \sigma^2_{acd} + k_c''' \sigma^2_{ac} + k_d''' \sigma^2_{ad}\right] + \frac{1}{A-1}\sum_a R_{a\ldots} a_a^2$	$a_a = 0$	$F'^*(\mathcal{A})$ 1)
Line "$\mathcal{B}$" analogous to line "$\mathcal{A}$": Interchange respective symbols.			
$\mathcal{C}$	$\sigma_r^2 + R_{\ldots.}\frac{1-k_c'''}{C-1}\left[k_d''' \sigma^2_{cd} + \sigma^2_c\right]$	$\sigma^2_c = 0$	$F'(\mathcal{C})$ 2)
Line "$\mathcal{D}$" analogous to line "$\mathcal{C}$": Interchange respective symbols.			
$\mathcal{AB}$	$\sigma_r^2 + \frac{R_{\ldots.}}{(A-1)(B-1)}\left[k_c''' k_d''' \sigma^2_{abcd} + k_c''' \sigma^2_{abc} + k_d''' \sigma^2_{abd}\right] + \frac{1}{(A-1)(B-1)}\sum_a\sum_\beta R_{a\beta..} ab^2_{a\beta}$	$ab_{a\beta} = 0$	$F'^*(\mathcal{AB})$ 3)
$\mathcal{AC}$	$\sigma_r^2 + \frac{R_{\ldots.}(1-k_c''')}{(A-1)(C-1)}\left[k_d''' \sigma^2_{acd} + \sigma^2_{ac}\right]$	$\sigma^2_{ac} = 0$	$F'^*(\mathcal{AC})$ 4)
Lines "$\mathcal{AD}$", "$\mathcal{BC}$" and "$\mathcal{BD}$" analogous to line "$\mathcal{AC}$": Interchange respective symbols.			
$\mathcal{CD}$	$\sigma_r^2 + R_{\ldots.}\frac{(1-k_c''')(1-k_d''')}{(C-1)(D-1)}\sigma^2_{cd}$	$\sigma^2_{cd} = 0$	$F = \frac{MS(\mathcal{CD})}{MS(\mathcal{R})}$
$\mathcal{ABC}$	$\sigma_r^2 + \frac{R_{\ldots.}(1-k_c''')}{(A-1)(B-1)(C-1)}\left[k_d''' \sigma^2_{abcd} + \sigma^2_{abc}\right]$	$\sigma^2_{abc} = 0$	$F'^*(\mathcal{ABC})$ 5)
Line "$\mathcal{ABD}$" analogous to line "$\mathcal{ABC}$": Interchange respective symbols.			
$\mathcal{ACD}$	$\sigma_r^2 + \frac{R_{\ldots.}(1-k_c''')(1-k_d''')}{(A-1)(C-1)(D-1)}\sigma^2_{acd}$	$\sigma^2_{acd} = 0$	$F = \frac{MS(\mathcal{ACD})}{MS(\mathcal{R})}$
Line "$\mathcal{BCD}$" analogous to line "$\mathcal{ACD}$": Interchange respective symbols.			
$\mathcal{ABCD}$	$\sigma_r^2 + \frac{R_{\ldots.}(1-k_c''')(1-k_d''')}{(A-1)(B-1)(C-1)(D-1)}\sigma^2_{abcd}$	$\sigma^2_{abcd} = 0$	$F = \frac{MS(\mathcal{ABCD})}{MS(\mathcal{R})}$
$\mathcal{R}$	σ_r^2	—	1) – 5): See next page.

4.$\mathcal{CD}.\alpha\beta\gamma\delta$

[4.$\mathcal{CD}$.$\alpha\beta\gamma\delta$] continued.

1) $F'^{*}(\mathcal{A}) = MS_1(\mathcal{A})/MS_2(\mathcal{A})$, where

$$MS_1(\mathcal{A}) = MS(\mathcal{A}) + \frac{(C-1)(D-1)k_c^{'''}k_d^{'''}\,MS(\mathcal{ACD})}{(1-k_c^{'''})(1-k_d^{'''})}$$

$$MS_2(\mathcal{A}) = \frac{(C-1)k_c^{'''}MS(\mathcal{AC})}{1-k_c^{'''}} + \frac{(D-1)k_d^{'''}MS(\mathcal{AD})}{1-k_d^{'''}} + \frac{(Ck_c^{'''}-1)(Dk_d^{'''}-1)MS(\mathcal{R})}{(1-k_c^{'''})(1-k_d^{'''})}$$

with effective degrees of freedom for $MS_1(\mathcal{A})$ and $MS_2(\mathcal{A})$, respectively:

$$f_1 = [MS_1(\mathcal{A})]^2 \Bigg/ \left\{ \frac{[MS(\mathcal{A})]^2}{A-1} + \frac{\left[\dfrac{(C-1)(D-1)k_c^{'''}k_d^{'''}MS(\mathcal{ACD})}{(1-k_c^{'''})(1-k_d^{'''})}\right]^2}{(A-1)(C-1)(D-1)} \right\}$$

$$f_2 = [MS_2(\mathcal{A})]^2 \Bigg/ \left\{ \frac{\left[\dfrac{(C-1)k_c^{'''}MS(\mathcal{AC})}{1-k_c^{'''}}\right]^2}{(A-1)(C-1)} + \frac{\left[\dfrac{(D-1)k_d^{'''}MS(\mathcal{AD})}{1-k_d^{'''}}\right]^2}{(A-1)(D-1)} + \frac{\left[\dfrac{(Ck_c^{'''}-1)(Dk_d^{'''}-1)MS(\mathcal{R})}{(1-k_c^{'''})(1-k_d^{'''})}\right]^2}{R_{\cdots\cdot}-ABCD} \right\}$$

2) $F'(\mathcal{C}) = MS_1(\mathcal{C})/MS(\mathcal{CD})$, where

$$MS_1(\mathcal{C}) = \frac{(1-k_d^{'''})MS(\mathcal{C})}{(D-1)k_d^{'''}} + \frac{(Dk_d^{'''}-1)MS(\mathcal{R})}{(D-1)k_d^{'''}}$$

with effective degrees of freedom

$$f_1 = [MS_1(\mathcal{C})]^2 \Bigg/ \left\{ \frac{\left[\dfrac{(1-k_d^{'''})MS(\mathcal{C})}{(D-1)k_d^{'''}}\right]^2}{C-1} + \frac{\left[\dfrac{(Dk_d^{'''}-1)MS(\mathcal{R})}{(D-1)k_d^{'''}}\right]^2}{R_{\cdots\cdot}-ABCD} \right\}$$

3) $F'^{*}(\mathcal{AB}) = MS_1(\mathcal{AB})/MS_2(\mathcal{AB})$ as in [4.$\mathcal{ABCD}$.$\alpha\beta\gamma\delta$], page 50.

4) $F'^{*}(\mathcal{AC}) = MS_1(\mathcal{AC})/MS(\mathcal{ACD})$, where

$$MS_1(\mathcal{AC}) = \frac{(1-k_d^{'''})MS(\mathcal{AC})}{(D-1)k_d^{'''}} + \frac{(Dk_d^{'''}-1)MS(\mathcal{R})}{(D-1)k_d^{'''}}$$

with effective degrees of freedom

$$f_1 = [MS_1(\mathcal{AC})]^2 \Bigg/ \left\{ \frac{\left[\dfrac{(1-k_d^{'''})MS(\mathcal{AC})}{(D-1)k_d^{'''}}\right]^2}{(A-1)(C-1)} + \frac{\left[\dfrac{(Dk_d^{'''})MS(\mathcal{R})}{(D-1)k_d^{'''}}\right]^2}{R_{\cdots\cdot}-ABCD} \right\}$$

5) $F'^{*}(\mathcal{ABC}) = MS_1(\mathcal{ABC})/MS(\mathcal{ABCD})$ as in [4.$\mathcal{ABCD}$.$\alpha\beta\gamma\delta$], page 50.

[4.$\mathcal{CD}$.$\alpha\beta\gamma\delta$] continued

$[4.\mathcal{C}\mathfrak{D}.\alpha\beta\gamma]$ and $[4.\mathcal{C}\mathfrak{D}.\alpha\beta]$: $\mathfrak{A}$ and $\mathfrak{B}$ fixed, $\mathcal{C}$ and $\mathfrak{D}$ random ; $R_{\alpha\beta\gamma\delta} = \dfrac{R_{\alpha\ldots}R_{.\beta..}R_{..\gamma.}}{DR^2_{....}}$ and $R_{\alpha\beta\gamma\delta} = \dfrac{R_{\alpha\ldots}R_{.\beta..}}{CDR_{....}}$, respectively.

For definition of symbols used see [4-way classification. Table I, II and III], pages 45-47.

	H_o	Testing H_o: $[4.\mathcal{C}\mathfrak{D}.\alpha\beta\gamma]$. $\left(R_{...\delta} = \dfrac{R_{....}}{D}\right)$: $R_{\alpha\beta\gamma\delta} = \dfrac{R_{\alpha\ldots}R_{.\beta..}R_{..\gamma.}}{DR^2_{....}}$	Testing H_o: $[4.\mathcal{C}\mathfrak{D}.\alpha\beta]$. $\left(R_{..\gamma\delta} = \dfrac{R_{....}}{CD}\right)$: $R_{\alpha\beta\gamma\delta} = \dfrac{R_{\alpha\ldots}R_{.\beta..}}{CDR_{....}}$
$\mathfrak{A}$	$a_\alpha = 0$	$F'^*(\mathfrak{A})$ 1)	$F'^*(\mathfrak{A})$ 1)
Line "$\mathfrak{B}$" analogous to line "$\mathfrak{A}$": Interchange respective symbols.			
$\mathcal{C}$	$\sigma^2_c = 0$	$F = MS(\mathcal{C})/MS(\mathcal{C}\mathfrak{D})$	$F = MS(\mathcal{C})/MS(\mathcal{C}\mathfrak{D})$
$\mathfrak{D}$	$\sigma^2_d = 0$	$F'(\mathfrak{D})$ 2)	$F = MS(\mathfrak{D})/MS(\mathcal{C}\mathfrak{D})$
$\mathfrak{A}\mathfrak{B}$	$ab_{\alpha\beta} = 0$	$F'^*(\mathfrak{A}\mathfrak{B})$ 3)	$F'^*(\mathfrak{A}\mathfrak{B})$ 3)
$\mathfrak{A}\mathcal{C}$	$\sigma^2_{ac} = 0$	$F^* = MS(\mathfrak{A}\mathcal{C})/MS(\mathfrak{A}\mathcal{C}\mathfrak{D})$	$F^* = MS(\mathfrak{A}\mathcal{C})/MS(\mathfrak{A}\mathcal{C}\mathfrak{D})$
$\mathfrak{A}\mathfrak{D}$	$\sigma^2_{ad} = 0$	$F'^*(\mathfrak{A}\mathfrak{D})$ 4)	$F^* = MS(\mathfrak{A}\mathfrak{D})/MS(\mathfrak{A}\mathcal{C}\mathfrak{D})$
$\mathfrak{B}\mathcal{C}$	$\sigma^2_{bc} = 0$	$F^* = MS(\mathfrak{B}\mathcal{C})/MS(\mathfrak{B}\mathcal{C}\mathfrak{D})$	$F^* = MS(\mathfrak{B}\mathcal{C})/MS(\mathfrak{B}\mathcal{C}\mathfrak{D})$
$\mathfrak{B}\mathfrak{D}$	$\sigma^2_{bd} = 0$	$F'^*(\mathfrak{B}\mathfrak{D})$ 4)	$F^* = MS(\mathfrak{B}\mathfrak{D})/MS(\mathfrak{B}\mathcal{C}\mathfrak{D})$
$\mathcal{C}\mathfrak{D}$	$\sigma^2_{cd} = 0$	$F = MS(\mathcal{C}\mathfrak{D})/MS(\mathfrak{R})$	$F = MS(\mathcal{C}\mathfrak{D})/MS(\mathfrak{R})$
$\mathfrak{A}\mathfrak{B}\mathcal{C}$	$\sigma^2_{abc} = 0$	$F^* = MS(\mathfrak{A}\mathfrak{B}\mathcal{C})/MS(\mathfrak{A}\mathfrak{B}\mathcal{C}\mathfrak{D})$	$F^* = MS(\mathfrak{A}\mathfrak{B}\mathcal{C})/MS(\mathfrak{A}\mathfrak{B}\mathcal{C}\mathfrak{D})$
$\mathfrak{A}\mathfrak{B}\mathfrak{D}$	$\sigma^2_{abd} = 0$	$F'^*(\mathfrak{A}\mathfrak{B}\mathfrak{D})$ 5)	$F^* = MS(\mathfrak{A}\mathfrak{B}\mathfrak{D})/MS(\mathfrak{A}\mathfrak{B}\mathcal{C}\mathfrak{D})$
$\mathfrak{A}\mathcal{C}\mathfrak{D}$	$\sigma^2_{acd} = 0$	$F = MS(\mathfrak{A}\mathcal{C}\mathfrak{D})/MS(\mathfrak{R})$	$F = MS(\mathfrak{A}\mathcal{C}\mathfrak{D})/MS(\mathfrak{R})$
Line "$\mathfrak{B}\mathcal{C}\mathfrak{D}$" analogous to line "$\mathfrak{A}\mathcal{C}\mathfrak{D}$": Interchange respective symbols.			
$\mathfrak{A}\mathfrak{B}\mathcal{C}\mathfrak{D}$	$\sigma^2_{abcd} = 0$	$F = MS(\mathfrak{A}\mathfrak{B}\mathcal{C}\mathfrak{D})/MS(\mathfrak{R})$	$F = MS(\mathfrak{A}\mathfrak{B}\mathcal{C}\mathfrak{D})/MS(\mathfrak{R})$
$\mathfrak{R}$	—	—	—
		1) and 3): As in $[4.\mathcal{C}\mathfrak{D}.\alpha\beta\gamma\delta]$, page 60, with $k'''_d = \frac{1}{D}$. 2), 4), 5): Analogous (interchange respective symbols!) to $F'^*(\mathcal{C})$, $F'^*(\mathfrak{A}\mathcal{C})$, $F'^*(\mathfrak{A}\mathfrak{B}\mathcal{C})$, respectively, in $[4.\mathcal{C}\mathfrak{D}.\alpha\beta\gamma\delta]$, page 60.	1) and 3): As in $[4.\mathcal{C}\mathfrak{D}.\alpha\beta\gamma\delta]$, page 60, with $k'''_c = \frac{1}{C}$ and $k'''_d = \frac{1}{D}$.
		For mean square expectations put $k'''_d = \frac{1}{D}$ in those of $[4.\mathcal{C}\mathfrak{D}.\alpha\beta\gamma\delta]$, page 59.	For mean square expectations put $k'''_c = \frac{1}{C}$ and $k'''_d = \frac{1}{D}$ in those of $[4.\mathcal{C}\mathfrak{D}.\alpha\beta\gamma\delta]$, page 59.

$\boxed{4.\mathcal{C}\mathfrak{D}.\alpha\beta\gamma}$ + $\boxed{4.\mathcal{C}\mathfrak{D}.\alpha\beta}$

$[4.\mathcal{CD}.R]$ and $[4.\mathcal{CD}.1]$: $\mathcal{A}$ and $\mathcal{B}$ fixed, $\mathcal{C}$ and $\mathcal{D}$ random; $R_{\alpha\beta\gamma\delta} = R$ and $R_{\alpha\beta\gamma\delta} = 1$, respectively.

For definition of symbols used see [4-way classification. Table I, II and III], pages 45-47.

	EMS($[4.\mathcal{CD}.R]$)†	H_o	Testing H_o: $[4.\mathcal{CD}.R]$: $R_{\alpha\beta\gamma\delta} = R$	Testing H_o: $[4.\mathcal{CD}.1]$: $R_{\alpha\beta\gamma\delta} = 1$
$\mathcal{A}$	$\sigma_r^2 + \frac{RAB}{A-1}\sigma_{acd}^2 + \frac{RABD}{A-1}\sigma_{ac}^2 + \frac{RABC}{A-1}\sigma_{ad}^2 + \frac{RBCD}{A-1}\sum_{\alpha} a_{\alpha}^2$	$a_{\alpha} = 0$	$F'^{*}(\mathcal{A})$ 1)	
Line "$\mathcal{B}$" analogous to line "$\mathcal{A}$": Interchange respective symbols.				
$\mathcal{C}$	$\sigma_r^2 + RAB\sigma_{cd}^2 + RABD\sigma_c^2$	$\sigma_c^2 = 0$	$F = MS(\mathcal{C})/MS(\mathcal{CD})$	
Line "$\mathcal{D}$" analogous to line "$\mathcal{C}$": Interchange respective symbols.				
$\mathcal{AB}$	$\sigma_r^2 + \frac{RAB}{(A-1)(B-1)}\sigma_{abcd}^2 + \frac{RABD}{(A-1)(B-1)}\sigma_{abc}^2 + \frac{RABC}{(A-1)(B-1)}\sigma_{abd}^2 + \frac{RCD}{(A-1)(B-1)}\sum_{\alpha}\sum_{\beta} ab_{\alpha\beta}^2$	$ab_{\alpha\beta} = 0$	$F'^{*}(\mathcal{AB})$ 2)	
$\mathcal{AC}$	$\sigma_r^2 + \frac{RAB}{A-1}\sigma_{acd}^2 + \frac{RABD}{A-1}\sigma_{ac}^2$		$F^{*} = MS(\mathcal{AC})/MS(\mathcal{ACD})$	
Lines "$\mathcal{AD}$", "$\mathcal{BC}$" and "$\mathcal{BD}$" analogous to line "$\mathcal{AC}$": Interchange respective symbols.				
$\mathcal{CD}$	$\sigma_r^2 + RAB\sigma_{cd}^2$	$\sigma_{cd}^2 = 0$	$F = \frac{MS(\mathcal{CD})}{MS(\mathcal{R})}$	If $\sigma_{abcd}^2 = 0$ can be assumed: $F = MS(\mathcal{CD})/MS(\mathcal{ABCD})$
$\mathcal{ABC}$	$\sigma_r^2 + \frac{RAB}{(A-1)(B-1)}\sigma_{abcd}^2 + \frac{RABD}{(A-1)(B-1)}\sigma_{abc}^2$	$\sigma_{abc}^2 = 0$	$F^{*} = MS(\mathcal{ABC})/MS(\mathcal{ABCD})$	
Line "$\mathcal{ABD}$" analogous to line "$\mathcal{ABC}$": Interchange respective symbols.				
$\mathcal{ACD}$	$\sigma_r^2 + \frac{RAB}{A-1}\sigma_{acd}^2$	$\sigma_{acd}^2 = 0$	$F = \frac{MS(\mathcal{ACD})}{MS(\mathcal{R})}$	If $\sigma_{abcd}^2 = 0$ can be assumed: $F = MS(\mathcal{ACD})/MS(\mathcal{ABCD})$
Line "$\mathcal{BCD}$" analogous to line "$\mathcal{ACD}$": Interchange respective symbols.				
$\mathcal{ABCD}$	$\sigma_r^2 + \frac{RAB}{(A-1)(B-1)}\sigma_{abcd}^2$	$\sigma_{abcd}^2 = 0$	$F = \frac{MS(\mathcal{ABCD})}{MS(\mathcal{R})}$	Not possible
$\mathcal{R}$	σ_r^2	—	1) and 2): See next page.	

†) For EMS of $[4.\mathcal{CD}.1]$ put $R = 1$ and delete line "$\mathcal{R}$".

$4.\mathcal{CD}.R$ + $4.\mathcal{CD}.1$

[4.$\mathcal{CD}$.R] and [4.$\mathcal{CD}$.1] continued.

1) $F'^{*}(\mathcal{A}) = \mathrm{MS}_1(\mathcal{A})/\mathrm{MS}_2(\mathcal{A})$, where

$$\mathrm{MS}_1(\mathcal{A}) = \mathrm{MS}(\mathcal{A}) + \mathrm{MS}(\mathcal{ACD})$$

$$\mathrm{MS}_2(\mathcal{A}) = \mathrm{MS}(\mathcal{AC}) + \mathrm{MS}(\mathcal{AD})$$

with effective degrees of freedom for $\mathrm{MS}_1(\mathcal{A})$ and $\mathrm{MS}_2(\mathcal{A})$, respectively:

$$f_1 = [\mathrm{MS}_1(\mathcal{A})]^2 \Big/ \left\{ \frac{1}{A-1}[\mathrm{MS}(\mathcal{A})]^2 + \frac{1}{(A-1)(C-1)(D-1)}[\mathrm{MS}(\mathcal{ACD})]^2 \right\}$$

$$f_2 = [\mathrm{MS}_2(\mathcal{A})]^2 \Big/ \left\{ \frac{1}{(A-1)(C-1)}[\mathrm{MS}(\mathcal{AC})]^2 + \frac{1}{(A-1)(D-1)}[\mathrm{MS}(\mathcal{AD})]^2 \right\}$$

2) $F'^{*}(\mathcal{AB}) = \mathrm{MS}_1(\mathcal{AB})/\mathrm{MS}_2(\mathcal{AB})$ as in [4.$\mathcal{ABCD}$.R] + [4.$\mathcal{ABCD}$.1], page 53.

[4.$\mathcal{CD}$.R] + [4.$\mathcal{CD}$.1] continued

$[4.\mathfrak{D}.a\beta\gamma\delta]$ and $[4.\mathfrak{D}.a\beta\gamma]$: $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ fixed, $\mathfrak{D}$ random; $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha\ldots}R_{.\beta..}R_{..\gamma.}R_{...\delta}}{R^3_{....}}$ and $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha\ldots}R_{.\beta..}R_{..\gamma.}}{DR^2_{....}}$, respectively.

For definition of symbols used see [4-way classification. Table I, II and III], pages 45-47.

	EMS ($[4.\mathfrak{D}.a\beta\gamma\delta]$)†	H_o	Testing H_o: $[4.\mathfrak{D}.a\beta\gamma\delta]$: $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha\ldots}R_{.\beta..}R_{..\gamma.}R_{...\delta}}{R^3_{....}}$	Testing H_o: $[4.\mathfrak{D}.a\beta\gamma]$, $\left(R_{...\delta} = \frac{R_{....}}{D}\right)$: $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha\ldots}R_{.\beta..}R_{..\gamma.}}{DR^2_{....}}$
$\mathfrak{A}$	$\sigma_r^2 + \frac{R_{....}k_d^{m}}{A-1}\sigma^2_{ad} + \frac{1}{A-1}\sum_{\alpha} R_{\alpha\ldots}a_\alpha^2$	$a_\alpha = 0$	$F'^*(\mathfrak{A})$ [1]	$F^* = \frac{MS(\mathfrak{A})}{MS(\mathfrak{A}\mathfrak{D})}$
Lines "$\mathfrak{B}$" and "$\mathfrak{C}$" analogous to line "$\mathfrak{A}$" : Interchange respective symbols.				
$\mathfrak{D}$	$\sigma_r^2 + R_{....}\frac{1-k_d^{m}}{D-1}\sigma_d^2$	$\sigma_d^2 = 0$	$F = MS(\mathfrak{D})/MS(\mathfrak{R})$	
$\mathfrak{A}\mathfrak{B}$	$\sigma_r^2 + \frac{R_{....}k_d^{m}}{(A-1)(B-1)}\sigma^2_{abd} + \frac{1}{(A-1)(B-1)}\sum_\alpha\sum_\beta R_{\alpha\beta..}ab^2_{\alpha\beta}$	$ab_{\alpha\beta} = 0$	$F'^*(\mathfrak{A}\mathfrak{B})$ [2]	$F^* = \frac{MS(\mathfrak{A}\mathfrak{B})}{MS(\mathfrak{A}\mathfrak{B}\mathfrak{D})}$
Lines "$\mathfrak{A}\mathfrak{C}$" and "$\mathfrak{B}\mathfrak{C}$" analogous to line "$\mathfrak{A}\mathfrak{B}$" : Interchange respective symbols.				
$\mathfrak{A}\mathfrak{D}$	$\sigma_r^2 + \frac{R_{....}(1-k_d^{m})}{(A-1)(D-1)}\sigma^2_{ad}$	$\sigma^2_{ad} = 0$	$F = MS(\mathfrak{A}\mathfrak{D})/MS(\mathfrak{R})$	
Lines "$\mathfrak{B}\mathfrak{D}$" and "$\mathfrak{C}\mathfrak{D}$" analogous to line "$\mathfrak{A}\mathfrak{D}$" : Interchange respective symbols.				
$\mathfrak{A}\mathfrak{B}\mathfrak{C}$	$\sigma_r^2 + \frac{R_{....}k_d^{m}}{(A-1)(B-1)(C-1)}\sigma^2_{abcd} + \frac{1}{(A-1)(B-1)(C-1)}\sum_\alpha\sum_\beta\sum_\gamma R_{\alpha\beta\gamma.}abc^2_{\alpha\beta\gamma}$	$abc_{\alpha\beta\gamma} = 0$	$F'^*(\mathfrak{A}\mathfrak{B}\mathfrak{C})$ [3]	$F^* = \frac{MS(\mathfrak{A}\mathfrak{B}\mathfrak{C})}{MS(\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D})}$
$\mathfrak{A}\mathfrak{B}\mathfrak{D}$	$\sigma_r^2 + \frac{R_{....}(1-k_d^{m})}{(A-1)(B-1)(D-1)}\sigma^2_{abd}$	$\sigma^2_{abd} = 0$	$F = MS(\mathfrak{A}\mathfrak{B}\mathfrak{D})/MS(\mathfrak{R})$	
Lines "$\mathfrak{A}\mathfrak{C}\mathfrak{D}$" and "$\mathfrak{B}\mathfrak{C}\mathfrak{D}$" analogous to line "$\mathfrak{A}\mathfrak{B}\mathfrak{D}$" : Interchange respective symbols.				
$\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}$	$\sigma_r^2 + \frac{R_{....}(1-k_d^{m})}{(A-1)(B-1)(C-1)(D-1)}\sigma^2_{abcd}$	$\sigma^2_{abcd} = 0$	$F = MS(\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D})/MS(\mathfrak{R})$	
$\mathfrak{R}$	σ_r^2	———	1)–3) : See next page.	

†) For EMS of $[4.\mathfrak{D}.a\beta\gamma]$ put $k_d^{m} = \frac{1}{D}$.

$\boxed{4.\mathfrak{D}.a\beta\gamma\delta}$ + $\boxed{4.\mathfrak{D}.a\beta\gamma}$

$[4.\mathcal{D}.\alpha\beta\gamma\delta]$ continued

1) $F'^{*}(\mathcal{A}) \quad = \mathrm{MS}_1(\mathcal{A}) / \mathrm{MS}(\mathcal{A}\mathcal{D})$, where

$$\mathrm{MS}_1(\mathcal{A}) = \frac{(1-k_d'''')\mathrm{MS}(\mathcal{A})}{(D-1)k_d''''} + \frac{(Dk_d''''-1)\mathrm{MS}(\mathcal{R})}{(D-1)k_d''''}$$

with effective degrees of freedom

$$f_1 = [\mathrm{MS}_1(\mathcal{A})]^2 \Bigg/ \left\{ \frac{\left[\dfrac{(1-k_d'''')\mathrm{MS}(\mathcal{A})}{(D-1)k_d''''}\right]^2}{A-1} + \frac{\left[\dfrac{(Dk_d''''-1)\mathrm{MS}(\mathcal{R})}{(D-1)k_d''''}\right]^2}{R\ldots\ldots ABCD} \right\}$$

2) $F'^{*}(\mathcal{A}\mathcal{B}) \quad = \mathrm{MS}_1(\mathcal{A}\mathcal{B}) / \mathrm{MS}(\mathcal{A}\mathcal{B}\mathcal{D})$, where

$$\mathrm{MS}_1(\mathcal{A}\mathcal{B}) = \frac{(1-k_d'''')\mathrm{MS}(\mathcal{A}\mathcal{B})}{(D-1)k_d''''} + \frac{(Dk_d''''-1)\mathrm{MS}(\mathcal{R})}{(D-1)k_d''''}$$

with effective degrees of freedom

$$f_1 = [\mathrm{MS}_1(\mathcal{A}\mathcal{B})]^2 \Bigg/ \left\{ \frac{\left[\dfrac{(1-k_d'''')\mathrm{MS}(\mathcal{A}\mathcal{B})}{(D-1)k_d''''}\right]^2}{(A-1)(B-1)} + \frac{\left[\dfrac{(Dk_d''''-1)\mathrm{MS}(\mathcal{R})}{(D-1)k_d''''}\right]^2}{R\ldots\ldots ABCD} \right\}$$

3) $F'^{*}(\mathcal{A}\mathcal{B}\mathcal{C}) = \mathrm{MS}_1(\mathcal{A}\mathcal{B}\mathcal{C}) / \mathrm{MS}(\mathcal{A}\mathcal{B}\mathcal{C}\mathcal{D})$ as in $[4.\mathcal{A}\mathcal{B}\mathcal{C}\mathcal{D}.\alpha\beta\gamma\delta]$, page 50.

[4.$\mathfrak{D}$.R] and [4.$\mathfrak{D}$.1]: $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ fixed, $\mathfrak{D}$ random; $R_{\alpha\beta\gamma\delta} \equiv R$ and $R_{\alpha\beta\gamma\delta} \equiv 1$, respectively.
For definition of symbols used see [4-way classification. Table I, II and III], pages 45-47.

	EMS ([4.$\mathfrak{D}$.R])†	H_o	Testing H_o	
			[4.$\mathfrak{D}$.R]: $R_{\alpha\beta\gamma\delta} \equiv R$	[4.$\mathfrak{D}$.1]: $R_{\alpha\beta\gamma\delta} \equiv 1$
$\mathfrak{A}$	$\sigma_r^2 + \frac{RABC}{A-1}\sigma_{ad}^2 + \frac{RBCD}{A-1}\sum_\alpha a_\alpha^2$	$a_\alpha \equiv 0$	$F^* = \mathrm{MS}(\mathfrak{A})/\mathrm{MS}(\mathfrak{AD})$	
Lines "$\mathfrak{B}$" and "$\mathfrak{C}$" analogous to line "$\mathfrak{A}$": Interchange respective symbols.				
$\mathfrak{D}$	$\sigma_r^2 + RABC\ \sigma_d^2$	$\sigma_d^2 = 0$	$F = \frac{\mathrm{MS}(\mathfrak{D})}{\mathrm{MS}(\mathfrak{R})}$	If $\sigma_{abcd}^2 = 0$ can be assumed: $F = \mathrm{MS}(\mathfrak{D})/\mathrm{MS}(\mathfrak{ABCD})$
$\mathfrak{AB}$	$\sigma_r^2 + \frac{RABC}{(A-1)(B-1)}\sigma_{abd}^2 + \frac{RCD}{(A-1)(B-1)}\sum_\alpha\sum_\beta ab_{\alpha\beta}^2$	$ab_{\alpha\beta} \equiv 0$	$F^* = \mathrm{MS}(\mathfrak{AB})/\mathrm{MS}(\mathfrak{ABD})$	
Lines "$\mathfrak{AC}$" and "$\mathfrak{BC}$" analogous to line "$\mathfrak{AB}$": Interchange respective symbols.				
$\mathfrak{AD}$	$\sigma_r^2 + \frac{RABC}{A-1}\sigma_{ad}^2$	$\sigma_{ad}^2 = 0$	$F = \frac{\mathrm{MS}(\mathfrak{AD})}{\mathrm{MS}(\mathfrak{R})}$	If $\sigma_{abcd}^2 = 0$ can be assumed: $F = \mathrm{MS}(\mathfrak{AD})/\mathrm{MS}(\mathfrak{ABCD})$
Lines "$\mathfrak{BD}$" and "$\mathfrak{CD}$" analogous to line "$\mathfrak{AD}$": Interchange respective symbols.				
$\mathfrak{ABC}$	$\sigma_r^2 + \frac{RABC}{(A-1)(B-1)(C-1)}\sigma_{abcd}^2 + \frac{RD}{(A-1)(B-1)(C-1)}\sum_\alpha\sum_\beta\sum_\gamma abc_{\alpha\beta\gamma}^2$	$abc_{\alpha\beta\gamma} \equiv 0$	$F^* = \mathrm{MS}(\mathfrak{ABC})/\mathrm{MS}(\mathfrak{ABCD})$	
$\mathfrak{ABD}$	$\sigma_r^2 + \frac{RABC}{(A-1)(B-1)}\sigma_{abd}^2$	$\sigma_{abd}^2 = 0$	$F = \frac{\mathrm{MS}(\mathfrak{ABD})}{\mathrm{MS}(\mathfrak{R})}$	If $\sigma_{abcd}^2 = 0$ can be assumed: $F = \mathrm{MS}(\mathfrak{ABD})/\mathrm{MS}(\mathfrak{ABCD})$
Lines "$\mathfrak{ACD}$" and "$\mathfrak{BCD}$" analogous to line "$\mathfrak{ABD}$": Interchange respective symbols.				
$\mathfrak{ABCD}$	$\sigma_r^2 + \frac{RABC}{(A-1)(B-1)(C-1)}\sigma_{abcd}^2$	$\sigma_{abcd}^2 = 0$	$F = \frac{\mathrm{MS}(\mathfrak{ABCD})}{\mathrm{MS}(\mathfrak{R})}$	Not possible
$\mathfrak{R}$	σ_r^2	—	—	—

†For EMS of [4.$\mathfrak{D}$.1] put $R = 1$ and delete line "$\mathfrak{R}$".

4.$\mathfrak{D}$.R + 4.$\mathfrak{D}$.1

[4..$\alpha\beta\gamma\delta$] and [4..1]: $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ fixed; $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha\ldots}R_{.\beta..}R_{..\gamma.}R_{\ldots\delta}}{R^3_{\ldots.}}$ and $R_{\alpha\beta\gamma\delta} \equiv 1$, respectively.

For definition of symbols used see [4-way classification. Table I, II and III], pages 45-47.

	EMS([4..$\alpha\beta\gamma\delta$])†	H_o	Testing H_o: [4..$\alpha\beta\gamma\delta$]: $R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha\ldots}R_{.\beta..}R_{..\gamma.}R_{\ldots\delta}}{R^3_{\ldots.}}$	Testing H_o: [4..1]: $R_{\alpha\beta\gamma\delta} \equiv 1$
$\mathfrak{A}$	$\sigma_r^2 + \frac{1}{A-1}\sum_\alpha R_{\alpha\ldots}a_\alpha^2$	$a_\alpha \equiv 0$	$F = \frac{MS(\mathfrak{A})}{MS(\mathfrak{R})}$	If $abcd_{\alpha\beta\gamma\delta} \equiv 0$ can be assumed: $F = MS(\mathfrak{A})/MS(\mathfrak{ABCD})$
Lines "$\mathfrak{B}$", "$\mathfrak{C}$" and "$\mathfrak{D}$" analogous to line "$\mathfrak{A}$": Interchange respective symbols.				
$\mathfrak{AB}$	$\sigma_r^2 + \frac{1}{(A-1)(B-1)}\sum_\alpha\sum_\beta R_{\alpha\beta..}ab_{\alpha\beta}^2$	$ab_{\alpha\beta} \equiv 0$	$F = \frac{MS(\mathfrak{AB})}{MS(\mathfrak{R})}$	If $abcd_{\alpha\beta\gamma\delta} \equiv 0$ can be assumed: $F = MS(\mathfrak{AB})/MS(\mathfrak{ABCD})$
Lines "$\mathfrak{AC}$", "$\mathfrak{AD}$", "$\mathfrak{BC}$", "$\mathfrak{BD}$" and "$\mathfrak{CD}$" analogous to line "$\mathfrak{AB}$": Interchange respective symbols.				
$\mathfrak{ABC}$	$\sigma_r^2 + \frac{1}{(A-1)(B-1)(C-1)}\sum_\alpha\sum_\beta\sum_\gamma R_{\alpha\beta\gamma.}abc_{\alpha\beta\gamma}^2$	$abc_{\alpha\beta\gamma} \equiv 0$	$F = \frac{MS(\mathfrak{ABC})}{MS(\mathfrak{R})}$	If $abcd_{\alpha\beta\gamma\delta} \equiv 0$ can be assumed: $F = MS(\mathfrak{ABC})/MS(\mathfrak{ABCD})$
Lines "$\mathfrak{ABD}$", "$\mathfrak{ACD}$" and "$\mathfrak{BCD}$" analogous to line "$\mathfrak{ABC}$": Interchange respective symbols.				
$\mathfrak{ABCD}$	$\sigma_r^2 + \frac{1}{(A-1)(B-1)(C-1)(D-1)}\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta R_{\alpha\beta\gamma\delta}abcd_{\alpha\beta\gamma\delta}^2$	$abcd_{\alpha\beta\gamma\delta} \equiv 0$	$F = \frac{MS(\mathfrak{ABCD})}{MS(\mathfrak{R})}$	Not possible
$\mathfrak{R}$	σ_r^2	—	—	—

†) For EMS of [4..1] put $R_{\alpha\beta\gamma\delta} \equiv 1$ and delete line "$\mathfrak{R}$".

4..$\alpha\beta\gamma\delta$	+	4..1

APPENDIX A

Description of the method used for the derivations of mean square expectations.

1. Definition of a generalized expectation.

In deriving the mean square expectations as they are listed in this report use was made of a generalized expectation which will be defined below. This expectation operation is applied to the "cell" responses which are "true" in the sense that they constitute one of the two additive components in the assumed underlying model for the analysis of variance. (The second component is the residual or error term from a normal population with expectation zero and variance σ_r^2.)

The generalized expectation (as well as the entire method) will be explained with the example of the two-way crossed classification. Here, one has the model

$$x_{\alpha\beta\rho} = X_{\alpha\beta} + r_{\alpha\beta\rho}$$

where $x_{\alpha\beta\rho}$ is the ρth observation in cell "$\alpha\beta$", or: the ρth observation at level α of factor $\mathfrak{A}$ and at level β of factor $\mathfrak{B}$, with $\alpha = 1, \ldots, A$, $\beta = 1, \ldots, B$ and $\rho = 1, \ldots, R_{\alpha\beta}$. $R_{\alpha\beta}$ is the number of replicated observations in cell "$\alpha\beta$" and is assumed to meet the proportionality condition $R_{\alpha\beta} = \dfrac{R_{\alpha.}R_{.\beta}}{R_{..}}$, (see Appendix B.)

$r_{\alpha\beta\rho}$ is the above-mentioned error term with the assumption $r_{\alpha\beta\rho} \sim \mathrm{NID}(0, \sigma_r^2)$.

Finally, $X_{\alpha\beta}$ denotes the "true" response of the random variable under consideration in cell "$\alpha\beta$", no matter whether random or fixed effects are represented by factors $\mathfrak{A}$ and $\mathfrak{B}$. In the following, $X_{\alpha\beta}$ and its generalized expectation is defined for the three cases: 1a. $\mathfrak{A}$ and $\mathfrak{B}$ random, 1b. $\mathfrak{A}$ and $\mathfrak{B}$ fixed, 1c. $\mathfrak{A}$ fixed, $\mathfrak{B}$ random.

1a. Case of both factors random

Here $X_{\alpha\beta}$ is defined as follows:

$$X_{\alpha\beta} = f(x,y) \Big|_{x = x_\alpha,\, y = y_\beta},$$

where $f(x,y)$ is a unique but unknown function of the two "carrier variables" x and y. These carrier variables are continuous but otherwise unspecified random variables, relating to the $\mathfrak{A}$- and $\mathfrak{B}$- classification, respectively, with joint probability density function $p(x,y)$. Thus, level α of factor $\mathfrak{A}$ is uniquely defined by the sampled value $x = x_\alpha$, and level β of factor $\mathfrak{B}$ correspondingly by the sampled value $y = y_\beta$. Now, using the notation $X_{(\alpha)\beta} = f(x, y_\beta)$, the expectation of $X_{\alpha\beta}$ with respect to the carrier variable x for any given $y = y_\beta$ is defined as:

$$E\left[X_{(\alpha)\beta} \mid y = y_\beta\right] = \int_{-\infty}^{+\infty} f(x,y \mid y = y_\beta)\, p\,(x,y \mid y = y_\beta)\, dx \; .$$

The left hand side will be abbreviated to $\underset{\alpha}{E}\left[X_{\alpha\beta}\right]$ because the above operation is analogous to an averaging over subscript α for any given $y = y_\beta$. If further the marginal probability density function of y,

$$p_\beta(y) = \int_{-\infty}^{+\infty} p\,(x,y)dx$$

is introduced, one gets

$$\underset{\alpha}{E}\left[X_{\alpha\beta}\right] = \int_{-\infty}^{+\infty} f(x,y_\beta)\,\frac{p(x,y_\beta)}{p_\beta(y_\beta)}\, dx \; .$$

Because this will hold for any sampled value $y = y_\beta$, the definition can finally be written

$$\underset{\alpha}{E}\left[X_{\alpha\beta}\right] = \int_{-\infty}^{+\infty} f(x,y)\,\frac{p(x,y)}{p_\beta(y)}\, dx \; .$$

Correspondingly, one has

$$\underset{\beta}{E}\left[X_{\alpha\beta}\right] = \int_{-\infty}^{+\infty} f(x,y)\,\frac{p\,(x,y)}{p_\alpha(x)}\, dy \; .$$

The expectation of $X_{\alpha\beta}$ with respect to both carrier variables x and y is defined by attaching both subscripts α and β to the expectation symbol:

$$\underset{\alpha\beta}{E}\left[X_{\alpha\beta}\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)p\,(x,y)\,dxdy \; .$$

In the derivations of expected mean squares repeated use must be made of the following relation between the three above-defined expectations:

$$\underset{\alpha\beta}{E}\left[X_{\alpha\beta}\right] = \underset{\alpha}{E}\left[\underset{\beta}{E}\left[X_{\alpha\beta}\right]\right] = \underset{\beta}{E}\left[\underset{\alpha}{E}\left[X_{\alpha\beta}\right]\right] .$$

This property holds if the product $f(x,y)\,p(x,y)$ is continuous for $(-\infty < x < +\infty, -\infty < y < +\infty)$. This can readily be assumed, so that the relation will be generally valid.

<u>Proof</u>. One has

$$E\left[\underset{\beta}{E}\left[X_{\alpha\beta}\right]\right] = E\ [\varphi(x)],$$

where the random variable

$$\varphi(x) = \int_{-\infty}^{+\infty} f(x,y)\frac{p(x,y)}{p_\alpha(x)}\,dy$$

has probability density

$$p_\alpha(x) = \int_{-\infty}^{+\infty} p(x,y)dy \quad .$$

The expectation of $\varphi(x)$, being that with respect to the carrier variable x, will be expressed according to the present notation by attaching the subscript α to the expectation symbol:

$$\underset{\alpha}{E}\left[\varphi(x)\right] = \int_{-\infty}^{+\infty} \varphi(x)p_\alpha(x)dx$$

$$= \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(x,y)\frac{p(x,y)}{p_\alpha(x)}\,dy\right]p_\alpha(x)dx$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)p(x,y)dxdy,$$

because the sequence of integration can be interchanged if the integrand is everywhere continuous. The last term by definition is equal to $\underset{\alpha\beta}{E}\left[X_{\alpha\beta}\right]$, so that

$$\underset{\alpha}{E}\left[\underset{\beta}{E}\left[X_{\alpha\beta}\right]\right] = \underset{\alpha\beta}{E}\left[X_{\alpha\beta}\right] .$$

Correspondingly it can be shown that

$$\underset{\beta}{E}\left[\underset{\alpha}{E}\left[X_{\alpha\beta}\right]\right] = \underset{\alpha\beta}{E}\left[X_{\alpha\beta}\right] \quad , \quad \text{q. e. d.}$$

1b. Case of both factors fixed

Here the definition of $X_{\alpha\beta}$ is:

$$X_{\alpha\beta} = g(x_\alpha, y_\beta)$$

with x_α and y_β being discrete but otherwise unspecified carrier variables (non-random) which take on the values and only the values $x_1, \ldots, x_\alpha, \ldots, x_A$ and $y_1, \ldots, y_\beta, \ldots, y_B$, respectively. $g(x_\alpha, y_\beta)$ is another unique but unknown function of the carrier variables. Because these are non-random, there will be no possibility of taking the usual expectations of $X_{\alpha\beta}$ in this case. By analogy to the usual expectations and for later use in the model, however, the weighted averages of $X_{\alpha\beta}$ are formed, making use of the proportionality condition $R_{\alpha\beta} = \frac{R_{\alpha.}R_{.\beta}}{R_{..}}$:

$$\frac{\sum\limits_{\alpha} R_{\alpha\beta} g(x_\alpha, y_\beta)}{\sum\limits_{\alpha} R_{\alpha\beta}} = \frac{\sum\limits_{\alpha} R_{\alpha.} X_{\alpha\beta}}{R_{..}} ,$$

$$\frac{\sum\limits_{\beta} R_{\alpha\beta} g(x_\alpha, y_\beta)}{\sum\limits_{\beta} R_{\alpha\beta}} = \frac{\sum\limits_{\beta} R_{.\beta} X_{\alpha\beta}}{R_{..}} ,$$

$$\frac{\sum\limits_{\alpha}\sum\limits_{\beta} R_{\alpha\beta} g(x_\alpha, y_\beta)}{\sum\limits_{\alpha}\sum\limits_{\beta} R_{\alpha\beta}} = \frac{\sum\limits_{\alpha}\sum\limits_{\beta} R_{\alpha\beta} X_{\alpha\beta}}{R_{..}} .$$

1c. Case of one factor fixed ($\mathcal{A}$, say), the other ($\mathcal{B}$) random. (Mixed model).

As a combination of cases 1a and 1b, here $X_{\alpha\beta}$ is defined as:

$$X_{\alpha\beta} = h(x_\alpha, y)\Big|_{y = y_\beta}$$

where $y = y_\beta$ is a sampled value of the continuous random variable y and where x_α takes on the values and only the values $x_1, \ldots\ldots, x_\alpha, \ldots\ldots, x_A$. y has probability density function $p(y)$, say. $h(x_\alpha, y)$ is a third unique but unknown function of the carrier variables. Then the expectation with respect to y is defined to be

$$\underset{\beta}{E}\left[X_{\alpha\beta}\right] = \int_{-\infty}^{+\infty} h(x_\alpha, y)\, p(y)\, dy \quad .$$

With respect to the non-random variable x_α, the weighted average of $X_{\alpha\beta}$ is formed for later use in the model:

$$\frac{\sum_\alpha R_{\alpha\beta} h(x_\alpha, y)}{\sum_\alpha R_{\alpha\beta}} = \frac{\sum_\alpha R_{\alpha .} X_{\alpha\beta}}{R_{..}} \quad .$$

By analogy of taking expectations with respect to both carrier variables the following term is formed for later use:

$$\int_{-\infty}^{+\infty} \frac{\sum_\alpha R_{\alpha\beta} h(x_\alpha, y)}{\sum_\alpha R_{\alpha\beta}}\, p(y)\, dy = \frac{\sum_\alpha R_{\alpha .} \underset{\beta}{E}\left[X_{\alpha\beta}\right]}{R_{..}} \quad .$$

The expectations and their analogues as introduced above for the three models will be used for the definition of the model components (section 2, below).

In deriving the properties of the model components expectations will have to be taken also of the error term $r_{\alpha\beta\rho}$. In accordance with the notation used before, in these cases the subscript ρ will be attached to the expectation symbol. It is obvious that, for example,

$$\underset{\beta\rho}{E}\left[b_\beta r_{\alpha\beta\rho}\right] = \underset{\beta}{E}\left[\underset{\rho}{E}\left[b_\beta r_{\alpha\beta\rho}\right]\right] \quad .$$

In this connection it may also be mentioned that the expectation "with respect to ρ" applied to the actual observation $x_{\alpha\beta\rho}$ as expressed by the basic model, $x_{\alpha\beta\rho} = X_{\alpha\beta} + r_{\alpha\beta\rho}$, leads to the "true" cell response:

$$\underset{\rho}{E}\left[x_{\alpha\beta\rho}\right] = X_{\alpha\beta} \quad .$$

Finally, the definition $E\left[r^2_{\alpha\beta\rho}\right] = \sigma_r^2$ is used for all three models.

2. Definitions and properties of model components.

In the following, the definition and properties of the model components are exemplified with the two-way classification for the random, fixed and mixed effects case. The generalization for three- and more-way classifications can easily be made.

Generally (for all three models in the two-way classification), the true cell response $X_{\alpha\beta}$ will be split up into a general mean μ, plus the effect of level α of factor $\mathfrak{A}$, plus the effect of level β of factor $\mathfrak{B}$, plus the interaction effect $ab_{\alpha\beta}$. Therefore, the general model for the two-way classification will read:

$$x_{\alpha\beta\rho} = X_{\alpha\beta} + r_{\alpha\beta\rho} = \mu + a_\alpha + b_\beta + ab_{\alpha\beta} + r_{\alpha\beta\rho} \quad .$$

2a. Case of both factors random

Here the following definitions are made:

$$\mu = \underset{\alpha\beta}{E}\left[X_{\alpha\beta}\right]$$

$$a_\alpha = \underset{\beta}{E}\left[X_{\alpha\beta}\right] - \underset{\alpha\beta}{E}\left[X_{\alpha\beta}\right]$$

$$b_\beta = \underset{\alpha}{E}\left[X_{\alpha\beta}\right] - \underset{\alpha\beta}{E}\left[X_{\alpha\beta}\right]$$

$$ab_{\alpha\beta} = X_{\alpha\beta} - \underset{\beta}{E}\left[X_{\alpha\beta}\right] - \underset{\alpha}{E}\left[X_{\alpha\beta}\right] + \underset{\alpha\beta}{E}\left[X_{\alpha\beta}\right]$$

It will be noted that

$$\mu + a_\alpha + b_\beta + ab_{\alpha\beta} = X_{\alpha\beta}$$

Also it can be seen, inserting the above definitions of the model terms and making use of the theorem on the expectation operations as proven in section 1, that:

$$\underset{\alpha}{E}\left[a_\alpha\right] = 0$$

$$\underset{\beta}{E}\left[b_\beta\right] = 0$$

$$\underset{\alpha}{E}\left[ab_{\alpha\beta}\right] = \underset{\beta}{E}\left[ab_{\alpha\beta}\right] = \underset{\alpha\beta}{E}\left[ab_{\alpha\beta}\right] = 0.$$

In the same way, the following covariances are shown to be zero:

$$\underset{\alpha\beta}{E}\left[a_\alpha b_\beta\right] = \underset{\alpha}{E}\left[a_\alpha\right]\underset{\beta}{E}\left[b_\beta\right] = 0$$

$$\underset{\alpha\beta}{E}\left[a_\alpha ab_{\alpha\beta}\right] = \underset{\alpha}{E}\left[a_\alpha \underset{\beta}{E}\left[ab_{\alpha\beta}\right]\right] = 0$$

$$\underset{\alpha\beta\rho}{E}\left[a_\alpha r_{\alpha\beta\rho}\right] = \underset{\alpha\beta}{E}\left[a_\alpha \underset{\rho}{E}\left[r_{\alpha\beta\rho}\right]\right] = 0.$$

Correspondingly:

$$\underset{\alpha\beta}{E}\left[b_\beta ab_{\alpha\beta}\right] = \underset{\alpha\beta\rho}{E}\left[b_\beta r_{\alpha\beta\rho}\right] = \underset{\alpha\beta\rho}{E}\left[ab_{\alpha\beta} r_{\alpha\beta\rho}\right] = 0..$$

Considering the fact that the values x_α and y_β of the carrier variables x and y in

$$X_{\alpha\beta} = f(x,y)\Big|_{x = x_\alpha, y = y_\beta}$$

are randomly sampled from continuous distributions, the covariances $\underset{\alpha\neq\alpha'}{\mathrm{Cov}}(x_\alpha, x_{\alpha'})$ and $\underset{\beta\neq\beta'}{\mathrm{Cov}}(y_\beta, y_{\beta'})$ are zero by definition. This in turn implies that

$$E\left[a_\alpha a_{\alpha'}\right] = E\left[b_\beta b_{\beta'}\right] = 0$$

for $\alpha\neq\alpha'$ and $\beta\neq\beta'$, respectively, and

$$E\left[ab_{\alpha\beta}\, ab_{\alpha'\beta'}\right] = 0$$

for $\alpha\beta\neq\alpha'\beta'$.

In the last statements the expectation operation is the usual one, i.e., it is clear with respect to which random variables the expectations will have to be taken. Definitions of these expectation operations by attaching subscripts to the expectation symbols, analogous to the definitions introduced before, would lead to further but actually unnecessary formulas. The usual expectation symbol will also be applied in all further derivations of this appendix wherever its meaning is obvious.

Finally for this case of both factors random, the following definitions are used for the expectations of the squared model terms as they appear in the expectations of the mean squares:

$$E\left[a_\alpha^2\right] = \sigma_a{}^2$$

$$E\left[b_\beta^2\right] = \sigma_b{}^2$$

$$E\left[ab_{\alpha\beta}^2\right] = \sigma_{ab}^2$$

For the F-tests in the analysis of variance only, the assumption is made that the a_α, b_β and $ab_{\alpha\beta}$ are normally distributed. (That they are also independently distributed with expectations zero followed from their definitions as shown above.)

2b. Case of both factors fixed

For this case the model components are defined as follows:

$$\mu = \frac{\sum\limits_\alpha \sum\limits_\beta R_{\alpha\beta} X_{\alpha\beta}}{R_{\cdot\cdot}}$$

$$a_\alpha = \frac{\sum\limits_\beta R_{\cdot\beta} X_{\alpha\beta}}{R_{\cdot\cdot}} - \frac{\sum\limits_\alpha \sum\limits_\beta R_{\alpha\beta} X_{\alpha\beta}}{R_{\cdot\cdot}}$$

$$b_\beta = \frac{\sum\limits_\alpha R_{\alpha\cdot} X_{\alpha\beta}}{R_{\cdot\cdot}} - \frac{\sum\limits_\alpha \sum\limits_\beta R_{\alpha\beta} X_{u\beta}}{R_{\cdot\cdot}}$$

$$ab_{\alpha\beta} = X_{\alpha\beta} - \frac{\sum\limits_\beta R_{\cdot\beta} X_{\alpha\beta}}{R_{\cdot\cdot}} - \frac{\sum\limits_\alpha R_{\alpha\cdot} X_{\alpha\beta}}{R_{\cdot\cdot}} + \frac{\sum\limits_\alpha \sum\limits_\beta R_{\alpha\beta} X_{\alpha\beta}}{R_{\cdot\cdot}}$$

Again, one has

$$\mu + a_\alpha + b_\beta + ab_{\alpha\beta} = X_{\alpha\beta} \quad .$$

As can readily be verified, the following relations hold:

$$\sum_\alpha R_{\alpha.} a_\alpha = 0$$

$$\sum_\beta R_{.\beta} b_\beta = 0$$

$$\sum_\alpha R_{\alpha.} ab_{\alpha\beta} = \sum_\beta R_{.\beta} ab_{\alpha\beta} = 0 \quad .$$

2c. Case of one factor ($\mathfrak{A}$) fixed, the other ($\mathfrak{B}$) random. (Mixed model).

Making proper use of both types of definitions one has for this case:

$$\mu = \frac{\sum_\alpha R_{\alpha.} \underset{\beta}{E}\left[X_{\alpha\beta}\right]}{R_{..}}$$

$$a_\alpha = \underset{\beta}{E}\left[X_{\alpha\beta}\right] - \frac{\sum_\alpha R_{\alpha.} \underset{\beta}{E}\left[X_{\alpha\beta}\right]}{R_{..}}$$

$$b_\beta = \frac{\sum_\alpha R_{\alpha.} X_{\alpha\beta}}{R_{..}} - \frac{\sum_\alpha R_{\alpha.} \underset{\beta}{E}\left[X_{\alpha\beta}\right]}{R_{..}}$$

$$ab_{\alpha\beta} = X_{\alpha\beta} - \underset{\beta}{E}\left[X_{\alpha\beta}\right] - \frac{\sum_\alpha R_{\alpha.} X_{\alpha\beta}}{R_{..}} + \frac{\sum_\alpha R_{\alpha.} \underset{\beta}{E}\left[X_{\alpha\beta}\right]}{R_{..}}$$

As before,

$$\mu + a_\alpha + b_\beta + ab_{\alpha\beta} = X_{\alpha\beta} \quad .$$

It will also be noticed that a_α is no random variable, whereas b_β is one and $ab_{\alpha\beta}$ is one for any given α.

Here one has the following relations:

$$\sum_{\alpha} R_{\alpha.}\, a_{\alpha} = 0$$

$$\sum_{\alpha} R_{\alpha.}\, ab_{\alpha\beta} = 0$$

$$\underset{\beta}{E}\left[b_{\beta}\right] = 0$$

$$\underset{\beta}{E}\left[ab_{\alpha\beta}\right] = 0$$

It is of importance that $\sum_{\beta} R_{.\beta} b_{\beta} \neq 0$ and $\sum_{\beta} R_{.\beta} ab_{\alpha\beta} \neq 0$. Further it is worthwhile to mention that the term $\underset{\beta}{E}\left[b_{\beta} ab_{\alpha\beta}\right]$ does not appear in the derivations of expected mean squares for this case. Naturally,

$$\underset{\beta\cdot\rho}{E}\left[b_{\beta}\, r_{\alpha\beta\rho}\right] = \underset{\beta\cdot\rho}{E}\left[ab_{\alpha\beta}\, r_{\alpha\beta\rho}\right] = 0.$$

Finally, the variances are defined for this case as follows:

$$E\left[b_{\beta}^{2}\right] = \sigma_{b}{}^{2}$$

$$E\left[ab_{\alpha\beta}^{2}\right] = \sigma_{ab}^{2} \ ,$$

the latter under the assumption that $E\left[ab_{\alpha\beta}^{2}\right]$ is constant for all A levels of the fixed factor $\mathfrak{A}$.

For the F-tests only it is further assumed that the b_{β} and the $ab_{\alpha\beta}$ (the latter for all A levels of $\mathfrak{A}$) are normally distributed.

For this case of the mixed model only it will be demonstrated below that the differences between the estimates of any two main effects and the second differences between estimates of interaction effects are unbiased. (The same property can be shown to be valid for the estimates in the other two models and, equally, for those in three- and more-way classifications.)

As is well known, one gets as least squares estimates:

$$\hat{a}_\alpha = \bar{x}_{\alpha..} - \bar{x}_{...}$$

$$\hat{b}_\beta = \bar{x}_{.\beta.} - \bar{x}_{...}$$

$$\widehat{ab}_{\alpha\beta} = \bar{x}_{\alpha\beta.} - \bar{x}_{\alpha..} - \bar{x}_{.\beta.} + \bar{x}_{...}$$

The contrasts which have to be shown to be unbiased are consequently ($\alpha \neq \alpha'$, $\beta \neq \beta'$):

$$\hat{a}_\alpha - \hat{a}_{\alpha'} = \bar{x}_{\alpha..} - \bar{x}_{\alpha'..}$$

$$\hat{b}_\beta - \hat{b}_{\beta'} = \bar{x}_{.\beta.} - \bar{x}_{.\beta'.}$$

$$\widehat{ab}_{\alpha\beta} - \widehat{ab}_{\alpha'\beta} - \widehat{ab}_{\alpha\beta'} + \widehat{ab}_{\alpha'\beta'} = \bar{x}_{\alpha\beta.} - \bar{x}_{\alpha'\beta.} - \bar{x}_{\alpha\beta'.} + \bar{x}_{\alpha'\beta'.}$$

Making use of the model $x_{\alpha\beta\rho} = X_{\alpha\beta} + r_{\alpha\beta\rho}$ and of the fact that for the present mixed model the expectation of the first contrast has to be taken "with respect to β and ρ" and the expectations of the other two contrasts only "with respect to ρ", one gets:

$$\underset{\beta\rho}{E}\left[\hat{a}_\alpha - \hat{a}_{\alpha'}\right] = \underset{\beta\rho}{E}\left[\frac{\sum\limits_\beta \sum\limits_\rho \left(X_{\alpha\beta} + r_{\alpha\beta\rho}\right)}{R_{\alpha.}} - \frac{\sum\limits_\beta \sum\limits_\rho \left(X_{\alpha'\beta} + r_{\alpha'\beta\rho}\right)}{R_{\alpha'.}}\right]$$

$$= \underset{\beta}{E}\left[X_{\alpha\beta}\right] - \underset{\beta}{E}\left[X_{\alpha'\beta}\right]$$

$$= a_\alpha - a_{\alpha'} \;;$$

$$\underset{\rho}{E}\left[\hat{b}_\beta - \hat{b}_{\beta'}\right] = \underset{\rho}{E}\left[\frac{\sum\limits_\alpha \sum\limits_\rho \left(X_{\alpha\beta} + r_{\alpha\beta\rho}\right)}{R_{.\beta}} - \frac{\sum\limits_\alpha \sum\limits_\rho \left(X_{\alpha\beta'} + r_{\alpha\beta'\rho}\right)}{R_{.\beta'}}\right]$$

$$= \frac{\sum\limits_\alpha R_{\alpha.} X_{\alpha\beta}}{R_{..}} - \frac{\sum\limits_\alpha R_{\alpha.} X_{\alpha\beta'}}{R_{..}}$$

$$= b_\beta - b_{\beta'} \;;$$

and, correspondingly:

$$\underset{\rho}{E}\left[\hat{ab}_{\alpha\beta} - \hat{ab}_{\alpha'\beta} - \hat{ab}_{\alpha\beta'} + \hat{ab}_{\alpha'\beta'}\right] = X_{\alpha\beta} - X_{\alpha'\beta} - X_{\alpha\beta'} + X_{\alpha'\beta'}$$

$$= ab_{\alpha\beta} - ab_{\alpha'\beta} - ab_{\alpha\beta'} + ab_{\alpha'\beta'}$$

3. <u>Derivation of EMS-formulas.</u>

In this section the derivation of the formulas for the mean square expectations will be exemplified with the case of the mixed effects model, i.e., the case of factor $\mathfrak{A}$ fixed, factor $\mathfrak{B}$ random. This example will show all essential principles of the derivations which have been applied to obtain all expectations contained in the table of this report.

First, the average values of the actual observations $x_{\alpha\beta\rho}$ are expressed in terms of the underlying model,

$$x_{\alpha\beta\rho} = \mu + a_\alpha + b_\beta + ab_{\alpha\beta} + r_{\alpha\beta\rho} ,$$

using the dot-notation for the summation over a subscript:

$$\bar{x}_{\alpha\beta.} = \mu + a_\alpha + b_\beta + ab_{\alpha\beta} + \frac{r_{\alpha\beta.}}{R_{\alpha\beta}}$$

$$\bar{x}_{\alpha..} = \mu + a_\alpha + \frac{\sum_\beta R_{.\beta} b_\beta}{R_{..}} + \frac{\sum_\beta R_{.\beta} ab_{\alpha\beta}}{R_{..}} + \frac{r_{\alpha..}}{R_{\alpha.}}$$

$$\bar{x}_{.\beta.} = \mu + b_\beta + \frac{r_{.\beta.}}{R_{.\beta}}$$

$$\bar{x}_{...} = \mu + \frac{\sum_\beta R_{.\beta} b_\beta}{R_{..}} + \frac{r_{...}}{R_{..}}$$

(In these expressions use has already been made of the relations $\sum_\alpha R_{\alpha.} a_\alpha = 0$ and $\sum_\alpha R_{\alpha.} ab_{\alpha\beta} = 0$ as they were found in paragraph 2c, above.)

Substituting the above expressions for the averages in the mean square terms one gets:

$$E\left[\mathrm{MS}(\mathfrak{A})\right] = E\left[\frac{1}{A-1}\sum_a R_{a\cdot}(\bar{x}_{a\cdot\cdot} - \bar{x}_{\cdot\cdot\cdot})^2\right]$$

$$= E\left[\frac{1}{A-1}\sum_a R_{a\cdot}\left(a_a + \frac{\sum_\beta R_{\cdot\beta}\, ab_{a\beta}}{R_{\cdot\cdot}} + \frac{r_{a\cdot\cdot}}{R_{a\cdot}} - \frac{r_{\cdot\cdot\cdot}}{R_{\cdot\cdot}}\right)^2\right]$$

After performing the squaring and summation and in using the definitions and relations from paragraph 2c, above, one gets:

$$E\left[\mathrm{MS}(\mathfrak{A})\right] = \sigma_r^2 + \frac{1}{A-1}\frac{\sum_\beta R^2_{\cdot\beta}}{R_{\cdot\cdot}}\sigma_{ab}^2 + \frac{1}{A-1}\sum_a R_{a\cdot}a_a^2$$

$$= \sigma_r^2 + \frac{R_{\cdot\cdot}k_b''}{A-1}\sigma_{ab}^2 + \frac{1}{A-1}\sum_a R_{a\cdot}a_a^2 \quad ,$$

with $k_b'' = \frac{1}{R^2_{\cdot\cdot}}\sum_\beta R^2_{\cdot\beta}$.

The last form is that which is given for case $[2\mathfrak{B}\cdot a\beta]$ in the table of this report.

Further one has:

$$E\left[\mathrm{MS}(\mathfrak{B})\right] = \underset{\beta\rho}{E}\left[\frac{1}{B-1}\sum_\beta R_{\cdot\beta}(\bar{x}_{\cdot\beta\cdot} - \bar{x}_{\cdot\cdot\cdot})^2\right]$$

$$= \underset{\beta\rho}{E}\left[\frac{1}{B-1}\sum_\beta R_{\cdot\beta}\left(b_\beta - \frac{\sum_\beta R_{\cdot\beta}b_\beta}{R_{\cdot\cdot}} + \frac{r_{\cdot\beta\cdot}}{R_{\cdot\beta}} - \frac{r_{\cdot\cdot\cdot}}{R_{\cdot\cdot}}\right)^2\right]$$

$$= \sigma_r^2 + \frac{1}{B-1}\left(R_{\cdot\cdot} - \frac{\sum_\beta R^2_{\cdot\beta}}{R_{\cdot\cdot}}\right)\sigma_b^2$$

$$= \sigma_r^2 + \frac{R_{\cdot\cdot}(1-k_b'')}{B-1}\sigma_b^2 .$$

Correspondingly, the other two expectations are obtained:

$$E\left[\mathrm{MS}(\mathcal{A}\mathcal{B})\right] = \sigma_r^{\,2} + \frac{R_{\cdot\cdot\cdot}(1-k_b'')}{(A-1)(B-1)}\,\sigma_{ab}^2$$

$$E\left[\,\mathrm{MS}(\mathcal{R})\,\right] = \sigma_r^{\,2}$$

APPENDIX B

Proportionality conditions for cell numbers in crossed classifications.

In order to obtain an orthogonal decomposition of the total sum of squares in the analysis of variance for crossed classifications with unequal numbers of observations in the cells, all sums of products have to be identically zero. This only is achieved when the so-called proportionality condition holds for the cell numbers. The condition will be derived below for the two-way crossed classification; the generalization to three- and more-way classifications can easily be obtained.

In the notation of this report one has for the decomposition of the total sum of squares in a two-way classification with unequal cell numbers $R_{\alpha\beta}$:

$$\sum_{\alpha}\sum_{\beta}\sum_{\rho}(x_{\alpha\beta\rho} - \bar{x}_{...})^2 = \text{(orthogonal sums of squares)}$$
$$+ \text{(sums of products)}.$$

In the second part on the right hand side it is sufficient to consider the sum

$$2\sum_{\alpha}\sum_{\beta} R_{\alpha\beta}(\bar{x}_{\alpha..} - \bar{x}_{...})(\bar{x}_{.\beta.} - \bar{x}_{...}) .$$

The basic condition for this sum to be identically zero is that the two summations over α and β can be performed independently, i.e., it must be

$$R_{\alpha\beta} = R_1(\alpha)\,R_2(\beta) .$$

However, $\sum_{\alpha} R_1(\alpha)(\bar{x}_{\alpha..} - \bar{x}_{...})$, say, is identically zero only if $R_1(\alpha) = CR_{\alpha.}$, with C = const., because of

$$\sum_{\alpha} R_1(\alpha)(\bar{x}_{\alpha..} - \bar{x}_{...}) = C\sum_{\alpha} R_{\alpha.}\left(\frac{x_{\alpha..}}{R_{\alpha.}} - \frac{x_{...}}{R_{..}}\right) = 0 .$$

Then, from

$$R_{\alpha\beta} = C\,R_{\alpha.}\,R_2(\beta)$$

it follows that

$$R_{.\beta} = C\,R_{..}\,R_2(\beta) ,$$

or:

$$R_2(\beta) = \frac{R_{.\beta}}{CR_{..}} \quad .$$

Finally, therefore, the condition reads:

$$R_{\alpha\beta} = \frac{R_{\alpha.}R_{.\beta}}{R_{..}} \quad .$$

This, in fact, is a "proportionality condition" because it is equal to

$$\frac{R_{\alpha\beta}}{R_{\alpha.}} = \frac{R_{.\beta}}{R_{..}}$$, or in words:

"The number $R_{\alpha\beta}$ of observations in cell '$\alpha\beta$' must be to the marginal total $R_{\alpha.}$ as the marginal total $R_{.\beta}$ is to the total number $R_{..}$ of all observations."

Correspondingly, the conditions for the three- and four-way classifications can be shown to be, respectively:

$$R_{\alpha\beta\gamma} = \frac{R_{\alpha..}R_{.\beta.}R_{..\gamma}}{R_{...}^{\;2}}$$

and

$$R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha...}R_{.\beta..}R_{..\gamma.}R_{...\delta}}{R_{....}^{\;3}}$$

In general, one will have for the n-way classification:

$$R_{\varphi_1\varphi_2\cdots\varphi_n} = \frac{R_{\varphi_1(\cdots\cdot)}R_{\cdot\varphi_2(\cdots\cdot)}\cdots R_{(\cdots\cdot)\varphi_n}}{R^{n-1}_{(\cdots\cdot)}}$$

where $\varphi_1, \varphi_2, \ldots, \varphi_n$ denote the level subscripts for factors $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$, respectively.

APPENDIX C

DISTRIBUTION

Special Projects Office Department of the Navy Washington 25, D. C. Attention: SP-12	2
Bureau of Naval Weapons	
DLI-31	1
R-12 Attention: Dr. Lamar	1
R-14 Attention: Dr. Burington	1
RM-12 Attention: Dr. Tanczos	1
Commander Armed Services Technical Information Agency Arlington Hall Station Arlington 12, Virginia Attention: TIPDR	10
Commanding General Aberdeen Proving Ground Aberdeen, Maryland Attention: Technical Information Section Development and Proof Services	2
Commander, Operational Development Force U. S. Atlantic Fleet, U. S. Naval Base Norfolk 11, Virginia	1
Commander Naval Ordnance Laboratory White Oak, Maryland Attention: Technical Library	4
Chief, Bureau of Ships Department of the Navy Washington 25, D. C.	2
Director David Taylor Model Basin Washington 7, D. C. Attention: Library	2
Superintendent U. S. Naval Postgraduate School Monterey, California Attention: Library, Technical Reports Section	1

Chief of Naval Research Department of the Navy Washington 25, D. C. Attention: Logistics and Mathematical Statistics Branch Attention: Technical Library	 2 2
Commanding Officer Office of Naval Research Branch Office The John Crerar Library Building 86 Randolph Street Chicago 1, Illinois	 1
Commanding Officer Office of Naval Research Branch Office 346 Broadway New York 13, New York	 1
Commanding Officer Office of Naval Research Branch Office 1030 Geary Street San Francisco 9, California	 1
Commanding Officer Office of Naval Research Branch Office Navy #100, Fleet Post Office New York, New York	 2
Director Naval Research Laboratory Washington 25, D. C. Attention: Code 2021 (Library)	 5
Office of Technical Services Department of Commerce Washington 25, D. C.	 1
Commander U. S. Naval Ordnance Test Station China Lake, California Attention: Code 753 (Library Div.)	 3
Commandant U. S. Coast Guard 1300 E Street, N. W. Washington, D. C.	 1

Superintendent U. S. Naval Academy Annapolis, Maryland Attention: Library	1
Director of Research National Aeronautics and Space Administration 1512 H Street, N. W. Washington 25, D. C.	3
National Science Foundation 1520 H Street, N. W. Washington, D. C. Attention: Mathematical Sciences Division	1
Director National Bureau of Standards Washington 25, D. C. Attention: Dr. C. Eisenhart Attention: Technical Library	1 2
U. S. Naval Observatory Washington 25, D. C. Attention: Dr. G. M. Clemence Attention: Technical Library	1 1
Bureau of Naval Weapons Representative P.O. Box 504 Sunnyvale, California	1
Commander Naval Ordnance Test Station Pasadena Annex 3202 Foothill Boulevard Pasadena, California	1
Commanding General Army Ballistic Missile Agency Redstone Arsenal Huntsville, Alabama Attention: Technical Library	2
Lockheed Aircraft Corporation Missiles and Space Division Sunnyvale, California	1

RAND Corporation 1700 Main Street Santa Monica, California Attention: Dr. G. B. Dantzig Attention: Librarian	 1 2
Johns Hopkins University Operations Research Office 6935 Arlington Road Bethesda 14, Maryland	 1
Applied Physics Laboratory Johns Hopkins University Silver Spring, Maryland Attention: Librarian	 1
Professor K. A. Brownlee Statistical Research Center Eckhart Hall University of Chicago Chicago 37, Illinois	 1
Professor W. G. Cochran Statistics Department Harvard University Cambridge 38, Massachusetts	 1
Professor John W. Tukey Princeton University Box 708, Fine Hall Princeton, New Jersey	 1
Lockheed Propulsion Company P. O. Box 111 Redlands, California Attention: Mr. M. M. Goff	 1
Los Alamos Scientific Laboratory University of California P. O. Box 1663 Los Alamos, New Mexico Attention: Dr. R. H. Moore	 1
Commanding Officer Naval Radiological Defense Laboratory San Francisco, California Attention: Technical Library	 2

Commanding Officer U. S. Naval Explosive Ordnance Disposal Facility U. S. Naval Propellant Plant Indian Head, Maryland Attention: Technical Library	2
American Statistical Association 810 18th Street, N. W. Washington 6, D. C.	1
Commanding Officer Atlantic Missile Range Cape Canaveral, Florida	1
Commander Pacific Missile Range Point Mugu, California	1
Lockheed Missile and Space Division Palo Alto California	1
Jet Propulsion Laboratory California Institute of Technology 4800 Oak Grove Drive Pasadena, California	1
The Bureau of the Census Washington 25, D. C. Attention: Technical Library	2
Statistical Laboratory Iowa State University Ames, Iowa Attention: Professor O. Kempthorne Attention: Library	 1 2
Research Triangle Institute Durham, North Carolina Attention: Professor Gertrude M. Cox Attention: Professor H. Cramér Attention: Librarian	 1 1 2

Agricultural Experiment Station University of Florida Gainesville, Florida	
Attention: Professor A. E. Brandt	1
Attention: Library	2
Statistics Department University of North Carolina Chapel Hill, North Carolina	
Attention: Professor H. Hotelling	1
Attention: Library	2
Applied Mathematics and Statistics Laboratory Stanford University Stanford, California	
Attention: Professor A. H. Bowker	1
Attention: Professor G. J. Lieberman	1
Attention: Library	2
Institute of Statistics North Carolina State College Raleigh, North Carolina	
Attention: Professor R. L. Anderson	1
Attention: Library	2
Department of Statistics University of California Berkeley 4, California	
Attention: Professor J. Neyman	1
Attention: Professor H. Scheffé	1
Attention: Library	2
Statistical Laboratory Case Institute of Technology 10900 Euclid Avenue Cleveland 6, Ohio	
Attention: Dr. F. C. Leone	1
Attention: Librarian	2
Professor W. H. Kruskal Department of Statistics University of Chicago Chicago 37, Illinois	1

Local:

D	1
T	1
K	1
K-1	1
K-3	1
KP	1
KR	1
KRM	250
KRT	1
KRW	1
KXF	1
KXK	1
KYD	1
KYD-1	1
WWI	1
ACL	5

LIBRARY CATALOGING INPUT
PRNC-NWL-5070/15 (7-62)

BIBLIOGRAPHIC INFORMATION

DESCRIPTOR	CODE	DESCRIPTOR	CODE
SOURCE NAVAL WEAPONS LABORATORY	NPGA	SECURITY CLASSIFICATION AND CODE COUNT UNCLASSIFIED	U019
REPORT NUMBER 1833	1833	CIRCULATION LIMITATION	
REPORT DATE 18 OCTOBER 1962	1062	CIRCULATION LIMITATION OR BIBLIOGRAPHIC	
		BIBLIOGRAPHIC (*Suppl., Vol., etc.*)	

SUBJECT ANALYSIS OF REPORT

DESCRIPTOR	CODE	DESCRIPTOR	CODE	DESCRIPTOR	CODE
PROPORTIONAL	PRTI	DERIVATION	DERI		
CELL	CELL	RATIOS	RATI		
NUMBERS	NUMB	STATISTICS	STAT		
RANDOM	RAND	ORTHOGONAL	ORTG		
MIXED	MIXE	PROBLEM	PRBL		
FIXED	FIXE				
MODELS	MODE				
TABLE	TABL				
MEAN	AVER				
SQUARES	SQAR				
ANALYSIS	ANAL				
VARIANCE	VARB				
CROSSED	CROS				
CLASSIFICATIONS	CLAS				

Naval Weapons Laboratory, Dahlgren, Virginia.
(NWL Report No. 1833)
TABLE OF EXPECTATIONS OF MEAN SQUARES IN THE ANALYSIS OF VARIANCE FOR CROSSED CLASSIFICATIONS, by K. Abt. 18 Oct 1962. 67, 16 p.
UNCLASSIFIED

This report contains a table of the mean square expectations and the variance ratios for testing all nullhypotheses in the analysis of variance for the two-way, three-way and four-way crossed classifications with both proportional (unequal) and equal cell numbers. Rules are given to construct the formulas for the general n-way classification. The formulas are listed for all models (random, mixed, fixed). The report also contains a discussion of the table and an outline of the method which was used to derive the formulas.

1. Proportional cell numbers
2. Random, mixed, fixed models
3. Variance ratios
I. Abt, K.
II. Title

UNCLASSIFIED

Naval Weapons Laboratory, Dahlgren, Virginia.
(NWL Report No. 1833)
TABLE OF EXPECTATIONS OF MEAN SQUARES IN THE ANALYSIS OF VARIANCE FOR CROSSED CLASSIFICATIONS, by K. Abt. 18 Oct 1962. 67, 16 p.
UNCLASSIFIED

This report contains a table of the mean square expectations and the variance ratios for testing all nullhypotheses in the analysis of variance for the two-way, three-way and four-way crossed classifications with both proportional (unequal) and equal cell numbers. Rules are given to construct the formulas for the general n-way classification. The formulas are listed for all models (random, mixed, fixed). The report also contains a discussion of the table and an outline of the method which was used to derive the formulas.

1. Proportional cell numbers
2. Random, mixed, fixed models
3. Variance ratios
I. Abt, K.
II. Title

UNCLASSIFIED

Naval Weapons Laboratory, Dahlgren, Virginia.
(NWL Report No. 1833)
TABLE OF EXPECTATIONS OF MEAN SQUARES IN THE ANALYSIS OF VARIANCE FOR CROSSED CLASSIFICATIONS, by K. Abt. 18 Oct 1962. 67, 16 p.
UNCLASSIFIED

This report contains a table of the mean square expectations and the variance ratios for testing all nullhypotheses in the analysis of variance for the two-way, three-way and four-way crossed classifications with both proportional (unequal) and equal cell numbers. Rules are given to construct the formulas for the general n-way classification. The formulas are listed for all models (random, mixed, fixed). The report also contains a discussion of the table and an outline of the method which was used to derive the formulas.

1. Proportional cell numbers
2. Random, mixed, fixed models
3. Variance ratios
I. Abt, K.
II. Title

UNCLASSIFIED

Naval Weapons Laboratory, Dahlgren, Virginia.
(NWL Report No. 1833)
TABLE OF EXPECTATIONS OF MEAN SQUARES IN THE ANALYSIS OF VARIANCE FOR CROSSED CLASSIFICATIONS, by K. Abt., 18 Oct 1962. 67, 16 p.
UNCLASSIFIED

This report contains a table of the mean square expectations and the variance ratios for testing all nullhypotheses in the analysis of variance for the two-way, three-way and four-way crossed classifications with both proportional (unequal) and equal cell numbers. Rules are given to construct the formulas for the general n-way classification. The formulas are listed for all models (random, mixed, fixed). The report also contains a discussion of the table and an outline of the method which was used to derive the formulas.

1. Proportional cell numbers
2. Random, mixed, fixed models
3. Variance ratios
I. Abt, K.
II. Title

UNCLASSIFIED