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# BOUNDARY-VALUE PROBLEMS FOR THE MAXWELL'S EQUATIONS

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Abstract

This report contains the proofs of the uniqueness and existence theorems for an electromagnetic field when the normal component of both the electric and magnetic fields are given on a smooth surface. The truth of the above theorems was suggested by V. Rumsey. The results are obtained for an exterior domain. However, the same method can be used for the interior problems. Whereas one synthesizes an electromagnetic field by a surface current when either the tangential electric or magnetic field is given, we synthesize our electromagnetic field by means of the electric and magnetic surface charges.

We also show that solutions to Maxwell's equations can be expressed in terms of solutions to a second order partial differential equation in certain coordinate systems when the parameters $\varepsilon$ and $\mu$ are allowed to have a certain anisotropic property. This result represents an extension of those obtained by C. Müller and by B. Friedman.

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# Introduction

This report will be concerned with some boundary-value problems for Maxwell's equations; in particular, those concerning the normal component of the electric and magnetic fields. The space is the Euclidean 3-dimensional space and the boundary will be a simply connected surface. In electromagnetic theory explicit solutions of boundary-value problems are known only in cases where some components of the electric or magnetic field satisfy a second order partial differential equation. Excluding some specially symmetric boundary-value problems, only in the cases when the surface is a sphere, a cylinder (circular, elliptic, or parabolic) or a plane can the solutions be expressed in explicit forms. In all these cases the electromagnetic field can be decomposed into "T.E." (transverse electric) and "T.M." (transverse magnetic) fields and the solutions are synthesized through solutions of a second order partial differential equation, i.e., the Helmholtz's equation $(\Delta + k^2)\varphi = 0$. General separability problems in orthogonal coordinates have been investigated by Bromwich [1919], Müller [1949] and Friedman [1955] and in general coordinates by Itch [1959]. In general, it is possible to use the Stratton-Chu formula [Stratton, p.466] to represent the solution of the Maxwell's

equations in terms of surface integrals if the tangential components of the electric field and magnetic field are given on a surface. Existence and uniqueness of the solution of Maxwell's equations when the tangential components of either the electric field or the magnetic field are given on a smooth surface is known. Proofs of these results have been given by Saunders [1951], Müller [1957] and recently by Werner [1962]. In a recent paper, Rumsey [1959] suggested that the electromagnetic field would be determined uniquely when the normal components of the electric and magnetic fields are given on a smooth surface. This proposition is obviously true in the cases when the boundary surfaces are infinite planes, cylinders, or spheres, because in these cases the problem reduces to that of uniqueness and existence of a scalar boundary-value problem.

In Chapter I we shall collect some known formulas of electromagnetic theory. In particular, the Stratton-Chu representations of an electromagnetic field are recorded. Using the Stratton-Chu formulas we can prove a rather trivial uniqueness theorem of an electromagnetic field. However, in order to use the Stratton-Chu representations to calculate an electromagnetic field, one would have to know the tangential components of both the electric and magnetic fields on the surface. It is known that only the tangential electric field or magnetic field is sufficient to

determine an electromagnetic field uniquely. We therefore cannot prescribe the values of the electric and magnetic fields arbitrarily on a boundary and expect the surface representations to satisfy Maxwell's equations. By the existence theorem, we know that given a tangential electric or magnetic field on a surface, there exists an electromagnetic field which satisfies the Maxwell's equations and takes on the appropriate boundary values. Saunders [1951] had actually shown the existence of the "Green's Matrix" by means of which one can calculate the electric or magnetic field by knowing its tangential values on a smooth surface. The Green's matrix cannot be explicit, however. But, in special cases (and important cases) when the boundary conditions call for the vanishing of the tangential electric field or magnetic field, the Stratton-Chu representations can be used directly to calculate the electric or magnetic field. In other cases, one usually tries to synthesize solutions to Maxwell's equations by means of scalar functions satisfying a partial differential equation.

Chapter II is devoted to the synthesis of the explicit solutions to Maxwell's equations in terms of a function satisfying a second order partial differential equation. The possibility of expressing solutions to Maxwell's equations in terms of scalar functions satisfying a second order linear partial differential equation had been

investigated by Bromwich [1919], Müller [1949], Friedman [1955] etc., in the case of orthogonal coordinates. Itoh [1959] has investigated the possibility by means of tensor calculus in the general coordinates. In all cases, except Friedman, the $\varepsilon$ and $\mu$ are assumed to be constants. Bromwich imposed conditions on the metric elements and showed that the conditions were met by spherical coordinates. Müller showed that the only coordinate systems satisfying the conditions imposed by Bromwich were rectangular, cylindrical, and spherical. Itoh treated the general non-orthogonal coordinate system and arrived at the same conclusion as Müller's. Friedman has investigated the case when the dielectric constant takes on different constant values in different orthogonal coordinate directions. He also imposed the Bromwich conditions. In this chapter, we follow the work of Friedman and extend the results of Müller and show that the conclusions of Müller are still true when two of the coordinate directions are not mutually orthogonal but the third is perpendicular to the other two. We shall express the solutions to Maxwell's equations in terms of solutions of a second order differential equation. The present results can be used in the case of electromagnetic wave propagation over the particular anisotropic media.

Chapter III is devoted to the proofs of two uniqueness

theorems. One of the theorems states that if the normal component of the electric and magnetic fields vanish on a regular surface, and if the electromagnetic field satisfies the radiation condition, then the electromagnetic field vanishes identically in the exterior of S. This uniqueness theorem furnishes a proof of Rumsey's assertion that the normal component of the electric and the magnetic fields determine a field, if there exists one, uniquely It seems "unnatural" to prescribe normal components of the electric and the magnetic fields, since the known representation theorems, such as the Stratton-Chu formulas, call for known tangential components. However, it is well known that in the electromagnetic boundary-value problems involving a spherical surface, the most fruitful treatments are to "separate" the Maxwell's equations into "T E." and "T.M." modes, as was done in Chapter II. The boundary values for the two scalar functions are closely related to the normal component of electric and magnetic fields.

Chapter IV will contain the proof of the existence theorem for the case when the normal component of the electric and magnetic fields are given on a smooth surface. We shall construct an existence proof by means of a system of integral equations. The proof is carried out for the exterior domain; however, the method can be used for interior problems. Whereas one synthesizes an electromagnetic field by a surface current when either the

tangential electric or magnetic field is given on a surface, we synthesize our electromagnetic field by means of the electric and magnetic surface charges. The starting formulas are the Stratton-Chu formulas. Using the given boundary values on the normal component of the electric and magnetic fields, we form a system of two Fredholm integral equations, the solutions of which give us the desired surface charges. Using the known charges, we can derive the surface currents. We then use these charges and currents in the Stratton-Chu representations for our electric and magnetic fields.

# I.

## A. Maxwell's Equations

The mathematical theory of electromagentics consists of the study of the four vectors $\vec{E}$, $\vec{D}$, $\vec{H}$, and $\vec{B}$ satisfying the Maxwell's equations;

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = o, \quad (1a) \qquad \nabla \cdot \vec{B} = o, \quad (1c)$$

$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}, \quad (1b) \qquad \nabla \cdot \vec{D} = \rho, \quad (1d)$$

together with

$$\vec{D} = \varepsilon \vec{E} \text{ and } \vec{B} = \mu \vec{H},$$

where $\vec{E}$ is the electric intensity, $\vec{H}$ the magnetic intensity, $\vec{D}$ the electric displacement and $\vec{B}$ the magnetic induction. $\vec{J}$ is the volume current density and $\rho$ is the volume charge density. $\varepsilon$ is called the electric inductive capacity and $\mu$ the magnetic inductive capacity. $\varepsilon$ and $\mu$ are in general tensor functions of position. However, in many important applications, $\mu$ and $\varepsilon$ are constants. The ratio of $\varepsilon$ in a medium to that of free space is sometimes referred to as dielectric constant and denoted also by $\varepsilon$.

The most important and understood case of the Maxwell's equation is that when the variation with respect to time $t$ enters as $e^{-i\omega t}$, where $\omega$ is a constant called

circular frequency. In this case the Maxwell's equations become

$$\nabla \times \vec{E} - i\omega\mu\vec{H} = o, \quad (2a) \qquad \nabla \cdot \vec{B} = o, \quad (2c)$$

$$\nabla \times \vec{H} + i\omega\varepsilon\vec{E} = \vec{J}, \quad (2b) \qquad \nabla \cdot D = \rho, \quad (2d)$$

if we let $\vec{E}$, $\vec{H}$ etc. to represent the time independent parts of the quantities in the Maxwell's equations (1a) - (1d). Except in Chapter II we shall take $\varepsilon$, $\mu$ and $\omega$ to be constants.

The interesting and difficult problems of electromagnetic theory are the boundary-value problems in which "scattering" objects are present in the otherwise homogeneous, isotropic medium. In such cases one synthesizes the solutions to (2) by adding to a particular solution of (2) the solutions with $\vec{J} = o$. This requires finding a solution of:

$$\nabla \times \vec{E} - i\omega\mu\vec{H} = o, \qquad (3a)$$

$$\nabla \times \vec{H} + i\omega\epsilon\vec{E} = o. \qquad (3b)$$

with prescribed boundary values. When the region considered extends to infinity, it is necessary, for both physical as well as mathematical reasons, to introduce a condition regarding the behavior of the solution at infinity. Assuming differentiability and that $\varepsilon$, $\mu$, and $\omega$ are constants, we can derive from (3a) and (3b)

the equations

$$\nabla \times \nabla \times \vec{E} - k^2\vec{E} = o, \tag{4a}$$

$$\nabla \times \nabla \times \vec{H} - k^2\vec{H} = o, \tag{4b}$$

where

$$k^2 = \omega^2\epsilon\mu .$$

From (4a) and (4b), we get $\nabla\cdot\vec{E} = \nabla\cdot\vec{H} = o$. Using the vector identity*

$$\nabla \times \nabla \times \vec{F} = \nabla(\nabla\cdot\vec{F}) - \Delta\vec{F},$$

we get

$$(\Delta + k^2)\vec{E} = o, \tag{6a}$$

$$(\Delta + k^2)\vec{H} = o. \tag{6b}$$

One can see that each of the rectangular components of the electric field $\vec{E}$ and the magnetic field $\vec{H}$ satisfies the scalar "wave" equation

$$(\Delta + k^2)\varphi = o. \tag{7}$$

Therefore, many properties of the solutions to the Maxwell's equations can be derived from those of the sclutions to the scalar wave equation. In the next section, we shall collect some formulas which will be of use for the later chapters. Proofs for many of them can be found in the standard text by Stratton [1941] and in

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* $\Delta$ is the Laplacian differential operator

a recent book by Müller [1957]. Therefore we omit many and just sketch some of them.

B.

Radiation Conditions and Representations

In this section, we shall use the terms regular curv s, regular surfaces, and regular regions. Their definitions can be found in Kellogg [1929; Chap IV sections 8 and 9] or in Müller [1957,p.20]. We shall need

Definition 1. A complex function $\varphi$ in $E^3$ (3-dimensional euclidean space) is said to satisfy the Sommerfeld radiation condition if

$$\lim_{r\to\infty} r\left|\frac{\partial\varphi}{\partial r} - ik\varphi\right| = o \qquad \lim_{r\to\infty} |\varphi| = O\left(\frac{1}{r}\right)$$

uniformly with respect to directions.

A similar radiation condition for the electromagnetic field in unbounded medium is the Silver-Müller radiation condition.

Definition 2. An electromagnetic field in $E^3$ is said to satisfy the vector radiation condition (Silver-Müller) if

$$\lim_{r\to\infty} r|\hat{r} \times \nabla \times \vec{E} + ik\vec{E}| = o \quad \text{and} \quad \lim_{r\to\infty} |\vec{E}| = O\left(\frac{1}{r}\right)$$

or

$$\lim_{r\to\infty} r|\hat{r} \times \nabla \times \vec{H} + ik\vec{H}| = o \quad \text{and} \quad \lim_{r\to\infty} |\vec{H}| = O(\tfrac{1}{r}),$$

where $r$ is the distance from a fixed point chosen as the origin and $\hat{r}$ is the unit radial vector. $|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}^*}$, where $*$ denotes complex conjugate.

Remark: The conditions $\lim_{r\to\infty} |\varphi| = O(\frac{1}{r})$ and $\lim_{r\to\infty} |\vec{E}| = O(\frac{1}{r})$ or $\lim_{r\to\infty} |\vec{H}| = O(\frac{1}{r})$ can be dropped. Proofs of these can be found in Wilcox [1956,1957].

The following representation theorems are well known.

<u>Theorem 1a</u>. Let $G$ be a finite, regular region and $S$ its boundary; let $\varphi \in C^2(G)$ be a solution to

$$\Delta\varphi + k^2\varphi = o$$

and $\varphi \in C^1(\bar{G})$*,

then for $\vec{x} \in G$, we have

$$\varphi(x) = \frac{1}{4\pi}\int_S \left[\varphi(y)\frac{\partial}{\partial n_y}\frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x-y|}}{|x-y|}\frac{\partial}{\partial n_y}\varphi(y)\right] dS_y \text{**}$$

where $\vec{n}$ is the exterior unit surface normal, and $\frac{\partial}{\partial n_y}$ is the differentiation along the exterior normal.

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* $\bar{G}$ is the closure of $G$; it is the region $G$ plus its boundary. $C^k(G)$ is the space of functions whose derivatives up to and including $k^{th}$ order are continuous.

** We denote $x = \vec{x} = (x^1, x^2, x^3)$ as the point having rectangular coordinates $x^1, x^2$, and $x^3$ respectively.

For point $x \notin G$ we have

<u>Theorem 1b</u>. Let

i) $G$ be a regular region, $S$ its boundary and $\varphi$ a solution to $(\Delta+k^2)\varphi = o$.

ii) $\varphi$ is of class $C^1$ in the closure of the exterior of $G$.

iii) $\varphi$ satisfy the Sommerfeld radiation condition.

Then for $x \notin \bar{G}$

$$\varphi(x) = \frac{1}{4\pi}\int_S [\varphi(y) \frac{\partial}{\partial n_y} \frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x-y|}}{|x-y|} \frac{\partial\varphi(y)}{\partial n_y}] \, dS_y ,$$

where $\vec{n}$ is the surface normal pointing <u>into</u> $G$.

The vector analog of the above two representation theorems are

<u>Theorem 2a</u> [Müller p.134]. Let $\vec{E}$ and $\vec{H}$ be of $C^2(G)$ in a finite regular region $G$ enclosed by a regular surface $S$. $\vec{E},\vec{H} \in C^1(\bar{G})$ and satisfy the Maxwell's equations

$$\nabla \times \vec{E} - i\omega\mu\vec{H} = o,$$

$$\nabla \times \vec{H} + i\omega\varepsilon\vec{E} = o.$$

then for $x \in G$

$$\vec{E}(x) = -\frac{1}{4\pi}\int_S [i\omega\mu(\vec{n} \times \vec{H}) \Phi + (\vec{n} \times \vec{E}) \times \nabla\Phi + (\vec{n}\cdot\vec{E}) \nabla\Phi] \, dS_y,$$

$$\vec{H}(x) = \frac{1}{4\pi}\int_S [i\omega\varepsilon(\vec{n} \times \vec{E})\Phi - (\vec{n} \times \vec{H}) \times \nabla\Phi - (\vec{n}\cdot\vec{H})\nabla\Phi] dS_y,$$

where $\Phi = \frac{e^{ik|x-y|}}{|x-y|}$ and $\vec{n}$ is the exterior surface normal to the surface S.

<u>Theorem 2b</u>. Let G be a finite, regular region. Let $\vec{E}$ and $\vec{H}$ satisfy the Maxwell's equation in Theorem 2a in the exterior of G. If $\vec{E}$ and $\vec{H}$ satisfy the vector radiation condition, then for x in the exterior of S, the formulas in Theorem 2a hold if we replace the exterior normal by an interior unit normal (pointing into G).

Theorems 1 and 2 give representations of the solutions to the scalar wave equation and the Maxwell's equations in terms of their boundary data. These representations furnish a means of constructing solutions to the appropriate equation when the boundary data are given.

<u>Theorem 3</u>. [Müller p.156]. Let S be a regular surface element which is bounded by a regular curve c. Suppose $\vec{v}$ is a continuously differentiable surface field, then

$$\int_S \nabla_o \cdot \vec{v}\, ds = \int_c \vec{v} \cdot \vec{n}_o\, dl,$$

where $\nabla_o \cdot$ is the surface divergence operator, $\vec{n}_o$ is a unit surface tangential vector which is normal to c and points away from S. Using Theorem 3 and the Maxwell's equations, Müller showed that

$$\nabla \cdot \vec{K} - i\omega\rho_o = o\,,$$

$$\nabla_o \cdot \vec{K}' - i\omega\rho_o' = o,$$

where $\vec{K} = -\vec{n} \times \vec{H}$, $\vec{K}' = \vec{n} \times \vec{E}$,

$$\rho_o = -\varepsilon(\vec{n}\cdot\vec{E}), \qquad \rho_o' = \mu\,(\vec{n}\cdot\vec{H}).$$

We can therefore write the representations for $\vec{E}$ and $\vec{H}$ as

$$\vec{E}(x) = \frac{1}{4\pi}\int_S [i\omega\mu\vec{K}\Phi - \vec{K}' \times \nabla\Phi + \frac{1}{\varepsilon}\rho_o\nabla\Phi]\, ds_y, \tag{8a}$$

$$\vec{H}(x) = \frac{1}{4\pi}\int_S [i\omega\varepsilon\vec{K}'\Phi + \vec{K} \times \nabla\Phi + \frac{1}{\mu}\rho_o' \nabla\Phi]\, ds_y, \tag{8b}$$

where

$$\nabla_o\cdot\vec{K} = i\omega\rho_o\,, \quad \nabla\cdot\vec{K}_o = i\omega\rho_o'.$$

conversely, it can be shown [Müller pp.210-212] that if $\vec{K}, \rho_o, \vec{K}'$ and $\rho_o'$ are given by the above relations the surface integrals represent an electromagnetic field; that is, satisfy the Maxwell's equations for points not on S. These representations of $\vec{E}$ and $\vec{H}$ in terms of the boundary values are known as the Stratton-Chu representations. In Chapter IV, we shall use the Stratton-Chu formula to construct an electromagnetic field when the normal component of the electric and the magnetic field are given on a smooth surface.

The above representations give the following two trivial uniqueness theorems.

<u>Theorem 4</u>. Let $\varphi$ satisfy the equation

$$(\Delta + k^2)\varphi = 0$$

---

μ* We use $\vec{K}$ and $\vec{K}'$ in conformity with Stratton; Müller used $\vec{J}$ and $\vec{J}'$.

in the whole space, and let $\varphi$ satisfy the radiation condition

$$\lim_{r\to\infty} r\left|\frac{\partial\varphi}{\partial r} - ik\varphi\right| = o, \qquad |r\varphi| < c,$$

Then $\varphi \equiv 0$.

Proof: We can use the representation in Theorem 1a for $\varphi$, where we can take $x = o$ and $S$ being a spherical surface. The radiation condition will insure the vanishing of the surface integral when the radius of the sphere goes to infinity. Hence we get $\varphi(x) = \varphi(o) = 0$. Since $x$ is an arbitrary point, we conclude that $\varphi(x) \equiv o$ over the whole space.

Remark: This Theorem is the analog to the statement that the only regular potential function over the whole space is a constant.

<u>Theorem 5</u>. If $\vec{E}$ and $\vec{H}$ satisfy the Maxwell's equations

$$\nabla \times \vec{E} - i\omega\mu\vec{H} = o,$$
$$\nabla \times \vec{H} + i\omega\epsilon\vec{E} = o,$$

over the whole space and the radiation condition

$$\lim_{r\to\infty} r|\hat{r} \times \nabla \times \vec{E} - ik\vec{E}| = o, \ |r\vec{E}| < c, \text{ and } r|\vec{H}| < c,$$

then $\vec{E} = \vec{H} \equiv o$.

Proof: We take the representations for the $\vec{E}$ and $\vec{H}$ as in Theorem 2a. Take $x = o$ and $S$ a spherical surface of radius $R$.

then

$$\lim_{R\to\infty} [i\omega\mu(\hat{r} \times \vec{H})\Phi + (\hat{r} \times \vec{E}) \times \nabla\Phi + (\vec{n}\cdot\vec{E})\ \nabla\Phi]$$

$$= \quad \Phi[i\omega\mu\hat{r} \times \vec{H} + ik\vec{E}] - \Phi\frac{\vec{E}}{R} = o(\frac{1}{R^2}),$$

where $\Phi = \dfrac{e^{ikR}}{R}$ .

Hence the surface integral for the electric field is zero as $R\to\infty$ . Similarly $\vec{H}$ is zero.

# II

## Representation of Solutions to Maxwell's Equations in Terms of Scalar Potentials in a Nonhomogeneous Medium

This chapter is devoted to the synthesis of solutions to Maxwell's equations in certain coordinate systems and for an anisotropic dielectric constant $\mathcal{E}$. We shall assume $\mu$ = constant and $\mathcal{E}$ to be a function of position and anisotropic as prescribed in Theorem 1a.

We shall employ the differential geometrical quantities as used in Stratton [1941 Chapter I]. Specifically, we let $u^1$, $u^2$, and $u^3$ be parameters. We shall denote a space point by

$$\vec{r} = \vec{r}(u^1,u^2,u^3)\,, \tag{1}$$

$$\vec{r}_i = \vec{a}_i = \frac{\partial \vec{r}}{\partial u^i}\,, \quad i = 1,2,3 \tag{2}$$

and assume that $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ are linearly independent. We also define

$$g_{ij} = \vec{a}_i \cdot \vec{a}_j \quad \text{and} \quad g = \det g_{ij}\,, \tag{3}$$

$$\vec{a}^1 = \frac{1}{\sqrt{g}}(\vec{a}_2 \times \vec{a}_3),\ \vec{a}^2 = \frac{1}{\sqrt{g}}(\vec{a}_3 \times \vec{a}_1),\ \vec{a}^3 = \frac{1}{\sqrt{g}}(\vec{a}_1 \times \vec{a}_2), \tag{4}$$

$g$ can also be expressed in

$$\sqrt{g} = \vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)\,.$$

Since $\vec{a}_i$, $i = 1,2,3$ are assumed to be linearly

independent, $g$ is not zero. From the definition of $\vec{a}^i$, we have:

$$\vec{a}_i \cdot \vec{a}^j = \delta_i^j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{5}$$

Any space vector can be expressed as *

$$\vec{F} = f_j \vec{a}^j = f^i \vec{a}_i , \tag{6}$$

where $$f^i = \vec{F} \cdot \vec{a}^i, \; f_j = \vec{F} \cdot \vec{a}_j .$$

Introducing $g^{ij}$ such that

$$g_{ij} g^{jk} = \delta_i^k ,$$

we can express $f_i$ in terms of $f^j$ as follows

$$f_j = g_{ji} f^i , \; f^i = g^{ij} f_j . \tag{7}$$

The $f^i$ and $f_j$ in (6) are the contravariant and the covariant components of the vector $\vec{F}$ respectively. In terms of the above notations, we shall name:

<u>Theorem 1</u>. Let $\vec{E} = e^i \vec{a}_i = e_j \vec{a}^j$ and $\vec{H} = h^i \vec{a}_i = h_j \vec{a}^j$ denote the electric and magnetic fields. Let $\mu$ = constant and $\varepsilon$ be such that

$$\varepsilon \vec{E} = \varepsilon_1 e_1 \vec{a}_1 + \varepsilon_2 e_2 \vec{a}_2 + \varepsilon_3 e_3 \vec{a}_3 .$$

---

* We follow the summation convention and write

$$a^i b_i = a^1 b_1 + a^2 b_2 + a^3 b_3$$

Let $g_{33} = g_{33}(u^3)$, $g_{13} = g_{23} = o$,

$\frac{g_{12}}{g_{11}}$ and $\frac{g_{12}}{g_{22}}$ be independent of $u^3$.

Let $\varepsilon_1 = \varepsilon_2 = \varepsilon(u^3)$; then a possible T.E. electromagnetic field is given by

$$h_1 = \frac{1}{\sqrt{g_{33}}} \frac{\partial^2 \varphi}{\partial u^3 \partial u^1},$$

$$h_2 = \frac{1}{\sqrt{g_{33}}} \frac{\partial^2 \varphi}{\partial u^3 \partial u^2},$$

$$h_3 = \frac{1}{\sqrt{g_{33}}} \frac{\partial^2 \varphi}{\partial u^3 \partial u^3} + k^2 \sqrt{g_{33}}\, \varphi,$$

$$e_1 = i\omega\mu \frac{\sqrt{g_{33}}}{\sqrt{g}} \left\{ \frac{\partial \varphi}{\partial u^2} g_{11} - g_{12} \frac{\partial \varphi}{\partial u^1} \right\},$$

$$e_2 = i\omega\mu \frac{\sqrt{g_{33}}}{\sqrt{g}} \left\{ g_{21} \frac{\partial \varphi}{\partial u^2} - g_{22} \frac{\partial \varphi}{\partial u^1} \right\},$$

$$e_3 = o,$$

where $\varphi$ is a solution to the following equation

$$\frac{1}{g_{33}} \frac{\partial^2 \varphi}{\partial u^3 \partial u^3} + k^2 \varphi + \frac{1}{\sqrt{g}} \left\{ \frac{\partial}{\partial u^2} \left[ \sqrt{g}\, g^{22} \frac{\partial \varphi}{\partial u^2} + \sqrt{g}\, g^{12} \frac{\partial \varphi}{\partial u^1} \right] + \frac{\partial}{\partial u^1} \left[ \sqrt{g}\, g^{12} \frac{\partial \varphi}{\partial u^2} + \sqrt{g}\, g^{11} \frac{\partial \varphi}{\partial u^1} \right] \right\} = o,$$

where $k^2 = \omega^2 \mu \varepsilon$.

In curvilinear coordinates the curl of a vector $\vec{F}$ can be expressed as [Stratton p.47]

$$\nabla \times \vec{F} = \frac{1}{\sqrt{g}} \left[ \left(\frac{\partial f_3}{\partial u^2} - \frac{\partial f_2}{\partial u^3}\right) \vec{a}_1 + \left(\frac{\partial f_1}{\partial u^3} - \frac{\partial f_3}{\partial u^1}\right) \vec{a}_2 + \left(\frac{\partial f_2}{\partial u^1} - \frac{\partial f_1}{\partial u^2}\right) \vec{a}_3 \right] \tag{7a}$$

Using (7a) we can express Maxwell's equations in the medium satisfying the conditions in Theorem 1a as

$$\frac{1}{\sqrt{g}} \left[\frac{\partial e_3}{\partial u^2} - \frac{\partial e_2}{\partial u^3}\right] = i\mu\omega h^1 ; \tag{8a}$$

$$\frac{1}{\sqrt{g}} \left[\frac{\partial e_1}{\partial u^3} - \frac{\partial e_3}{\partial u^1}\right] = i\mu\omega h^2 , \tag{8b}$$

$$\frac{1}{\sqrt{g}} \left[\frac{\partial e_2}{\partial u^1} - \frac{\partial e_1}{\partial u^2}\right] = i\mu\omega h^3 , \tag{8c}$$

$$\frac{1}{\sqrt{g}} \left[\frac{\partial h_3}{\partial u^2} - \frac{\partial h_2}{\partial u^3}\right] = -i\,\varepsilon_1\omega e^1 , \tag{9a}$$

$$\frac{1}{\sqrt{g}} \left[\frac{\partial h_1}{\partial u^3} - \frac{\partial h_3}{\partial u^1}\right] = -i\,\varepsilon_2\omega e^2 , \tag{9b}$$

$$\frac{1}{\sqrt{g}} \left[\frac{\partial h_2}{\partial u^1} - \frac{\partial h_1}{\partial u^2}\right] = -i\,\varepsilon_3\omega e^3 , \tag{9c}$$

Let us assume $e^3 = o$, (10)

from (9c) we get

$$\frac{\partial h_2}{\partial u^1} - \frac{\partial h_1}{\partial u^2} = o . \tag{11}$$

If we assume that all components have continuous first derivatives, (11) implies that there exists a function $\varphi$ having second continuous derivatives such that

$$h_1 = \frac{1}{\sqrt{g_{33}}} \frac{\partial^2 \varphi}{\partial u^3 \partial u^1}, \quad h_2 = \frac{1}{\sqrt{g_{33}}} \frac{\partial^2 \varphi}{\partial u^3 \partial u^2} . \tag{12}$$

Using (6) we can write

$$\frac{\partial e_3}{\partial u^2} = \frac{\partial}{\partial u^2} [g_{3j} e^j], \quad \frac{\partial e_2}{\partial u^3} = \frac{\partial}{\partial u^3} [g_{2j} e^j] . \tag{13}$$

By the assumptions of the Theorem and $e_3 \equiv 0$, we have [from (8a) and (8b)]

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial u^3} (g_{21} e^1 + g_{22} e^2) = i\mu\omega g^{11} h_1 + i\omega\mu g^{12} h_2 , \tag{14a}$$

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial u^3} (g_{11} e^1 + g_{12} e^2) = i\mu\omega g^{21} h_1 + i\omega\mu g^{22} h_2 . \tag{14b}$$

Using (12) and the assumptions on the metric coefficients $g_{ij}$*, we can integrate (14a) and (14b) to get

$$e^1 = i\omega\mu \frac{\sqrt{g_{33}}}{\sqrt{g}} \frac{\partial \varphi}{\partial u^2} , \tag{15a}$$

$$e^2 = -i\omega\mu \frac{\sqrt{g_{33}}}{\sqrt{g}} \frac{\partial \varphi}{\partial u^1} . \tag{15b}$$

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*We note that $g^{11} = \vec{a}^1 \cdot \vec{a}^1 = \frac{1}{g} |\vec{a}_2 \times \vec{a}_3|^2 = \frac{1}{g} [g_{22} g_{33} - g_{23}^2] = \frac{1}{g} g_{22} g_{33}$. $g^{12} = \frac{1}{g} \{(\vec{a}_2 \cdot \vec{a}_3)(\vec{a}_3 \cdot \vec{a}_1) - (\vec{a}_2 \cdot \vec{a}_1)(\vec{a}_3 \cdot \vec{a}_3)\} = \frac{1}{g} g_{21} g_{33}$. $g^{22} = \frac{1}{g} g_{11} g_{33}$.

From (15a) and (15b) we get

$$e_1 = g_{1i}e^i = i\omega\mu\left\{g_{11}\frac{\partial\varphi}{\partial u^2} - g_{12}\frac{\partial\varphi}{\partial u^1}\right\}\sqrt{\frac{g_{33}}{g}}\ , \tag{16a}$$

$$e_2 = g_{2i}e^i = i\omega\mu\left\{g_{21}\frac{\partial\varphi}{\partial u^2} - g_{22}\frac{\partial\varphi}{\partial u^1}\right\}\sqrt{\frac{g_{33}}{g}}\ . \tag{16b}$$

Substituting (16a) and (16b) into (8c) and using

$$g^{11} = \frac{1}{g}\, g_{22}g_{33}\ ,$$
$$g^{12} = -\frac{1}{g}\, g_{21}g_{33}\ ,$$
$$g^{22} = \frac{1}{g}\, g_{11}g_{33}\ ,$$

we get

$$i\mu\omega h^3 = i\omega\mu\frac{1}{\sqrt{g_{33}}}\left\{-\frac{\partial}{\partial u^1}\left[\sqrt{g}\, g^{12}\frac{\partial\varphi}{\partial u^2} + \sqrt{g}\, g^{11}\frac{\partial\varphi}{\partial u^1}\right]\right.$$
$$\left. - \frac{\partial}{\partial u^2}\left[\sqrt{g}\, g^{22}\frac{\partial\varphi}{\partial u^2} - \sqrt{g}\, g^{12}\frac{\partial\varphi}{\partial u^1}\right]\right\}. \tag{17}$$

From (9a) we get

$$\frac{1}{\sqrt{g}}\left[\frac{\partial h_3}{\partial u^2} - \frac{\partial h_2}{\partial u^3}\right] = -i\varepsilon\omega e^1 = k^2\frac{\sqrt{g_{33}}}{\sqrt{g}}\frac{\partial\varphi}{\partial u^2}\ ,\ k^2 = \omega^2\mu\varepsilon\ . \tag{18}$$

On the assumption that $g_{33}$ depends only on $u^3$, (18) can be integrated with respect to $u^2$ to yield

$$h_3 = \frac{1}{\sqrt{g_{33}}}\frac{\partial^2\varphi}{\partial u^3\partial u^3} + k^2\sqrt{g_{33}}\,\varphi\ . \tag{19}$$

One can see from (19), (12) and (15b) that (9b) is satisfied. From (17), (19) and the relation $h_3 = g_{33}h^3$, we obtain the differential equation satisfied by $\varphi$,

$$\frac{1}{g_{33}}\frac{\partial^2\varphi}{\partial u^3\partial u^3} + k^2\varphi + \frac{1}{\sqrt{g}}\left\{\frac{\partial}{\partial u^2}\left[\sqrt{g}\, g^{22}\frac{\partial\varphi}{\partial u^2} + \sqrt{g}\, g^{12}\frac{\partial\varphi}{\partial u^1}\right] + \right.$$

$$\left.\frac{\partial}{\partial u^1}\left[\sqrt{g}\, g^{12}\frac{\partial\varphi}{\partial u^2} + \sqrt{g}\, g^{11}\frac{\partial\varphi}{\partial u^1}\right]\right\} = 0,\ k^2 = \omega^2\mu\varepsilon\ . \tag{20}$$

The assertion of Theorem 1a is expressed in (12), (19), (16a), (16b), (10) and (20).

Entirely similar to the proof of Theorem 1a, we have

<u>Theorem 1b.</u> If the conditions of Theorem 1a are satisfied, if in addition, $\varepsilon_1 = \varepsilon_2 = \varepsilon = \text{const.}$, $\varepsilon_3 = \varepsilon_3(u^1,u^2,u^3)$, a possible T.M. wave is given by:

$$e_1 = \frac{1}{\sqrt{g_{33}}}\frac{\partial^2\psi}{\partial u^3\partial u^1}\ ,$$

$$e_2 = \frac{1}{\sqrt{g_{33}}}\frac{\partial^2\psi}{\partial u^3\partial u^1}\ ,$$

$$e_3 = \frac{1}{\sqrt{g_{33}}}\left[\frac{\partial^2\psi}{(\partial u^3)^2} + k^2 g_{33}\,\psi\right]\ ,$$

$$h_1 = -i\omega\varepsilon\sqrt{\frac{g_{33}}{g}}\left\{\frac{\partial\psi}{\partial u^2}\, g_{11} - g_{12}\frac{\partial\psi}{\partial u^1}\right\}\ ,$$

$$h_2 = -i\omega\varepsilon\sqrt{\frac{g_{33}}{g}}\left\{\frac{\partial\psi}{\partial u^2}\, g_{21} - g_{22}\frac{\partial\psi}{\partial u^1}\right\}\ ,$$

$$h_3 = 0,$$

where $\psi$ is a solution of

$$\frac{1}{g_{33}}\frac{\partial^2\varphi}{\partial u^3\partial u^3} + k^2\varphi + \frac{\varepsilon}{\varepsilon_3}\frac{1}{\sqrt{g}}\left\{\frac{\partial}{\partial u^2}\left[\sqrt{g}\, g^{22}\frac{\partial\varphi}{\partial u^2} + \sqrt{g}\, g^{12}\frac{\partial\varphi}{\partial u^1}\right]\right.$$

$$\left. + \frac{\partial}{\partial u^1}\left[\sqrt{g}\, g^{12}\frac{\partial\varphi}{\partial u^2} + \sqrt{g}\, g^{11}\frac{\partial\varphi}{\partial u^1}\right]\right\} = 0. \tag{21}$$

Remark: If $\varepsilon_1 = \varepsilon_2 = \varepsilon(u^3)$, we can still reduce the finding of a T.M. wave to the equivalent problem as specified

in Theorem 1a.

If we try to express the results of Theorems 1a and 1b in orthogonal coordinates, we introduce* [Assume $g_{33} = 1$]

$$h_1 = \sqrt{g_{11}},\ h_2 = \sqrt{g_{22}},\ h_3 = \sqrt{g_{33}} = 1,$$

$$\sqrt{g} = h_1 h_2 h_3,\ f^i = \frac{1}{h_i} F_i,\ f_i = h_i F_i,\ g^{ii} = \frac{1}{h_i^2},\ \text{(no sum)}$$

$$g^{11}\sqrt{g} = \frac{h_2 h_3}{h_1},\quad g^{22}\sqrt{g} = \frac{h_1 h_3}{h_2},\quad g^{ik} = g_{ik} = 0,\ i \neq k,$$

and we obtain

<u>Theorem 2a</u>. If the conditions of Theorem 1a are satisfied, i.e.,

i) $\varepsilon_1 = \varepsilon_2 = \varepsilon(u^3),\ \varepsilon_3 = \varepsilon_3(u^1, u^2, u^3),\ \mu = \text{const.}$

ii) $h_3 = 1,\ \frac{h_1}{h_2}$ independent of $u^3$,

iii) $E_3 \equiv 0$,

we have a possible solution to the Maxwell's equations given by

$$H_1 = \frac{1}{h_1}\frac{\partial^2\varphi}{\partial u^3 \partial u^1};\ H_2 = \frac{1}{h_2}\frac{\partial^2\varphi}{\partial u^3 \partial u^2};\ H_3 = \frac{\partial^2\varphi}{\partial u^3 \partial u^3} + k^2\varphi$$

$$E_1 = i\omega\mu\frac{1}{h_2}\frac{\partial\varphi}{\partial u^2};\ E_2 = -i\omega\mu\frac{1}{h_1}\frac{\partial\varphi}{\partial u^1};\ E_3 \equiv 0,$$

where $\varphi$ satisfies the following equation

$$\frac{\partial^2\varphi}{\partial u^3 \partial u^3} + k^2\varphi + \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u^1}\left(\frac{h_2}{h_1}\frac{\partial\varphi}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{h_1}{h_2}\frac{\partial\varphi}{\partial u^2}\right)\right] = 0.$$

<u>Theorem 2b</u>. If the conditions of Theorem 1b are satisfied,

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*The $h_i$'s here are metric coefficients and not components of the magnetic field.

a possible electromagnetic field is given as follows:

$$E_1 = \frac{1}{h_1}\frac{\partial^2\psi}{\partial u^3\partial u^1};\quad E_2 = \frac{1}{h_2}\frac{\partial^2\psi}{\partial u^3\partial u^2};\quad E_3 = \frac{\partial^2\psi}{\partial u^3\partial u^3} + k^2\psi;$$

$$H_1 = -i\omega\varepsilon\,\frac{1}{h_2}\frac{\partial\psi}{\partial u^2};\quad H_2 = i\omega\varepsilon\,\frac{1}{h_1}\frac{\partial\psi}{\partial u^1};\quad H_3 = 0;$$

where $\psi$ satisfies

$$\frac{\partial^2\psi}{\partial u^3\partial u^3} + k^2\psi\,\frac{\varepsilon}{\varepsilon_3}\,\frac{1}{h_1h_2}\left[\frac{\partial}{\partial u^1}\left(\frac{h_2}{h_1}\frac{\partial\psi}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{h_1}{h_2}\frac{\partial\psi}{\partial u^2}\right)\right] = 0,$$

where $k^2 = \omega^2\varepsilon\mu$.

In both Theorems 2a and 2b, the electric and magnetic fields are expressed as

$$\vec{E} = E_1\hat{e}_1 + E_2\hat{e}_2 + E_3\hat{e}_3,$$

$$\vec{H} = H_1\hat{e}_1 + H_2\hat{e}_2 + H_3\hat{e}_3,$$

where $\hat{e}_i = \frac{\partial\vec{r}}{\partial u^i} \Big/ \left|\frac{\partial\vec{r}}{\partial u^i}\right|$.

We would like to discuss the geometric significance of the assumptions in Theorem 1a on the metric coefficients $g_{ij}$. The assumption $g_{i3} = 0$, $i \neq 3$, implies that the "$u^3$-axis" is perpendicular to the $u^1$-$u^2$ "plane". To analyze the significance of the assumptions that

$g_{33} = g_{33}(u^3)$,

$\frac{g_{12}}{g_{11}}$ and $\frac{g_{11}}{g_{22}}$ are independent of $u^3$,

we follow exactly the same treatment by Müller [ 1949 ] in the case when $g_{12} = 0$; we arrive at the following two

possible cases.

i) The "$u^3$-axis" (i.e. the direction of $\frac{\partial \vec{r}}{\partial u^3}$) is perpendicular to a family of concentric spherical surface. This gives rise to a spherical coordinate system.

ii) The $u^3$-axis is perpendicular to a family of parallel planes. This gives rise to the rectangular coordinate system or the cylindrical system with the z-axis being the $u^3$-axis.

Remark: We also note that spherical coordinate system can be used to solve boundary-value problems in which the boundary is a circular cone; the base of which is a part of the spherical surface with center at the vertex.

Theorems 1a, 1b, 2a and 2b state the sufficient conditions that a T.E. (no $e_3$) or a T.M. (no $h_3$) electromagnetic field exists. If the conditions are satisfied, the T.M. and T.E. electromagnetic fields will exist independent of each other. Therefore we must prove that an electromagnetic field has unique T.E. and T.M. decomposition. The unique decomposition theorem is not known except in the special case when $\varepsilon$ and $\mu$ are constants [see Wilcox (1957)]. However, if $\mu_1 = \mu_2 = \mu_3 = \mu(r)$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon(r)$, then equation (20) takes the following form in spherical coordinates

$$\frac{\partial^2 \Phi}{\partial r^2} + k^2 \Phi + \frac{1}{r^2} \left[ \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 \Phi}{\partial \varphi^2} \right] = 0 .$$

Given $E_r = \frac{\partial^2\Phi}{\partial r^2} + k^2\Phi$ at $r = a$, we can solve for $\Phi$ [see Chapter IV] over the whole spherical surface if $\int E_r dS = o$. If we now let $\Phi = r\Psi$, we see that

$$(\Delta + k^2)\Psi = o.$$

Hence if we restrict ourselves to a certain class of $k^2 = \omega^2\mu\varepsilon$, we can solve $\Psi$ in the exterior of a spherical surface when $\Psi$ is prescribed on the surface. Results of this nature can be obtained by means of the Green's function shown to exist by F. Odeh (1960) for a certain class of $k^2$.

We have shown in Theorems 1 and 2 the sufficient conditions that one can construct a solution to the Maxwell's equation by means of solutions to the scalar wave equation with variable $k^2$ depending on $u^3$ alone (this can be done by rescaling of the $u^3$ coordinate). In this section we give a necessity argument. This argument can only be regarded as heuristic instead of a rigorous proof. We shall "prove" that when $\varepsilon_1$ and $\mu_1$ satisfy conditions assumed in Theorems 1 and 2, we can have T.E. or T.M. wave only if the metric coefficients satisfy the conditions there. Specifically, we shall show

<u>Theorem</u> 3. If $\varepsilon_1 = \varepsilon_2 = \varepsilon(u^3)$, $\mu_1 = \mu_2 = \mu(u^3)$,
$\varepsilon_3 = \varepsilon_3\ (u^3)$, $\mu_3 = \mu\ (u^3)$

and if $g_{13} = g_{23} = 0$,

we can have T.E. or T.M. wave only if

i) $g_{33}$ depends only on $u^3$.

ii) $\sqrt{g}\, g^{11}$, $\sqrt{g}\, g^{12}$, and $\sqrt{g}\, g^{22}$, depend on $u^3$ in such a way that the dependence enters as a product with a function of $u^3$.

[These are equivalent to the conditions in Theorem 1a]

Let us assume $e^3 = o$. From $\nabla\cdot\vec{D} = o$ (source free), we get

$$o = \nabla\cdot\vec{D} = \frac{1}{\sqrt{g}}\left\{\frac{\partial}{\partial u^1}(\mathcal{E}_1 e^1\sqrt{g}) + \frac{\partial}{\partial u^2}(\mathcal{E}_2 e^2\sqrt{g}) + \frac{\partial}{\partial u^3}(\mathcal{E}_3 e^3\sqrt{g})\right\}. \tag{22}$$

But $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}(u^3)$, hence

$$\frac{\partial}{\partial u^1}(e^1\sqrt{g}) + \frac{\partial}{\partial u^2}(e^2\sqrt{g}) = o. \tag{23}$$

(23) implies that there exists a $\varphi$ such that

$$e^1\sqrt{g} = \frac{\partial\varphi}{\partial u^2}, \tag{24}$$

$$e^2\sqrt{g} = \frac{\partial\varphi}{\partial u^1}. \tag{25}$$

From (8a) and (8b),

$$i\mu_1\omega h^2 = \frac{1}{\sqrt{g}}\frac{\partial e_1}{\partial u^3} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial u^3}(g_{11}e^1 + g_{12}e^2); \tag{26}$$

$$i\mu_2\omega h^3 = \frac{1}{\sqrt{g}}\frac{\partial e_2}{\partial u^3} = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial u^3}(g_{21}e^1 + g_{22}e^2). \tag{27}$$

From (9c),

$$\frac{\partial}{\partial u^1}[g_{21}h^1 + g_{22}h^2] = \frac{\partial}{\partial u^2}[g_{11}h^1 + g_{12}h^2]. \tag{28}$$

We substitute (24) and (25) into (26) and (27) and then (26) and (27) into (28). Since $\varphi$ is quite arbitrary, we

equate the coefficients of various derivatives of $\varphi$ in (28)* ; then we get the following results;

i) Equating the coefficients of $\varphi_{32}$, we get

$$\left(\frac{1}{g_{33}}\right)_1 = o. \tag{29a}$$

ii) Equating the coefficients of $\varphi_{31}$, we get

$$\left(\frac{1}{g_{33}}\right)_2 = o. \tag{29b}$$

iii) Equating the coefficients of $\varphi_{22}$, we get

$$-g_{11}\left(\frac{g_{21}}{g_{11}}\right)_3 + g_{12}\left(\frac{g_{11}}{\sqrt{g}}\right)_3 = \frac{g_{11}^2}{\sqrt{g}}\left(\frac{g_{21}}{g_{11}}\right)_3 = o. \tag{29c}$$

iv) Equating coefficients of $\varphi_{11}$, we get

$$\left(\frac{g_{12}}{g_{22}}\right)_3 = o. \tag{29d}$$

v) Equating the coefficients of $\varphi_{21}$, we get

$$\frac{1}{\sqrt{g}}(g_{22}(g_{11})_3 - g_{11}(g_{22})_3) = o \quad \text{or}$$

$$\left(\frac{g_{11}}{g_{22}}\right)_3 = o, \quad \text{since } g_{22} \neq o. \tag{29e}$$

vi) Equating the coefficients of $\varphi_2$, we get

$$\left\{ \frac{1}{\sqrt{g}}\left[g_{22}\left(\frac{g_{11}}{\sqrt{g}}\right)_3 - (g_{21})\left(\frac{g_{21}}{\sqrt{g}}\right)_3\right]\right\}_1 = o. \tag{29f}$$

vii) Equating the coefficients $\varphi_1$, we get

$$\left\{ \frac{1}{\sqrt{g}}\left[g_{11}\left(\frac{g_{22}}{\sqrt{g}}\right)_3 - g_{12}\left(\frac{g_{12}}{\sqrt{g}}\right)_3\right]\right\}_2 = o. \tag{29g}$$

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* $\varphi_{ij} = \dfrac{\partial^2\varphi}{\partial u^i \partial u^j}$ $\qquad \varphi_k = \dfrac{\partial\varphi}{\partial u^k}$

(29a) and (29b) imply $g_{33}$ depends only on $u^3$;

(29c) implies that $\frac{g_{21}}{g_{11}} = -\frac{g^{12}}{g^{22}}$ is independent of $u^3$;

(29d) implies that $\frac{g_{12}}{g_{22}} = -\frac{g^{12}}{g^{11}}$ is independent of $u^3$;

(29e) implies that $\frac{g_{11}}{g_{22}} = \frac{g^{22}}{g^{11}}$ is independent of $u^3$;

(29a) - (29e) are consistent and imply that

$$g_{11} = f_{11}(u^1,u^2)\ G(u^3), \tag{30a}$$

$$g_{12} = f_{12}(u^1,u^2)\ G(u^3), \tag{30b}$$

$$g_{22} = f_{22}(u^1,u^2)\ G(u^3), \tag{30c}$$

$$g_{33} = g_{33}(u^3). \tag{30d}$$

Hence

$$\sqrt{g} = [g_{33}\ G(f_{11}f_{22} - f_{12}^2)]^{1/2}; \tag{31}$$

consequently, we see that (29f) and (29g) are satisfied. Equations (30a)-(30d) are the equivalent statements of the theorem.

# III

## Uniqueness Theorems for Maxwell's Equations if Normal Component of the Electric and Magnetic Fields are Prescribed.

We shall devote this chapter to the proofs of two uniqueness theorems concerning the solutions of Maxwell's equations if the normal component of the electric and magnetic fields are prescribed on a surface. Specifically, we shall establish

<u>Theorem 1.</u> Let $S$ be a closed regular surface; let $\vec{E}$ and $\vec{H}$ satisfy

$$\nabla \times \vec{E} - i\omega\mu \vec{H} = 0$$

$$\nabla \times \vec{H} + i\omega\varepsilon\vec{E} = 0$$

in the exterior and on $S$. Here, $\mu$ and $\omega$ are taken to be positively real and $\varepsilon$ can be positively real or $\varepsilon = \varepsilon_0 + i\varepsilon_1$ with $\varepsilon_0, \varepsilon_1 > 0$. Let $\vec{E}$ and $\vec{H}$ be piecewise continuously differentiable on the regular points of $S$ and in the exterior of $S$. If

$$\vec{n}\cdot\vec{H} = \vec{n}\cdot\vec{E} = 0$$

on the regular points of $S$ and if the vector radiation condition

$$\lim_{r\to\infty} r|\hat{r} \times \nabla \times \vec{E} + ik\vec{E}| = 0 , \quad \lim_{r\to\infty} r|\vec{E}| \leq c$$

is satisfied, then

$$\vec{E} = \vec{H} \equiv 0 \text{ in the exterior of } S.$$

Let $S$ be a smooth closed surface which encloses a simply connected region. Near the surface, we can introduce a set of local coordinates such that $u^3$ is a parameter characterizing the distance along the exterior surface normal. Let $u^1$ and $u^2$ be parameters characterizing points on the surface $S$. Let us write a point in space as $\vec{r}(u^1,u^2,u^3) = \vec{x}(u^1,u^2) + u^3\,\vec{n}$, where $\vec{x}$ is a point of the surface. The curves $u^1$ and $u^2$ can be chosen so that the curves for constant $u^i$ form an orthogonal coordinate system. Let $h_1$, $h_2$, and $h_3$ be the metric elements as introduced in Theorem 2a of Chapter II. We can prove

<u>Theorem 2.</u> Let $S$ be a closed surface of class $C^2$ which encloses a simply connected region. Let $\vec{E}$ and $\vec{H}$ be solutions to Maxwell's equations in the exterior and on $S$. Let $H_3 = H_n = \vec{H}\cdot\vec{n} = 0$ and $\frac{\partial}{\partial u^3}(h_1h_2E_3) = 0$ on $S$. Let the vector radiation condition be satisfied. Then $\vec{E} = \vec{H} \equiv 0$ on and in the exterior of $S$.

Remark: The conditions for $\vec{E}$ and $\vec{H}$ can be interchanged. Theorem 1 asserts that in the exterior problem there is only one electromagnetic field with the prescribed normal component of the electric and magnetic fields on a surface. In the interior problem, the assertion is not true because of the possibility that "mode" solutions may exist. However, if we assume that $I - \varepsilon > 0$, Lemma 2 below will assure that

there is no non-trivial interior field. The proof of theorem 1 is based on a new vector identity, while the proof of theorem 2 is based on the maximum principle theorem for solutions of an elliptic partial differential equation, a result due to E. Hopf [see Hellwig p.86]. We shall first state a few lemmas.

<u>Lemma 1</u> [Rellich (1943)]. Let $\varphi$ be a solution of

$$(\Delta + k^2)u = 0 \qquad k > 0$$

for $r > R_o$ with $R_o$ fixed,

then there exists a positive number $P$ such that for all large enough $R$, the following inequality holds:

$$\int_{R_o < R_1 < r < R} |u|^2 \, dv \geqslant P R$$

<u>Corollary:</u> Let $\vec{E}$ and $\vec{H}$ be solutions to the Maxwell's equations

$$\nabla \times \vec{E} - i\omega\mu\vec{H} = 0$$

$$\nabla \times \vec{H} + i\omega\varepsilon\vec{E} = 0$$

for $r > R_o$ with $R_o$ fixed.

Then there exists a positive number $P$ such that for all large enough $R$, the following inequality holds:

$$\int_{R_o < R_1 < r < R} |\vec{E}|^2 \, dv \geqslant P R$$

The truth of this corollary follows from the fact that each

rectangular component of the electric and magnetic field satisfies the scalar wave equation.

<u>Lemma 2</u>. Let the surface $S$, the electric field $\vec{E}$, and the magnetic field $\vec{H}$ satisfy the conditions in Theorem 1. Then if

$$\int_S \vec{n}\cdot(\vec{E}^* \times \vec{H})\, ds = o,$$

we must have

$$\vec{E} = \vec{H} \equiv o \quad \text{in the exterior of } S.$$

Proof: [Müller p.284]

i) Suppose $\omega, \mu > o$, $\mathcal{E} = \mathcal{E}_o + i\,\mathcal{E}_1$ with $\mathcal{E}_o, \mathcal{E}_1 > o$. Using the divergence theorem, Maxwell's equations, the radiation conditions and the condition of the lemma, we get

$$\int_{D_R} [i\omega\mathcal{E}|\vec{E}|^2 - i\omega\mu|\vec{H}|^2]\, dv = \frac{k^*}{\omega\mathcal{E}^*} \int_{r=R} |\vec{E}|^2\, ds + o(1)$$

where $*$ denotes complex conjugate and $D_R$ is the region in the exterior of $S$ but inside a large sphere of radius $R$. Because $\epsilon_1 > o$, the real part of the left-hand side of the above equation is negative and the real part of the right-hand side is positive, hence

$$\int_{D_R} |\vec{E}|^2\, dv = o$$

This implies that $\vec{E} \equiv o$ in the exterior of $S$.

ii) [Wilcox 1956] Suppose $k > o$

The radiation condition implies that

$$o = \lim_{R\to\infty} \int_{r=R} |\hat{r} \times (\nabla \times \vec{E}) + ik\vec{E}|^2 \, ds.$$

Expand the integrand and use some vector identities to get

$$\lim_{R\to\infty} \int_{r=R} |\vec{E}|^2 \, ds = o(1)$$

By Rellich's lemma (corollary of lemma 1)

$$\vec{E} \equiv o \quad \text{for} \quad r > R.$$

This implies $\vec{E} \equiv o$ in the exterior of $S$ by continuation.

<u>Lemma 3.</u> Let $S$ be a regular closed surface; $f$ and $g$ be piecewise $C^2$ functions on $S$. If $\vec{n}$ is the unit surface normal on $S$ at its regular points and $\nabla_t f$ denotes the surface gradient, then

$$\int_S \vec{n}\cdot(\nabla_t f \times \nabla_t g) \, ds = o$$

Proof: The assumption on the surface $S$ permits one to decompose the surface into finitely many regular surface elements, in each of which a coordinate system can be introduced.

If we write a space point in the neighborhood of $S$ as*

$$\vec{r} = \vec{x}(u^1,u^2) + u^3 \, \vec{n}(u^1,u^2), \qquad u^3 > o$$

---

* $\vec{a}_i = \dfrac{\partial \vec{r}}{\partial u^i}$, $\vec{a}^j = g^{ji} \, \vec{a}_i$

then

$$\nabla_t f = \vec{a}^1 \frac{\partial f}{\partial u^1} + \vec{a}^2 \frac{\partial f}{\partial u^2} ,$$

$$\nabla_t g = \vec{a}^1 \frac{\partial g}{\partial u^1} + \vec{a}^2 \frac{\partial g}{\partial u^2} ,$$

$$\vec{n} = \vec{a}_3 .$$

We find

$$(\nabla_t f \times \nabla_t g)\cdot\vec{n} = \left(\frac{\partial f}{\partial u^1}\frac{\partial g}{\partial u^2} - \frac{\partial f}{\partial u^2}\frac{\partial g}{\partial u^1}\right) \vec{a}^1 x \vec{a}^2 \cdot \vec{a}_3$$

But $ds = \sqrt{g}\ du^1 du^2$;

$\vec{a}^i = g^{ij}\ \vec{a}_j$,

Hence

$$\vec{a}^1 \times \vec{a}^2 \cdot \vec{a}_3 = g^{1i} g^{2j} \vec{a}_i \times \vec{a}_j \cdot \vec{a}_3 = (g^{11}g^{22} - g^{12}g^{12})$$

$$(\vec{a}_1 \times \vec{a}_2)\cdot\vec{a}_3 = g^{-1}\sqrt{g}\ \Big|_{u^3} = 0 ,$$

Therefore

$$\int_S \vec{n}\cdot(\nabla_t f \times \nabla_t g)\ ds = \int_S \left(\frac{\partial f}{\partial u^1}\frac{\partial g}{\partial u^2} - \frac{\partial f}{\partial u^2}\frac{\partial g}{\partial u^1}\right) du^1 du^2$$

$$= \sum_{\alpha=1}^{N} \int_{S_\alpha} \left(\frac{\partial f}{\partial u^1}\frac{\partial g}{\partial u^2} - \frac{\partial f}{\partial u^2}\frac{\partial g}{\partial u^1}\right) du^1 du^2,$$

where $S_\alpha$ are the surface elements into which $S$ is decomposed. The integration is over the corresponding plane areas in the $u^1$-$u^2$ plane. We observe that

$$\frac{\partial f}{\partial u^1}\frac{\partial f}{\partial u^2} - \frac{\partial f}{\partial u^2}\frac{\partial g}{\partial u^1} = \frac{\partial}{\partial u^1}\left(f\frac{\partial g}{\partial u^2}\right) - \frac{\partial}{\partial u^2}\left(f\frac{\partial g}{\partial u^1}\right)$$

using the two dimensional Green's Theorem, namely

$$\int_C (P(x,y)dx + Q(x,y)dy\ ) = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x}\right) dxdy,$$

we can therefore write

$$\int_S \vec{n}\cdot(\nabla_t f \times \nabla_t g)\, ds = \Sigma \int_{C_\alpha} \left(f \frac{\partial g}{\partial u^1} \frac{du^1}{dl} + f \frac{\partial g}{\partial u^2} \frac{du^2}{dl}\right) dl$$

where $C_\alpha$ is the bounding curve of $S_\alpha$ and $l$ is the arc length. Since the sense of the line integral on the common edge (or common division curve) of two neighboring surface elements is opposite, the sum of the surface integrals will be zero. Hence the result is proved.

<u>Lemma 4:</u> [Hopf, see Hellwig (1960, p.86) ]

Let the differential operator $A$ be defined as

$$Au = \sum_{i,k=1}^{n} a_{ik} u_{x^i x^k} + \sum_{i=1}^{n} a_i u_{x^i} \qquad x = (x^1, x^2, \ldots x^n)$$

in an n-dimensional space. Let

$A_{ik}(x)$, $a_i(x) \in C$ in a closed region $\bar{G}$. Let $A$ be elliptic in $\bar{G}$, i.e.

$$\sum_{i,k=1}^{n} a_{ik}(x)\, y^i y^k > o \quad \text{with equality holding only}$$

if $y^i = y^k = o$. Then if $u \in C^2$ is a solution of $Au = o$, $u$ takes its maximum and minimum on the boundary. This implies that if $x_o \in G$ and $u(x_o) \geqslant u(x)$ for $x \in \bar{G}$, $u(x) \equiv u(x_o)$ in $\bar{G}$ for each $x$.

With these four lemmas, we can now give the proofs of the theorems stated at the beginning of the chapter.

Proof of Theorem 1.

By means of Stoke's Theorem, we get for any closed regular curve $C_\Sigma$ enclosing a regular surface element $\Sigma$ on $S$

$$o = \int_\Sigma \vec{n}\cdot\vec{H}^* ds = \frac{1}{\omega\mu} \int_\Sigma \vec{n}\cdot\nabla \times \vec{E}^* ds = \frac{1}{\omega\mu} \int_{C_\Sigma} \vec{E}^*\cdot d\vec{s}$$

Since $C_\Sigma$ is any closed curve on $S$, the above equation implies that there exists a function $f(\vec{x})$ such that

$$f(\vec{x}) - f(\vec{x}_o) = \int_{x_o}^{x} \vec{E}\cdot dl$$

Similarly, from the condition $\vec{n}\cdot\vec{H} = o$ on the surface, there exists a function $g(\vec{x})$ such that

$$g(\vec{x}) - g(\vec{x}_o) = \int_{x_o}^{x} \vec{H}^*\cdot d\vec{l}$$

These two equations are the same as

$$\vec{E} = \nabla_t f \quad , \quad \vec{H}^* = \nabla_t g \qquad \text{on } S$$

By the assumption of Theorem 1 we see that $f$ and $g$ will be of class $C^2$ (piecewise) over $S$. Therefore by Lemma 3

$$\int_S \vec{n}(\vec{E} \times \vec{H}^*)\, ds = \int_S \vec{n}\cdot(\nabla_t f \times \nabla_t g)\, ds = o$$

and Lemma 2 implies that $E = H = o$ on and in the exterior of $S$.

Remark: The statement of Theorem 1 is not true for the interior problem because of the possibility of the existence of mode solutions (solutions such that

$\vec{n}\cdot\vec{E} = \vec{n}\cdot\vec{H} = o$ on the boundary). As an easy example, let us consider the interior of a closed circular cylinder of length $l$ and radius a. It is known that there exists an electromagnetic field satisfying

$$(\Delta + k^2)\, H_z = o$$

$$\sqrt{\frac{\varepsilon}{\mu}}\, E_\rho = i\, \frac{1}{k\rho}\, \frac{\partial H_z}{\partial \varphi}$$

$$\sqrt{\frac{\varepsilon}{\mu}}\, E_\varphi = -\, \frac{1}{k}\, \frac{\partial H_z}{\partial \rho}$$

$$E_z = H_\rho = H_\varphi = o$$

where $z, \rho, \varphi$ are the cylindrical coordinates, and the z-axis is along the axis of the cylinder. If we look for a solution which is independent of $\varphi$, we find that

$$H_z = \sin\left(\frac{\pi n}{l}\, z\right) J_o\,(k_n \rho)$$

where $n$ = integer, $k_n = \sqrt{k^2 - \left(\frac{\pi n}{l}\right)^2}$, and $J_o(x)$ is the Bessel function of zero order.

This is a nontrivial infinitely differentiable solution inside the cylinder which vanishes at $z = o$ and $z = l$. This electromagnetic field will have $n\cdot\vec{E} = n\cdot\vec{H} = o$ on the boundary if a is so chosen that $k_n a$ is a root of the Bessel function. However, if $\mathrm{Im}\, k > o$, $k_n$ will be complex and it is known that the Bessel functions of real order have no complex zeros.

Proof of Theorem 2

From the divergenceless of $\vec{E}$ we get

$$o = \nabla\cdot\vec{E} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u^1}(h_2h_3E_1) + \frac{\partial}{\partial u^2}(h_3h_1E_2) + \frac{\partial}{\partial u^3}(h_1h_2E_3)\right].$$

If we choose $u^3$ such that $h_3 = 1$ on the surface S, then the above equation and the condition of Theorem 2 give

$$\frac{\partial}{\partial u^1}(h_2E_1) + \frac{\partial}{\partial u^2}(h_1E_2) = o. \tag{A}$$

From the argument in the proof of Theorem 1 we see that $n\cdot\vec{H} = o$ implies that there exists a function $\varphi$ such that

$$\vec{E}_t = \nabla_t\varphi = E_1\hat{e}_1 + E_2\hat{e}_2 = \frac{1}{h_1}\frac{\partial\varphi}{\partial u^1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\varphi}{\partial u^2}\hat{e}_2 \tag{B}$$

where $\hat{e}_1$ and $\hat{e}_2$ are mutually orthogonal unit surface vectors. From (A) and (B) we get

$$\frac{\partial}{\partial u^1}\left(\frac{h_2}{h_1}\frac{\partial\varphi}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{h_1}{h_2}\frac{\partial\varphi}{\partial u^1}\right) = o$$

this equation holds on any point of S in which a local orthogonal coordinate system is introduced. At any point of S we can choose a coordinate system such that $h_1$ and $h_2$ are greater than zero. Now $\varphi$, being a solution to the equation, is continuous

over $S$ (a compact set). Hence $\varphi$ must attain its maximum on $S$. Let it attain its maximum at a point $x_o \in S$. We apply Hopf's lemma to get $\varphi(x) = \varphi(x_o)$ in the neighborhood of $x_o$. Repeated applications of Hopf's lemma will give $\varphi = \text{const.}$ over $S$. This implies $\vec{E}_t \equiv o$ on $S$. By the uniqueness theorem, which states that an electromagnetic field in an exterior domain is uniquely determined by the tangential electric field on its boundary, we see that $\vec{E} \equiv o$ in the exterior of $S$. Hence, also $\vec{H} \equiv o$ and the theorem is proved.

# IV.

## Existence Theorems for Maxwell's Equations if the Normal Component of the Electric and Magnetic Fields are Given on a Smooth Surface

In this chapter we shall construct a solution to the Maxwell's equations in the exterior domain when the normal component of the electric and the magnetic fields are given on a smooth boundary surface. Then by Theorem 1 of chapter III, a solution so found will be <u>the</u> solution, In particular, we state our result in the

<u>Theorem.</u> Let

i) $S$ be a $C^4$ closed surface which encloses a simply connected region,

ii) $e(y)$ , $h(y)$ $(y \in S)$ be Hölder continuous functions on $S$ satisfying the conditions

$$\int_S e(y)\, ds = \int_S h(y)\, ds = 0 \;;$$

then there exist $\vec{E}$ and $\vec{H}$ satisfying

a) The Maxwell's equations

$$\nabla \times \vec{E} - i\omega\mu\vec{H} = 0 \,,$$

$$\nabla \times \vec{H} + i\omega\varepsilon\vec{E} = 0 \,,$$

in the exterior of $S$.

b) The vector radiation condition

$$\lim_{r\to\infty} r\left|\hat{r} \times \nabla \times E + ik\vec{E}\right| = 0 \qquad \text{and}$$

c) $n\cdot\vec{E} = e(x)$ , $\vec{n}\cdot\vec{H} = h(x)$ for $x \in S$,

Define a map $T$ implicitly as follows:

Let $\varphi$ be the solution of

$$\Delta_t \varphi = i\omega\rho$$

such that $\int_S \varphi \, ds = o$,

where $\Delta_t$ is the "surface Laplacian" (see also eq. (4.3)).

Put

$$T\rho = \nabla_t \varphi = \vec{K}.$$

Similarly define $\varphi'$ and $\vec{K}'$ for a function $\rho'$.

Then $\rho$ and $\rho'$ are solutions to the following equations:

$$e(x) = \frac{\rho(x)}{2\varepsilon} + \frac{1}{4\pi} \int_S [i\omega\mu(\vec{n}(x)\cdot T\rho(y))\Phi + (\vec{n}(y)-\vec{n}(x))\cdot (T\rho'(y)) \times \nabla\Phi - \frac{1}{\varepsilon}\rho(y)\frac{\partial}{\partial n_x}\Phi] \, ds_y ,$$

$$h(x) = \frac{\rho'(x)}{2\mu} + \frac{1}{4\pi} \int_S [i\omega\varepsilon(\vec{n}(x)\cdot T\rho'(y))\Phi - (\vec{n}(y)-\vec{n}(x)) (T\rho(y)) \times \nabla\Phi - \frac{1}{\mu}\rho'(y)\frac{\partial}{\partial n_x}\Phi] \, ds_y$$

and the explicit expressions for $\vec{E}$ and $\vec{H}$ are given by the formulas:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi} \int_S [i\omega\mu\vec{K}\Phi - \vec{K}' \times \nabla\Phi + \frac{1}{\varepsilon}\rho \, \nabla\Phi] \, ds_y , \quad \nabla = \nabla_y$$

$$\vec{H}(\vec{r}) = \frac{1}{4\pi} \int_S [i\omega\varepsilon\vec{K}'\Phi + \vec{K} \times \nabla\Phi + \frac{1}{\mu}\rho'\nabla\Phi] \, ds_y .$$

Remark: The restriction ii) on $e(y)$ and $h(y)$ is necessary because if $\vec{E}$ and $\vec{H}$ are solutions to the Maxwell's equations, we have by Stoke's Theorem

$$\int_S \vec{n}\cdot\vec{E}\, ds = -\frac{1}{i\omega\varepsilon} \int_S \vec{n}\cdot\nabla \times \vec{H}\, ds = o$$

for a smooth surface $S$.

Similarly,

$$\int_S \vec{n}\cdot\vec{H}\, ds = \frac{1}{i\omega\mu} \int_S \vec{n}\cdot\nabla \times \vec{E}\, ds = o.$$

Our starting formula is the famous Stratton-Chu formulas (Eqs. (8a), (8b) of Chapter I):

$$\vec{E}(\vec{r}) = \frac{1}{4\pi} \int_S [i\omega\mu\vec{K}\Phi - \vec{K}' \times \nabla\Phi + \frac{1}{\varepsilon}\rho\, \nabla\Phi]\, ds, \tag{4.1a}$$

$$\vec{H}(\vec{r}) = \frac{1}{4\pi} \int_S [i\omega\varepsilon\vec{K}'\Phi + \vec{K} \times \nabla\Phi + \frac{1}{\mu}\rho'\, \nabla\Phi]\, ds, \tag{4.1b}$$

where

$$\vec{K} = \vec{n} \times \vec{H}, \quad \vec{K}' = -\vec{n} \times \vec{E},$$

$$\nabla_t\cdot\vec{K} = i\omega\rho, \quad \nabla_t\cdot\vec{K}' = i\omega\rho', \tag{4.1c}$$

$$\Phi = \Phi(\vec{r},\vec{y}) = \frac{e^{ik|\vec{r}-\vec{y}|}}{|\vec{r}-\vec{y}|},$$

$\vec{n}$ is the surface normal pointing into the interior of $S$,

$\vec{r}$ is any exterior point,

$\nabla_t\cdot$ is the surface divergence operator.

Observe that if we take $\vec{K}$ and $\vec{K}'$ to be known and define $\rho$ and $\rho'$ by (4.1c), we can show that (4.1a) and (4.1b) represent two vector fields $\vec{E}$ and $\vec{H}$, which satisfy

Maxwell's equations for points $\vec{r}$ not on S. Furthermore, $\vec{E}$ and $\vec{H}$ in (4.1a) and (4.1b) satisfy the radiation condition. Only the boundary conditions are not yet seen to be satisfied. We are given two relations; consequently there should be two unknowns to be determined. We therefore regard $\rho$ and $\rho'$ as unknowns in (4.1a) and (4.1b) and restrict $\vec{K}$ and $\vec{K}'$ to be the surface gradients of $\varphi$ and $\varphi'$ respectively where the functions $\varphi$ and $\varphi'$ are related to $\rho$ and $\rho'$ as follows:

$$\Delta_t \varphi = i\omega\rho \quad , \quad \vec{K} = \nabla_t \varphi , \qquad (4.2a)$$

$$\Delta_t \varphi' = i\omega\rho' , \quad \vec{K}' = \nabla_t \varphi' .$$

Here $\Delta_t$ is the "surface Laplacian" defined by

$$\Delta_t \varphi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \left( g^{ij} \sqrt{g} \frac{\partial \varphi}{\partial u^j} \right) \qquad (4.3) \quad \text{(sum)}$$

with $g_{ij}$ being the metric coefficients on the surface.

If we can prove that for each $\rho$ there exists one and only one $\vec{K}$ satisfying (4.2a), (and similar result for $\rho'$ and $\vec{K}'$ ), we see that (4.1a) and (4.1b) actually contain two unknowns $\rho$ and $\rho'$. That there is at most one $\vec{K}$ for a given $\rho$ is clear from the results of Theorem 2 of last chapter, since the only continuous solution to $\Delta_t \varphi = o$ is a constant. We therefore complete our solutions by finding $\rho$ and $\rho'$ from*

---

* In the following we shall also write

$$r = \vec{r}, \; x = \vec{x} \text{ and } y = \vec{y}.$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi} \int_S [i\omega\mu\vec{K}\Phi - \vec{K}' \times \nabla\Phi + \frac{1}{\varepsilon} \rho \nabla\Phi] \, ds_y, \; \nabla = \nabla_y \tag{4.4a}$$

$$\vec{H}(\vec{r}) = + \frac{1}{4\pi} \int_S [i\omega\varepsilon\vec{K}'\Phi + \vec{K} \times \nabla\Phi + \frac{1}{\mu} \rho' \nabla\Phi] \, ds_y , \tag{4.4b}$$

where $\vec{K} = \nabla_t \varphi$ , $\vec{K}' = \nabla_t \varphi'$, (4.4c)

$$\Delta_t \varphi = i\omega\rho \; , \; \Delta_t \varphi' = i\omega\rho'$$

and $\nabla_t \varphi$ is the surface gradient of $\varphi$.

Remark: The condition $S \in C^4$ is needed for the validity of the inequality to be given in lemma 2. (See (4.10))

We shall need*

<u>Lemma 1</u>. Let $\mu$ be continuous on $S$ and put

$$P(\vec{r}) = \vec{n}(x) \cdot \int_S \mu(y) \, \nabla_r \Phi(\vec{r},\vec{y}) \, ds_y, \; \vec{r} = \vec{x} + \vec{n}d, \; \vec{x} \in S.$$

Define

$$P_e(\vec{x}) = \lim P(\vec{r}) \quad \text{as} \quad d \to o, \; d > o;$$

$$P_i(\vec{x}) = \lim P(\vec{r}) \quad \text{as} \quad d \to o, \; d < o.$$

Then

$$P_e(x) = -2\pi\mu(x) + P(x) ,$$

$$P_i(x) = 2\pi\mu(x) + P(x) ,$$

where

$$P(x) = \int_S \mu(y) \vec{n}(x) \cdot \nabla_x \Phi(\vec{x},\vec{y}) \, ds_y = \int_S \mu(y) \frac{\partial \Phi(x,y)}{\partial n_x} \, ds_y .$$

This lemma can be proved in the same way as Kellogg [1929, p.164] for $\Phi(\vec{r},\vec{y}) = \dfrac{1}{|\vec{r}-\vec{y}|}$ .

Let us take the scalar product of (4.4a) and (4.4b) with $\vec{n}(x)$ and let $d \to +o$ ,

*From now on $\vec{n}$ will denote the exterior surface normal.

we get

$$\lim_{d\to o^+} \vec{n}(x)\cdot\vec{E}(\vec{x} + d\vec{n}) = e(x) = \frac{\rho(x)}{2\varepsilon} + \lim_{d\to o} \frac{1}{4\pi} \vec{n}\cdot\int_S [i\omega\mu\vec{K}\Phi -$$

$$\vec{K}' \times \nabla\Phi]ds - \frac{1}{4\pi} \int_S \frac{1}{\varepsilon}\rho(y)\frac{\partial}{\partial n_x} \Phi(x,y) \quad ds_y \ , \qquad (4.5a)$$

$$\lim_{d\to o^+}\vec{n}(x)\cdot\vec{H}(\vec{x} + d\vec{n}) = h(x) = \frac{\rho'}{2\mu} + \lim_{d\to o^+}\frac{1}{4\pi} \vec{n}(x)\cdot \int_S [i\omega\mu\vec{K}'\Phi +$$

$$\vec{K} \times \nabla\Phi] \ ds - \frac{1}{4\pi} \int_S \frac{1}{\mu} \rho'(y)\nabla\Phi(x,y) \ ds_y \ . \qquad (4.5b)$$

Since $\Phi(x,y) = O(\frac{1}{|x-y|})$ as $x\to y$, we have

$$\lim_{d\to o} \int_S \vec{n}\cdot\vec{K}(y) \ \Phi(r,y) \ ds_y = \int_S (\vec{n}(x)\cdot\vec{K}(y))\Phi(x,y) \ ds_y \ , \qquad (4.6a)$$

$$\lim_{d\to o} \vec{n}\cdot\int \vec{K} \times \nabla\Phi(r,y) \ ds_y = \lim_{d\to o} \int \vec{n}(x)\cdot\vec{K}(y) \times \nabla\Phi(\vec{x}+d\vec{n},\vec{y}) \ ds_y$$

$$= +\int \vec{n}(x)\cdot\vec{K}(y) \times \nabla_y\Phi(\vec{x}+d\vec{n},y) \ ds_y +$$

$$-\int_S [\vec{n}(y) - \vec{n}(x)]\cdot\vec{K}(y) \times \nabla_y\Phi \quad ds_y \ . \qquad (4.6b)$$

But $K(y) = \nabla_t\varphi(y)$ and

$$\vec{n}(y)\cdot\nabla_t\varphi(y) \times \nabla_y\Phi(\vec{x}+d\vec{n},y) = \vec{n}(y)\cdot\nabla_t\varphi(y) \times \nabla_t\Phi(x+d\vec{n},y);$$

by lemma 3 of Chapter III we have

$$\int \vec{n}\cdot\vec{K}(y) \times \nabla_y\Phi(\vec{x}+d\vec{n},y) \ ds_y = o.$$

Hence

$$\lim_{d\to o} n(x)\cdot \int_S \vec{K}(y) \times \nabla\Phi(\vec{x}+d\vec{n},\vec{y}) \ ds_y = \int_S [\vec{n}(y)-\vec{n}(x)]\cdot\vec{K}(y)\times \nabla_y\Phi \quad ds_y. \qquad (4.7)$$

Using (4.6a) and (4.7), we can write (4.5a) and (4.5b) as

follows:

$$e(x) = \frac{\rho(x)}{2\varepsilon} + \frac{1}{4\pi}\int_S [i\omega\mu(\vec{n}(x)\cdot\vec{K}(y))\Phi + (\vec{n}(y) - \vec{n}(x))\cdot \vec{K}' \times \nabla_y\Phi - \frac{1}{\varepsilon}\rho(y)\frac{\partial}{\partial n_x}\Phi]\, ds_y \quad (4.8a)$$

and

$$h(x) = \frac{\rho'(x)}{2\mu} + \frac{1}{4\pi}\int_S [i\omega\varepsilon(\vec{n}(x)\cdot\vec{K}'(y))\Phi - (\vec{n}(y)-\vec{n}(x))\cdot \vec{K}(y) \times \nabla_y\Phi - \frac{1}{\mu}\rho'(y)\frac{\partial}{\partial n_x}\Phi]\, ds_y\,, \quad (4.8b)$$

where

$$\vec{K} = \nabla_t\varphi \quad , \quad \Delta_t\varphi = i\omega\rho, \quad (4.8c)$$

$$\vec{K} = \nabla_t\varphi' \quad , \quad \Delta_t\varphi' = i\omega\rho'. \quad (4.8d)$$

Equations (4.8a)-(4.8d) can be regarded as four equations for the four unknowns $\rho$, $\rho'$, $\varphi$, and $\varphi'$. However, if we know that we can solve for $\varphi$ and $\varphi'$ from (4.8c) and (4.8d) for given $\rho$ and $\rho'$, we can regard (4.8a) and (4.8b) as two equations for two unknowns. In the language of operator theory we shall prove that the map of $\rho$ to $\vec{K}$ ($\rho'$ to $\vec{K}'$) is a bounded linear operator from the space of continuous functions on $S$ to the space of continuous tangential fields. The first two integral operators in (4.8a) and (4.8b) are completely continuous operators that map continuous tangential fields into cintinuous functions. The last integral operator in (4.8a) and (4.8b) is a completely continuous operator which maps continuous functions on $S$ into continuous functions on $S$. Therefore, if there

exist no non-trivial solutions to the homogeneous integral equations (the conditions in Lemma 7 will insure this), we can conclude that there exist unique $\rho$ and $\rho'$ to the system of integral equations.

To prove the above assertions we shall need

<u>Lemma 2.</u> Suppose $\rho$ is a real, continuous function on $S$ and $\int_S \rho \, ds = o$. There exists a unique $\varphi$ satisfying

$$\Delta_t \varphi = \rho \tag{4.9}$$

almost everywhere with the property

$$\max_{x \in S} |\nabla_t \varphi(x)| \leqslant C \max_{x \in S} |\rho(x)| , \tag{4.10}$$

where $C$ is a constant depending on the surface $S$ only. The existence of a function $\varphi$ satisfying equation (4.9) is known from the theory of harmonic integrals [C. B. Morrey, Jr. and James Eells, Jr. (1955), p.124]. We shall sketch a proof for the existence of the function $\varphi$ and the inequality (4.10) in the appendix.

<u>Lemma 3.</u> Let $K(x,y)$ be defined on $S$ and continuous for $x \neq y$. Assume that positive numbers $A$, $\alpha$, $\tau_0$ exist such that

$$|K(x,y)| \leqslant \frac{A}{|x-y|}$$

for $x, y \in S$ and

$$|K(x_1 y) - K(x_2, y)| \leqslant \frac{B|x_1 - x_2|}{\tau^{\alpha}}$$

for all $\tau$ with $\tau \leq \tau_0$

and all $x_1, x_2, y \in S$ with $|x_1 - x_2| \leq \tau$ and $|x_1 - y| \geq 2\tau$.

Let $\mu(x)$ be continuous on $S$. Define

$\|\mu\| = \max_{x \in S} |\mu(x)|$. If

$$K\mu(x) = \int_S \mu(y)\, K(x,y)\, ds_y,$$

then

$$|K\mu(x)| \leq C \|\mu\|$$

$$|K\mu(x_1) - K\mu(x_2)| \leq C\, |x_1 - x_2|^{\frac{1}{1+\alpha}}$$

for all $x_1, x_2 \in S$ with $|x_1 - x_2| \leq \min [1, \tau_0^{1+\alpha}, (\frac{\alpha}{3})^{1+\tau}]$.

A proof of this lemma can be found in Werner [1961,p.10]. A similar result can also be found in Müller [1957,p.307].

Let us denote the space $B = \{\rho: \rho \in C$ in $S$ and $\int_S \rho\, ds = 0\}$*

and introduce the norm $\|\rho\| = \max_{x \in S} |\rho(x)|$, where $|\,|$ denotes the absolute value of a complex number. The introduction of this norm makes $B$ a Banach space of continuous functions on $S$. Similarly we define $B' = \{\vec{K}: \vec{K} \in C$ in $S\}$ With norm $\|\vec{K}\| = \max |\vec{K}| = \max \sqrt{\vec{K} \cdot \vec{K}^*}$. With this norm

* This shorthand notation means that $B$ consists of all complex continuous functions on $S$ such that the continuous functions satisfy the condition

$$\int_S \rho\, ds = 0 .$$

$B'$ becomes the Banach space of continuous tangential vector fields on $S$. We now prove

<u>Lemma 4.</u> The map $T:\rho\to\vec{K}$ from the Banach space $B$ to $B'$ by means of

$$\Delta_0\varphi = i\omega\rho\,, \qquad \Delta_0\varphi = \vec{K}$$

is linear and bounded.

Proof: Linearity is clear. Since the coefficients of the partial differntial equation are real, both the real and imaginary parts of $\varphi$ must be solutions to the same equation with the right hand side being the real and imaginary parts of $\rho$ respectively.

Let us write

$$\varphi = \varphi_r + i\,\varphi_i\,,$$

$$\rho = \rho_r + i\rho_i\,.$$

By lemma 2 we have

$$\max_{x\in S} |\nabla_t\,\varphi_r| \le C \max_{x\in S} |\rho_r(x)|\,,$$

$$\max_{x\in S} |\nabla_t\,\varphi_i| \le C \max_{x\in S} |\rho_i(x)|\,,$$

these two together give

$$||\nabla_t\varphi|| = ||\vec{K}|| = ||T\rho|| \le C||\rho||\,.$$

Hence $T$ is bounded.

<u>Lemma 5.</u> The following maps have kernels satisfying the conditions of lemma 3.

i) $K_1:B'\to B_* = \{\rho:\rho\in C \text{ in } S\}$ where $K_1(\vec{K}) = \int_S \vec{n}(x)\cdot \vec{K}(y)\Phi(x,y)\,ds_y$ ,

ii) $K_2$: $B' \to B_*$ where $K_2(\vec{K}) = \int_S \vec{K}(y)\cdot[\vec{n}(y)-\vec{n}(x)] \times \nabla_y \Phi \; ds_y$ ,

iii) $K_3$: $B_* \to B_*$ where $K_3(\rho) = \int_S \rho(y) \frac{\partial}{\partial n} \Phi(x,y) \; ds_y$ .

Proof:

i) Since $\Phi(x,y)$ has a singularity of the order $\frac{1}{|x-y|}$ as $x \to y$, we see that there exists $A_1$ such that

$$|\vec{n}(x)\Phi(x,y)| \leq \frac{A_1}{|x-y|}, \quad x,y \in S .$$

ii) Since we assume $S \in C^4$, there exists $A_2'$ such that

$$|\vec{n}(x)-\vec{n}(y)| \leq A_2' \; |x-y|, \; x,y \in S$$

and therefore there exists $A_2$ such that

$$|(\vec{n}(y)-\vec{n}(x)) \times \nabla_y \Phi(x,y)| \leq |\vec{n}(x)-\vec{n}(y)| \; |\nabla_y \Phi(x,y)| \leq \frac{A_2}{|x-y|} .$$

iii) $\frac{\partial}{\partial n_x}\Phi(x,y) = \vec{n}(x)\cdot\nabla_x\Phi(x,y) = \vec{n}(x)\cdot\frac{(\vec{x}-\vec{y})}{|x-y|^3}$ .

If we introduce a rectangular coordinate system with $x$ as origin, and consider the tangent plane through $x$ as the $u^1$-$u^2$ plane, we can represent the surface points in the neighborhood of $x$ as*

$$\vec{y} = u^1\vec{e}_1 + u^2\vec{e}_2 + \vec{n}(x) \; f(u^1,u^2) ,$$

where $\vec{e}_i$ are unit vectors in the tangent plane and $\frac{\partial f(o,o)}{\partial u^i} = o$.

We have

$$\vec{n}(x)\cdot(\vec{x}-\vec{y}) = O((u^1)^2 + (u^2)^2) = O(|x-y|^2)$$

*This is usually referred to as tangent-normal system.

and hence we get

$$\left|\frac{\partial}{\partial n_y}\Phi(x,y)\right| \leq \frac{A_3}{|x-y|}, \quad x,y \in S.$$

The proofs that $K_1$, $K_2$ and $K_3$ have kernels satisfying the second condition of lemma 3 are similar, hence we shall only give the proof for the kernel of $K_1$.

Because of the assumption on $S$, the number $\tau_0$ which is required in lemma 3 always exists. We note that

$$|\vec{n}(x_1)\Phi(x_1,y) - \vec{n}(x_2)\Phi(x_2,y)| \leq |\Phi(x_1,y)-\Phi(x_2,y)| + |\Phi(x_2,y)||\vec{n}(x_1)-\vec{n}(x_2)|.$$

If

$|x_1-x_2| \leq \tau$ and $|x_1-y| \geq 2\tau$, we get

$$|\Phi(x_2,y)||\vec{n}(x_1)-\vec{n}(x_2)| \leq \frac{A_4}{\tau}|\vec{x}_1-\vec{x}_2|.$$

Applying the mean value theorem, we get

$$|\Phi(x_1,y) - \Phi(x_2,y)| \leq \frac{A_5}{\tau^2}|\vec{x}_1-\vec{x}_2|.$$

Consequently, for $\tau \leq \tau_0 < 1$, we get

$$|\vec{n}(x_1)\Phi(x_1,y) - \vec{n}(x_2)\Phi(x_2,y)| \leq \frac{A_5}{\tau^2}|x_1-x_2| \qquad (4.11)$$

Equation (4.11) is the second condition of lemma 3. Hence the proof of lemma 3 is completed.

We now define

$$B \times B = \{(\rho_1,\rho_2) : \rho_i \in B\}.$$

By introducing the norm of $\vec{\rho} = (\rho_1,\rho_2)$ as follows:

$$||\vec{\rho}|| = (\max_{x\in S}|\rho_1| + \max_{x\in S}|\rho_2|) = ||\rho_1|| + ||\rho_2||$$

the set $B \times B$ becomes a Banach space. We can write (4.8a) and (4.8b) as

$$\begin{pmatrix} 2\mathcal{E}\, e(x) \\ 2\mu h(x) \end{pmatrix} = \begin{pmatrix} \rho(x) \\ \rho'(x) \end{pmatrix} + \begin{pmatrix} L_1T + L_3 & L_2T \\ -L_2T & L_1T+L_3 \end{pmatrix} \begin{pmatrix} \rho(x) \\ \rho'(x) \end{pmatrix} \tag{4.12}$$

or symbolically as

$$\vec{e} = \vec{\rho} + T\vec{\rho} = (I + T)\vec{\rho} \tag{4.13}$$

We now prove

<u>Lemma 6.</u> The operator $T$ is a completely continuous linear operator from

$$B \times B \to B \times B,$$

where

$$B = \left\{ \rho : \rho \in C \text{ in } S \text{ and } \int_S \rho \, ds = o \right\}.$$

Linearity is clear from (4.12). In order to prove that an operator is completely continuous, we must show that it maps a bounded sequence into a compact sequence (a sequence which has a convergen subsequence). We first show that $T$ maps $B \times B \to B \times B$. For this we show that

$$\int_S ds_x \overbrace{\left\{ \int_S [i\omega\mu(\vec{n}(x)\cdot\vec{K}(y))\Phi + (\vec{n}(y)-\vec{n}(x))\cdot\vec{K}' \times \nabla_y\Phi - \frac{1}{\mathcal{E}}\rho(y)\frac{\partial}{\partial n_x}\Phi)ds_y \right\}}^{F(x)} = o$$

and

$$\int_S ds_x \overbrace{\left\{ \int_S [i\omega\mathcal{E}(\vec{n}(x)\cdot\vec{K}'(y))\Phi - (\vec{n}(y)-\vec{n}(x))\nabla_y\Phi - \frac{1}{\mu}\rho'(y)\frac{\partial}{\partial n_x}\Phi]ds_y \right\}}^{G(x)} = o,$$

if

$$\int \rho \, ds = \int \rho' ds = o.$$

From the steps by which we derived (4.8a) and (4.8b),

we see that

$$\frac{1}{4\pi}\int_S F(x)\ ds_x = \lim_{d\to o^+}\int_S \vec{n}(x)\cdot\vec{E}(\vec{x}+d\vec{n})ds_x + \int_S \frac{\rho(x)}{2\varepsilon}\ ds_x\ , \qquad (4.14)$$

where $\vec{E}$ is given by (4.1a). It can be shown* that $\vec{E}$ and $\vec{H}$ as given in (4.1a) and (4.1b) satisfy the Maxwell's equations for points not on S. Hence

$$\lim_{d\to o^+}\int_S \vec{n}(x)\cdot\vec{E}(\vec{x}+d\vec{n})ds = \lim_{d\to o^+} \frac{1}{i\omega\varepsilon}\int_S \vec{n}(x)\cdot\nabla \times \vec{H}(\vec{x}+d\vec{n})ds_x = 0. \qquad (4.15)$$

But we are also given

$$\int\rho(x)\ ds_x = \int\rho'(x)\ ds_x = o\ .$$

Hence from (4.14) and (4.15) we get

$$\int_S ds_x\ F(x) = o\ .$$

Similarly,

$$\int_S dx_x\ G(x) = o\ .$$

Having shown that T maps $B \times B \to B \times B$, we now show it is completely continuous. From lemmas 3, 4 and 5, we see that T will map a bounded sequence in $B \times B$ into a bounded, equi-Hölder continuous sequence. This means that the resulting sequence is uniformly bounded and equicontinuous over S. By the theorem of Ascoli [see Kellogg p.265], the resulting sequence will contain a convergent subsequence. The limit of this uniformly convergent subsequence will also

*See Appendix II

be continuous and belongs to $B \times B$. The proof that $T$ is completely continuous is therefore completed.

Since $T$ is completely continuous, we can apply the Fredholm alternatives to discuss the solutions for (4.13). Since we have not succeeded in finding the explicit expression for the adjoint of $T$ in (4.13), we shall limit ourselves to the case when the homogeneous integral equation has no non-trivial solution. The condition of the following lemma will insure that there is no non-trivial solution to the homogeneous integral equation.

<u>Lemma 7.</u> If we assume that $\mu$ and $\omega$ are positive constants and $\mathcal{E} = \mathcal{E}_0 + i\,\mathcal{E}_1$ with $\mathcal{E}_0, \mathcal{E}_1 > 0$*, the integral equation (4.13) [which is equivalent to the system in (4.8a) and (4.8b)] has no non-trivial solution.

Proof: Let $\vec{\rho} = (\rho, \rho')$ be a solution of the homogeneous equation (4.13). We form the electric and magentic fields by means of (4.4a) and (4.4b). $\rho$ and $\rho'$ being solutions of the homogeneous system (4.8a) and (4.8b) imply that

$$(\vec{n}\cdot\vec{E})_e = (\vec{n}\cdot\vec{H})_e = 0 \quad \text{on } S\,;$$

where $(\ )_e$ denotes the values obtained when the points of $S$ are approached from the exterior. By the uniqueness theorem (theorem 1, Chapter III), we see that

* Physically, this means that the medium is lossy.

the electromagnetic field in the exterior of S vanishes identically; i. e.

$$\vec{E}_e = \vec{H}_e = o^{**}. \tag{4.16}$$

From the jump condition, we have

$$\vec{n} \times [\vec{E}_e - \vec{E}_i] = -\vec{K}' = -\nabla_t \varphi', \tag{4.17a}$$

$$\vec{n} \times [\vec{H}_e - \vec{H}_i] = \vec{K} = \nabla_t \varphi . \tag{4.17b}$$

From (4.12) we therefore get

$$\vec{n} \times \vec{E}_i = \vec{K}' = \nabla_t \varphi', \tag{4.18a}$$

$$\vec{n} \times \vec{H}_i = -\vec{K} = -\nabla_t \varphi ; \tag{4.18b}$$

consequently ,

$$\int \vec{n} \cdot (\vec{E}_i \times \vec{H}_i^{\,*}) ds = -\int \vec{n} \cdot (\nabla_t \varphi' \times \nabla_t \varphi^{\,*}) ds = o \tag{4.19}$$

by lemma 3 of Chapter III.

But if the condition on the dielectric constant $\varepsilon$ in the lemma is satisfied, (4.15) implies that $\vec{E}_i = \vec{H}_i = o$.

Now

$$\vec{n} \cdot [\vec{E}_e - \vec{E}_i] = \frac{\rho}{\varepsilon} , \tag{4.20a}$$

$$\vec{n} \cdot [\vec{H}_e - \vec{H}_i] = \frac{\rho'}{\mu} ; \tag{4.20b}$$

therefore we get

$$\rho = \rho' = o$$

as asserted.

If the conditions of lemma 7 are satisfied, the

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** $\vec{E}_e$ is the electric field in the exterior of S.
$\vec{E}_i$ is the electric field in the interior of S.

homogeneous equation (4.13) (or (4.8a) and (4.8b)) has only a trivial solution. Hence by Fredholm's first alternative, there exists a unique solution $\vec{\rho} \in B \times B$ to (4.13) for a given $\vec{e} \in B \times B$. This is the same as saying that the system (4.8a)-(4.8d) have unique solutions if conditions of lemma 7 are satisfied. Having solved for $\rho$ and $\rho'$, we can obtain $\vec{K}$ and $\vec{K}'$; substituting these four quantities into the representations (4.1a) and (4.1b), we have the desired solutions to the Maxwell's equations.

We shall remark briefly on the limiting values of the electric and the magnetic fields as we approach from the exterior or the interior of S. By the assumption that e(x) and h(x) are Hölder continuous, continuous solutions for $\rho$ and $\rho'$ from (4.8a) and (4.8b) are then Hölder continuous; $\vec{K}$ and $\vec{K}'$ are Hölder continuous if $\rho$ and $\rho'$ are continuous. Therefore we conclude (see theorem 48 p.217 of Müller) that the electric and the magnetic fields given in the statement of the existence theorem are continuous up to and on S, as we approach from either side of S.

Given a smooth surface S, the results of this chapter show that we can always decompose an electromagnetic field into two fields, one of which has no normal component of the electric field while the other has no normal component of the magnetic field on the surface S. This should be compared with the conclusions in Chapter II that only in

spherical, cylindrical or rectangular coordinates can we have T.E. or T.M. wave.

# Appendix I

We shall sketch the proof of lemma 2 of Chapter IV. Specifically, we shall show

<u>Theorem.</u> Let $\rho$ be a continuous function on $S$ and $\int_S \rho \, ds = o$. Let $S \in C^4$. Then there exists a unique $\varphi$ satisfying the following equation

$$\Delta_t \varphi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \left( g^{ij} \sqrt{g} \frac{\partial \varphi}{\partial u^j} \right) = -\rho, \quad 1 \leq i, j \leq 2 \qquad (A.1)$$

and

$$\int_S \varphi \, ds = o.$$

Furthermore, there exists a constant $C$ depending on the surface $S$ such that

$$\max_{x \in S} |\nabla_t \varphi(x)| \leq C \max_{x \in S} |\rho(x)|.$$

Proof: With no loss of generality, we may assume $\rho(x)$ to be real. We shall use the lemma of Lax and Milgram to prove that there exists a "weak" solution to (A.1). Then we shall show that this weak solution satisfies (A.1) almost everywhere.

By the assumption on the surface $S$, we can, by the Heine-Borel Theorem, cover it with a finite number of open coordinate patches. We denote a covering by $\mathcal{U} = \bigcup_{\alpha=1}^{N} U_\alpha \supset S$. Each of the coordinate patches is taken to be small enough that there exists a coordinate system

$x_\alpha^1 - x_\alpha^2$ such that the following lemma holds:

<u>Lemma α.</u> There exist positive $R_\alpha, r_\alpha, M_\alpha$, and $\mathcal{M}_\alpha$ such that

i) $R_\alpha > \sqrt{g(x)} > r_\alpha$, for $x \in U_\alpha$ ;

ii) $g^{ij}(y_\alpha) = \delta^{ij}$, for a $y_\alpha \in U_\alpha$ ;

iii) $\mathcal{M}_\alpha(\xi_1^2 + \xi_2^2) \leq g^{ij}(x)\, \xi_i\, \xi_j \leq M_\alpha(\xi_1^2 + \xi_2^2)$,

for $x \in U_\alpha$, $1 \leq \alpha \leq N$, and $|\xi_1^2| + |\xi_2^2| \neq o$.

We now choose such a covering $\mathcal{U}$ for the surface S and define a space $P_{20}$ of functions.

<u>Definition:</u> The function $f \in P_2$ if and only if

i) $f \in L_2(S)$;

ii) There exists a sequence of $C^1$ functions such that

$$\lim_{n\to\infty} ||f - f_n||^2 = \lim_{n\to\infty} \int_S (f - f_n)^2\, ds = o,$$

$$\lim_{n,m\to\infty} ||Df_n - Df_m||_{L_2} = o,$$

where D is any first derivative.

We define a scalar product in $P_2$ as follows:

$$\langle \varphi, \psi \rangle = \sum_{\alpha=1}^{N} \int_{U_\alpha} [\varphi\psi + \frac{\partial\varphi}{\partial x^1}\frac{\partial\psi}{\partial x^1} + \frac{\partial\varphi}{\partial x^2}\frac{\partial\psi}{\partial x^2}]\, dx^1 dx^2$$

$$= \int_{\mathcal{U}} [\varphi\psi + \nabla\varphi \cdot \nabla\psi]\, dx.$$

The norm of $\varphi \in P_2$ is defined as follows:

$$||\varphi||^2 = \langle \varphi, \varphi \rangle .$$

If in addition,

iii) $\int_S f\, ds = o$,

we say $f \in P_{20}$.

With this definition of scalar product, $P_{20}$ becomes a Hilbert space of functions with properties satisfying i), ii), and iii). Furthermore, we note that the functions of $P_{20}$ are limits of sequences of $C^1$ functions in S. Therefore, for calculations we can take the functions to be of class $C^1$.

<u>Lemma β.</u> (Poincaré's inequality) If $u \in P_{20}$, then there exists a constant $C_1 > o$ such that

$$C_1 \int_{\mathcal{U}} [(\frac{\partial u}{\partial x^1})^2 + (\frac{\partial u}{\partial x^2})^2\, dx \geq \int_{\mathcal{U}} u^2\, dx.$$

Here, $C_1$ depends only on the surface S.

Proof: We shall prove this inequality by contradiction. Suppose the inequality is not true; then there exists a sequence of functions $\{f_n\} \in P_{20}$ such that

$$\int_{\mathcal{U}} f_n^2\, dx = 1 \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathcal{U}} [(\frac{\partial f_n}{\partial x^1})^2 + (\frac{\partial f_n}{\partial x^2})^2]\, dx = o.$$

This means that the sequence $\{f_n\}$ is bounded [in the sense of the norm we introduced for functions in $P_{20}$]. Consequently, the sequence $\{f_n\}$ contains a subsequence, which will also be denoted by $\{f_n\}$, converging weakly to a function $f \in P_{20}$; i.e. there exists $f \in P_{20}$ such that*

$$\lim_{n\to\infty} \left\{ \int_{\mathcal{U}} g f_n\, dx + \int_{\mathcal{U}} \nabla g \cdot \nabla f_n\, dx \right\} = \int_{\mathcal{U}} fg\, dx + \int_{\mathcal{U}} \nabla f \cdot \nabla g\, dx, \qquad (A.2)$$

for all $g \in P_{20}$. But it can be proved [see Morrey (1956)]

*We denote $\nabla f \cdot \nabla g = \frac{\partial f}{\partial x^1}\frac{\partial g}{\partial x^1} + \frac{\partial f}{\partial x^2}\frac{\partial g}{\partial x^2}$

that if $\{f_n\}$ converges weakly to $f$ in $P_{20}$, $\{f_n\}$ converges strongly to $f$ in $L_2$. This means

$$\lim_{n\to\infty} \int_{\mathcal{U}} |f_n - f|^2 \, dx = o. \tag{A.3}$$

But (A.3) implies

$$\lim_{n\to\infty} \int_{\mathcal{U}} |f_n|^2 \, dx = \int_{\mathcal{U}} f^2 \, dx. \tag{A.4}$$

This being the case, we get from (A.2)

$$\lim_{n\to\infty} \int_{\mathcal{U}} \nabla g \cdot \nabla f_n \, dx = \int_{\mathcal{U}} \nabla g \cdot \nabla f \, dx. \tag{A.5}$$

But by the assumption on $\nabla f_n$, we get

$$\lim_{n\to\infty} \left| \int_{\mathcal{U}} \nabla g \cdot \nabla f_n \, dx \right| \leq \lim_{n\to\infty} \sqrt{\int_{\mathcal{U}} |\nabla g|^2 dx \int_{\mathcal{U}} |\nabla f_n|^2 \, dx} \to o.$$

Hence

$$\int_{\mathcal{U}} \nabla f \cdot \nabla g \, dx = o, \quad \text{for all} \quad g \in P_{20}. \tag{A.5a}$$

If now we choose $g = f$, we see that (A.5a) implies that

$$f = \text{constant}.$$

But by the requirement that $\int_S f \, ds = o$, we have

$$f = o \quad \text{on} \quad S.$$

Since we have shown in (A.4) that

$$\int_{\mathcal{U}} |f|^2 \, dx = \lim_{n\to\infty} \int_{\mathcal{U}} |f_n|^2 \, dx = 1. \tag{A.6}$$

(A.6) says that $f$ cannot vanish identically on $S$. This contradiction proves the lemma.

By means of lemma $\alpha$ we can immediately show

<u>Lemma $\gamma$.</u> There exist positive constants $A_1$, $a_1$, $A_2$, $a_2$, $A_3$, and $a_3$ such that for $\varphi \in P_2$,

$$a_1 \int_S \varphi^2 \, ds \leq \int_{\mathcal{U}} \varphi^2 \, dx \leq A_1 \int_S \varphi^2 \, ds,$$

$$a_2 \int_S |\nabla_t \varphi|^2 \, ds \leq \int_{\mathcal{U}} |\nabla \varphi|^2 \, dx \leq A_2 \int_S |\nabla_t \varphi|^2 \, ds,$$

$$a_3 \int_S [\varphi^2 + |\nabla_t \varphi|^2] ds \leq ||\varphi||^2 \leq A_3 \int_S [\varphi^2 + [\nabla_t \varphi|^2] ds.$$

<u>Lemma $\lambda$.</u> [Lax and Milgram]. Let $H$ be a Hilbert space, $B(u,v)$ a bilinear functional in $H$ such that

i) $|B(u,v)| \leq K \, ||u|| \, ||v||$,

ii) $K_1 ||u||^2 \leq |B(u,u)|$,

for some constants $K, K_1 > o$. Then for every $u \in H$, there exists a $u' \in H$ and conversely for every $u' \in H$, there exists a $u \in H$ such that

$$B(u,v) = \langle u', v \rangle,$$

for all $v \in H$.

For a proof to this lemma, we refer the reader to Hellwig [1960, p.203].

Using the Lax-Milgram lemma, we show that there exists a function $\varphi$ such that

$$\int_S (\nabla_t \varphi \cdot \nabla_t \psi) \, ds = \int_S \psi \, \rho \, ds, \tag{A.7}$$

for all $\psi \in P_{20} = H$.

In order to show this we define $B(\varphi, \psi)$ for $\varphi, \psi \in P_{20}$ as follows:

$$B(\varphi, \psi) = \int_S (\nabla_t \varphi \cdot \nabla_t \psi)\, ds. \tag{A.8}$$

By Schwarz's inequality and Lemma $\gamma$ we get

$$|B(\varphi,\psi)| = |\int_S (\nabla_t \varphi \cdot \nabla_t \psi)\, ds| \leqslant \sqrt{\int_S |\nabla_t \varphi|^2 ds \int_S |\nabla_t \psi|^2 ds}$$

$$\leqslant K\, ||\varphi||\, ||\psi||. \tag{A.9}$$

By Poincaré's inequality and Lemma $\gamma$, we get

$$||\varphi||^2 = \int_{\mathcal{U}} (\varphi^2 + |\nabla \varphi|^2)\, dx \leqslant C \int_{\mathcal{U}} |\nabla \varphi|^2\, dx \tag{A.10}$$

$$\leqslant \frac{1}{K_1} |B(\varphi, \varphi)|$$

or

$$|B(\varphi, \varphi)| \geqslant K_1\, ||\varphi||^2. \tag{A.11}$$

(A.9) and (A.11) are the conditions satisfied by our bilinear functional $B(u,v)$. Therefore, for each $u' \in P_{20}$ there exists a $u \in P_{20}$ such that

$$B(v,u) = \langle v,u' \rangle, \text{ for all } v \in P_{20}. \tag{A.12}$$

But the right-hand side of (A.7) is a bounded linear functional on $P_{20}$ for a given $\rho \in P_{20}$, since

$$|\int_S v\rho\, ds| \leqslant C\, ||\rho||\, ||v||.$$

Therefore, there exists a $u' \in P_{20}$ such that

$$\int_S v\rho\, ds = \langle v,u' \rangle, \quad \text{for all } v \in P_{20}. \tag{A.13}$$

From (A.12) and (A.13) we see that there exists a $u$ such that

$$B(v,u) = \int_S \nabla_t v \cdot \nabla_t u\, ds = \int_S v\rho\, ds, \quad \text{for all } v \in P_{20}. \tag{A.14}$$

We observe from (A.14) that if $u$ has second continuous derivatives, we can integrate by parts (using Theorem 3 of Chapter I) to get

$$\int_S v\, \Delta_t\, u\, ds = -\int_S v\rho\, ds, \tag{A.14a}$$

for all $v \in P_{20}$. That $u$ has continuous second derivatives everywhere cannot be proved unless we put more restrictions on the behavior of $\rho$ in (A.1). Since we assume $S \in C^4$, the coefficients of the operator are at least $C^2$. From Theorem (4.3) of Morrey (1956, p.47) and Theorem (4.7) of Morrey (1954, p.129), we conclude that the function $u$ as given in (A.14) will have Hölder continuous first derivatives and the derivatives belong to $P_2$. Hence (A.14a) holds for $u$. (A.14a) implies that

$$\Delta_t u = -\rho + C \tag{A.14b}$$

almost everywhere. The constant $C$ in (A.14b) is zero since

$\int_S \Delta_t\, u\, ds = \int_S \nabla_t \cdot (\nabla_t u)\, ds = o$ and by assumption

$\int_S \rho\, ds = o.$

Consequently, we obtain a function $\varphi$ satisfying (A.1) almost everywhere.

To prove the inequality

$$\max_{x \in S} |\nabla_t \varphi| \leq C \max_{x \in S} |\rho|, \tag{A.14c}$$

we may use the Heine-Borel Theorem to cover the surface $S$

a finite union of open sets $u_\alpha (\alpha=1,\ldots N)$. Let $\bar{D}_\alpha$ be a closed set such that $\bar{D}_\alpha \subset U_\alpha$ and such that $\bigcup_{\alpha=1}^{N} \bar{D}_\alpha \supset S$.

Let us consider one of the sets $\bar{D}_\alpha$'s and call it $\bar{D} = \bar{D}_\alpha \subset U_\alpha = U$. Let $\tilde{D}$ be another closed set such that $\bar{D} \subset U' \subset \tilde{D} \subset U$, where $U'$ is an open set in $U$. Let $u^1$ and $u^2$ be parameters for the surface element $U$. We choose $u^1$ and $u^2$ such that $g^{ij}(\xi^1, \xi^2) = \delta^{ij}$ for a point $(\xi^1, \xi^2) \in \bar{D}$. If $U$ is small enough, there exists a Green's function $G(u^1,u^2; \xi^1, \xi^2)$ which is a solution to

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \left(g^{ij} \sqrt{g} \quad \frac{\partial \varphi}{\partial u^j}\right) = -\delta(\xi^1 - u^1)\,\delta(\xi^2 - u^2). \qquad (A.15)$$

The existence of a function $G(u^1,u^2; \xi^1, \xi^2)$ satisfying (A.15) has been proved by E. E. Levi (1907) by using the method of the parametrix if $g^{ij} \in C^3$; this means that the surface has to be of class $C^4$. Also, F. John [1950] has proved, by the Cauchy-Kowalewski Theorem, the existence of a Green's function for linear elliptic differential equations with analytic coefficients for a small region. For our equation, the Green s function will behave like:

$$G(u^1,u^2; \xi^1, \xi^2) \approx \frac{1}{2\pi} \log \sqrt{(u^1 - \xi^1)^2 + (u^2 - \xi^2)^2}$$

in the neighborhood of the point $(\xi^1, \xi^2)$. We construct a function $\zeta$ such that $\zeta \in C^2$ and

$$\zeta(x) = \begin{cases} 1 & \text{for } x \in \bar{D} \\ 0 & \text{for } x \in U - \tilde{D} \end{cases}$$

We use the Green's identity on a region $U-\Sigma_\varepsilon$, where $\Sigma_\varepsilon$ is a small circle excluding the point $(\xi^1, \xi^2)$, to get

$$\int_{U-\Sigma_\varepsilon} [\zeta(x)G(x,\xi)\Delta_t\varphi(x) - \varphi(x)\Delta_t(\zeta(x)G(x,\xi))]\, ds_x$$
$$= \left(\int_C + \int_{C_\varepsilon}\right)\left[\varphi(x)\frac{\partial}{\partial n_o}(\zeta(x)G(x,\xi)) - \frac{\partial\varphi}{\partial n_o}\zeta(x)G(x,\xi)\right]dl_x, \tag{A.16}$$

where $C$ is the boundary curve of $u$, $C_\varepsilon$ is the boundary curve of $\Sigma_\varepsilon$ and $\vec{n}_o$ is the surface tangent vector which is normal to the boundary and which points away from the region $U-\Sigma_\varepsilon$. Since $\zeta(x) = o$ in the neighborhood of $C$, as $\varepsilon \to o$, we get from (A.16)

$$\varphi(\xi) = \int_U \zeta(x)G(x,\xi)\rho(x)ds_x - \int_{U-\bar{D}} \varphi(x)\Delta_t(\zeta(x)G(x,\xi))ds_x, \tag{A.17}$$

for $\xi = (\xi^1, \xi^2) \in \bar{D}$. From (A.17) we get

$$|D_\xi \varphi(\xi)| \leq \left|\int_U \zeta(x)\rho(x)\, D_\xi G(x,\xi)\, ds_x\right| +$$
$$\int_{U-\bar{D}} \varphi(x)\, D_\xi\, \Delta_t\, (\zeta(x)G(x,\xi))\, ds_x\Big|, \tag{A.18}$$

where $D_\xi$ is any derivative. If the region $U$ is small enough, there exists a constant $K$ such that for $\xi = (\xi^1, \xi^2) \in \bar{D}$, we have

$$|\nabla_t \varphi(\xi)|^2 \leq K\,[D_{\xi^1}\varphi(\xi))^2 + (D_{\xi^2}\varphi(\xi))^2]. \tag{A.19}$$

But from (A.18) we get

$$|D_{\xi^1}\varphi(\xi)| \leq K_1 \max_{x\in U} |\rho(x)| + K_2 \max_{x\in U} |\varphi(x)|. \tag{A.20}$$

(A.20) is true for all $\xi \in \bar{D}$; hence from (A.19) we have

$$\max_{x \in \bar{D}} |\nabla_t \varphi(\xi)| \leq K_4 \max_{x \in U} |\rho(x)| + K_5 \max_{x \in U} |\varphi(x)|. \qquad (A.21)$$

In each of the open sets, the union of which covers S, (A.21) holds with appropriate constants. This implies that there exist $K_6$ and $K_7$ such that

$$\max_{x \in S} |\nabla_t \varphi(x)| \leq K_6 \max_{x \in S} |\rho(x)| + K_7 \max_{x \in S} |\varphi(x)|, \qquad (A.22)$$

where $K_6$ and $K_7$ are constants depending on the surface and on a decomposition of the surface.

For any function $_n \in C$ such that $\int_S \rho_n ds = o$, we denote by $\varphi_n$ the corresponding solution such that $\int_S \varphi_n ds = o$. Suppose there exists no constant C such that

$$\max_{x \in S} |\nabla_t \varphi(x)| \leq C \max|\rho(x)|$$

holds; then there exists a sequence of functions $\{\rho_n\}$ with $\rho_n \in C$ and $\int_S \rho_n ds = o$ for which $\lim_{n \to \infty} |\rho_n(x)| = o$ and such that the sequence of functions $\{\varphi_n\}$ satisfying $\max_{x \in S} |\nabla_t \varphi_n| = 1$. The sequence $\{\varphi_n\}$ is therefore uniformly bounded and equicontinuous; consequently, it contains a subsequence converging uniformly to a function $\psi$; this function $\psi$ will be a solution to (A.1) with $\rho = o$. [See Morrey (1956) p.45]. But we have proved in Theorem 2 of Chapter III that $\psi \equiv o$ on S. This means that a subsequence of $\{\varphi_n\}$ [also denoted by $\{\varphi_n\}$] converges uniformly to zero. Consequently, for $n > N_o$, we have $\varepsilon > |\varphi_n(x)|$ and

$\varepsilon_1 > |\rho_n(x)|$ for all $x \in S$. Therefore, [see (A.22)]

$$1 = \max_{x \in S} |\nabla_o \varphi_n(x)| \leqslant K_6 \max_{x \in S} |\rho_n(x)| + K_7 \max_{x \in S} |\varphi_n(x)|$$

cannot hold for $n > N_o$. This contradiction proves the inequality of the Theorem.

## Appendix II

We want to show in this Appendix that $\vec{E}$ and $\vec{H}$ as given in (4.4a)-(4.4c) satisfy Maxwell's equations (Müller p.211) for points not on S. We have

$$\vec{E} = \frac{1}{4\pi} \int_S [i\omega\mu\, \vec{K} - \vec{K}' \times \nabla\Phi + \frac{1}{\varepsilon}\, \rho\nabla\Phi]\, ds_y \;.\; \nabla = \nabla_y \qquad (B.1)$$

$$\vec{H} = \frac{1}{4\pi} \int_S [i\omega\varepsilon\, \vec{K}' + \vec{K} \times \nabla\Phi + \frac{1}{\mu}\, \rho'\, \nabla\Phi]\, ds_y \;; \qquad (B.2)$$

where

$$\vec{K} = \nabla_t \varphi \quad , \; \vec{K}' = \nabla_t \varphi',$$

$$\Delta_t \varphi = \nabla_t \cdot \vec{K} = i\omega\rho \;,\; \Delta_t \varphi' = \nabla_t \cdot \vec{K}' = i\omega\rho'.$$

In the above formulas the variable point is denoted by $\vec{r}$ or r, and the integration variable is denoted by $\vec{y}$ or y. We take the curl of $\vec{E}$ to get

$$\nabla_r \times \vec{E} = \frac{1}{4\pi} \int_S i\mu\omega \nabla_r \times (\vec{K}(y)\Phi(r,y))ds_y - \nabla_r \times (\nabla_r \times \int_S \vec{K}'(y)\Phi(r,y)ds_y) \;. \qquad (B.3)$$

The last term of (B.1) has no curl since it is the gradient of some function. Using $\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla\cdot\vec{v}) - \Delta\vec{v}$, and noting that

$$\Delta_r \Phi(r,y) = -k^2\Phi(r,y) \quad \text{for} \quad r \neq y,$$

we may write (B.3) as follows:

$$\nabla_r \times \vec{E} = \frac{i\omega\mu}{4\pi} \int_S (i\omega\varepsilon\, \vec{K}'(y)\Phi(r,y) + \vec{K}(y) \times \nabla_y\Phi(r,y))ds_y - \frac{1}{4\pi} \nabla_r(\nabla_r \cdot \int_S \vec{K}'(y)\Phi(r,y)ds_y) \;. \qquad (B.4)$$

Using Theorem 3 of Chapter I, we see that

$$\nabla_r \cdot \int_S \vec{K}'(y)\Phi(r,y)ds_y = \int_S \vec{K}'(y)\cdot\nabla_r\Phi(r,y)ds_y = -\int_S \vec{K}'\cdot\nabla_{ty}\Phi(r,y)ds_y$$

$$= -\int_S \nabla_{ty}\cdot\rho\vec{K}\, ds_y + \int_S \Phi(r,y)\nabla_{ty}\cdot\vec{K}'(y)ds_y =$$

$$i\omega \int_S \rho'(y)\Phi(r,y)ds_y \ .$$

Hence we get

$$\nabla_r(\nabla_r\cdot\int_S \vec{K}'\Phi\, ds_y) = -i\omega\int_S \rho'\, \nabla_y\Phi\, ds_y \ . \qquad (B.5)$$

Substituting (B.5) into (B.4) we get

$$\nabla_r \times \vec{E} = \frac{i\omega\mu}{4\pi}\int_S [i\omega\epsilon\vec{K}'(y)\Phi(r,y) + \vec{K}(y) \times \nabla\Phi(r,y) +$$

$$\frac{1}{\mu}\rho'\nabla\Phi(r,y)]\, ds_y = i\omega\mu\, \vec{H} \ .$$

Similarly we can prove

$$\nabla \times \vec{H} + i\omega\epsilon\vec{E} = 0 \ .$$

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