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A METHOD FOR USING MEASURED RATE TO IMPROVE RADAR RANGE DATA

BY

WILLIAM L. SHEPHERD

USA ERDA-4

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NEW MEXICO
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DA Project 1-G-2-50206-D-241

INSTRUMENTATION DEPARTMENT
U. S. ARMY ELECTRONICS RESEARCH AND DEVELOPMENT ACTIVITY
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NEW MEXICO
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Pulse type radars are presently being built which incorporate Doppler techniques for measuring range rate. These measurements will be independent of and in addition to measurements of range. It is anticipated that measured range rates will be much more precise than measured ranges.

In this report we develop two least squares methods which will permit the use of range rate information to obtain improved estimates of range.

The first method consists of fitting a least squares approximating polynomial to range rate data without reference to range, integrating the polynomial, and determining the constant of integration so as to obtain the integral curve which best fits the range data in the sense of least squares. It is to be noted that this does not involve numerical integration from a tie-in point.

The second method is an adaptation of the interpolating polynomial which fits the entire collection of range and range rate data. A simple weighted least squares approximating polynomial is fitted to the entire collection of data.

The first method permits one to smooth range rate information without degradation caused by imprecise range. The introduction of the simple weights in the second method permits some flexibility in adapting the method to particular cases where measures of precision for the range and for the range rate are available.

For greater ease of presentation only quadratic fits to range are considered.
Suppose we have a set of measured values of range from a radar site of a moving point, and a corresponding set of independently measured values of range rate, according to the following table.

<table>
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<tr>
<th>Time</th>
<th>Range</th>
<th>Range Rate</th>
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</thead>
<tbody>
<tr>
<td>-kh,</td>
<td>r_{-k}, f_{-k}</td>
<td></td>
</tr>
<tr>
<td>(-k + 1)h</td>
<td>r_{-k + 1}, f_{-k + 1}</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>0</td>
<td>r_{0}, f_{0}</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>(k - 1)h</td>
<td>r_{k - 1}, f_{k - 1}</td>
<td></td>
</tr>
<tr>
<td>kh</td>
<td>r_{k}, f_{k}</td>
<td></td>
</tr>
</tbody>
</table>

Notice we have a table with an odd number of columns. This restriction, though not necessary, permits desirable simplifications of form.

Let the required approximating polynomial be

\[ r(t) = a_0 + a_1 t + a_2 t^2, \]

with derivative

\[ f(t) = a_1 + 2a_2 t. \]

We shall determine \( a_1 \) and \( a_2 \) so that \( f(t) \) is the least squares fit to the rate data, then determine \( a_0 \) so that \( r(t) \) is the least squares fit to the range data.

Set

\[ g(a_1, a_2) = \frac{1}{2} \sum_{i=-k}^{k} (a_1 + 2a_2 h_i f_i - f_i)^2. \]

The normal equations for (2) are

\[ \frac{\partial g(a_1, a_2)}{\partial a_1} = \frac{\partial g(a_1, a_2)}{\partial a_2} = 0. \]
Thus
\[ \sum_{i=-k}^{k} (a_i + 2a_2 h_i - f_i) = 0, \] (3)
\[ \sum_{i=-k}^{k} (a_i + 2a_2 h_i - f_i) 2h_i = 0. \] (4)

Set
\[ \sum_{i=-k}^{k} i f_i = g(k). \]

Using \( g(k) = 0 \) and solving (3) and (4) for \( a_1 \) and \( a_2 \), we get
\[ a_1 = \frac{1}{2k+1} \sum_{i=-k}^{k} f_i \] (5)
\[ a_2 = \frac{1}{2h} \frac{g_2(k)}{s_2} \sum_{i=-k}^{k} i f_i. \] (6)

We now determine \( a_0 \) so that \( r(t) \) is the least squares fit to the range data under the constraint that \( a_1 \) and \( a_2 \) are given by (5) and (6).

Set
\[ f(a_0) = \frac{1}{2} \sum_{i=-k}^{k} (a_0 + a_1 ih + a_2 i^2 h^2 - r_i)^2. \]

The normal equation for (1) is
\[ \frac{\partial f}{\partial a_0} = \sum_{i=-k}^{k} (a_0 + a_1 ih + a_2 i^2 h^2 - r_i) = 0. \] (7)
Then
\[ a_0 = \frac{1}{2k+1} \sum_{i=-k}^{k} r_i - \frac{h^2 s^{(k)}}{2k+1} a_2. \]  

(8)

Equation (1), with \( a_1 \) and \( a_2 \) given by (5) and (6) and \( a_0 \) given by (8) gives a formula for smoothed range whose derivative must be the smoothed range rate obtained from the independently measured range rate.

To obtain \( r(0) \) and \( \dot{r}(0) \), the values for smoothed range and range rate at the mid-point of the smoothing interval, we set \( t = 0 \) in (1) and (2) and get

\[ \dot{r}(0) = a_1 \]  

(9)

\[ r(0) = a_0 \]  

(10)

Inspection of equations (10), (9), (8), (6), and (5) shows that \( r(0) \) and \( \dot{r}(0) \) may each be expressed as a linear combination of \( x_{-k}, \ldots, x_k \), \( \dot{x}_{-k}, \ldots, \dot{x}_k \).

APPROXIMATING QUADRATIC FOR THE INTERPOLATING POLYNOMIAL

We again refer to the table and equations (1) and (2). Introduce the common weighting factor, \( c \), for the residuals for range and the common factor, \( d \), for the residuals for range rate, and set

\[ f(a_0, a_1, a_2) = c^2 \sum_{i=-k}^{k} \left[ r(ih) - r_1 \right]^2 + d^2 \sum_{i=-k}^{k} \left[ \dot{r}(ih) - \dot{r}_1 \right]^2. \]  

(11)
The normal equations

\[ \frac{\partial f}{\partial a_j} = 0 \quad j = 1, 2, 3 \]

may be written after slight simplification as

\[ \sum_{i=-k}^{k} (a_o + a_1 h_i + a_2 h^2 i^2 - r_i) = 0 , \quad (12) \]

\[ \sqrt{h} \sum_{i=-k}^{k} (a_o + a_1 h_i + a_2 h^2 i^2 - r_i) i + \sqrt{d} \sum_{i=-k}^{k} (a_1 + 2a_2 h_i - f_i) = 0 , \quad (13) \]

\[ \sqrt{h} \sum_{i=-k}^{k} (a_o + a_1 h_i + a_2 h^2 i^2 - r_i) i^2 + 2\sqrt{d} \sum_{i=-k}^{k} (a_1 + 2a_2 h_i - f_i) i = 0 . \quad (14) \]

Using

\[ s_p^{(k)} = 0, \quad p = 1, 3 \]

and collecting coefficients of \( a_o, a_1, a_2, \) in equations (12), (13) and (14), we get

\[ s_0^{(k)} a_o + \sqrt{h} s_2^{(k)} a_2 = \sum_{i=-k}^{k} r_i , \quad (15) \]

\[ (\sqrt{h} s_2^{(k)} + d \sqrt{h} s_1^{(k)}) a_1 = \sqrt{h} \sum_{i=-k}^{k} i r_i + d \sum_{i=-k}^{k} i f_i , \quad (16) \]

\[ \sqrt{h} e_2^{(k)} a_o + (e_4^{(k)} + 4d e_2^{(k)}) a_2 = e_2 \sum_{i=-k}^{k} i^2 r_i + 2d e_2 \sum_{i=-k}^{k} i f_i . \quad (17) \]
\[
D = \begin{vmatrix}
    s_0(k) & s_2(k) \\
    h_c s_2(k) & mh_s s_4(k) + rh_d s_8(k)
\end{vmatrix} = h^2 \begin{vmatrix}
    s_0(k) & s_2(k) \\
    h_c s_2(k) & mh_c s_4(k) + rh_d s_8(k)
\end{vmatrix}
\]

\[= h^2 \left\{ h_c s_2 \left[ s_0(k) s_4(k) - (s_2(k))^2 \right] + rh_d s_0(k) s_2(k) \right\}. \quad (18)\]

It can be shown that \(D > 0\) for \(k > 1\). Hence, the normal equations have a unique solution:

\[
a_o = \frac{1}{D} \begin{vmatrix}
    \sum_{i=-k}^{k} r_i & h_c s_2(k) \\
    c_h^2 \sum_{i=-k}^{k} i^2 r_i + 2d_h \sum_{i=-k}^{k} i f_i & h_c^2 s_4(k) + rh_d s_8(k)
\end{vmatrix}, \quad (19)\]

\[
a_1 = \left( h_c^2 \sum_{i=-k}^{k} i r_i + d^2 \sum_{i=-k}^{k} f_i \right) \div (h_c^2 s_2(k) + d^2 s_0(k)), \quad (20)\]

\[
a_2 = \frac{1}{D} \begin{vmatrix}
    s_0(k) & \sum_{i=-k}^{k} r_i \\
    h_c s_2(k), c_h^2 \sum_{i=-k}^{k} i^2 r_i + 2d_h \sum_{i=-k}^{k} i f_i
\end{vmatrix}. \quad (21)\]

Clearly \(a_o\), \(a_1\), and \(a_2\) can be expressed as explicit linear combinations of \((r_i, f_i)\). Also, \(a_o\), \(a_1\), and \(a_2\) are homogeneous and of degree 0 in \(c, d\). Hence, only the ratio of \(c\) and \(d\) need be specified.
The equations
\[ r(0) = a_0 \tag{22} \]
\[ f(0) = a_1 \tag{23} \]

with \( a_0 \) and \( a_1 \) given by equations (19) and (20) give the mid-point smoothed range and range rate. For mid-point range and range rate only, it is not necessary to obtain \( a_2 \).

**SUMMARY**

In this report two methods of treating a combination of measured radar range and range rate are discussed. Each uses least squares theory, but the first permits independent smoothing of range rate. Each method anticipates a situation likely to occur in radar instrumentation.
Approval. Technical Report USA ERDA-4 has been reviewed and approved for publication:

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FOR THE COMMANDER:

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Adjutant
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