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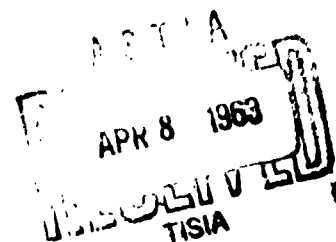
DYNAMIC PROGRAMMING
AND MATHEMATICAL ECONOMICS

Richard Bellman

The RAND *Corporation*

SANTA MONICA • CALIFORNIA

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PREFACE

In this Memorandum, the author describes the uses and contributions of the mathematical theory known as dynamic programming to certain problems in economics. Examples of these are the optimal allocation of resources, and multistage decision processes that involve planning and learning in the face of uncertainty (i.e., adaptive control processes).

SUMMARY

The functions of a mathematical theory in a scientific field are to furnish a systematic means of formulation of classes of problems, to indicate various techniques for their analysis, and to provide methods for obtaining numerical answers to numerical questions.

At one point in the nineteenth century, serious doubt was expressed that problems in the field of economics could ever be handled mathematically. The introduction of the digital computer changed the situation drastically. Nevertheless, much remains to be done, and many new approaches must be devised, before we can consider ourselves to have a firm hold in the domain of mathematical economics.

In this Memorandum we outline briefly some of the principal contributions of the theory of dynamic programming to the formulation, analysis, and computational treatment of economic processes.

CONTENTS

DYNAMIC PROGRAMMING AND MATHEMATICAL ECONOMICS

1. INTRODUCTION

The functions of a mathematical theory in a scientific field are to furnish a systematic means of formulation of classes of problems, to indicate various techniques for their analysis, and to provide methods for obtaining numerical answers to numerical questions.

At one point in the nineteenth century, serious doubt was expressed that problems in the field of economics could ever be handled mathematically. The introduction of the digital computer changed the situation drastically. Nevertheless, much remains to be done, and many new approaches must be devised, before we can consider ourselves to have a firm hold in the domain of mathematical economics.

In what follows, we outline briefly some of the principal contributions of the theory of dynamic programming to the formulation, analysis, and computational treatment of economic processes. Further references, examples, and detailed discussion will be found in [1,2,3].

2. ALLOCATION PROCESSES

A fundamental problem in economics is that of the allocation of resources. Assuming that we possess the basic utility functions, the mathematical problem in

many cases is that of maximizing the function

$$(2.1) \qquad R_N = \sum_{i=1}^{N} g_i(x_{i1}, x_{i2}, \ldots, x_{iM}),$$

subject to the constraints

$$(2.2) \qquad \sum_{i=1}^{N} x_{ij} \leq c_j, \quad j = 1, 2, \ldots, M,$$

and $x_{ij} \geq 0$.

The functional-equation technique of dynamic programming transforms this problem into that of the solution of the functional equation

$$(2.3) \qquad f_N(c_1, c_2, \ldots, c_M) = \max_{R} \, [\, g_N(x_{N1}, x_{N2}, \ldots, x_{NM})$$

$$+ f_{N-1}(c_1 - x_{N1}, \ldots, c_M - x_{NM})\,],$$

where R is determined by the constraints

$$(2.4) \qquad 0 \leq x_{Nj} \leq c_j, \quad j = 1, 2, \ldots, M.$$

In a few cases, this equation can be treated analytically. In general, we are interested in computational algorithms which are _independent_ of the analytic structure of the functions g_i. If the solution can be obtained numerically, we have the optimal allocation policy as a function of the number of activities and the quantities of resources available.

3. DIRECT COMPUTATIONAL APPROACHES

In view of the rapid–access storage capacity of the biggest and fastest modern digital computer, a count of required storage shows that one– and two–dimensional allocation processes are readily resolved in a direct fashion. Three–dimensional allocation processes are on the border line; see [2], Chaps. 1 and 2.

A very simple analytic device, that of the Lagrange multiplier, enables us to increase the power of the method. In place of the M constraints of (2.2), we impose K constraints,

$$(3.1) \qquad \sum_{i=1}^{N} x_{ij} \leq c_j, \quad j = 1, 2, \ldots, K,$$

and replace the criterion function by

$$(3.2) \qquad R_N(\lambda) = \sum_{i=1}^{N} g_i - \sum_{j=R+1}^{M} \lambda_j \left(\sum_{i=1}^{N} x_{ij} \right).$$

As is well known, the economic interpretation of the λ_j is that of a _price_. The dynamic–programming approach now furnishes the optimal allocation as a function of prices and resources.

We can now consider up to four or five different kinds of resources without difficulty.

4. ANALYTIC NOTES

If the functions g_i are linear, the foregoing provides a dynamic–programming approach to the general

linear programming, an approach from which a number of results can be deduced; see Appendix 2 of [2]. These methods can also be applied to nonlinear programming.

For an analytic approach to allocation processes based upon the transform

$$(4.1) \qquad F(x) = \max_{y \geq 0} \, [f(y) - xy],$$

see [4], [5].

5. SOPHISTICATED COMPUTATIONAL APPROACHES

Let us now consider two techniques for circumventing multidimensional difficulties, namely, successive approximations and polynomial approximation; see [6], [2]. As usual, the method of successive approximations cannot guarantee convergence to the absolute maximum, but it can provide monotonicity. This is important, since we can use it to test any proposed solution and thus to improve anything which is being done currently.

To illustrate an application of the method, consider the problem of maximizing

$$(5.1) \qquad \sum_{i=1}^{N} g_i(x_i, y_i)$$

over all x_i and y_i satisfying the constraints x_i, $y_i \geq 0$ and

$$(5.2) \qquad \sum_i x_i = c_1, \quad \sum_i y_i = c_2.$$

Let $\{x_i^{(0)}, y_i^{(0)}\}$ be an initial approximation.
Determine a new approximation by holding the $y_i^{(0)}$
fixed and maximizing

$$(5.3) \qquad \sum_{i=1}^{N} g_i(x_i, y_i^{(0)})$$

over $x_i \geq 0$, $\sum_i x_i = c_1$, a one-dimensional problem.
This yields a set $\{x_i^{(0)}\}$. Keep this set fixed and
maximize

$$(5.4) \qquad \sum_{i=1}^{N} g_i(x_i^{(1)}, y_i)$$

over $y_i \geq 0$, $\sum_i y_i = c_2$, and so on. It is clear that
the maximum return increases monotonically.

This technique can be applied to allocation
processes with a considerable number of resources. In
particular, it has been used with success to treat a
Hitchcock—Koopmans—Kantorovich scheduling problem
involving nonlinear shipping costs; see [2].

The dimensionality difficulties encountered in the
direct approach referred to in Sec. 3 were due to the
storage of the values of $f_N(c_1, c_2, \ldots, c_M)$ at lattice
points in c—space. Taking advantage of the smooth
behavior of f_N which we can expect in general, we can
store the functional values by means of a polynomial
approximation

$$(5.5) \qquad f_N \cong \sum_{j=1}^{R} a_j^{(N)} g_j(c),$$

where the $g_j(c)$ are polynomials. If the $g_j(c)$ are fixed in advance, storage of $f_N(c)$ requires only storage of the coefficients $a_j^{(N)}$. By means of a few hundred, or a few thousand, coefficients, we can obtain excellent approximations. In this way, we trade time, which we possess, for space, which we do not. See [2] and [6] for detailed discussion and applications.

6. MULTISTAGE DECISION PROCESSES

In the main, dynamic programming is most useful in the study of multistage processes requiring sequences of decisions. In the parlance of control theory, it is most powerful in the treatment of feedback control processes. Processes of this type arise frequently in economics, multistage investment, purchasing, inspection, and repair of equipment, multistage production processes, inventory theory; see [1], [2]. Many of these processes are Markovian decision processes, such as those arising in inventory and repair theory; see the various volumes edited by Arrow, Karlin, and Scarf [7] for numerous examples of the analysis of dynamic programming processes. Many further references will be found in [1,2,3].

The basic functional equation takes the form

$$(6.1) \qquad f_N(p) = \max_q \left[g(p,q) + \sum_1 w_1 f_{N-1}(T_1(p,q)) \right].$$

An advantage of this formulation is that deterministic

and stochastic processes are treated in the same
conceptual, analytic, and computational fashion.

The computational techniques described above can
all be applied to this equation, as well as that of
"approximation in policy space." (See [1,2,3].)

7. ADAPTIVE PROCESSES

In many processes involving planning in the face of
uncertainty, the classical techniques are not immediately
applicable due to the absence of information which we
usually take for granted, e.g., distributions of random
variables, cause and effect, and so on; see [3].

In these cases, it is necessary to make decisions
and to learn about the process at the same time. The
theory of dynamic programming provides a uniform approach
to decision processes of this type, which we call
adaptive. Detailed discussion and examples are
presented in [3]. The simple and intuitive Principle of
Optimality provides the conceptual framework upon which
analytic approaches can be based. This principle can be
stated as follows:

Principle of Optimality. An optimal policy has the
property that whatever the initial state and initial
decision are, the remaining decisions must constitute an
optimal policy with regard to the state resulting from
the first decision.

As before, computational difficulties arise because of the presence of functions of many variables and, indeed, functions of functions.

8. SIMULATION PROCESSES

One of the advantages of the emphasis on the concept of policy in dynamic programming is that multistage decision processes with no criterion for optimal outcome can still be fruitfully studied. Combining digital computers with mathematical formulation, we can employ simulation, or "gaming" techniques; see [8].

9. CONCLUSION

At the present time, we possess a number of powerful mathematical techniques for the analytic and computational solution of classes of problems in the field of mathematical economics. What is now needed is a systematic exploitation of these methods to provide a backlog of solved problems which will guide our subsequent research.

REFERENCES

1. Bellman, R., _Dynamic Programming_, Princeton University Press, Princeton, New Jersey, 1957.

2. Bellman, R., and S. Dreyfus, _Applied Dynamic Programming_, Princeton University Press, Princeton, New Jersey, 1962.

3. Bellman, R., _Adaptive Control Processes: A Guided Tour_, Princeton University Press, Princeton, New Jersey, 1961.

4. Bellman, R., and W. Karush, "Mathematical Programming and the Maximum Transform," _J. Soc. Indust. Appl. Math._, Vol. 10, 1962, pp. 550–567.

5. Bellman, R., "On the Reduction of Dimensionality for Classes of Dynamic Programming Processes," _J. Math. Anal. Appl._, Vol. 3, 1961, pp. 358–360.

6. Bellman, R., R. Kalaba, and B. Kotkin, _Polynomial Approximation—A New Computational Technique in Dynamic Programming—I: Allocation Processes_, The RAND Corporation, RM-3084-PR, April 1962; also to appear in _Mathematics of Computation_.

7. Arrow, K. D., S. Karlin, and H. Scarf, _Mathematical Theory of Inventory and Production_, Stanford University Press, Stanford, California, 1960.

8. Bellman, R., C. Clark, C. Craft, D. Malcolm, and F. Ricciardi, "On the Construction of a Multi-person, Multistage Business Game," _Operations Research_, Vol. 5, 1957, pp. 469–503.