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STRENGTH OF AERONAUTICAL STRUCTURES
(SELECTED ARTICLES)

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THE CALCULATION OF CONICAL SHELLS BY THE VARIATIONAL METHOD OF V. Z. VLASOV

B. A. Konovalov, Candidate of Technical Sciences

The purpose of the present investigation is to obtain equations for calculating slightly conical shells of constant thickness with allowance for deplanation of the cross sections during both torsion and bending and on the basis of these equations to show the possibility of constructing more accurate solutions for shells of the aircraft-wing type.

The theoretical basis of the present article is Prof. V. Z. Vlasov's general variational method of reducing complex two-dimensional contact problems of the theory of plates and shells to one-dimensional problems. The entire substance of the article is divided into three sections, each of which is self-contained.

Given in the first section is the derivation of a system of differential equations for calculating a slightly conical shell of constant thickness with allowance for deformation of the contour (i.e., according to the theory of moments). These equations are a direct generalization of equations obtained previosuly [4]. On the basis of the results of the first section, in the second and third sections, respectively, more accurate solutions for the cases of bending and torsion of a simply closed conical shell are constructed with the proviso that the contour be indeformable.

Thus the equations in the first section are most general, and, what is especially important to note, on the basis of them sufficiently accurate solutions for a very wide class of problems can be obtained. The degree of accuracy of the solutions will depend essentially on the number of approximating functions which figure in the argumentation and satisfy the physical meaning of the problem.

The article is devoted mainly to a study of the stressed and deformed states of a slightly conical shell with allowance for deplanation of the cross sections during bending by crosscutting forces. Such a formulation is entirely justified, since failure to take into account the effect of a constraint of the deplanation of the cross sections during bending in prismatic shells can result in considerable errors in the calculation. Professor I. F. Obraztsov (Doctor of Tech. Sciences) studied theoretically and proved experimentally [8, 9] the fact of the appearance of considerable bimomental stresses in prismatic shells as a result of a constraint of the deplanation of the cross sections during bending. It was precisely this circumstance that moved us to study the effect of a constraint of the deplanation occurring during bending in conical shells.

The final results for stresses and displacements in the general form for any shell parameters and for external loads of sufficiently general type are obtained in the article.

In order to compare the solution to the problem of torsion of a conical shell with the results obtained by L. I. Balabukh [1] and

B. P. Tsibulya [14]*, a numerical calculation of the normal stresses was made and showed qualitative agreement with the results.

The solutions for bending and torsion of straight conical shells can be applied to the calculation of swept-back conical shells. In this case some of the boundary conditions must be written in a cross section along a slanting edge [8].

There are no fundamental difficulties involved in taking the elasticity of the embedding into account, i.e., in considering the combined operation of a swept-back conical shell and a center-section (subfuselage) shell [8].

The proposed method of calculating for bending and torsion can also be extended to multiply closed conical shells [9].

The author deems it his duty to express his gratitude to A. N. Yelpat'yevskiy, Cand. of Tech. Sci. and Senior Scientific Worker at the Institute of Mechanics of the Academy of Sciences of the USSR, for a number of valuable suggestions and recommendations.

Derivation of the Equations for Calculating Conical Shells by Approximating the Displacements with the Aid of Power Functions

Let us consider a conical shell of constant thickness related to a system of coordinates z, S (Fig. 1),

where z is the longitudinal coordinate and determines the position of any transverse coordinate;

S is the contour coordinate of a point in the plane of this cross section of the shell.

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* Translator's Note: Reference [14], although cited in the text, is not found in the list of references at the end of the article. Possibly reference [13] was the one the author wished to cite.

In accordance with the basic idea of Prof. V. Z. Vlasov's variational method, let us represent the longitudinal u(z, S) and the transverse v(z, S) displacement of a point M(z, S) in the form of the following finite expansions:

$$\left.\begin{aligned} u(z, S) &= \sum_i U_i(z)\varphi_i(S) \quad (i=1, 2 \ldots m); \\ v(z, S) &= \sum_k V_k(z)\psi_k(S) \quad (k=1, 2 \ldots n), \end{aligned}\right\} \tag{1}$$

where $U_i(z)$ and $V_k(z)$ are the unknown generalized longitudinal and transverse displacements;

$\varphi_i(S)$ and $\psi_k(S)$ are distribution functions of the generalized longitudinal and transverse displacements along the contour of the cross section of the shell and are chosen beforehand.

Let us consider a system of $\underline{i}$ functions $\varphi_i(S)$, each of which is a power function of the order α with respect to the contour coordinate S. We shall assume that a system of $\underline{k}$ functions $\psi_k(S)$ consists of power functions of the order β with respect to the contour coordinate S. The approximating functions of which the systems $\varphi_i(S)$ and $\psi_k(S)$ are composed must satisfy the condition of linear independence and the condition of continuity of the longitudinal and transverse displacements at all points of the contour.

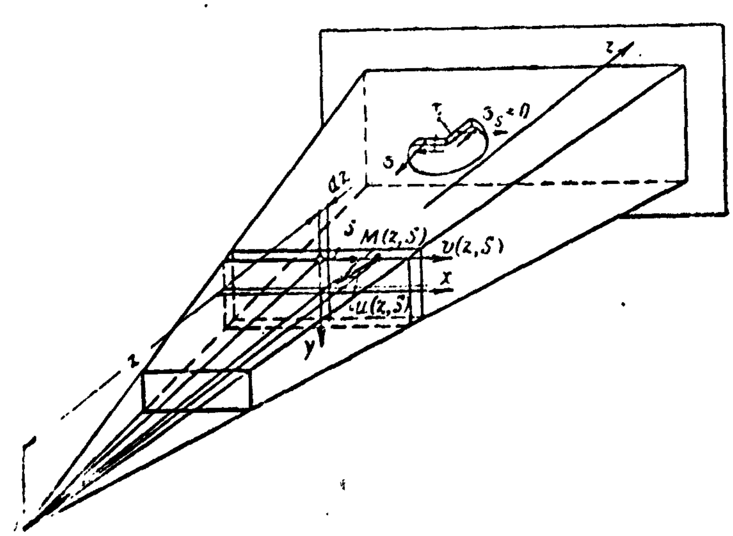

Fig. 1. Over-all view of a conical shell.

Neglecting the normal stresses acting along the coordinate lines S, let us write Hooke's law for the shell under consideration in the following form:

$$\left.\begin{aligned} \sigma &= E\frac{\partial u}{\partial z};\\ \tau &= G\left(\frac{\partial u}{\partial S}+\frac{\partial v}{\partial z}\right). \end{aligned}\right| \quad (2)$$

In order to determine the unknown functions $U_i(z)$ and $V_k(z)$, let us apply the principle of possible displacements.

Let us take from a shell with the cross sections z = const and z + dz = const an elementary strip (frame) of width dz. The isolated strip (Fig. 2) is under the action of normal and tangential stresses exerted in the cross sections and given surfaces by longitudinal p(z, S) and transverse q(z, S) forces and, from the geometrical point of view, possesses m longitudinal (from the plane of the cross section of the shell) and n transverse (in the plane of the cross section of the shell) degrees of freedom. Let us write the work of the forces acting on the isolated strip during a possible displacement from the plane of the cross section $u_j = \varphi_j(S)$, when $U_j(z) = 1$, and in the plane of the cross section $v_h = \psi_h(S)$, when $V_h(z) = 1$:

$$\left.\begin{aligned} &\oint\limits_{\left(S+\frac{dS}{dz}dz\right)}\left(\sigma+\frac{\partial\sigma}{\partial z}dz\right)\varphi_j(S)\delta dS-\oint\limits_{(S)}\sigma\varphi_j(S)\delta dS-\\ &-\oint\limits_{(S)}\tau\frac{\partial\varphi_j(S)}{\partial S}dz\delta dS+\oint\limits_{(S)}p(z,S)\varphi_j(S)dz\,dS=0\quad(j=1,2\dots m);\\ &\oint\limits_{\left(S+\frac{dS}{dz}dz\right)}\left(\tau+\frac{\partial\tau}{\partial z}dz\right)\psi_h(S)\delta dS-\oint\limits_{(S)}\tau\psi_h(S)\delta dS-\\ &-\oint\limits_{(S)}\frac{M(z,S)\,dz M_h(s)\,dS}{EJ}+\oint\limits_{(S)}q(z,S)\psi_h(S)dz\,dS=0\quad(h=1,2\dots n). \end{aligned}\right| \quad (3)$$

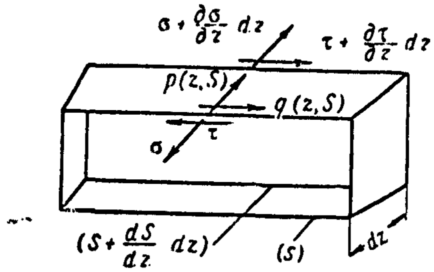

Fig. 2. For the derivation of the equilibrium equations. The forces acting on an isolated elementary frame.

Let us replace the variable of integration S in Eqs. (3) by $s\frac{z}{l}$. As a result of this substitution the integrals with respect to the variable contours S in these equations are transformed into integrals with respect to contours s of fixed cross section $z = l$. Another consequence of this substitution is the possibility of comparing the values of any function of the systems $\varphi_i(S)$ and $\psi_k(S)$ in a moving and a fixed cross section.

Let us calculate the integrands in the function of the new variable $s\,\frac{z}{l}$.

For each of the functions of the systems $\varphi_i(S)$ and $\psi_k(S)$ the following equalities hold true:

$$\left.\begin{aligned} \varphi_i\left(s\frac{z}{l}\right) &= \left(\frac{z}{l}\right)^{\alpha} \varphi_i(s); \\ \psi_k\left(s\frac{z}{l}\right) &= \left(\frac{z}{l}\right)^{\beta} \psi_k(s). \end{aligned}\right\} \tag{4}$$

The derivative of each function of the system $\varphi_i(S)$ with respect to the contour is determined from the formula

$$\varphi_i'\left(s\frac{z}{l}\right) = \left(\frac{z}{l}\right)^{\alpha-1} \varphi_i'(s). \tag{5}$$

The linear bending moment along the contour of the shell due to a displacement $v(z, S) = \sum_k V_k(z)\,\psi_k(S)$ is determined by the equality

$$M(z, S) = \sum_k V_k(z)\, M_k(S). \tag{6}$$

Comparing the displacements in a moving and a fixed cross section, we obtain a formula for the linear bending moment of the shell $M_k(S)$ ($V_k(z) = 1$) for the new variable s

$$M_k\left(s\frac{z}{l}\right)=\left(\frac{z}{l}\right)^{\beta-2}M_k(s). \tag{7}$$

The normal and tangential stresses and their derivatives with respect to the coordinate <u>z</u> along the generatrices, on the basis of (1), (2), (4) and (5), are determined from the formulas

$$\left.\begin{aligned}
\sigma &= E\sum_i U_i'(z)\varphi_i\left(s\frac{z}{l}\right);\\
\frac{\partial\sigma}{\partial z} &= E\sum_i U_i''(z)\varphi_i\left(s\frac{z}{l}\right);\\
\tau &= G\left[\sum_i U_i(z)\varphi_i'\left(s\frac{z}{l}\right)+\sum_k V_k'(z)\psi_k\left(s\frac{z}{l}\right)\right];\\
\frac{\partial\tau}{\partial z} &= G\left[\sum_i U_i'(z)\varphi_i'\left(s\frac{z}{l}\right)+\sum_k V_k''(z)\psi_k\left(s\frac{z}{l}\right)\right].
\end{aligned}\right\} \tag{8}$$

Let us assume that $\varphi_j(S)$ and $\psi_h(S)$ are power functions of the contour coordinate S with the exponents λ and η. Since α and β include the power values of the contour coordinate S of all the functions of which the systems $\varphi_i(S)$ and $\psi_k(S)$ are composed, they naturally pass through the values λ and η.

Substituting (4)-(8) in Eqs. (3), we obtain, respectively, the following equalities:

$$\begin{gathered}
E\oint_{(s)}\left[\sum_i (U_i'+U_i''dz)\left(\frac{z+dz}{l}\right)^{\alpha+\lambda+1}-\sum_i U_i'\left(\frac{z}{l}\right)^{\alpha+\lambda+1}\right]\times\\
\times\varphi_i(s)\varphi_j(s)\delta\,ds-G\oint_{(s)}\left[\sum_i U_i\varphi_i'(s)\left(\frac{z}{l}\right)^{\alpha+\lambda-1}+\right.\\
\left.+\sum_k V_k'\psi_k(s)\left(\frac{z}{l}\right)^{\beta+\lambda}\right]\varphi_j'(s)\,dz\,\delta\,ds+\\
+\int_{(s)} p(z,S)\varphi_j(s)\left(\frac{z}{l}\right)^{\lambda+1}dz\,ds=0 \qquad (j=1,2\ldots m);\\
G\oint_{(s)}\left\{\left[\sum_i (U_i+U_i'dz)\left(\frac{z+dz}{l}\right)^{\alpha+\eta}-\sum_i U_i\left(\frac{z}{l}\right)^{\alpha+\eta}\right]\varphi_i'(s)+\right.\\
\left.+\left[\sum_k (V_k'+V_k''dz)\left(\frac{z+dz}{l}\right)^{\beta+\eta+1}-\sum_k V_k'\left(\frac{z}{l}\right)^{\beta+\eta+1}\right]\psi_k(s)\right\}\times
\end{gathered}$$

$$\times \psi_h(s)\delta\, ds - \oint_{(s)} \frac{\sum_k V_k M_k(s)\left(\frac{z}{l}\right)^{\beta+\eta-3} M_h(s)\, dz\, ds}{EJ} +$$

$$+\oint_{(s)} q(z, s)\psi_h(s)\left(\frac{z}{l}\right)^{\eta+1} dz\, ds = 0 \qquad (h = 1, 2 \ldots n).$$

Neglecting terms higher than the first infinitesimal order and cancelling out dz, we obtain finally the following equations:

$$\left.\begin{aligned}
&\gamma \sum_i a_{ji} \frac{d}{dz}\left[\left(\frac{z}{l}\right)^{\alpha+\lambda+1} U_i'\right] - \sum_i \left(\frac{z}{l}\right)^{\alpha+\lambda-1} b_{ji} U_i - \\
&\quad - \sum_k \left(\frac{z}{l}\right)^{\beta+\lambda} c_{jk} V_k' + \left(\frac{z}{l}\right)^{\lambda+1} \frac{1}{G} p_j = 0 \\
&\qquad\qquad\qquad\qquad (j = 1, 2 \ldots m); \\
&\sum_i c_{hi} \frac{d}{dz}\left[\left(\frac{z}{l}\right)^{\alpha+\eta} U_i\right] + \sum_k r_{hk} \frac{d}{dz}\left[\left(\frac{z}{l}\right)^{\beta+\eta+1} V_k'\right] - \\
&\quad - \gamma \sum_k \left(\frac{z}{l}\right)^{\beta+\eta-3} S_{hk} V_k + \left(\frac{z}{l}\right)^{\eta+1} \frac{1}{G} q_h = 0 \\
&\qquad\qquad\qquad\qquad (h = 1, 2 \ldots n),
\end{aligned}\right\} \tag{9}$$

where $\gamma = \frac{E}{G}$;

$$\left.\begin{aligned}
a_{ji} &= \oint_{(s)} \varphi_j(s)\varphi_i(s)\delta\, ds; & c_{hi} &= \oint_{(s)} \psi_h(s)\varphi_i'(s)\delta\, ds; \\
b_{ji} &= \oint_{(s)} \varphi_j'(s)\varphi_i'(s)\delta\, ds; & r_{hk} &= \oint_{(s)} \psi_h(s)\psi_k(s)\delta\, ds; \\
c_{jk} &= \oint_{(s)} \varphi_j'(s)\psi_k(s)\delta\, ds; & S_{hk} &= \frac{1}{E}\oint_{(s)} \frac{M_h(s)M_k(s)\, ds}{EJ}.
\end{aligned}\right\} \tag{10}$$

The constant coefficients determined by formulas (10) are calculated for any arbitrarily chosen fixed cross section.

The formulas for the coefficients (10) can also be extended to shells reinforced with longitudinal elements. In this case the quadratures (10) must be understood in the sense of Stieltjes integrals. The free terms in Eqs. (9) are determined from the formulas

$$\left.\begin{aligned}
p_j &= \oint_{(s)} p(z, S)\varphi_j(s)\, ds; \\
q_h &= \oint_{(s)} q(z, S)\psi_h(s)\, ds.
\end{aligned}\right\} \tag{11}$$

Formulas (11) are calculated for each specific case of the distribution of the surface loads p(z, S) and q(z, S) along the contour S. For example, in the case of a quadratic variation of the surface loads p(z, S) and q(z, S) along the contour we obtain

$$p(z) = \frac{z^2}{l^2}\, p(z, s);$$
$$q(z) = \frac{z^2}{l^2}\, q(z, s) \quad \text{etc.}$$

Equations (9) constitute a nonhomogeneous system of m + n linear differential equations with variable coefficients for the unknown generalized displacements. This system is obtained by approximating the displacements along the contour with the aid of an arbitrary number of power functions of any order of the contour coordinate S. Therefore the system of equations (9) may, with complete justification, be called a general system of differential equations for calculating slightly conical shells of constant thickness and arbitrary cross section. Assuming that each of the sets of functions $\varphi_i(S)$ and $\psi_k(S)$ is orthogonal, we obtain

$$\left.\begin{aligned} a_{ji} &= \oint_{(s)} \varphi_j(s)\,\varphi_i(s)\,dF(s) = 0, \quad \text{if} \quad j \neq i; \\ r_{hk} &= \oint_{(s)} \psi_h(s)\,\psi_k(s)\,dF(s) = 0, \quad \text{if} \quad h \neq k. \end{aligned}\right\} \tag{12}$$

Under conditions (12) the system of equations (9) assumes the form

$$\left.\begin{aligned} &\gamma a_{jj} \frac{d}{dz}\left[\left(\frac{z}{l}\right)^{2\lambda+1} U'_j\right] - \sum_i \left(\frac{z}{l}\right)^{\alpha+\lambda-1} b_{ji} U_i - \\ &- \sum_k \left(\frac{z}{l}\right)^{\beta+\lambda} c_{jk} V'_k + \left(\frac{z}{l}\right)^{\lambda+1} \frac{1}{G}\, p_j = 0 \\ &\qquad (j = 1, 2 \ldots m); \\ &\sum_i c_{hi} \frac{d}{dz}\left[\left(\frac{z}{l}\right)^{\alpha+\eta} U_i\right] + r_{hh} \frac{d}{dz}\left[\left(\frac{z}{l}\right)^{2\eta+1} V'_h\right] - \\ &- \gamma \sum_k \left(\frac{z}{l}\right)^{\beta+\eta-3} S_{hk} V_k + \left(\frac{z}{l}\right)^{\eta+1} \frac{1}{G}\, q_h = 0 \\ &\qquad (h = 1, 2 \ldots n). \end{aligned}\right\} \tag{13}$$

For a system of $\underline{i}$ functions $\varphi_i(S)$ consisting of power functions of the order $\alpha = \lambda = \text{const}$ and a system of $\underline{k}$ functions $\psi_k(S)$ of the order $\beta = \eta = \alpha - 1 = \text{const}$, on the basis of (9), we obtain

$$\left.\begin{array}{c}
\gamma \sum_i a_{ji} \frac{d}{dz}\left[\left(\frac{z}{l}\right)^{2\alpha+1} U_i'\right] - \sum_i \left(\frac{z}{l}\right)^{2\alpha-1} b_{ji} U_i - \\
- \sum_k \left(\frac{z}{l}\right)^{2\alpha-1} c_{jk} V_k' + \left(\frac{z}{l}\right)^{\alpha+1} \frac{1}{G} p_j = 0 \\
(j = 1, 2 \ldots m); \\
\sum_i c_{hi} \frac{d}{dz}\left[\left(\frac{z}{l}\right)^{2\alpha-1} U_i\right] + \sum_k r_{hk} \frac{d}{dz}\left[\left(\frac{z}{l}\right)^{2\alpha-1} V_k'\right] - \\
- \gamma \sum_k \left(\frac{z}{l}\right)^{2\alpha-3} S_{hk} V_k + \left(\frac{z}{l}\right)^{\alpha} \frac{1}{G} q_h = 0 \\
(h = 1, 2 \ldots n).
\end{array}\right| \qquad (14)$$

The system of equations (14) for determining the unknown generalized displacements is obtained by approximating the displacements with the aid of a power function of any order α of the contour coordinate S.

The system of equations (14) is a generalization of a system of equations [4], where $\alpha = 1$, i.e., for functions that are linear $\varphi_i(S)$ and constant (of zero degree) $\psi_k(S)$ with respect to the contour coordinate S. For shells with a rigid contour (when $S_{hk} = 0$) Eqs. (14) are simplified and assume the form

$$\left.\begin{array}{c}
\gamma \sum_i a_{ji} \frac{d}{dz}\left[\left(\frac{z}{l}\right)^{2\alpha+1} U_i'\right] - \sum_i \left(\frac{z}{l}\right)^{2\alpha-1} b_{ji} U_i - \\
- \sum_k \left(\frac{z}{l}\right)^{2\alpha-1} c_{jk} V_k' + \left(\frac{z}{l}\right)^{\alpha+1} \frac{1}{G} p_j = 0 \\
(j = 1, 2 \ldots m); \\
\sum_i c_{hi} \frac{d}{dz}\left[\left(\frac{z}{l}\right)^{2\alpha-1} U_i\right] + \sum_k r_{hk} \frac{d}{dz}\left[\left(\frac{z}{l}\right)^{2\alpha-1} V_k'\right] + \\
+ \left(\frac{z}{l}\right)^{\alpha} \frac{1}{G} q_h = 0 \\
(h = 1, 2 \ldots n).
\end{array}\right| \qquad (15)$$

Equations (15) constitute a system of differential equations with variable coefficients of Euler type with respect to the function $U_1(z)$ and $V_k'(z)$ with a right member in the presence of surface loads.

The unknown functions, which are the generalized displacements $U_1(z)$ and $V_k(z)$ of the above-mentioned systems of equations (9) and (13)-(15), must satisfy the boundary conditions.

The latter are given in the form of geometrical, static, and mixed conditions, depending on the nature of the attachment of the edges of the shell, and serve to determine the arbitrary constants of integration. Let us show by examples the use of the systems of equations that have been obtained.

A Study of Cross-Sectional Deplanation During Bending for the Case Where the Deplanation is Approximated by Power Functions

In this section we shall seek a solution for the stressed and deformed states of a slightly conical shell of constant thickness with allowance for cross-sectional deplanation during bending.

The deplanation along the contour of the cross section may be given by one function or a system of functions of any degree relative to the contour coordinate S, provided that these functions are in keeping with the physical conditions of the problem posed.

Let us study two problems in general form.

1. The deplanation is approximated by linear functions.
2. The deplanation is approximated by quadratic functions.

A conical shell with a rigid contour ($S_{hk} = 0$) is loaded with a concentrated crosscutting force Q_0 and a transverse linear load $q_{(z)}$ given by a linear law (Fig. 3).

The concentrated crosscutting force Q_0, in keeping with the method being used, is understood as a generalized transverse force, the effect of which is taken into account in the formulation of the boundary conditions.

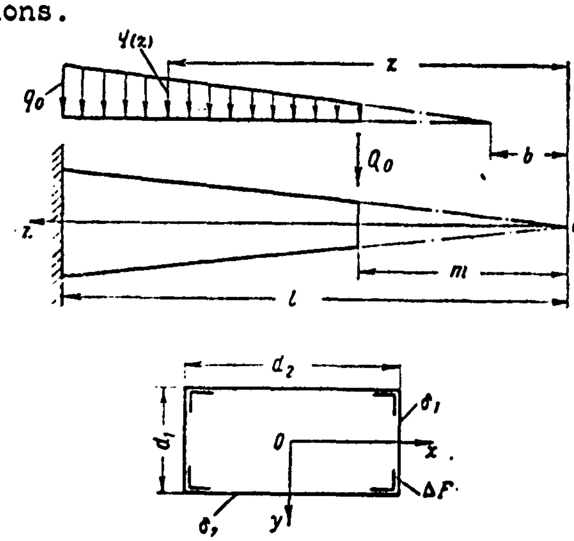

Fig. 3. Loading scheme and geometrical parameters of a conical caisson.

$$q_{(z)} = \frac{q_0}{l-b}(z-b)\,[-\infty < b < m];$$
$$F_1 = \delta_1 d_1; \quad F_2 = \delta_2 d_2.$$

Cross section of shell (z = I)

The transverse linear load $q_{(z)}$ is obtained by integrating along the contour of the surface load over the possible displacement in the plane of the cross section of the shell.

1. The Approximation of Deplanation Along the Contour with the Aid of Linear Functions

Let us represent the longitudinal and transverse displacements of the shell in the form of the following finite expansions:

$$\left.\begin{aligned} u(z, S) &= U_1(z)\varphi_1(S) + U_2(z)\varphi_2(S); \\ v(z, S) &= V_1(z)\psi_1(S). \end{aligned}\right\} \qquad (16)$$

The functions $\varphi_1(S)$, $\varphi_2(S)$, and $\psi_1(S)$ shown in Fig. 4 are chosen as follows:

$$\left.\begin{aligned} &\varphi_1(S)=y(S); \\ &\varphi_2(S)=\mp\left[-\frac{d_2}{2}+x(S)\right]+cy(S); \\ &\psi_1(S)=y'(S). \end{aligned}\right\} \qquad (17)$$

In formulas (17) the coordinates $x(S)$ and $y(S)$ and the parameter d_2 refer to a moving cross section $\underline{z}$. The functions $\varphi_1(S)$ and $\varphi_2(S)$ are linear, while $\psi_1(S)$ is constant with respect to the contour coordinate S, and we can use the system of equations (15) for the solution of the problem formulated.

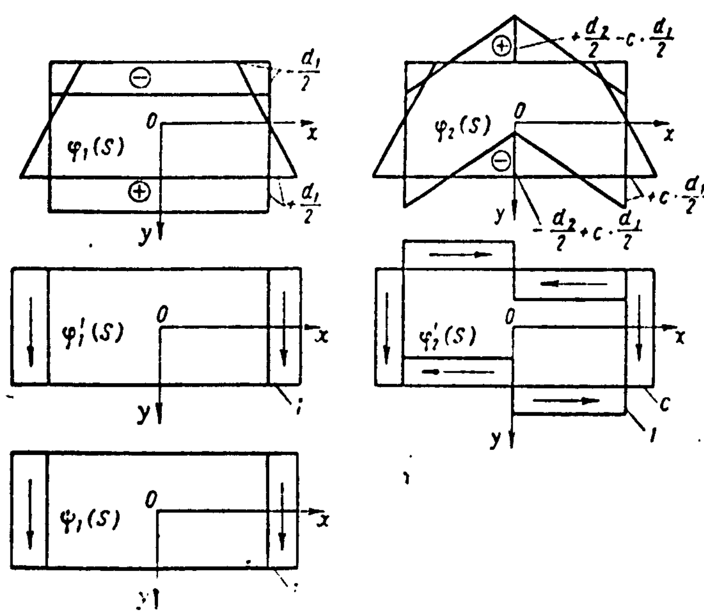

Fig. 4. Diagrams of approximating functions along the contour of a caisson.

The system of equations (15) will be simplified, if the functions $\varphi_1(S)$ and $\varphi_2(S)$ are chosen orthogonal. The condition of orthogonality of the functions $\varphi_1(S)$ and $\varphi_2(S)$ for a moving cross section has the form

$$a_{12}=\oint_{(S)}\varphi_1(S)\varphi_2(S)\,dF(S)=0.$$

Evaluating this condition, we find the coefficient of orthogonalization for a moving cross section

$$c = \frac{d_1 d_2 F_2}{4 J_x}.$$

where J_x is the moment of inertia of the cross section $z = l$ with respect to the x-axis;

d_1, d_2, and F_2 (cf. Fig. 3) are also calculated in the cross section $z = l$.

From the system of equations (15) we obtain for the given loads and displacements (16)

$$\left.\begin{aligned}
&\gamma a_{11} \frac{d}{dz}\left(\frac{z^3}{l^3} U_1'\right) - \frac{z}{l}\left(b_{11} U_1 + b_{12} U_2 + c_{11} V_1'\right) = 0;\\
&\gamma a_{22} \frac{d}{dz}\left(\frac{z^3}{l^3} U_2'\right) - \frac{z}{l}\left(b_{21} U_1 + b_{22} U_2 + c_{21} V_1'\right) = 0;\\
&c_{11} \cdot \frac{d}{dz}\left(\frac{z}{l} U_1\right) + c_{12} \cdot \frac{d}{dz}\left(\frac{z}{l} U_2\right) +\\
&\quad + r_{11} \cdot \frac{d}{dz}\left(\frac{z}{l} V_1'\right) = -\frac{q_0 l}{G(l-b)}\left[\left(\frac{z}{l}\right)^{\gamma+1} - \frac{b}{l}\right].
\end{aligned}\right| \qquad (18)$$

Equations (18) can be obtained from the corresponding system of equations given by Yelpat'yevskiy and Konovalov [4].

Bearing in mind the functions chosen and the symbols used (cf. Fig. 4), we determine the coefficients figuring in system (18) in a fixed cross section $z = l$ from the formulas

$$\left.\begin{aligned}
&a_{11} = J_x = \oint_{(s)} \varphi_1^2(s)\, dF(s) = d_1^2\left(\frac{F_1}{6} + \frac{F_2}{2} + \Delta F\right);\\
&a_{22} = J_{1\varphi} = \oint_{(s)} \varphi_2^2(s)\, dF(s) =\\
&= \frac{F_2}{3}\left[\frac{d_2^2}{2} - \frac{3}{2} c d_1 (d_2 - c d_1)\right] + c^2 d_1^2\left(\frac{F_1}{6} + \Delta F\right);\\
&b_{11} = \int_{(s)} \varphi_1'^2(s)\, dF(s) = 2F_1;\\
&b_{12} = b_{21} = \oint_{(s)} \varphi_1'(s)\, \varphi_2'(s)\, dF(s) = 2cF_1;\\
&b_{22} = \oint_{(s)} \varphi_2'^2(s)\, dF(s) = 2(c^2 F_1 + F_2);
\end{aligned}\right| \qquad (19)$$

$$
\left.
\begin{aligned}
&c_{11}=\oint\limits_{(s)}\varphi_1'(s)\,\psi_1(s)\,dF(s)=2F_1;\\
&c_{12}=c_{21}=\oint\limits_{(s)}\psi_1(s)\,\varphi_2'(s)\,dF(s)=2cF_1;\\
&r_{11}=\oint\limits_{(s)}\psi_1^2(s)\,dF(s)=2F_1,
\end{aligned}
\right|
\qquad (19)
$$

where ΔF is the area of the cross section of a longitudinal element of the shell (longeron belt, stringer);

$J_{1\varphi}$ is the bimoment of bending inertia.

Let us rewrite the system of equations (18) with the coefficients (19) as follows:

$$
\left.
\begin{aligned}
&\frac{z^3}{l^3}U_1''+3\frac{z^2}{l^3}U_1'-\frac{z}{l}\frac{2F_1}{\gamma J_x}(U_1+cU_2+V_1')=0;\\
&\frac{z^3}{l^3}U_2''+3\frac{z^2}{l^3}U_2'-\frac{z}{l}\left(\frac{2cF_1}{\gamma J_{1\varphi}}U_1+\frac{b_{22}}{\gamma J_{1\varphi}}U_2+\frac{2cF_1}{\gamma J_{1\varphi}}V_1'\right)=0;\\
&\frac{z}{l}U_1'+\frac{1}{l}U_1+c\left(\frac{z}{l}U_2'+\frac{1}{l}U_2\right)+\frac{z}{l}V_1''+\frac{1}{l}V_1'=\\
&\qquad=-\frac{q_0 l}{2F_1G(l-b)}\left[\left(\frac{z}{l}\right)^{n+1}-\frac{b}{l}\right].
\end{aligned}
\right|
\qquad (20)
$$

Equations (20) constitute a nonhomogeneous system of three ordinary linear differential equations with variable coefficients of Euler type with respect to $U_1(z)$, $U_2(z)$, and $V_1'(z)$.

Let us replace the variable <u>z</u> by <u>t</u> according to the formula:

$$\frac{z}{l}=e^t.$$

Bearing in mind that $U_1(z)=U_1(t)$, $U_2(z)=U_2(t)$, and $V_1'(z)=V_1'(t)$, we reduce system (20), after performing the necessary operations, to a system of ordinary differential equations with constant coefficients

$$
\left.
\begin{aligned}
&U_1''(t)+2U_1'(t)-NU_1(t)-cNU_2(t)-NV_1'(t)=0;\\
&U_2''(t)+2U_2'(t)-LU_1(t)-MU_2(t)-LV_1'(t)=0;\\
&U_1'(t)+U_1(t)+c[U_2'(t)+U_2(t)]+V_1''(t)+V_1'(t)=\\
&\qquad=-\frac{q_0 l^2}{2F_1G(l-b)}\left(e^t-\frac{b}{l}\right).
\end{aligned}
\right|
\qquad (21)
$$

where N, L, and M are coefficients determined by the formulas

$$N = \frac{2F_1 l^2}{\gamma J_x};$$

$$L = \frac{2cF_1 l^2}{\gamma J_{1\varphi}};$$

$$M = \frac{b_{22} l^2}{\gamma J_{1\varphi}}.$$

Let us represent system (21) in the form of Table 1, where D and D^2 denote, respectively, the first and second derivatives with respect to the independent variable t of the functions in the upper line.

TABLE 1

$U_1(t)$	$U_2(t)$	$V_1'(t)$	Right Member
D^2+2D-N	$-cN$	$-N$	0
$-L$	D^2+2D-M	$-L$	0
$D+1$	$c(D+1)$	$D+1$	$-\frac{qcl^2}{2F_1 G(l-b)}\left(t-\frac{b}{l}\right)$

Let us introduce into our study a new function f(t) such that:

$$\left.\begin{aligned} U_1(t) &= \begin{vmatrix} -cN & -N \\ D^2+2D-M & -L \end{vmatrix} f(t); \\ U_2(t) &= \begin{vmatrix} D^2+2D-N & -N \\ -L & -L \end{vmatrix} f(t); \\ V_1'(t) &= \begin{vmatrix} D^2+2D-N & -cN \\ -L & D^2+2D-M \end{vmatrix} f(t). \end{aligned}\right\} \quad (22)$$

Then the first two equations in system (21) are satisfied identically, while the last equation gives the resolvent equation with respect to the newly introduced function f(t)

$$f^{V}+5f^{IV}+(8-P)f'''+(4-3P)f''-2Pf'= \\ =-\frac{1}{2F_1G}\frac{q_0l^2}{l-b}\left(e^t-\frac{b}{l}\right), \tag{23}$$

where

$$P=\frac{2F_2l^2}{\gamma J_{1\varphi}}.$$

Thus the solution to system (21) is equivalent to the solution of one nonhomogeneous linear differential equation of the fifth order with constant coefficients (23).

The solution to Eq. (23), as is known, consists of the general solution of the homogeneous equation plus the particular integral.

The homogeneous differential equation corresponding to (23) has the form

$$f^{V}+5f^{IV}+(8-P)f'''+(4-3P)f''-2Pf'=0. \tag{24}$$

Let us write the characteristic equation corresponding to (24)

$$n^5+5n^4+(8-P)n^3+(4-3P)n^2-2Pn=0. \tag{25}$$

Equation (25) is satisfied for $n_5 = 0$. Moreover, Eq. (25) has the general roots $n_1 = -1$ and $n_2 = -2$ for all shell parameters, which can be verified by direct substitution.

Calculating the remaining roots of the characteristic equation, we obtain finally

$$n_1=-1;\quad n_2=-2;\quad n_3=-1-k;\quad n_4=-1+k;\quad n_5=0,$$

where

$$k=\sqrt{1+\frac{2F_2l^2}{\gamma J_{1\varphi}}}.$$

We can now write the general solution of the homogeneous equation (24)

$$f^{(0)}(t)=c_1e^{-t}+C_2e^{-2t}+C_3e^{-(k+1)t}+C_4e^{(k-1)t}+C_5.$$

The particular integral of the nonhomogeneous equation (23) has the following value:

$$\tilde{f}(t) = \frac{q_0 l^2}{12F_1G(l-b)(P-3)} e^t - \frac{q_0 lb}{4F_1G(l-b)P} t.$$

The solution to Eq. (23) has the form

$$f(t) = f^{(0)}(t) + \tilde{f}(t). \tag{26}$$

On the basis of (22) and (26), we can obtain for the unknown functions $U_1(t)$, $U_2(t)$, and $V_1'(t)$

$$\begin{aligned}
U_1(t) &= -N\Big[k^2C_1e^{-t} + PC_2e^{-2t} + PC_5 + \frac{q_0l^2}{12F_1G(l-b)}e^t + \\
&\quad + \frac{q_0lb}{4F_1G(l-b)P}(2-Pt)\Big];\\
U_2(t) &= L\Big[-C_1e^{-t} + PC_3e^{-(k+1)t} + PC_4e^{(k-1)t} + \frac{q_0l^2}{4F_1G(l-b)(P-3)}e^t - \\
&\quad - \frac{q_0lb}{2F_1G(l-b)P}\Big];\\
V_1'(t) &= [1+N(P+1)+M]C_1e^{-t} + NPC_2e^{-2t} + \\
&\quad + P(P-M)C_3e^{-(k+1)t} + P(P-M)C_4e^{(k-1)t} + \\
&\quad + NPC_5 + \frac{q_0l^2[3(3-M)+N(P-3)]}{12F_1G(l-b)(P-3)}e^t + \\
&\quad + \frac{q_0lb}{4F_1G(l-b)P}[2(N+M)-NPt].
\end{aligned}$$

Returning to the variable z, we may write

$$\left.\begin{aligned}
U_1(z) &= -N\Big[k^2C_1\left(\frac{z}{l}\right)^{-1} + PC_2\left(\frac{z}{l}\right)^{-2} + PC_5 + \\
&\quad + \frac{q_0l^2}{12F_1G(l-b)}\left(\frac{z}{l}\right)^{+1} + \frac{q_0lb}{4F_1G(l-b)P}\left(2-P\ln\frac{z}{l}\right)\Big];\\
U_2(z) &= L\Big[-C_1\left(\frac{z}{l}\right)^{-1} + PC_3\left(\frac{z}{l}\right)^{-(k+1)} + PC_4\left(\frac{z}{l}\right)^{+k-1} + \\
&\quad + \frac{q_0l^2}{4F_1G(l-b)(P-3)}\left(\frac{z}{l}\right)^{+1} - \frac{q_0lb}{2F_1G(l-b)P}\Big];\\
V_1'(z) &= [1+N(P+1)+M]C_1\left(\frac{z}{l}\right)^{-1} + NPC_2\left(\frac{z}{l}\right)^{-2} + \\
&\quad + P(P-M)C_3\left(\frac{z}{l}\right)^{-(k+1)} + \\
&\quad + P(P-M)C_4\left(\frac{z}{l}\right)^{+k-1} + NPC_5 + \\
&\quad + \frac{q_0l^2[3(3-M)+N(P-3)]}{12F_1G(l-b)(P-3)}\left(\frac{z}{l}\right)^{+1} + \\
&\quad + \frac{q_0lb}{4F_1G(l-b)P}\left[2(N+M)-NP\ln\frac{z}{l}\right].
\end{aligned}\right\} \tag{27}$$

Integrating the last expression in (27), we obtain

$$V_1(z)=l\left\{\left[1+N(P-1)+M\right]C_1\ln\frac{z}{l}-NPC_2\left(\frac{z}{l}\right)^{-1}-\right.$$
$$-\frac{P}{k}(P-M)C_3\left(\frac{z}{l}\right)^{-k}+\frac{P}{k}(P-M)C_4\left(\frac{z}{l}\right)^{+k}+NPC_5\left(\frac{z}{l}\right)^{+1}+ \tag{28}$$
$$+C_6+\frac{q_0l^2[3(3-M)+N(P-3)]}{24F_1G(l-b)(P-3)}\left(\frac{z}{l}\right)^{+2}+$$
$$\left.+\frac{q_0lb}{4F_1G(l-b)P}\left[2(N+M)-NP\left(\ln\frac{z}{l}-1\right)\right]\left(\frac{z}{l}\right)^{+1}\right\}.$$

After the functions $U_1(z)$, $U_2(z)$, and $V_1(z)$ are found, we proce to the determination of the stresses.

The Determination of the Normal and Tangential Stresses in the Shell

The normal stresses in the shell occurring, on the basis of (2) during the displacements (16) are determined from the formula

$$\sigma(z,S)=E\left\{N\left[\frac{k^2}{l}C_1\left(\frac{z}{l}\right)^{-2}+\frac{2P}{l}C_2\left(\frac{z}{l}\right)^{-3}-\frac{q_0l}{12F_1G(l-b)}+\right.\right.$$
$$\left.+\frac{q_0b}{4F_1G(l-b)}\left(\frac{z}{l}\right)^{-1}\right]\varphi_1(S)+L\left[\frac{1}{l}C_1\left(\frac{z}{l}\right)^{-2}-\right.$$
$$-(k+1)\frac{P}{l}C_3\left(\frac{z}{l}\right)^{-k-2}+(k-1)\frac{P}{l}C_4\left(\frac{z}{l}\right)^{-k-2}+ \tag{29}$$
$$\left.\left.+\frac{q_0l}{4F_1G(l-b)(P-3)}\right]\varphi_2(S).\right.$$

The tangential stresses in the shell occurring, on the basis of Hooke's law, during the displacements (16) are determined from the formula

$$\tau(z,S)=G\left[U_1(z)\varphi_1'(S)+U_2(z)\varphi_2'(S)+V_1'(z)\psi_1(S)\right]. \tag{30}$$

A more accurate value of the tangential stresses may be obtained from the differential equation of equilibrium

$$E\sum_i\left\{\left(\frac{z}{l}\right)^{-(\kappa+1)}\frac{d}{dz}\left[\left(\frac{z}{l}\right)^{\kappa+1}U_i'\right]\varphi_i(S)\right\}\delta+\frac{\partial q}{\partial S}=0. \tag{31}$$

For the conical shells under consideration with $\alpha_{1,2} = 1$ the flux of tangential forces $q = \tau\delta$ can be found by integrating (31):

$$q(z, S) = -E\left(\frac{z}{l}\right)^{-2}\left\{\frac{d}{dz}\left[\left(\frac{z}{l}\right)^{+2}U_1'\right]\int_0^S \varphi_1(S)\delta\, dS + \right.$$
$$\left. + \frac{d}{dz}\left[\left(\frac{z}{l}\right)^{+2}U_2'\right]\int_0^S \varphi_2(S)\delta\, dS\right\} + q_0(z). \tag{32}$$

When determining the flux of tangential forces, it is necessary to make a cut in the cross section of the shell to serve as the initial point in calculating the integral terms (32). The flux of tangential forces $q_0(z)$ in each cross section of the shell in a simply closed contour is determined from the equation of the moments of the external and internal forces with respect to an arbitrary point in the cross section.

With the aid of cuts the cross section of a multiply closed shell should be transformed into an open contour, which makes it possible to determine the integral terms (32). The fluxes of tangential forces in each of the i contours $q_{0i}(z)$ are determined from the equation of the moments and a system of $i - 1$ equations of compatibility of the deformations.

The final expression for the flux of tangential forces has the form:

$$q(z, S) = \frac{E}{l^2}\left\{N\left[2PC_2\left(\frac{z}{l}\right)^{-4} + \frac{q_0 l^2}{6F_1 G(l-b)}\left(\frac{z}{l}\right)^{-1} - \right.\right.$$
$$\left. - \frac{q_0 b l}{4F_1 G(l-b)}\left(\frac{z}{l}\right)^{-2}\right]\int_0^S \varphi_1(S)\delta\, dS - \tag{33}$$
$$- Lk\left[(k+1)PC_3\left(\frac{z}{l}\right)^{-(k+3)} + (k-1)PC_4\left(\frac{z}{l}\right)^{+k-3} + \right.$$
$$\left.\left. + \frac{q_0 l^2}{2F_1 G(l-b)(P-3)k}\left(\frac{z}{l}\right)^{-1}\right]\int_0^S \varphi_2(S)\delta\, dS\right\} + q_0(z).$$

The transverse displacement (deflection), on the basis of (16) and (28), is determined by the formula

$$
\begin{aligned}
v(z, S) = l\Big\{&[1 + N(P+1) + M]\, C_1 \ln\frac{z}{l} - NPC_2\left(\frac{z}{l}\right)^{-1} - \\
&-\frac{P}{k}(P-M)\, C_3\left(\frac{z}{l}\right)^{-k} + \frac{P}{k}(P-M)\, C_4\left(\frac{z}{l}\right)^{+k} + NPC_5\left(\frac{z}{l}\right)^{+1} + \\
&+ C_6 + \frac{q_0 l^2\,[3(3-M) + N(P-3)]}{24F_1G(l-b)(P-3)}\left(\frac{z}{l}\right)^{+2} + \\
&+ \frac{q_0 bl}{4F_1G(l-b)P}\left[2(N+M) - NP\left(\ln\frac{z}{l} - 1\right)\right]\left(\frac{z}{l}\right)^{+1}\Big\}\,\psi_1(S).
\end{aligned}
\tag{34}
$$

The Determination of the Arbitrary Constants

For each particular case of attachment of the shell the arbitrary constants must be determined from the appropriate boundary conditions.

For example, for the shell shown in Fig. 3 the boundary conditions have the form

$$
\left.
\begin{array}{l}
\text{end section } z = l \text{ rigidly attached:} \\
\quad 1)\ U_1(z) = 0, \quad 2)\ U_2(z) = 0, \quad 3)\ V_1(z) = 0, \\
\text{end section } z = m \text{ freely deplanes:} \\
\quad 4)\ U_1'(z) = 0; \quad 5)\ U_2'(z) = 0, \\
\quad 6)\ G\,[c_{11}U_1 + c_{12}U_2 + r_{11}V_1']\,\dfrac{m}{l} = -Q_0.
\end{array}
\right\}
\tag{35}
$$

Evaluating the boundary conditions (35), we obtain a system of linear algebraic equations for the determination of the arbitrary constants. This system is shown in Table 2. It should be noted that the system of equations for determining the arbitrary constants has a very simple form and can be solved without any difficulties for any parameters of the shell.

If the transverse linear load acting on the shell is distributed uniformly, we must take b = $-\infty$ in formulas (27)-(29), (33), and (34) and in Table 2.

TABLE 2

System of Equations for Determining the Arbitrary Constants

Condition	C_1	C_2	C_3	C_4	C_5	C_6	Right Member
1	$+k^2$	$+P$	0	0	$+P$	0	$-\frac{q_0 l^2}{12F_1G(l-b)P}\left(P+6\frac{b}{l}\right)$
2	-1	0	$+P$	$+P$	0	0	$+\frac{q_0 l^2}{4F_1G(l-b)(P-3)P}\left[P\left(2\frac{b}{l}-1\right)-6\frac{b}{l}\right]$
3	0	$-NP$	$-\frac{P}{k}(P-M)$	$+\frac{P}{k}(P-M)$	$+NP$	$+1$	$-\frac{q_0 l^2[3(3-M)+N(P-3)]}{24F_1G(l-b)(P-3)}-\frac{q_0 lb[2(M+N)+NP]}{4F_1G(l-b)P}$
4	$+k^2$	$+2P\left(\frac{m}{l}\right)^{-1}$	0	0	0	0	$+\frac{q_0 l^2}{12F_1G(l-b)}\left[\left(\frac{m}{l}\right)^{+2}-3\frac{b}{l}\left(\frac{m}{l}\right)^{+1}\right]$
5	$+1$	0	$-(k+1)P\left(\frac{m}{l}\right)^{-k}$	$+(k-1)P\left(\frac{m}{l}\right)^{+k}$	0	0	$-\frac{q_0 l^2}{4F_1G(l-b)(P-3)}\left(\frac{m}{l}\right)^{+2}$
6	$+k^2$	0	0	0	0	0	$-\frac{Q_0}{2F_1G}+\frac{q_0 l^2}{4F_1G(l-b)}\left[\left(\frac{m}{l}\right)^{+2}-2\frac{b}{l}\left(\frac{m}{l}\right)^{+1}\right]$

Approximation of Deplanation Along the Contour with the Aid of Quadratic Functions

As was done previously, let us represent the longitudinal and transverse displacements of the shell in the form of the following finite expansions:

$$\left.\begin{aligned} u(z, S) &= U_1(z)\varphi_1(S)+U_2(z)\varphi_2(S); \\ v(z, S) &= V_1(z)\psi_1(S). \end{aligned}\right\} \qquad (36)$$

The functions $\varphi_1(S)$, $\psi_1(S)$ (cf. Fig. 4), and $\varphi_2(S)$ (Fig. 5) are chosen as follows:

$$\left.\begin{aligned}&\varphi_1(S)=y(S);\\&\varphi_2(S)=\mp\left[\frac{d_2^2}{4}-x^2(S)\right]+cy(S);\\&\psi_1(S)=y'(S).\end{aligned}\right\} \quad (37)$$

The functions $\varphi_1(S)$ and $\psi_1(S)$ remain the same as in Section 1. Let us write the condition of orthogonality of the functions $\varphi_1(S)$ and $\varphi_2(S)$ in a moving cross section

$$a_{12}=\oint_{(S)}\varphi_1(S)\varphi_2(S)\,dF(S)=0,$$

whence the coefficient of orthogonality of the functions $\varphi_1(S)$ and $\varphi_2(S)$

$$c=\frac{z}{l}\frac{d_1d_2^2F_2}{6J_x},$$

where J_x is the moment of inertia of the cross section $z = l$ with respect to the x-axis; d_1, d_2, and F_2 (cf. Fig. 4) are also calculated in the cross section $z = l$.

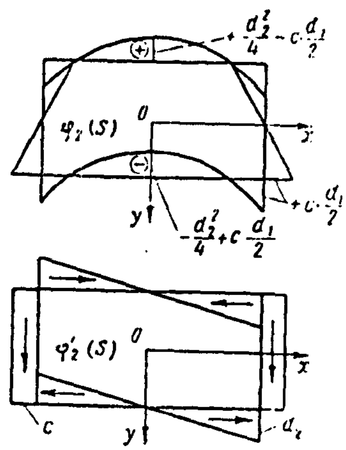

Fig. 5. Diagram of a quadratic deplanation function and its derivative with respect to the contour of a caisson.

In this case the function $\varphi_2(S)$ may be represented in the form

$$\varphi_2(S)=\frac{z^2}{l^2}\left\{\mp\left[\frac{d_2^2}{4}-x^2(s)\right]+\frac{d_1d_2^2F_2}{6J_x}y(s)\right\}, \quad (38)$$

where the coordinates $x(s)$ and $y(s)$ and the parameter d_2 refer to a fixed cross section $z = l$.

Formula (38) shows that $\varphi_2(S)$ is a quadratic function over the entire contour of the shell.

In other words, the transition from the function $\varphi_2(S)$ of a moving cross section to the function $\varphi_2(s)$ calculated in the fixed cross section $z = l$ can be made according to the formula

$$\varphi_2(S)=\frac{z^2}{l^2}\varphi_2(s).$$

For the solution of the problem formulated let us use the system of equations (9). For the given loads and displacements (36) we have

$$\left.\begin{aligned} &\gamma a_{11}\frac{d}{dz}\left(\frac{z^3}{l^3}U_1'\right)-\frac{z}{l}b_{11}U_1-\frac{z^2}{l^2}b_{12}U_2-\frac{z}{l}c_{11}V_1'=0;\\ &\gamma a_{22}\frac{d}{dz}\left(\frac{z^5}{l^5}U_2'\right)-\frac{z^2}{l^2}b_{21}U_1-\frac{z^3}{l^3}b_{22}U_2-\frac{z^2}{l^2}c_{21}V_1'=0;\\ &c_{11}\frac{d}{dz}\left(\frac{z}{l}U_1\right)+c_{12}\frac{d}{dz}\left(\frac{z^2}{l^2}U_2\right)+r_{11}\frac{d}{dz}\left(\frac{z}{l}V_1'\right)=\\ &\qquad=-\frac{q_0 l}{G(l-b)}\left[\left(\frac{z}{l}\right)^{+1}-\frac{b}{l}\right].\end{aligned}\right\} \tag{39}$$

The coefficients a_{11}, b_{11}, $b_{12} = b_{21}$, c_{11}, $c_{12} = c_{21}$, and r_{11} figuring in the system of equations (39) are determined from formulas (19).

For the coefficients a_{22} and b_{22} we have the following expressions:

$$\left.\begin{aligned} &a_{22}=J_{1\varphi}=\oint_{(s)}\varphi_2^2(s)\,dF(s)=\\ &\qquad=\frac{d_2^4F_2}{6}\left[\frac{2}{5}-\frac{d_1^2F_2}{3J_x}+\frac{d_1^4F_2(3F_2+F_1+6\Delta F)}{36J_x^2}\right];\\ &b_{22}=\oint_{(s)}\varphi_2'^2(s)\,dF(s)=\frac{2}{3}d_2^2F_2+2c^2F_1,\end{aligned}\right\} \tag{40}$$

where $J_{1\varphi}$ is the bimoment of bending inertia.

After the coefficients are calculated, let us rewrite the system of equations (39) as follows:

$$\left.\begin{aligned} &\frac{z^3}{l^3}U_1''+3\frac{z^2}{l^3}U_1'-\frac{z}{l}\frac{2F_1}{\gamma J_x}U_1-\frac{z^2}{l^2}\frac{2cF_1}{\gamma J_x}U_2-\frac{z}{l}\frac{2F_1}{\gamma J_x}V_1'=0;\\ &\frac{z^5}{l^5}U_2''+5\frac{z^4}{l^5}U_2'-\frac{z^2}{l^2}\frac{2cF_1}{\gamma J_{1\varphi}}U_1-\frac{z^3}{l^3}\frac{b_{22}}{\gamma J_{1\varphi}}U_2-\frac{z^2}{l^2}\frac{2cF_1}{\gamma J_{1\varphi}}V_1'=0;\\ &\frac{z}{l}U_1'+\frac{1}{l}U_1+c\left(\frac{z^2}{l^2}U_2'+2\frac{z}{l^2}U_2\right)+\frac{z}{l}V_1''+\frac{1}{l}V_1'=\\ &\qquad=-\frac{q_0 l}{2F_1G(l-b)}\left[\left(\frac{z}{l}\right)^{+1}-\frac{b}{l}\right].\end{aligned}\right\} \tag{41}$$

We have thus obtained a nonhomogeneous system of three ordinary linear differential equations with variable coefficients, each equation being of second order.

The system of equations (41) is distinguished by the fact that, in contrast to the system (20), the equations figuring in it are not of Euler type. Consequently we are unable to obtain a solution in terms of a resolvent function. The solution to the system (41) will be sought by successively eliminating the functions $U_1(z)$, $U_2(z)$, and $V_1'(z)$ figuring in it.

Let us introduce into our study a new variable t according to the formula

$$\frac{z}{l} = e^t.$$

Performing the necessary operations on the unknown functions, let us represent the system of equations (41) in the form

$$\left.\begin{aligned} &\ddot U_1(t) + 2\dot U_1(t) - NU_1(t) - N\dot V_1(t) - cNe^tU_2(t) = 0; \\ &-LU_1(t) - L\dot V_1(t) + e^t[\ddot U_2(t) + 4\dot U_2(t) - MU_2(t)] = 0; \\ &\dot U_1(t) + U_1(t) + \ddot V_1(t) + \dot V_1(t) + ce^t[\dot U_2(t) + 2U_2(t)] = \\ &= -\frac{q_0 l^2}{2F_1G(l-b)}\left(e^t - \frac{b}{l}\right). \end{aligned}\right| \qquad (42)$$

In equations (42) it is assumed that

$$N = \frac{2F_1 l^2}{\gamma J_x};$$
$$L = \frac{2cF_1 l^2}{\gamma J_{1\varphi}};$$
$$M = \frac{b_{22} l^2}{\gamma J_{1\varphi}}.$$

The solution of the nonhomogeneous system (42) consists of the general solution of the homogeneous system and the particular integral for $U_1(t)$, $U_2(t)$, and $V_1'(t)$ corresponding to the nature of the load under consideration.

Let us first consider the homogeneous system.

By eliminating the function $U_2(t)$ and its derivative from the first and third equations of the system (42), we can verify that the function $V_1(t)$ and the corresponding derivatives vanish simultaneously. The equation obtained will contain only derivatives of the function

$U_1(t)$

$$U_1^{\cdot\cdot} + 3U_1^{\cdot} + 2U_1^{'} = 0.$$

The general solution for $U_1(t)$ has the form

$$U_1^{(0)}(t) = C_1 e^{-2t} + C_2 e^{-t} + C_3. \tag{43}$$

After the function $U_1^{(0)}(t)$ has been determined, we eliminate $U_2(t)$ and its derivatives from the first and second equations of the same system. The resulting equation contains the functions $U_1(t)$ and $V_1(t)$ and the corresponding derivatives

$$\begin{gathered} U_1^{IV} + 4U_1^{\cdot\cdot} + (1-N-M)U_1^{\cdot} - 2(3+N+M)U_1^{'} + NRU_1 - \\ - NV_1^{\cdot\cdot} - 2NV_1^{\cdot} + NRV_1^{'} = 0, \end{gathered} \tag{44}$$

where

$$R = 3 + \frac{2d_2^2 F_2 l^2}{3\gamma J_{1\varphi}}.$$

On the basis of (43), Eq. (44) can be represented as:

$$V_1^{\cdot\cdot} + 2V_1^{\cdot} - RV_1^{'} = RC_1 e^{-2t} + \left(1 + R + \frac{4+M}{N}\right) C_2 e^{-t} + RC_3. \tag{45}$$

The general solution for $V_1^{'}(t)$ will consist of the general solution to the homogeneous equation corresponding to (45) and its particular integral and will have the form

$$\begin{gathered} V_1^{(0)'}(t) = -C_1 e^{-2t} - \left[1 + \frac{4+M}{(R+1)N}\right] C_2 e^{-t} - C_3 + \\ + C_4 e^{(-1+\sqrt{R+1})t} + C_5 e^{-(1+\sqrt{R+1})t}. \end{gathered}$$

$U_2^{(0)}(t)$ is determined from the first equation of the system (42):

$$cU_2^{(0)}(t) = \frac{cL}{(R+1)N} C_2 e^{-2t} - C_4 e^{(-2+\sqrt{R+1})t} - C_5 e^{-(2+\sqrt{R+1})t}.$$

The particular integrals for $U_1(t)$, $U_2(t)$, and $V_1^{'}(t)$ in the system of equations (42) will be sought in the form

$$\left.\begin{aligned} \tilde{U}_1(t) &= A_1 e^t + B_1 t; \\ \tilde{U}_2(t) &= A_2 + \tilde{B}_2 e^{-t}; \\ \tilde{V}_1^{'}(t) &= A_3 e^t - B_1 t + B_3. \end{aligned}\right\} \tag{46}$$

Substituting the expressions (46) in each equation of the system (42) and equating coefficients of like powers ($e^{\rho t}$) in the left and right members, we obtain the following systems of equations for determining the arbitrary constants figuring in (46):

$$\left.\begin{array}{l|l} (3-N)A_1 - cNA_2 - NA_3 = 0; & 2B_1 - cNB_2 - NB_3 = 0; \\ A_1 + \frac{M}{L}A_2 + A_3 = 0; & (3+M)B_2 + LB_3 = 0; \\ A_1 + cA_2 + A_3 = -\frac{q_0 l^2}{4F_1G(l-b)}; & cB_2 + B_3 = +\frac{q_0 lb}{2F_1G(l-b)}. \end{array}\right\} \tag{47}$$

Determining A_1, A_2, ... B_3 from the above, we find the particular integrals

$$\left.\begin{aligned} \tilde{U}_1(t) &= -\frac{q_0 l^2 N}{12F_1G(l-b)}\left(e^t - 3\frac{b}{l}t\right); \\ \tilde{U}_2(t) &= -\frac{q_0 l^2 L}{4F_1G(l-b)}\left(\frac{2}{R}\frac{b}{l}e^{-t} - \frac{1}{R-3}\right); \\ \tilde{V}_1'(t) &= \frac{q_0 l^2}{12F_1G(l-b)}\left[\left(N - \frac{3M}{R-3}\right)e^t - 3N\frac{b}{l}t + 6\frac{M+3}{R}\frac{b}{l}\right]. \end{aligned}\right\} \tag{48}$$

For the unknown functions of the nonhomogeneous system (42) we have the following formulas:

$$U_1(t) = C_1e^{-2t} + C_2e^{-t} + C_3 - \frac{q_0 l^2 N}{12F_1G(l-b)}\left(e^t - 3\frac{b}{l}t\right);$$

$$U_2(t) = \frac{L}{(R+1)N}C_2e^{-2t} - \frac{1}{c}C_4e^{(-2+\sqrt{R+1})t} -$$

$$-\frac{1}{c}C_5e^{-(2+\sqrt{R+1})t} + \frac{q_0 l^2 L}{4F_1G(l-b)}\left(\frac{1}{R-3} - \frac{2}{R}\frac{b}{l}e^{-t}\right);$$

$$V_1'(t) = -C_1e^{-2t} - \left[1 + \frac{4+M}{(R+1)N}\right]C_2e^{-t} - C_3 +$$

$$+ C_4e^{(-1+\sqrt{R+1})t} + C_5e^{-(1+\sqrt{R+1})t} +$$

$$+\frac{q_0 l^2}{12F_1G(l-b)}\left[\left(N - \frac{3M}{R-3}\right)e^t - 3N\frac{b}{l}t + 6\frac{M+3}{R}\frac{b}{l}\right].$$

Returning to the variable $\underline{z}$, we obtain

$$\left.\begin{aligned} U_1(z) &= C_1\left(\frac{z}{l}\right)^{-2} + C_2\left(\frac{z}{l}\right)^{-1} + C_3 - \\ &\quad -\frac{q_0 l^2 N}{12F_1G(l-b)}\left[\left(\frac{z}{l}\right)^{+1} - 3\frac{b}{l}\ln\frac{z}{l}\right]; \\ U_2(z) &= \frac{L}{(R+1)N}C_2\left(\frac{z}{l}\right)^{-2} - \frac{1}{c}C_4\left(\frac{z}{l}\right)^{-2+\sqrt{R+1}} - \end{aligned}\right\} \tag{49}$$

$$
\left.\begin{aligned}
&-\frac{1}{c} C_5\left(\frac{z}{l}\right)^{-(2+\sqrt{R+1})}+\\
&+\frac{q_0 l^2 L}{4F_1 G(l-b)}\left[\frac{1}{R-3}-\frac{2}{R}\,\frac{b}{l}\left(\frac{z}{l}\right)^{-1}\right];\\
&V_1'(z)=-C_1\left(\frac{z}{l}\right)^{-2}-\left[1+\frac{4+M}{(R+1)N}\right]C_2\left(\frac{z}{l}\right)^{-1}-\\
&-C_3+C_4\left(\frac{z}{l}\right)^{-1+\sqrt{R+1}}+C_5\left(\frac{z}{l}\right)^{-(1+\sqrt{R+1})}+\\
&+\frac{q_0 l^2}{12F_1 G(l-b)}\left[\left(N-\frac{3M}{R-3}\right)\left(\frac{z}{l}\right)^{+1}-3N\frac{b}{l}\ln\frac{z}{l}+\right.\\
&\left.+6\frac{M+3}{R}\,\frac{b}{l}\right].
\end{aligned}\right| \qquad (49)
$$

The function $V_1(z)$ is determined by integrating the last expression in (49)

$$
\begin{aligned}
V_1(z)=l\Big\{&C_1\left(\frac{z}{l}\right)^{-1}-\left[1+\frac{4+M}{(R+1)N}\right]C_2\ln\frac{z}{l}-C_3\left(\frac{z}{l}\right)^{+1}+\\
&+\frac{1}{\sqrt{R+1}}C_4\left(\frac{z}{l}\right)^{+\sqrt{R+1}}-\frac{1}{\sqrt{R+1}}C_5\left(\frac{z}{l}\right)^{-\sqrt{R+1}}+C_6+\\
&+\frac{q_0 l^2}{24F_1 G(l-b)}\left[\left(N-\frac{3M}{R-3}\right)\left(\frac{z}{l}\right)^{+2}-6N\frac{b}{l}\left(\ln\frac{z}{l}-1\right)\left(\frac{z}{l}\right)^{+1}+\right.\\
&\left.+12\frac{M+3}{R}\,\frac{b}{l}\left(\frac{z}{l}\right)^{+1}\right]\Big\}.
\end{aligned} \qquad (50)
$$

The Determination of the Normal and Tangential Stresses in the Shell

The normal stresses in the shell are determined from the formula

$$
\begin{aligned}
\sigma(z,S)=E\Big\{&-\frac{2}{l}C_1\left(\frac{z}{l}\right)^{-3}-\frac{1}{l}C_2\left(\frac{z}{l}\right)^{-2}-\frac{q_0 lN}{12F_1 G(l-b)}\left[1-\right.\\
&\left.-3\frac{b}{l}\left(\frac{z}{l}\right)^{-1}\right]\Big\}\varphi_1(S)+E\left[-\frac{2L}{(R+1)Nl}C_2\left(\frac{z}{l}\right)^{-3}+\right.\\
&+\frac{2-\sqrt{R+1}}{cl}C_4\left(\frac{z}{l}\right)^{-3+\sqrt{R+1}}+\frac{2+\sqrt{R+1}}{cl}C_5\left(\frac{z}{l}\right)^{-(3+\sqrt{R+1})}+\\
&\left.+\frac{q_0 bL}{2F_1 G(l-b)R}\left(\frac{z}{l}\right)^{-2}\right]\varphi_2(S).
\end{aligned} \qquad (51)
$$

The tangential stresses in the shell can be found according to Hooke's law from (30).

It is recommended that the tangential stresses (or the flux of tangential forces $q = \tau\delta$) be determined by integrating the differentia equation of equilibrium (31).

The flux of tangential forces in the case under consideration, where $\alpha_1 = 1$ and $\alpha_2 = 2$, is determined by the following expression:

$$
\begin{aligned}
q(z,S) = -\frac{E}{l^2}\Bigg\{ & \left[2C_1\left(\frac{z}{l}\right)^{-4} - \frac{q_0 l^2 N}{6F_1G(l-b)}\left(\frac{z}{l}\right)^{-1} + \right. \\
& \left. + \frac{q_0 b l N}{4F_1G(l-b)}\left(\frac{z}{l}\right)^{-2}\right]\int_0^S \varphi_1(S)\delta\, dS + \\
& + \left[\frac{(2-\sqrt{R+1})\sqrt{R+1}}{c} C_4\left(\frac{z}{l}\right)^{-4+\sqrt{R+1}} - \right. \\
& - \frac{(2+\sqrt{R+1})\sqrt{R+1}}{c} C_5\left(\frac{z}{l}\right)^{-(4+\sqrt{R+1})} + \\
& \left. + \frac{q_0 b l L}{2F_1G(l-b)R}\left(\frac{z}{l}\right)^{-3}\right]\int_0^S \varphi_2(S)\delta\, dS\Bigg\} + q_0(z).
\end{aligned}
\tag{52}
$$

The transverse displacement (deflection) is determined from the formula

$$
\begin{aligned}
v(z,S) = l\Bigg\{ & C_1\left(\frac{z}{l}\right)^{-1} - \left[1 + \frac{4+M}{(R+1)N}\right] C_2 \ln\frac{z}{l} - \\
& - C_3\left(\frac{z}{l}\right)^{+1} + \frac{1}{\sqrt{R+1}} C_4\left(\frac{z}{l}\right)^{+\sqrt{R+1}} - \\
& - \frac{1}{\sqrt{R+1}} C_5\left(\frac{z}{l}\right)^{-\sqrt{R+1}} + C_6 + \frac{q_0 l^2}{24F_1G(l-b)}\left[\left(N - \frac{3M}{R-3}\right)\left(\frac{z}{l}\right)^{+2} - \right. \\
& \left. - 6N\frac{b}{l}\left(\ln\frac{z}{l} - 1\right)\left(\frac{z}{l}\right)^{+1} + 12\frac{M+3}{R}\frac{b}{l}\left(\frac{z}{l}\right)^{+1}\right]\Bigg\}\psi_1(S).
\end{aligned}
\tag{53}
$$

The Determination of the Arbitrary Constants

If the end sections of the shell are attached as shown in Fig. 3, the boundary conditions (35) give a system of linear algebraic equations for determining the arbitrary constants. This system is shown in Table 3.

For a uniformly distributed transverse linear load we assume in formulas (49)-(53) and in Table 3 that $b = -\infty$.

Considering shells with specific parameters and load, we determine the arbitrary constants from Tables 2 and 3. Substituting the values of the arbitrary constants and of the corresponding coefficients (k, L, M, N, P, R) in formulas (29), (33), (34) and (51)-(53), we obtain the solutions for the stressed and deformed states of the shell. Thus the problem formulated is solved in the general form.

As can be seen, the solutions given in Section 2 are more difficult to obtain than the solution given in Section 1. However, it is to be expected that the stresses represented by formulas (51) and (52) will be closer to the actual stresses than those which can be determined from formulas (29) and (33). This circumstance was noted by Obraztsov [8] for an analogous problem in prismatic shells.

In practical calculations it is entirely permissable to use the solutions given in Section 1.

It would be of interest to simplify the solutions obtained without greatly prejudicing their accuracy and to make them more convenient for practical application.

The solutions given above can be made more accurate by retaining a greater number of terms in the expansions (1).

Torsion of a Conical Shell

It is known that in a conical shell normal stresses appear even when the shell is subjected to torsion by a constant moment and by free distortion of the end sections. The normal stresses in this case are a consequence of an internal constraint caused by the conicity of the shell.

TABLE 3

System of Equations for Determining the Arbitrary Constants

Condition	C_1	C_2	C_3	C_4	C_5	C_6	Right Member
1	$+1$	$+1$	$+1$	0	0	0	$+\dfrac{q_0 l^2 N}{12F_1G(l-b)}$
2	0	$\dfrac{cL}{(R+1)N}$	0	-1	-1	0	$-\dfrac{q_0 l^2 cL}{4F_1G(l-b)}\left(\dfrac{1}{R-3}-\dfrac{2}{R}\dfrac{b}{l}\right)$
3	$+1$	0	-1	$+\dfrac{1}{\sqrt{R+1}}$	$-\dfrac{1}{\sqrt{R+1}}$	$+1$	$-\dfrac{q_0 l^2}{24F_1G(l-b)}\left[N-\dfrac{3M}{R-3}+ +6\dfrac{b}{l}\left(N+2\dfrac{M+3}{R}\right)\right]$
4	$+2\left(\dfrac{m}{l}\right)^{-1}$	$+1$	0	0	0	0	$-\dfrac{q_0 l^2 N}{12F_1G(l-b)}\left[\left(\dfrac{m}{l}\right)^{+2}- -3\dfrac{b}{l}\left(\dfrac{m}{l}\right)^{+1}\right]$
5	0	$-\dfrac{2cL}{(R+1)N}$	0	$(2-\sqrt{R+1})\left(\dfrac{m}{l}\right)^{+\sqrt{R+1}}$	$(2+\sqrt{R+1})\left(\dfrac{m}{l}\right)^{-\sqrt{R+1}}$	0	$-\dfrac{q_0 lbcL}{2F_1G(l-b)R}\left(\dfrac{m}{l}\right)^{+1}$
6	0	$-\dfrac{1}{N}$	0	0	0	0	$-\dfrac{Q_0}{2F_1G}+\dfrac{q_0 l^2}{4F_1G(l-b)}\left[\left(\dfrac{m}{l}\right)^{+2}- -2\dfrac{b}{l}\left(\dfrac{m}{l}\right)^{+1}\right]$

Constrained torsion of straight and swept-back slightly conical shells was studied by Balabukh [1], Yelpat'yevskiy and Konovalov [4], and Novitskiy [7].

Let us solve the problem of constrained torsion of a straight slightly conical shell of constant thickness with a rigid contour on the basis of Prof. V. Z. Vlasov's variational method.

In the first approximation we shall assume that the displacements of the shell have only two degrees of freedom: one in the longitudinal direction (only one term is retained in the formula for the longitudinal displacement (1)) and one in the transverse direction.

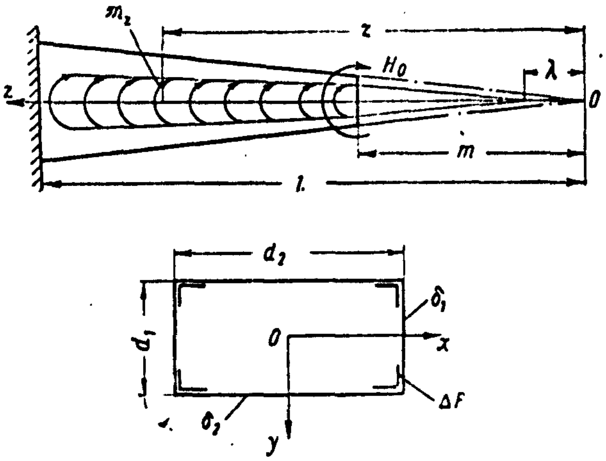

Fig. 6. Loading scheme of a conical caisson.

$$\mathfrak{M}_{(z)} = \frac{\mathfrak{M}_0}{l-\lambda}(z-\lambda) \mid -\infty < \lambda < m \mid;$$

$$F_1 = \delta_1 d_1; \quad F_2 = \delta_2 d_2.$$

Cross section of shell ($z = l$)

The shell is assumed to be loaded with a concentrated torque H_0 in the end cross section $z = m$ and a linear torque $\mathfrak{M}_{(z)}$ given by a linear law (Fig. 6).

Let us give the longitudinal and transverse displacements of the shell in the following form:

$$\left. \begin{aligned} u(z,S) &= U_1(z)\varphi_1(S); \\ v(z,S) &= V_1(z)\psi_1(S) \end{aligned} \right\} \qquad (54)$$

The functions $\varphi_1(S)$ and $\psi_1(S)$ represented in Fig. 7 are chosen as follows:

$$\left. \begin{aligned} \varphi_1(S) &= x(S)y(S); \\ \psi_1(S) &= h(S), \end{aligned} \right\} \qquad (55)$$

where h(S) is the length of the perpendicular dropped from the origin to the corresponding plate of the shell.

The function $\varphi_1(S)$ is quadratic, while the function $\psi_1(S)$ is linear with respect to the contour coordinate S.

For the solution of the problem formulated let us use the system of equations (15).

For the given loads and displacements (54) we obtain

$$\left. \begin{aligned} \gamma a_{11}\frac{d}{dz}\left(\frac{z^5}{l^5}U_1'\right) - \frac{z^3}{l^3}b_{11}U_1 - \frac{z^3}{l^3}c_{11}V_1' &= 0; \\ c_{11}\frac{d}{dz}\left(\frac{z^3}{l^3}U_1\right) + r_{11}\frac{d}{dz}\left(\frac{z^3}{l^3}V_1'\right) &= \\ = -\frac{\mathfrak{M}_0 l}{G(l-\lambda)}\left[\left(\frac{z}{l}\right)^{+1} - \frac{\lambda}{l}\right]. \end{aligned} \right\} \qquad (56)$$

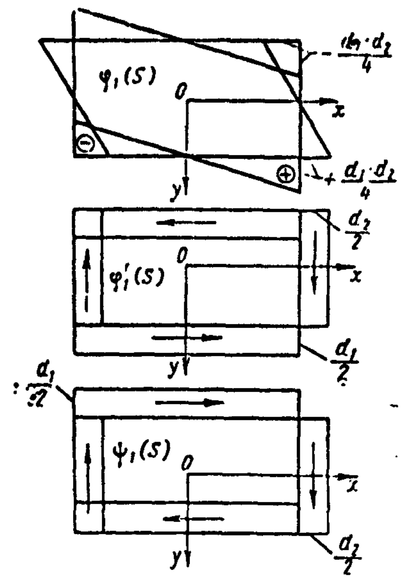

Fig. 7. Diagrams of approximating functions along the contour of a caisson.

The coefficients figuring in the equations of the system (56) are calculated in a fixed cross section $z = l$ from the formulas

$$a_{11} = J_{1\varphi} = \oint_{(s)} \varphi_1^2(s)\, dF(s) = \frac{1}{24} d_1^2 d_2^2 (F_1 + F_2 + 6\Delta F);$$

$$b_{11} = \oint_{(s)} \varphi_1'^2(s)\, dF(s) = \frac{1}{2}(d_2^2 F_1 + d_1^2 F_2);$$

$$c_{11} = \oint_{(s)} \varphi_1'(s)\psi_1(s)\, dF(s) = \frac{1}{2}(d_2^2 F_1 - d_1^2 F_2);$$

$$r_{11} = \oint_{(s)} \psi_1^2(s)\, dF(s) = \frac{1}{2}(d_2^2 F_1 + d_1^2 F_2),$$

where $J_{1\varphi}$ is the bimoment of torsional inertia.

Let

$$\left.\begin{aligned} a &= a_{11}E = \frac{1}{24} E d_1^2 d_2^2 (F_1 + F_2 + 6\Delta F); \\ b_1 &= b_{11}G = r_{11}G = \frac{1}{2} G (d_2^2 F_1 + d_1^2 F_2). \\ b_2 &= c_{11}G = \frac{1}{2} G (d_2^2 F_1 - d_1^2 F_2). \end{aligned}\right\} \qquad (57)$$

After certain transformations , let us write the system of equations (56) with the coefficients (57) in the form

$$\left.\begin{aligned} &\frac{a}{b_2 l^2}(z^2 U_1'' + 5zU_1') - \frac{b_1}{b_2} U_1 - V_1' = 0; \\ &b_2(zU_1' + 3U_1) + b_1(zV_1'' + 3V_1') = \\ &= -\frac{\mathfrak{M}_0 l^2}{l-\lambda}\left[\left(\frac{z}{l}\right)^{-1} - \frac{\lambda}{l}\left(\frac{z}{l}\right)^{-2}\right]. \end{aligned}\right\} \qquad (58)$$

We have obtained a nonhomogeneous system of two ordinary linear differential equations with variable coefficients of Euler type with respect to $U_1(z)$ and $V_1'(z)$, each equation being of second order.

We have already encountered the solution of such a system [system (20)].

After replacing the variable <u>z</u> by <u>t</u> according to the formula

$$\frac{z}{l} = e^{t},$$

performing the necessary operations on the unknown functions, let us write the system (58) in the form of Table 4, where D and D^2 denote, respectively, the first and second derivative with respect to the indpendent variable t of the functions in the upper line.

TABLE 4

$U_1(t)$	$V'_1(t)$	Right Member
$\frac{a}{b_2 l^2}(D^2+4D)-\frac{b_1}{b_2}$	-1	0
$b_2(D+3)$	$b_1(D+3)$	$-\frac{\mathfrak{M}_0 l^2}{l-\lambda}\left(e^{-t}-\frac{\lambda}{l}e^{-2t}\right)$

Let us introduce into our study a new variable f(t), while identically satisfying the first equation of the system (58)

$$\left.\begin{aligned} U_1(t) &= +f; \\ V'_1(t) &= \frac{a}{b_2 l^2}(f''+4f')-\frac{b_1}{b_2}f. \end{aligned}\right\} \qquad (59)$$

From the second equation in (58) we obtain a resolvent equation with respect to f(t)

$$f'''+7f''+(12-k^2)f'-3k^2 f = -\frac{\mathfrak{M}_0 l^4 b_2}{(l-\lambda)ab_1}\left(e^{-t}-\frac{\lambda}{l}e^{-2t}\right), \qquad (60)$$

where

$$k^2 = \frac{b_1^2-b_2^2}{ab_1}l^2. \qquad (61)$$

Equation (60) is a nonhomogeneous linear differential equation of the third order with constant coefficients.

The homogeneous differential equation corresponding to (60) has the form

$$f'''+7f''+(12-k^2)f'-3k^2 f=0.$$

Its characteristic equation is written as:

$$n^3+7n^2+(12-k^2)n-3k^2=0. \qquad (62)$$

Equation (62) for any shell parameters has the roots

$$n_1 = -3;\quad n_2 = -2 - \sqrt{4+k^2};\quad n_3 = -2 + \sqrt{4+k^2}.$$

Let us write the general solution for the function f(t):

$$f^{(0)}(t) = C_1 e^{-3t} + C_2 e^{-(2+\sqrt{4+k^2})t} + C_3 e^{(-2+\sqrt{4+k^2})t}.$$

The particular integral (60) has the following value:

$$\tilde{f}(t) = \frac{\mathfrak{M}_0 l^4 b_2}{(l-\lambda)ab_1}\left[\frac{1}{2(3+k^2)} e^{-t} - \frac{1}{4+k^2}\frac{\lambda}{l} e^{-2t}\right].$$

The solution to Eq. (60) has the form

$$f(t) = f^{(0)}(t) + \tilde{f}(t). \tag{63}$$

Let us pass, according to formulas (59), from f(t) to $U_1(t)$ and $V_1'(t)$; we replace the variable t by z and obtain expressions for the unknown functions

$$\left.\begin{aligned}
U_1(z) &= C_1\left(\frac{z}{l}\right)^{-3} + C_2\left(\frac{z}{l}\right)^{-(2+\sqrt{4+k^2})} + C_3\left(\frac{z}{l}\right)^{-2+\sqrt{4+k^2}} + \\
&\quad + \frac{\mathfrak{M}_0 l^4 b_2}{(l-\lambda)ab_1}\left[\frac{1}{2(3+k^2)}\left(\frac{z}{l}\right)^{-1} - \frac{1}{4+k^2}\frac{\lambda}{l}\left(\frac{z}{l}\right)^{-2}\right]; \\
V_1'(z) &= -\frac{3a+b_1 l^2}{b_2 l^2} C_1\left(\frac{z}{l}\right)^{-3} - \frac{b_2}{b_1} C_2\left(\frac{z}{l}\right)^{-(2+\sqrt{4+k^2})} - \\
&\quad - \frac{b_2}{b_1} C_3\left(\frac{z}{l}\right)^{-2+\sqrt{4+k}} + \\
&\quad + \frac{\mathfrak{M}_0 l^2}{(l-\lambda)ab_1}\left[-\frac{3a+b_1 l^2}{2(3+k^2)}\left(\frac{z}{l}\right)^{-1} + \frac{4a+b_1 l^2}{4+k^2}\frac{\lambda}{l}\left(\frac{z}{l}\right)^{-2}\right].
\end{aligned}\right\} \tag{64}$$

The function $V_1(z)$ is found by integrating the last expression in (64)

$$\begin{aligned}
V_1(z) = l\Bigg\{&\frac{3a+b_1 l^2}{2b_2 l^2} C_1\left(\frac{z}{l}\right)^{-2} + \frac{b_2}{b_1(1+\sqrt{4+k^2})} C_2\left(\frac{z}{l}\right)^{-(1+\sqrt{4+k^2})} + \\
&+ \frac{b_2}{b_1(1-\sqrt{4+k^2})} C_3\left(\frac{z}{l}\right)^{-1+\sqrt{4+k^2}} + C_4 + \\
&+ \frac{\mathfrak{M}_0 l^2}{(l-\lambda)ab_1}\left[-\frac{3a+b_1 l^2}{2(3+k^2)} \ln\frac{z}{l} - \frac{4a+b_1 l^2}{4+k^2}\frac{\lambda}{l}\left(\frac{z}{l}\right)^{-1}\right]\Bigg\}.
\end{aligned} \tag{65}$$

The Determination of the Normal and Tangential Stresses in the Shell

The normal stresses in the shell are determined from the formula

$$\sigma(z,S)=E\Big\{-\frac{3}{l}C_1\left(\frac{z}{l}\right)^{-4}-\frac{2+\sqrt{4+k^2}}{l}C_2\left(\frac{z}{l}\right)^{-(3+\sqrt{4+k^2})}-$$
$$-\frac{2-\sqrt{4+k^2}}{l}C_3\left(\frac{z}{l}\right)^{-3+\sqrt{4+k^2}}+ \qquad (66)$$
$$+\frac{\mathfrak{M}_0 l^3 b_2}{(l-\lambda)ab_1}\left[-\frac{1}{2(3+k^2)}\left(\frac{z}{l}\right)^{-2}+\frac{2}{4+k^2}\frac{\lambda}{l}\left(\frac{z}{l}\right)^{-3}\right]\Big\}\varphi_1(S).$$

The tangential stresses in the shell can be determined from Hooke's law.

As was done previously, it is recommended that the tangential stresses or the flux of tangential forces $q = \tau\delta$ be determined by integrating the differential equation of equilibrium (31).

In the case of the quadratic function $\varphi_1(s)$ $(\alpha = 2)$; after performing the necessary operations, we have for $q(z, S)$

$$q(z,S)=-\frac{E}{l^2}\Big[3C_1\left(\frac{z}{l}\right)^{-5}+(2+\sqrt{4+k^2})\sqrt{4+k^2}\,C_2\times$$
$$\times\left(\frac{z}{l}\right)^{-(4+\sqrt{4+k^2})}-(2-\sqrt{4+k^2})\sqrt{4+k^2}\,C_3\left(\frac{z}{l}\right)^{-4+\sqrt{4+k^2}}- \qquad (67)$$
$$-\frac{\mathfrak{M}_0 l^4 b_2}{2(l-\lambda)ab_1(3+k^2)}\left(\frac{z}{l}\right)^{-3}\Big]\int_0^S\varphi_1(S)\,\delta\,dS+q_0(z).$$

The longitudinal and transverse displacements, on the basis of (55), (64), and (65), are found from (54). In practical calculations for torsion we are interested in the angles of twist of the shell $\theta(z) = V_1(z)$, which are determined from (65).

The Determination of the Arbitrary Constants

Evaluating the boundary conditions for each particular case of attachment of the end sections of the shell, we obtain a system of linear algebraic equations for determining the arbitrary constants.

Let us consider the following two cases.

Case 1.

The shell (cf. Fig. 6) is rigidly attached in the cross section $z = l$ and freely deplanes in the section $z = m$. The arbitrary constants are determined from the boundary conditions

$$\left.\begin{array}{l} \underline{z=l} \\ 1.\ U_1(z)=0; \\ 2.\ V_1(z)=0. \\ \underline{z=m} \\ 3.\ U'_1(z)=0; \\ 4.\ \left(\frac{m}{l}\right)^3 (b_2 U_1 + b_1 V'_1) = -H_0. \end{array}\right\} \tag{68}$$

Case 2.

The end sections of the shell (cf. Fig. 6) $z = l$ and $z = m$ freely deplane. In order for the shell to be in equilibrium in the section $z = l$ under the action of the given loads, we apply the torque

$$H_z = H_0 + H_{\mathfrak{M}},$$

where $H_{\mathfrak{M}} = \frac{\mathfrak{M}_0 (l+m-2\lambda)}{2(l-\lambda)}(l-m)$ is a torque equalizing the linear torque $\mathfrak{M}_{(2)}$ over the length of the shell $L = l - m$.

The arbitrary constants are determined from the boundary conditions

$$\left.\begin{array}{ll} 1.\ z=l & U'_1(z)=0; \\ 2.\ z=m & U'_1(z)=0; \\ 3.\ z=z_0 & \frac{z_0^3}{l^3}(b_2 U_1 + b_1 V'_1) = -H_{z(z)}; \\ 4.\ z=z_0. & V_1(z)=0, \end{array}\right\} \tag{69}$$

where $H_{z(z)} = H_0 + \frac{\mathfrak{M}_0 + \mathfrak{M}_z}{2}(z-m)$ is the torque in a moving cross section of the shell.

It is enough to fulfill the third and fourth conditions of (69) in any cross section of the interval $m \leq z \leq l$. The cross section in which the fourth condition is fulfilled is the initial section when calculating the transverse displacements.

Let us write conditions (58) and (59) in the form of Tables 5 and 6.

Systems of Equations for Determining the Arbitrary Constants

It would be interesting to consider a shell loaded with a constant linear torque $\mathfrak{M}_0$.

In this case we assume that $\lambda = -\infty$ in formulas (64)-(67) and in Tables 5 and 6. The problem formulated, as in the case of bending, is solved in general form. For the purpose of comparing the solutions obtained by us with the results obtained by L. I. Balabukh [1] and B. P. Tsibulya [14], a calculation of the normal stresses in a conical shell with a rigid contour was made for the following parameter values in the end section z = 1:

$d_1 = 18$ cm; $d_2 = 60$ cm; $\delta_l = 0.2$ cm; $\delta_2 = 0.3$ cm; $\Delta F = 3.5$ cm^2; $l = 213.5$ cm; $m = 93.5$ cm.

The length of the shell $L = l - m = 120$ cm;

The coefficient $\gamma = \frac{E}{G} = 2.67$.

The conical shell in the end sections $z = l$ and $z = m$ is loaded with a torque H_0.

The coefficients (57) and (61) have the following values:

$$a=5{,}527\cdot 10^{+6}G;\ b_1=9396G;\ b_2=3564G;\ k^2=66{,}36.$$

The roots of the characteristic equation (62)

$$n_1=-3;\ n_2=-10{,}388;\ n_3=+6{,}388.$$

The normal stresses are determined along the edge of the shell

$$\left(x(S)=+\frac{d_2}{2};\ \ y(S)=+\frac{a_1}{2};\ \ \varphi_3(S)=x(S)y(S)\right)$$

from formulas (66). Graphs of the normal stresses for the two cases of attachment of the end sections of the shell considered above are shown in Fig. 8.

TABLE 5 (Case I)

Condition	C_1	C_2	C_3	C_4	Right Member
1	$+1$	$+1$	$+1$	0	$+\frac{\mathfrak{M}_0 l^4 b_2}{(l-\lambda)ab_1}\left[\frac{-1}{2(3+k^2)}+\frac{1}{4+k^2}\frac{\lambda}{l}\right]$
2	$\frac{3a+b_1 l^2}{2b_2 l^2}$	$\frac{b_2}{b_1(1+\sqrt{4+k^2})}$	$\frac{b_2}{b_1(1-\sqrt{4+k^2})}$	$+1$	$+\frac{\mathfrak{M}_0 \lambda l}{(l-\lambda)ab_1}\frac{4a+b_1 l^2}{4+k^2}$
3	$+3$	$(2+\sqrt{4+k^2})\left(\frac{m}{l}\right)^{+1-\sqrt{4+k^2}}$	$(2-\sqrt{4+k^2})\left(\frac{m}{l}\right)^{+1+\sqrt{4+k^2}}$	0	$+\frac{\mathfrak{M}_0 l^4 b_2}{(l-\lambda)ab_1}\left[-\frac{1}{2(3+k^2)}\left(\frac{m}{l}\right)^{+2}+\right.$ $\left.+\frac{2}{4+k^2}\frac{\lambda}{l}\left(\frac{m}{l}\right)^{+1}\right]$
4	$\frac{ab_1}{b_2 l^2}(3+k^2)$	0	0	0	$+H_0-\frac{\mathfrak{M}_0 l^2}{l-\lambda}\left[\frac{1}{2}\left(\frac{m}{l}\right)^{+2}-\frac{\lambda}{l}\left(\frac{m}{l}\right)^{+1}\right]$

TABLE 6 (Case II)

Condition	C_1	C_2	C_3	C_4	Right Member
1	$+3$	$2+\sqrt{4+k^2}$	$2-\sqrt{4+k^2}$	0	$+\frac{\mathfrak{M}_0 l^4 b_2}{(l-\lambda)ab_1}\left[\frac{-1}{2(3+k^2)}+\frac{2}{4+k^2}\frac{\lambda}{l}\right]$
2	$+3$	$(2+\sqrt{4+k^2})\left(\frac{m}{l}\right)^{+1-\sqrt{4+k^2}}$	$(2-\sqrt{4+k^2})\left(\frac{m}{l}\right)^{+1+\sqrt{4+k^2}}$	0	$+\frac{\mathfrak{M}_0 l^4 b_2}{(l-\lambda)ab_1}\left[-\frac{1}{2(3+k^2)}\left(\frac{m}{l}\right)^{+2}+\frac{2}{4+k^2}\frac{\lambda}{l}\left(\frac{m}{l}\right)^{+1}\right]$
3	$\frac{ab_1}{b_2 l^2}(3+k^2)$	0	0	0	$+H_0-\frac{\mathfrak{M}_0 l^2}{l-\lambda}\left[\frac{1}{2}\left(\frac{m}{l}\right)^{+2}-\frac{\lambda}{l}\left(\frac{m}{l}\right)^{+1}\right]$
4	$\frac{3a+b_1 l^2}{2b_2 l^2}$	$\frac{b_2}{b_1(1+\sqrt{4+k^2})}$	$\frac{b_2}{b_1(1-\sqrt{4+k^2})}$	$+1$	$+\frac{\mathfrak{M}_0\lambda l}{(l-\lambda)ab_1}\frac{4a+b_1 l^2}{4+k^2}$

Comment: The third and fourth conditions are written in the cross section $z = l$.

Curve a is obtained during free distortion of the end sections and indicates an internal constraint inherent in conical shells. Curve b is plotted for a shell with a rigid attachment of the end section $z = l$. The difference between the ordinates of curves a and b shows the effect of the embedding of the section $z = l$ of a conical shell.

Comparing the curves in Fig. 8 with analogous graphs [1, 14], we can conclude that there is an agreement in the nature of the distribution of the normal stresses over the length of conical shells. Such an agreement should not be regarded as accidental.

As has been shown [4], a calculation for bending of a conical shell on the basis of Prof. V. Z. Vlasov's variational method in the particular case corresponding to the hypothesis of plane cross sections gives expressions for the stresses and displacements which entirely coincide with the solutions of the resistance of materials when shear is taken into account.

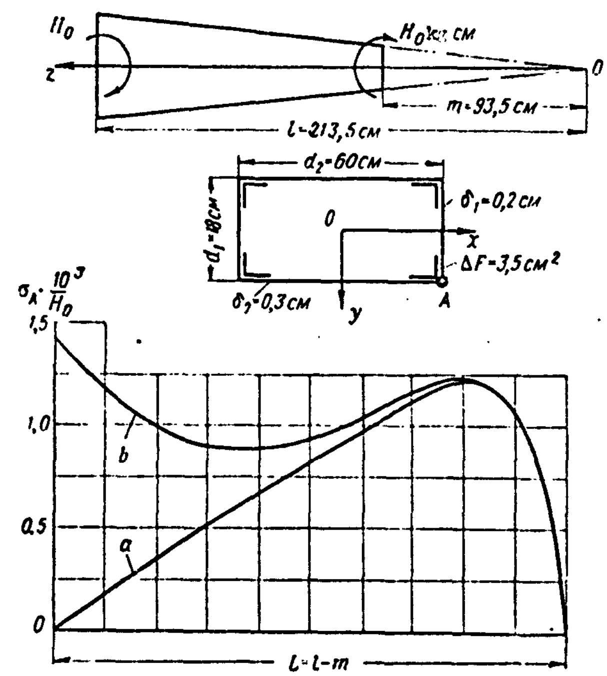

Fig. 8. The loading scheme of a caisson and its dimensions. Cross section of the shell ($z = l$). Graphs of the distribution of normal stresses along the edge of a conical caisson.

A more accurate calculation of the torsion of a conical shell by the proposed method in the second and subsequent approximations can be carried out by representing the displacements (1) in the form of several terms of a series, as was shown above in the solution of the bending problem

Thus the use of Prof. V. Z. Vlasov's variational method for calculating conical shells makes it possible to obtain more accurate solutions for the stressed and deformed states both in the case of bending and in the case of torsion.

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THE EQUATIONS OF AXISYMMETRICAL THREE-LAYER SHELLS WITH A LIGHT FILLER

V. F. Karavanov

In a previous article [1] the basic equations of axisymmetrical three-layer shells with a light filler were given without derivation in a linear formulation with assumptions based on neglecting the longitudinally directed stresses in the filler and the bending rigidity in the supporting layers.

In the present article we shall give the derivation of these equations in detailed form with the scheme and notations of E. Reissner [2] taken into account.

The equations obtained for axisymmetrical three-layer shells are similar in form to the equations of the theory of thin homogeneous axisymmetrical shells obtained by E. Meissner [3].

The equations for axisymmetrical conical, cylindrical, and spherical three-layer shells are obtained from the general equations as particular cases.

In certain practical cases terms depending on the transverse compressive deformation of the filler can be neglected. For this case the basic equations of axisymmetrical three-layer shells are given without taking the transverse compression of the filler into account.

The well-known equations for homogeneous thin axisymmetrical shells are obtained as a particular case of the equations of axisymmetrical three-layer shells.

Basic Assumptions

The wing and fuselage surfaces of present-day high-speed aircraft must remain smooth under a considerable load. There thus arises the question of the transition from a stringer covering to a covering of three-layer type, since the latter possesses great rigidity and strength and yet is light-weight. Great rigidity of coverings is of special value in the case of high-speed aircraft, since the question of sagging of the covering acquires great significance in connection with the considerable increase in flight speed. Three-layer structures with a light filler also possess good heat-insulation properties, soundproofing, good vibrational characteristics, and other qualities.

A three-layer structure consists of two strong thin outer layers connected to each other by a filler which ensures the concerted operation of the supporting layers. Foam plastics, honeycomb structures, corrugated sheet metal, and other materials may serve as the filler.

The supporting layers are made of metal, plastic, plywood, and delta wood. The calculation of three-layer plates and shells with allowance for deformation of the filler as a three-dimensional body is very cumbersome. Moreover, it is of little use in the case of light fillers, the bending rigidity, tensile rigidity, and longitudinal shear of which are small in relation to the rigidity of the supporting layers. In most cases approximate theories, based on various schemes acceptable in practice, are used in calculating such structures.

The present article is based on a scheme proposed by E. Reissner [2] with the following basic assumptions:

a) the bending rigidity of the supporting layers themselves is not taken into account (the nonuniformity of the distribution of stresses over the thickness of the supporting layers is neglected);

b) the filler undergoes only transverse shear and transverse compressive deformation, i.e., it is assumed that longitudinal stresses and moments are absorbed exclusively by the supporting layers, while the filler totally absorbs the transverse force (i.e., that the moduli of normal elasticity and shear of the filler are equal to zero in the longitudinal directions, but differ from zero in the transverse direction);

c) the filler is regarded as comparatively thick, light, elastic, and isotropic with a relatively small modulus of elasticity;

d) the dissimilarity between the lengths of the median surfaces of the inner and outer layers is taken into account.

The Geometry of the Shell

We shall determine the location of points on the median surface of an axisymmetrical three-layer shell by means of the angles θ and φ (Fig. 1).

Let θ be the angle between the normal to the median surface and the axis of the shell;

φ is the angle between the two meridional cross sections;

R_1 is the radius of curvature of the meridional cross section;

R_2 is the second principal radius of curvature.

By means of the two pairs of meridional and normal conical cross sections let us single out an element of an axisymmetrical three-layer shell with the dimensions dS_1 and dS_2 (Fig. 2) and apply to it all the internal and external stresses (Fig. 2).

Let us direct the coordinate axes as follows: the x-axis along the tangent to the meridian, the y-axis along the tangent to the parallel, the z-axis along the external normal to the median surface of the shell (cf. Fig. 3).

The square of a linear element of the median surface of the supporting layers

$$dS_{\pm}^2 = dS_{1\pm}^2 + dS_{2\pm}^2. \qquad (1)$$

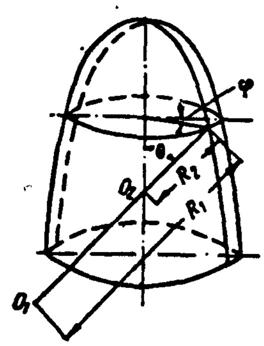

Fig. 1. Coordinate system of an axisymmetrical shell.

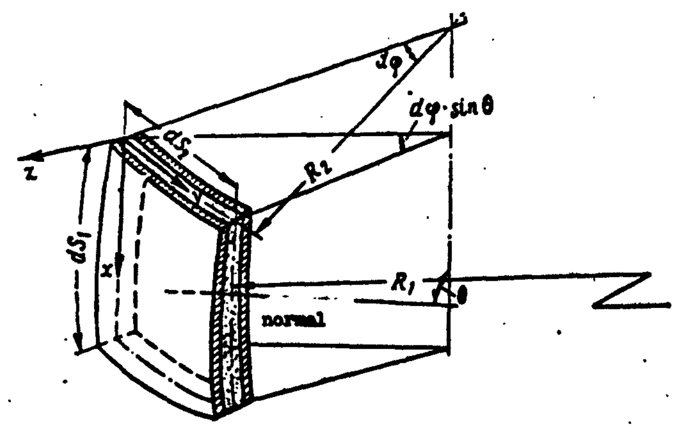

Fig. 2. An element of an axisymmetrical three-layer shell.

The subscript "+" refers to the outer supporting layer, while the subscript "-" refers to the inner supporting layer.

In turn,

$$\begin{aligned} dS_{1\pm} &= A_{1\pm}\, d\theta; \\ dS_{2\pm} &= A_{2\pm}\, d\varphi, \end{aligned} \qquad (2)$$

where $A_{1\pm}$ and $A_{2\pm}$ are the Lamé constants for an undeformed median surface of the supporting layers.

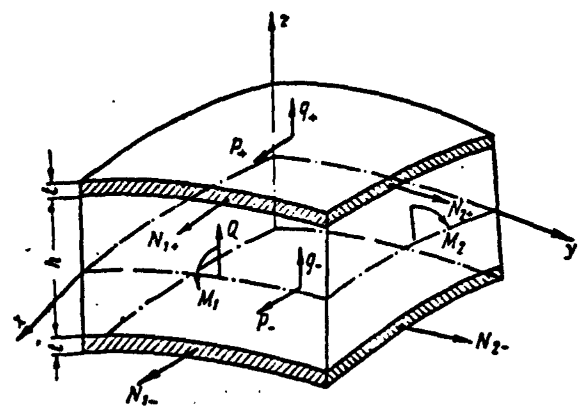

Fig. 3. Stresses, moments, and external loads acting on an element of an axisymmetrical three-layer shell.

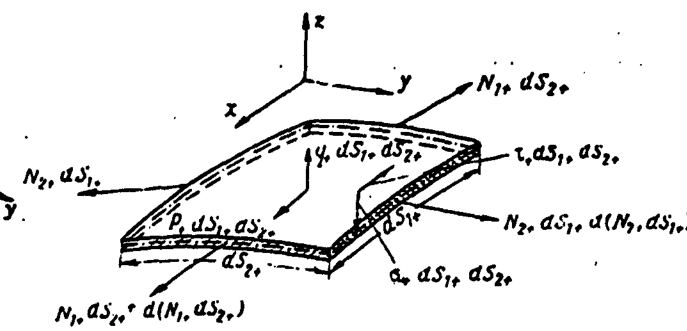

Fig. 4. An element of the outer supporting layer of a three-layer shell.

Taking into account the dissimilarity between the lengths of the median surfaces of the outer and inner supporting layers, the Lamé constants of a linear element on these surfaces are

$$A_{1\pm}=a_{1\pm}A_1;\quad A_{2\pm}=a_{2\pm}A_2,$$

where

$$a_{1\pm}=1\pm\frac{h+t}{2R_1};\quad a_{2\pm}=1\pm\frac{h+t}{2R_2},\qquad A_1=R_1;\quad A_2=R_2\sin\theta. \tag{3}$$

Here $\underline{t}$ is the thickness of the outer and inner supporting layers and is taken identical for both;

$\underline{h}$ is the thickness of the filler layer.

Let us resolve the intensity of the external load acting in the general case on the surface of the supporting layers into a force acting along the normal and a force acting along the tangent to the arc of the meridian $p_{\pm}$ and $q_{\pm}$ (cf. Fig. 3).

The third component of this intensity, owing to the condition of symmetry, goes to zero.

Equilibrium of the Supporting Layers

In order to obtain the complete system of equations for an axisymmetrical three-layer shell, it is necessary to consider the equilibrium of the supporting layers and the filler layer of the shell separately. By combining these equations we obtain the differential equations of equilibrium for the entire composite shell.

The equations of equilibrium for an element of the outer and inner supporting layer, respectively, of an axisymmetrical three-layer shell for small displacements of the median surface have the form

$$(N_{1\pm}a_{2\pm}R_2\sin\theta)' - N_2 a_{1\pm}R\cos\theta + (p_\pm \mp \tau_\pm)a_{1\pm}a_{2\pm}R_1R_2\sin\theta = 0; \tag{4}$$

$$\frac{N_{1\pm}}{a_{1\pm}R_1} + \frac{N_{2\pm}}{a_{2\pm}R_2} = q_\pm \mp \sigma_\pm. \tag{5}$$

Here $N_{1\pm}$ and $N_{2\pm}$ are, respectively, the specific normal meridional and annular stresses of the outer and inner supporting layers of a three-layer shell (kg/cm); $\sigma_\pm$ and $\tau_\pm$ are, respectively, the normal and tangential stresses in the transverse direction of the filler layer acting on the outer and inner supporting layers of a three-layer shell (kg/cm^2); $p_\pm$ and $q_\pm$ are, respectively, the components of the intensity of the external load acting on the outer and inner supporting layers of a three-layer shell (kg/cm^2); the prime indicates differentiation with respect to the variable θ.

Figure 4 shows an element of the outer supporting layer, while Fig. 5 shows an element of the inner supporting layer. All other internal factors, owing to the condition of symmetry, go to zero.

Distribution of the Stress in the Filler Layer

As was mentioned, the filler is regarded as light, elastic, isotropic, and homogeneous. Its bending rigidity, tension-compression rigidity, and shear in the longitudinal direction are negligibly small. Consequently it operates in the transverse direction during tension-compression and during shear (Fig. 6).

The differential equations of equilibrium of an element for this stressed state [4] will be the following:

$$\left[\left(1+\frac{z}{R_1}\right)^2\left(1+\frac{z}{R_2}\right)\tau\right]_z=0; \tag{6}$$

$$\left[R_1R_2\sin\theta\left(1+\frac{z}{R_1}\right)\left(1+\frac{z}{R_2}\right)\sigma\right]_z+\left[R_2\sin\theta\left(1+\frac{z}{R_2}\right)\tau\right]_\theta. \tag{7}$$

Here σ and τ are, respectively, the normal and tangential stresses in the transverse direction of the filler. The subscripts z and θ indicate differentiation with respect to the variables z and θ.

Integration of Eq. (6) gives

$$\tau=\frac{\tau_0}{\left(1+\frac{z}{R_1}\right)^2\left(1+\frac{z}{R_2}\right)}. \tag{8}$$

The subscript "0" indicates that the stresses σ and τ refer to the median surface of the filler ($z = 0$).

Let us integrate Eq. (7), after having first substituted expression (8) into this equation; we obtain

$$\begin{gathered}R_1R_2\sin\theta\left(1+\frac{z}{R_1}\right)\left(1+\frac{z}{R_2}\right)\sigma+\\ z+\left[R_2\sin\theta\frac{\tau_0}{1+\frac{z}{R_1}}\right]_\theta-R_1R_2\sin\theta\sigma_0=0.\end{gathered} \tag{9}$$

The resultant of the transverse shearing stress Q is obtained from Eq. (8) in the following form:

$$Q=\int_{-\frac{h+t}{2}}^{\frac{h+t}{2}}\tau\left(1+\frac{z}{R_2}\right)dz=\frac{(h+t)\tau_0}{1-\left(\frac{h+t}{2R_1}\right)^2}. \tag{10}$$

The integration extends over the thickness of the filler layer and also over half the thickness of the surface layers, as a result of the assumption that the stresses $\sigma_{\pm}$ and $\tau_{\pm}$ can act on the median surfaces of the supporting layers [5]. It follows from the meaning of integration with respect to $\pm\frac{h+t}{2}$ that the supporting layers are attached, as it were, to the filler by their median surfaces.*

From Eqs. (8) and (10) it follows that

$$\tau = \frac{Q}{h+t}. \tag{11}$$

This equation was obtained by discarding the terms $\frac{h+t}{2R}$ in comparison with unity.

From Eq. (11) it follows that the transverse tangential stresses are uniformly distributed over the thickness of the filler.

From Eqs. (8), (10), and (11) the following dependences can be obtained:

$$\frac{h+t}{2}(a_{1+}a_{2+}\tau_{+} + a_{1-}a_{2-}\tau_{-}) = Q; \tag{12}$$

$$a_{1+}a_{2+}\tau_{+} - a_{1-}a_{2-}\tau_{-} = -\frac{Q}{R_1}. \tag{13}$$

When h + t/2R << 1, using Eqs. (9), (10), and (11), we obtain

$$R_1R_2\sin\theta(a_{1+}a_{2+}\sigma_{+} - a_{1-}a_{2-}\sigma_{-}) = -(R_2\sin\theta Q)'; \tag{14}$$

$$a_{1+}a_{2+}\sigma_{+} + a_{1-}a_{2-}\sigma_{-} = 2\sigma_{0}. \tag{15}$$

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* Although this introduces a certain inaccuracy, it is so small that it has no significant effect on practical calculations in the case of comparatively thick fillers for which t/h < 0.1. The error of this assumption is all the greater, the greater the ratio between the thicknesses of the layers t/h.

Equilibrium of a Composite Shell

In order to obtain the equations of equilibrium of a composite three-layer shell, let us introduce the resulting expressions for the external and internal stresses of a three-layer shell [2].

The specific normal stresses

$$N_1 = a_{2+}N_{1+} + a_{2-}N_{1-};\quad N_2 = a_{1+}N_{2+} + a_{1-}N_{2-}. \tag{16}$$

The specific bending moments

$$M_1 = \frac{h+t}{2}(a_{2+}N_{1+} - a_{2-}N_{1-});\quad M_2 = \frac{h+t}{2}(a_{1+}N_{2+} - a_{1-}N_{2-}). \tag{17}$$

The components of the external force

$$p = a_{1+}a_{2+}p_+ + a_{1-}a_{2-}p_-;\quad q = a_{1+}a_{2+}q_+ + a_{1-}a_{2-}q_-. \tag{18}$$

The intensity of the moment formed by the action of tangential surface forces

$$m = \frac{h+t}{2}(a_{1+}a_{2+}p_+ - a_{1-}a_{2-}p_-), \tag{19}$$

and also

$$g = \frac{1}{2}(a_{1+}a_{2+}q_+ - a_{1-}a_{2-}q_-). \tag{20}$$

To the above relations let us add the expressions obtained previously.

The specific transverse force

$$Q = \frac{h+t}{2}(a_{1+}a_{2+}\tau_+ + a_{1-}a_{2-}\tau_-). \tag{12}$$

The normal transverse stress in the median surface of the filler

$$\sigma_0 = \frac{1}{2}(a_{1+}a_{2+}\sigma_+ + a_{1-}a_{2-}\sigma_-). \tag{15}$$

Using Eqs. (12)-(20), let us add and subtract Eqs. (4) and (5). After certain transformations we obtain the equations of equilibrium of composite axisymmetrical shells

$$(N_1R_2 \sin\theta)' - N_2R_1 \cos\theta + \left(p + \frac{Q}{R_1}\right)R_1R_2 \sin\theta = 0; \tag{21}$$

$$(M_1R_2 \sin\theta)' - M_2R_1 \cos\theta + (m - Q)R_1R_2 \sin\theta = 0; \tag{22}$$

$$\frac{N_1}{R_1} + \frac{N_2}{R_2} - q - \frac{1}{R_1R_2 \sin\theta}(QR_2 \sin\theta)' = 0; \tag{23}$$

$$\sigma_0 + \frac{1}{h+t}\left(\frac{M_1}{R_1} + \frac{M_2}{R_2}\right) - g = 0. \tag{24}$$

The latter equation becomes an identity, when the bending moments are expressed in terms of the stresses.

From Eqs. (9) and (10) it follows that

$$R_1R_2 \sin\theta\,\sigma = R_1R_2 \sin\theta\,\sigma_0 - \frac{z}{h+t}(QR_2 \sin\theta)'. \tag{25}$$

According to Eqs. (23) and (25), we obtain the law of change of the transverse normal stress over the thickness of the filler

$$\sigma = \sigma_0 - \frac{z}{h+t}\left(\frac{N_1}{R_1} + \frac{N_2}{R_2} - q\right). \tag{26}$$

Hence it can be seen that the transverse normal stress consists of two terms, one of which is constant, while the other varies linearly over the thickness of the filler layer.

The Potential Energy of Deformation of the Shell

The potential energy of deformation of an axisymmetrical three-layer shell is the sum of the potential energies of the deformations of the supporting layers and the filler layer

$$\Pi = \Pi_{\text{н}} + \Pi_{\text{з}}. \tag{27}$$

The supporting layers are characterized by the following elastic constants: the modulus of normal elasticity E_H, the Poisson coefficient $\mu = \mu_H$, and the shear modulus $G_H = \frac{E_H}{2(1 + \mu)}$.

The elastic constants of the filler are as follows: the modulus of transverse elasticity E_c; the Poisson coefficient μ_c is taken equal to zero on the basis of the assumption made concerning the operation of the filler layer; then the shear modulus of the filler $G_c = \frac{E_c}{2}$.

The supporting layers of the shell being considered undergo only tension-compression deformation; therefore the potential energy of deformation of the supporting layers

$$\Pi = \frac{1}{2}\iint \frac{1}{tE_n}(N_{1+}^2 + N_{2+}^2 - \\ -2\mu N_{1+}N_{2+})a_{1+}a_{2+}R_1R_2 \sin\theta\, d\theta\, d\varphi + \frac{1}{2}\iint \frac{1}{tE_n}(N_{1-}^2 + N_{2-}^2 - \\ -2\mu N_{1-}N_{2-})a_{1-}a_{2-}R_1R_2 \sin\theta\, d\theta\, d\varphi.$$

From Eqs. (16) and (17) we obtain

$$2a_{2+}N_{1+} = N_1 + \frac{2}{h+t}M_1;\quad 2a_{1+}N_{2+} = N_2 + \frac{2}{h+t}M_2; \\ 2a_{2-}N_{1-} = N_1 - \frac{2}{h+t}M_1;\quad 2a_{1-}N_{2-} = N_2 - \frac{2}{h+t}M_2. \tag{29}$$

Using the relations obtained and the condition h + t/2R << 1, expression (28) will assume the following form:

$$\Pi_n = \frac{1}{2}\iint\Big[\frac{1}{C^*}(N_1^2 + N_2^2 - 2\mu N_1 N_2) + \\ + \frac{1}{D^*}(M_1^2 + M_2^2 - 2\mu M_1 M_2)\Big]R_1R_2 \sin\theta\, d\theta\, d\varphi, \tag{30}$$

where C* is the tension-compression rigidity of a three-layer shell;

D* is the bending rigidity of a three-layer shell

$$C^* = 2E_n t; \\ D^* = {}^1/_2 E_n t(h+t)^2. \tag{31}$$

On the basis of the assumed distribution of the stress in the filler layer (cf. Fig. 6), the potential energy of deformation of this layer will be as follows:

$$\Pi_c = \frac{1}{2}\iiint\limits_{-\frac{h+t}{2}}^{+\frac{h+t}{2}}\left(\frac{\sigma^2}{E_c}+\frac{\tau^2}{G_c}\right)\left(1+\frac{z}{R_1}\right)\left(1+\frac{z}{R_2}\right)R_1R_2\sin\theta\, d\theta\, d\varphi\, dz.$$

As previously, we neglect the terms z/R in comparison with unity, while the values of σ and τ are taken from Eqs. (11) and (26).

Then, integrating with respect to z, we obtain

$$\Pi_c = \frac{1}{2}\iint\left\{\frac{h+t}{E_c}\left[\sigma_0^2+\frac{1}{12}\left(\frac{N_1}{R_1}+\frac{N_2}{R_2}-q\right)^2\right]+ + \frac{Q^2}{(h+t)\,G_c}\right\}R_1R_2\sin\theta\, d\theta\, d\varphi. \tag{32}$$

On the basis of (27), (30), and (32), the potential energy of deformation of an axisymmetrical three-layer shell

$$\Pi = \frac{1}{2}\iint\frac{1}{C^*}(N_1^2+N_2^2-2\mu N_1N_2)+\frac{1}{D^*}(M_1^2+M_2^2-2\mu M_1M_2)+ + \frac{h+t}{E_c}\left[\sigma_0^2+\frac{1}{12}\left(\frac{N_1}{R_1}+\frac{N_2}{R_2}-q\right)^2\right]+\frac{Q^2}{(h+t)\,G_c}\Big\}R_1R_2\sin\theta\, d\theta\, d\varphi. \tag{33}$$

Relationships Between the Stresses, Moments, and Displacements in the Case of a Composite Shell

According to Castigliano's principle, the stressed state actually arising in the shell differs from all statically possible states in that the potential energy of deformation of the shell Π, determined by Eq. (33), assumes a minimum value.

Thus it is necessary to find the minimum of a functional Π depending on six unknown functions N_1, N_2, M_1, M_2, Q, and σ_0, which satisfy the four equilibrium equations (21)-(24). The problem of finding this arbitrary extremum of the functional Π in the calculus of variations is usually replaced by the problem of finding the absolute extremum of a certain other functional τ.

In order to set up the functional τ, let us use the method of undetermined multipliers, which preserves the complete equivalence

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of the variables [6]

$$\tau=\iint\Phi(N_1, N_2, M_1, M_2, Q, \sigma_0)\,d\theta\,d\varphi. \qquad (34)$$

In turn $\Phi=F+\sum_{i=1}^{4}\nu_i v_i,$

where F is an integrand expression for the potential energy Π and is determined by expression (33); ν_i (θ,φ) $(i = 1, 2, 3, 4)$ are unknown nonvarying Lagrange multipliers;

v_i are the corresponding equilibrium equations determined by the expressions (21)-(24).

It can be shown that each of the four nonvarying Lagrange multipliers is nothing other than a displacement in the corresponding directions $\nu_1 = u$, $\nu_2 = \beta$, $\nu_3 = w$, $\nu_4 = k$,

where u and w are linear displacements of the median surface of a three-layer shell in the direction of the x- and z-axes;

β is the angle of rotation of the normal to the median surface of the shell in the direction of the x-axis;

k is a quantity proportional to the transverse compressive deformation.

Thus Eq. (21) is multiplied by u, Eq. (22) by β, Eq. (23) by w, and Eq. (24) by k.

Then the functional Φ will be written thus:

$$\begin{aligned}\Phi=\frac{1}{2}\Big\{&\frac{1}{C^*}(N_1^2+N_2^2-2\mu N_1N_2)+\frac{1}{D^*}(M_1^2+M_2^2-2\mu M_1M_2)+\\&+\frac{h+t}{E_c}\Big[\sigma_0^2+\frac{1}{12}\Big(\frac{N_1}{R_1}+\frac{N_2}{R_2}-q\Big)^2\Big]+\frac{Q^2}{(h+t)\,G_c}\Big\}R_1R_2\sin\theta+\\&+u\Big[(N_1R_2\sin\theta)'-N_2R_1\cos\theta+\Big(p+\frac{Q}{R_1}\Big)R_1R_2\sin\theta\Big]+\\&+\beta\left[(M_1R_2\sin\theta)'-M_2R_1\cos\theta+(m-Q)R_1R_2\sin\theta\right]+\\&+w\Big[(QR_2\sin\theta)'-\Big(\frac{N_1}{R_1}+\frac{N_2}{R_2}-q\Big)R_1R_2\sin\theta\Big]+\\&+k\Big[\sigma_0+\frac{1}{h+t}\Big(\frac{M_1}{R_1}+\frac{M_2}{R_2}\Big)-g\Big].\end{aligned}$$

The extremum of this functional occurs for the Euler condition

$$\Phi_{N_1} - \frac{\partial}{\partial\theta}\Phi_{\frac{\partial N_1}{\partial\theta}} = 0,$$
$$\cdots\cdots\cdots \tag{36}$$
$$\Phi_{\varepsilon_1} - \frac{\partial}{\partial\theta}\Phi_{\frac{\partial\varepsilon_1}{\partial\theta}} = 0,$$

where Φ_{N_1} is the partial variation of Φ with respect to N_1;

$\Phi_{\frac{\partial N_1}{\partial\theta}}$ is the partial variation of Φ with respect to $\frac{\partial N_1}{\partial\theta}$, etc.

Evaluating Eqs. (36), after elementary transformations, we obtain the following expressions:

$$\frac{N_1}{C^*}\left[1+\frac{(h+t)C^*}{12E_cR_1^2}\right]-\frac{N_2}{C^*}\left[\mu-\frac{(h+t)C^*}{12E_cR_1R_2}\right]=\frac{u'+w}{R_1}+\frac{(h+t)q}{12E_cR_1}; \tag{37}$$

$$\frac{N_2}{C^*}\left[1+\frac{(h+t)C^*}{12E_cR_2^2}\right]-\frac{N_1}{C^*}\left[\mu-\frac{(h+t)C^*}{12E_cR_1R_2}\right]=\frac{u\cot\theta+w}{R_2}+\frac{(h+t)q}{12E_cR_2}; \tag{38}$$

$$\frac{M_1-\mu M_2}{D^*}=\frac{\beta'}{R_1}-\frac{k}{(h+t)R_1^2R_2\sin\theta}; \tag{39}$$

$$\frac{M_2-\mu M_1}{D^*}=\frac{\beta}{R_2}\cot\theta-\frac{K}{(h+t)R_1R_2^2\sin\theta}; \tag{40}$$

$$\frac{Q}{(h+t)G_c}=\beta+\frac{w'-u}{R_1}; \tag{41}$$

$$\frac{\varepsilon_0}{E_c}=-\frac{K}{R_1R_2\sin\theta(h+t)}. \tag{42}$$

Let us reduce the system of equations (37)-(42) to a more convenient form.

Let us introduce the variable coefficients λ_1, λ_2, and λ_3

$$\lambda_1=\frac{1}{2}\frac{(h+t)t}{R_1^2}\frac{E_n}{E_c};$$
$$\lambda_2=\frac{1}{2}\frac{(h+t)t}{R_2^2}\frac{E_n}{E_c}, \tag{43}$$
$$\lambda_3=\frac{1}{2}\frac{(h+t)t}{R_1R_2}\frac{E_n}{E_c}.$$

Then Eqs. (37) and (38) will assume the form

$$\left(1+\frac{1}{3}\lambda_1\right)N_1-\left(\mu-\frac{1}{3}\lambda_3\right)N_2=C^*\left[\frac{u'+w}{R_1}+\frac{h+t}{12E_cR_1}q\right];$$
$$\left(1+\frac{1}{3}\lambda_2\right)N_2-\left(\mu-\frac{1}{3}\lambda_3\right)N_1=C^*\left[\frac{u\cot\theta+w}{R_2}+\frac{h+t}{12E_cR_2}q\right].$$

Let us eliminate k from Eqs. (39) and (40) by using relations (24) and (42); we obtain

$$(1+\lambda_1)M_1-(\mu-\lambda_3)M_2=D^*\left(\frac{\beta'}{R_1}+\frac{g}{E_cR_1}\right);$$

$$(1+\lambda_2)M_2-(\mu-\lambda_3)M_1=D^*\left(\frac{\beta\cot\theta}{R_2}+\frac{g}{R_2E_c}\right).$$

In transforming Eqs. (39) and (40) it is assumed that $\lambda_1 = \frac{D^*}{(h+t)E_cR_1^2}$ on the basis of the expressions for D^* and λ_1 from (31) and (43).

As a result we obtain a system of equations for an axisymmetrical three-layer shell: three equilibrium equations (21)-(23) and five relationships between the stresses and the displacements (37)-(41). From these eight equations eight unknowns are determined: three stresses N_1, N_2, and Q, two moments M_1 and M_2, and three displacements U, w, and β.

The normal transverse stress in the median surface of the filler σ_0 can be determined from Eq. (24). The action of the transverse shear of the filler figures only in Eq. (41), while the action of the transverse compressive deformation is reflected in Eqs. (37)-(40).

If in Eqs. (21)-(23) and (37)-(41) it is assumed that

$$G_c=E_c=\infty,\ \lambda_1=\lambda_2=\lambda_3=0,\quad D^*=\frac{Eh^3}{12},\ C^*=Eh$$

and m = 0, we obtain a system of equations for thin homogeneous axisymmetrical shells. Moreover, these equations completely coincide with the well-known equations [7].

Reduction of the System of Equations to Two Simultaneous Equations

Let us reduce the system of equations obtained (21)-(23) and (37)-(41) to a system of two second-order differential equations.

From Eqs. (40)-(41) let us determine the specific bending moments M_1 and M_2

$$M_1 = \frac{D^*}{\Omega}\left[(1+\lambda_2)\left(\frac{\beta'}{R_1}+\frac{g}{R_1E_c}\right)+(\mu-\lambda_3)\left(\frac{\beta\cot\theta}{R_2}+\frac{g}{R_2E_c}\right)\right],$$
$$M_2 = \frac{D^*}{\Omega}\left[(1+\lambda_1)\left(\frac{\beta\cot\theta}{R_2}+\frac{g}{R_2E_c}\right)+(\mu-\lambda_3)\left(\frac{\beta'}{R_1}+\frac{g}{R_1E_c}\right)\right]. \qquad (44)$$

Here

$$\Omega = (1+\lambda_1)(1+\lambda_2)-(\mu-\lambda_3)^2. \qquad (45)$$

Let us introduce Meissner's function $V = QR_2$. (46)

Let us introduce the values of M_1 and M_2 from (44) into Eq. (22).

As a result, after certain transformations, we obtain a deformation-compatibility equation containing the variables β and V

$$\frac{R_2}{R_1}\beta'' + \left[\left(\frac{R_2}{R_1}\right)' + \frac{R_2}{R_1}\left(\cot\theta+\gamma_3\lambda_2'-\frac{\Omega'}{\Omega}\right)\right]\beta' -$$
$$-\left[\gamma_2\frac{R_1}{R_2}\cot^2\theta+\left(\frac{\Omega'}{\Omega}\gamma_1+\lambda_3'\gamma_3\right)\cot\theta+\gamma_1\right]\beta-\frac{\Omega\gamma_3R_1}{D^*}V = G_1(\theta). \qquad (47)$$

Here

$$\gamma_1=\frac{\mu-\lambda_3}{1+\lambda_2},\quad \gamma_2=\frac{1+\lambda_1}{1+\lambda_2},\quad \gamma_3=\frac{1}{1+\lambda_2},$$
$$G_1(\theta)=-\left[\left(\frac{R_2}{R_1}\right)'+\left(\frac{R_2}{R_1}-\gamma_2\frac{R_1}{R_2}\right)\cot\theta+\frac{R_2}{R_1}\left(\gamma_3\lambda_2'-\frac{\Omega'}{\Omega}\right)-\right.$$
$$\left.-\left(\frac{\Omega'}{\Omega}\gamma_1+\gamma_3\lambda_3'\right)\right]\frac{g}{E_c}-\left(\frac{R_2}{R_1}+\gamma_1\right)\frac{g'}{E_c}-\gamma_3\Omega\frac{R_1R_2}{D^*}m. \qquad (48)$$

In order to obtain the second equation, let us consider the condition of equilibrium of the part of the shell cut off by a parallel circle of radius $r = R_2 \sin\theta$ (Fig. 7), i.e., let us project all the forces acting on the part of the shell and on the axis of symmetry of the shell. Then

$$\sum OO_1 = 2\pi R_2\sin\theta\,(N_1\sin\theta-Q\cos\theta)-2\pi C+$$
$$+\int 2\pi R_2\sin\theta\,(p\sin\theta-q\cos\theta)\,R_1\,d\theta = 0.$$

This equation can be conveniently rewritten as:

$$F(\theta) = (N_1\sin\theta-Q\cos\theta)\,R_2\sin\theta =$$
$$= C+\int R_2\sin\theta\,(q\cos\theta-p\sin\theta)\,R_1\,d\theta. \qquad (49)$$

Here $F(\theta)$ is a function of the load and the geometrical dimensions of the shell and is the axial component of the internal forces acting on one radian of the contour of the cross section;

$2\pi C = P_0$ is the axial component of the concentrated forces (not shown in Fig. 7).

Let us express N_1 and N_2 in terms of V and $F(\theta)$. Then from Eqs. (46) and (49) we obtain

$$N_1 = \frac{F(\theta)}{R_2 \sin^2\theta} + \frac{V}{R_2}\cot\theta. \tag{50}$$

From Eqs. (23), (46), and (50) we obtain

$$N_2 = \frac{V'}{R_1} - \frac{F(\theta)}{R_1 \sin^2\theta} + qR_2. \tag{51}$$

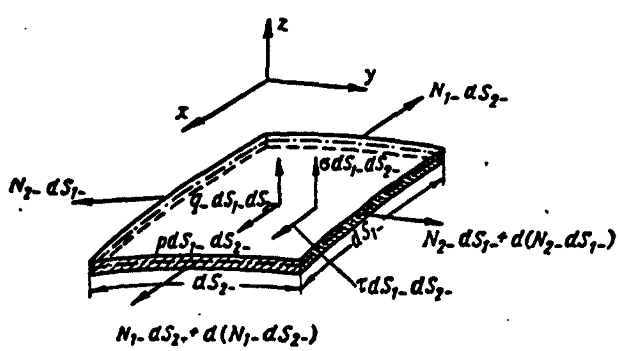

Fig. 5. An element of the inner supporting layer of a three-layer shell.

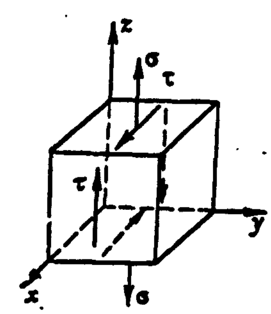

Fig. 6. Model of the stressed state of the filler.

Let us introduce the second load function

$$H(\theta) = \frac{F(\theta)}{\sin^2\theta} + qR_1R_2. \tag{52}$$

Equation (51) will assume the form

$$N_2 = \frac{V'}{R_1} - \frac{H(\theta)}{R_1}. \tag{53}$$

Equations (21) and (23) are identical to zero, if we substitute in them the values of N_1 and N_2 from (50) and (53).

Let us eliminate $\underline{w}$ from Eqs. (37) and (38). We obtain

$$N_1\left[\left(1+\frac{1}{3}\lambda\right)R_1+\left(\mu-\frac{1}{3}\lambda_3\right)R_2\right]-N_2\left[\left(\mu-\frac{1}{3}\lambda_3\right)R_1+\right.$$
$$\left.+\left(1+\frac{1}{3}\lambda_2\right)R_2\right]=C^*(u'-u\cot\theta). \tag{54}$$

Let us substitute in Eq. (54) the value of $\underline{u}$ from (41) with (46) taken into account. We obtain

$$u'-w'\cot\theta=\frac{1}{C^*}\left\{N_1\left[\left(1+\frac{1}{3}\lambda_1\right)R_1+\left(\mu-\frac{1}{3}\lambda_3\right)R_2\right]-\right.$$
$$\left.-N_2\left[\left(\mu-\frac{1}{3}\lambda_3\right)R_1+\left(1+\frac{1}{3}\lambda_2\right)R_2\right]\right\}+\beta R_1\cot\theta-$$
$$-\frac{VR_1}{(h+t)G_c}\frac{\cot\theta}{R_2}. \tag{55}$$

Let us represent Eq. (38) in the following form:

$$u\cot\theta+w=\frac{R_2}{C^*}\left[\left(1+\frac{1}{3}\lambda_2\right)N_2-\left(\mu-\frac{1}{3}\lambda_3\right)N_1\right]-\frac{h+t}{12E_c}q.$$

Differentiating once both sides of this equation and substituting in the equation obtained the value of $\underline{u}$ from (41) with (46) taken into account, we obtain

$$u-w'\cot\theta=\frac{1}{C^*}\left\{R_2'\tan\theta\left[\left(1+\frac{1}{3}\lambda_2\right)N_2-\left(\mu-\frac{1}{3}\lambda_3\right)N_1\right]+\right.$$
$$\left.+R_2\tan\theta\left[\left(1+\frac{1}{3}\lambda_2\right)N_2'-\left(\mu-\frac{1}{3}\lambda_3\right)N_1'+\frac{1}{3}\lambda_2'N_2+\frac{1}{3}\lambda_3'N_1\right]-\right.$$
$$\left.-\frac{h+t}{12E_c}\tan\theta q'+\left[\beta-\frac{V}{(h+t)G_cR_2}\right]\frac{2R_1}{\sin 2\theta}\right. . \tag{56}$$

Equating the left-hand sides of Eqs. (55) and (56) and substituting in the equation obtained the values of N_1 and N_2 from (50) and (53) and their derivatives N_1' and N_2', after a number of transformations, we obtain the second equation (the equilibrium equation)

$$\left(1+\frac{1}{3}\lambda_2\right)\frac{R_2}{R_1}V''+\left\{\left(1+\frac{1}{3}\lambda_2\right)\left[\left(\frac{R_2}{R_1}\right)'+\frac{R_2}{R_1}\cot\theta\right]+\right.$$
$$\left.+\frac{1}{3}\lambda_2'\frac{R_2}{R_1}\right\}V'-\left[\left(1+\frac{1}{3}\lambda_1\right)\frac{R_1}{R_2}\cot^2\theta+\frac{C^*}{(h+t)G_c}\frac{R_1}{R_2}-\right.$$
$$\left.-\mu+\frac{1}{3}\lambda_3-\frac{1}{3}\lambda_3'\cot\theta\right]V+R_1C^*\beta=G_2(\theta). \tag{57}$$

Here

$$G_2(\theta)=B_1(\theta)\frac{F(\theta)}{R_2\sin^2\theta\tan\theta}+B_2(\theta)\frac{H(\theta)}{R_1\tan\theta}+$$
$$+\left(\mu-\frac{1}{3}\lambda_3\right)R_2\left[\frac{F(\theta)}{R_2\sin^2\theta}\right]'+\left(1+\frac{1}{3}\lambda_2\right)R_2\left[\frac{H(\theta)}{R_1}\right]'+\frac{h+t}{12E_c}C^*q',$$

where

$$B_1(\theta)=\left(1+\frac{1}{3}\lambda_1\right)R_1+\left(\mu-\frac{1}{3}\lambda_3\right)R_2+\left(\mu-\frac{1}{3}\lambda_3\right)R_2'\tan\theta-$$
$$-\frac{1}{3}\lambda_3'R_2\tan\theta,$$
$$B_2(\theta)=\left(\mu-\frac{1}{3}\lambda_3\right)R_1+\left(1+\frac{1}{3}\lambda_2\right)R_2+\left(1+\frac{1}{3}\lambda_2\right)R_2'\tan\theta+\frac{1}{3}\lambda_2'R_2\tan\theta. \tag{58}$$

Thus Eqs. (47) and (57) are the basic differential equations of axisymmetrical three-layer shells in the case of small displacements.

Assuming in Eqs. (47) and (57) that

$$G_c=E_c=\infty,\quad \lambda_1=\lambda_2=\lambda_3=0,$$
$$\gamma_1=\mu,\ \gamma_2=\gamma_3=1,\ D^*=\frac{Eh^3}{12},\ C^*=Eh \text{ and } m=0,$$

we obtain the general equations of E. Meissner's theory of thin elastic homogeneous axisymmetrical shells [3].

After solving Eqs. (47) and (57) for the given axisymmetrical three-layer shell and the appropriate boundary conditions, let us determine the specific normal stresses N_1 and N_2 from formulas (50) and (53), the specific bending moments M_1 and M_2 from formulas (44), and the transverse force Q from expression (46). The specific normal stresses in the outer and inner supporting layers are determined on the basis of expression (29). Since the thicknesses of the outer and inner layer are identical, the normal stresses in these layers will be equal to the specific normal stresses divided by the thickness t.

The transverse tangential stress in the filler is calculated from formula (11), while the transverse normal stress is determined from Eqs. (24) and (26).

Particular Cases

From the basic differential equations for axisymmetrical three-layer shells (47) and (57) we obtain the equations for certain types of shells: conical, cylindrical, and spherical.

Conical Shell

In the conical shell the angle θ is constant, while the radius of curvature $R_1 = \infty$. Let us introduce a new independent variable <u>s</u>, the distance from the apex of the cone along its generatrix (Fig. 8). We have $ds_1 = R_1 d\theta$.

Let $\frac{d(\ )}{ds} = (\dot{\ })$, then $\frac{d(\ )}{d\theta} = R_1 \frac{d(\ )}{ds}$ and $\frac{d^2(\ )}{d\theta^2} = R_1^2 \frac{d^2(\ )}{ds^2}$.

Equations (47) and (57) will assume the form:

$$R_1\left\{R_2\ddot{\beta} + \left[\left(\frac{R_2}{R_1}\right)^{\cdot} + \frac{R_0}{R_1}\left(\cot\theta + \gamma_3\dot{\lambda}_2 R_1 - \frac{\dot{\Omega}}{\Omega}R_1\right)\right]\dot{\beta}\right\} -$$
$$-\left[\gamma_2 \frac{R_1}{R_2}\cot^2\theta + \left(\frac{\dot{\Omega}}{\Omega}R_1\gamma + \gamma_3\dot{\lambda}_3 R_1\right)\cot\theta + \gamma_1\right]\times$$
$$\times\beta - \gamma_3\Omega\frac{R_1}{D^*}V = G_1(s); \tag{59}$$

$$\left(1+\frac{1}{3}\lambda_2\right)R_1R_2\ddot{V} + \left\{\left(1+\frac{1}{3}\lambda_2\right)\left[\frac{R_2}{R_1}\cot\theta + R_1\left(\frac{R_2}{R_1}\right)^{\cdot}\right] +\right.$$
$$\left.+\frac{1}{3}\dot{\lambda}_2 R_2\right\}R_1\dot{V} - \left[\left(1+\frac{1}{3}\lambda_1\right)\frac{R_1}{R_2}\cot^2\theta + \frac{C^*}{(h+t)G_c}\frac{R_1}{R_2} -\right. \tag{60}$$
$$\left.-\mu + \frac{1}{3}\lambda_3 - \frac{1}{3}\dot{\lambda}_3 R_1\cot\theta\right]V + R_1C^*\beta = \dot{G}_2(s).$$

Here

$$G_1(s) = -\left[R_1\left(\frac{R_2}{R_1}\right)^{\cdot} + \left(\frac{R_2}{R_1} - \gamma_2\frac{R_1}{R_2}\right)\cot\theta + R_2\left(\gamma_3\dot{\lambda}_2 - \frac{\dot{\Omega}}{\Omega}\right) -\right.$$
$$\left.-\left(\gamma_1\frac{\dot{\Omega}}{\Omega} + \gamma_3\dot{\lambda}_3\right)\right]\frac{\varepsilon}{E_c} - \left(\frac{R_2}{R_1} + \gamma_1\right)R_1\frac{\dot{\varepsilon}}{E_c} - \Omega\gamma_3\frac{R_1R_2}{D^*}m,$$
$$G_2(s) = B_1(s)\frac{F(s)}{R_2\sin^2\theta\tan\theta} + B_2(s)\frac{H(s)}{R_1\tan\theta} + \tag{61}$$
$$+\left(\mu - \frac{1}{3}\lambda_3\right)R_1R_2\left[\frac{F(s)}{R_2\sin^2\theta}\right]^{\cdot} + \left(1+\frac{1}{3}\lambda_2\right)R_1R_2\left[\frac{H(s)}{R_1}\right]^{\cdot} +$$
$$+\frac{h+t}{12E_c}C^*R_1\dot{q}.$$

where

$$B_1(s)=\left(1+\frac{1}{3}\lambda_1\right)R_1+\left(\mu-\frac{1}{3}\lambda_3\right)R_2+\left(\mu-\frac{1}{3}\lambda_3\right)R_1R_2^{\cdot}\tan\theta-$$
$$-\frac{1}{3}\lambda_3^{\cdot}R_1R_2\tan\theta,$$
$$B_2(s)=\left(\mu-\frac{1}{3}\lambda_3\right)R_1+\left(1+\frac{1}{3}\lambda_2\right)R_2+\left(1+\frac{1}{3}\lambda_2\right)R_1R_2^{\cdot}\tan\theta+ \tag{62}$$
$$+\frac{1}{3}\lambda_2^{\cdot}R_1R_2\tan\theta.$$

For a conical shell it is necessary to assume that

$$R_1=\infty,\ R_2=s\cot\theta,\quad \lambda_1=\lambda_3=0,\quad \lambda_2=\frac{\omega\tan^2\theta}{s^2},$$
$$\omega=\frac{1}{2}\frac{(h+t)tE_z}{E_c},\quad \Omega=1+\lambda_2-\mu^2,\quad \gamma_1=\frac{\mu}{1+\frac{\omega\tan^2\theta}{s^2}}, \tag{63}$$
$$\gamma_2=\gamma_3=\frac{1}{1+\frac{\omega\tan^2\theta}{s^2}}.$$

Dividing both sides of Eqs. (59) and (60) by R_1 and substituting expressions (63) in these equations, after certain transformations, we obtain the basic equations for conical three-layer shells.

$$[(1-\mu^2)s^4+(2-\mu^2)\omega\tan^2\theta s^2+\omega^2\tan^4\theta]s\ddot{\beta}+$$
$$+[(1-\mu^2)s^4+(2+\mu)\omega\tan^2\theta s^2+\omega^2\tan^4\theta]\dot{\beta}-[(1-\mu^2)s^2+$$
$$+(1-2\mu)\omega\tan^2\theta]s\beta-[(1-\mu^2)^2s^4+2(1-\mu^2)\omega\tan^2\theta s^2+ \tag{64}$$
$$+\omega^2\tan^4\theta]\frac{V}{D^*}\tan\theta=G_1^*(s);$$

$$\left(1+\frac{\omega\tan^2\theta}{3s^2}\right)sV^{\cdot\cdot}+\left(1-\frac{\omega\tan^2\theta}{s^2}\right)V^{\cdot}-\left[1+\frac{C^*}{(h+t)G_c}\tan^2\theta\right]\frac{V}{s}+$$
$$+C^*\beta\tan\theta=G_2^*(s). \tag{65}$$

Here

$$G_1^*(s)=[(1-\mu^2)s^2+\omega\tan^2\theta-2\mu\omega\tan^2\theta]\frac{\tan\theta}{E_c}sg-$$
$$-[(1-\mu^2)s^2+\omega\tan^2\theta]\frac{\tan\theta}{E_c}s^2g^{\cdot}-s[(1-\mu^2)^2s^4+$$
$$+2(1-\mu^2)\omega\tan^2\theta s^2+\omega^2\tan^4\theta]\frac{m}{D^*}, \tag{66}$$
$$G_1^*(s)=\frac{F(s)}{s\sin\theta\cos\theta}-\mu sp-\left[s^2\cot\theta+\frac{\omega\tan\theta}{3}-\right.$$
$$\left.-\frac{(h+t)C^*}{12E_c}\tan\theta\right]q^{\cdot}-2qs\cot\theta.$$

If it is assumed in Eqs. (64), (65), and (66) that $E_c = G_c = \infty$, $\omega = m = 0$, $D^* = \frac{Eh^3}{12}$, and $C^* = Eh$, we obtain the well-known equations for thin homogeneous conical shells [8].

Cylindrical Shell

Let us divide Eqs. (59) and (60) by R_1 and assume that in these equations

$$\theta=\frac{\pi}{2},\ ds=dx,\ R_1=\infty,\ R_2=a,\ \lambda_1=\lambda_3=0,$$

$$\lambda_2=\frac{(h+t)t}{2a^2}\frac{E_z}{E_c},\ \gamma_1=\frac{\mu}{1+\lambda_2},\ \gamma_2=\gamma_3=\frac{1}{1+\lambda_2},$$

where $\underline{x}$ is the distance from the left edge of the cylindrical shell. We have

$$a\beta_{xx}-\frac{1+\lambda_2-\mu^2}{(1+\lambda_2)D^*}V=-\frac{\mu g_x}{(1+\lambda_2)E_c}-\frac{1+\lambda_2-\mu^2}{(1+\lambda_2)D^*}ma; \tag{67}$$

$$\left(1+\frac{1}{3}\lambda_2\right)aV_{xx}-\frac{V}{(h+t)aG_c}C^*+C^*\beta=-a^2\left(q_x+\frac{\mu p}{a}\right) \tag{68}$$

If it is assumed in Eqs. (67) and (68) that $E_c = G_c = \infty$, $D^* = \frac{Eh^3}{12}$, $C^* = Eh$, and $\lambda_2 = m = 0$, we obtain the equations for thin homogeneous cylindrical shells [8].

Spherical Shell

Let us assume that in Eqs. (47) and (57)

$$R_1=R_2=a=\text{const},\ \lambda_1=\lambda_2=\lambda_3=\lambda=\frac{1}{2}\frac{(h+t)t}{a^2}\frac{E_z}{E_c},$$

$$\gamma_1=\frac{\mu-\lambda}{1+\lambda},\quad \gamma_2=1,\quad \gamma_3=\frac{1}{1+\lambda}.$$

Then we obtain the equations for spherical three-layer shells

$$\beta''+\cot\theta\beta'-(\gamma_1+\cot^2\theta)\beta-\frac{a[(1+\lambda)^2-(\mu-\lambda)^2]}{(1+\lambda)D^*}V=$$
$$=-\frac{1+\gamma_1}{E_c}g'-\frac{(1+\lambda)^2-(\mu-\lambda)^2}{1+\lambda}\frac{a^2}{D^*}m; \tag{69}$$

$$V'' + \cot\theta V' - \cot^2\theta V + \left[\frac{\mu - \frac{1}{3}\lambda}{1+\frac{1}{3}\lambda} - \frac{C^*}{\left(1+\frac{1}{3}\lambda\right)(h+t)G_c}\right]V +$$

$$+ \frac{aC^*}{1+\frac{1}{3}\lambda}\beta = \left[\frac{(h+t)C^*}{12\left(1+\frac{1}{3}\lambda\right)E_c} - a^2\right]q' - \left(1 + \frac{\mu - \frac{1}{3}\lambda}{1+\frac{1}{3}\lambda}\right)pa^2. \tag{70}$$

If it is assumed in Eqs. (69) and (70) that $E_c = G_c = \infty$,

$$\gamma_1 = \mu,\ \gamma_2 = \gamma_3 = 1,\ \lambda = m = 0,\ D^* = \frac{Eh^3}{12} \text{ and } C^* = Eh,$$

we obtain the well-known equations for thin homogeneous spherical shells [8]. Equations (67)-(70) coincide with the corresponding equations of E. Reissner [2].

The Equations for Axisymmetrical Three-layer Shells Without Allowance for Transverse Compressive Deformation of the Filler

In solving the problems of transverse bending and over-all loss of stability of three-layer plates and shells, the effect of transverse compressive deformation of the filler is generally neglected. From Eqs. (47) and (57) it can be seen that the effect of transverse compression of the filler may be neglected, if

$$\frac{1}{2}\frac{(h+t)t}{R^2}\frac{E_s}{E_c} \ll 1.$$

Abiding by this criterion, we shall neglect terms depending on the transverse compressive deformation of the filler.

Then, assuming that in the general equations for axisymmetrical three-layer shells (47) and (57) $E_c = \infty$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$,

$$\gamma_1 = \mu,\quad \gamma_2 = \gamma_3 = 1 \text{ and } \Omega = 1 - \mu^2,$$

we obtain

$$\frac{R_2}{R_1}\beta'' + \left[\frac{R_2}{R_1}\cot\theta + \left(\frac{R_2}{R_1}\right)'\right]\beta' - \left(\frac{R_1}{R_2}\cot^2\theta + \mu\right)\beta - \frac{R_1}{D}V = -\frac{R_1R_2}{D}m; \tag{71}$$

$$\frac{R_2}{R_1}V'' + \left[\frac{R_2}{R_1}\cot\theta + \left(\frac{R_2}{R_1}\right)'\right]V' - \left(\frac{R_1}{R_2}\cot^2\theta - \mu\right)V - \frac{C^*}{(h+t)G_c}\frac{R_1}{R_2}V + R_1C^*\beta = \tau(\theta). \tag{72}$$

Here $D = \frac{D^*}{1-\mu^2}$ is the cylindrical rigidity of a three-layer shell,

$$\tau(\theta) = \frac{F(\theta)}{\sin^2\theta}\left[\left(\frac{R_1}{R_2} - \frac{R_2}{R_1}\right)\cot\theta + \right. \\ \left. + \left(\frac{R_2}{R_1}\right)'\right] - pR_2(R_2 + \mu R_1) - (qR_2^2)'. \tag{73}$$

In operator form Eqs. (71) and (72) will be

$$L(\beta) - \mu\beta - \frac{R_1}{D}V = -\frac{R_1 R_2}{D}m; \tag{71*}$$

$$L(V) - \left[\frac{C^*}{(h+t)G_c}\frac{R_1}{R_2} - \mu\right]V + R_1 C^*\beta = \tau(\theta), \tag{72*}$$

where the operator

$$L(\ ) = \frac{R_2}{R_1}(\ )'' + \left[\frac{R_2}{R_1}\cot\theta + \left(\frac{R_2}{R_1}\right)'\right](\ )' - \frac{R_1}{R_2}\cot^2\theta(\ ). \tag{74}$$

Let us obtain the equations for the particular cases.

<u>Conical Shell</u>

Assuming that in Eqs. (64), (65), and (66) $E_c = \infty$ and $\omega = 0$, we obtain

$$s\beta'' + \beta' - \frac{\beta}{s} - \frac{V}{D}\tan\theta = -\frac{m}{D}s; \tag{75}$$

$$sV'' + V' - \left[1 + \frac{C^*}{(h+t)G_c}\tan^2\theta\right]\frac{V}{s} + C^*\beta\tan\theta = K(s), \tag{76}$$

where

$$K(s) = \frac{F(s)}{s\sin\theta\cos\theta} - \mu sp - s^2\cot\theta q' - 2qs\cot\theta. \tag{77}$$

In operator form Eqs. (75) and (76) will be written:

$$L_1(\beta)-\frac{V}{D}\tan\theta=-\frac{m}{D}s;$$

$$L_1(V)-\frac{C^*}{(h+t)G_c}\tan^2\theta\frac{V}{s}+C^*\tan\theta\beta=K(s),$$

where the operator

$$L_1(\ )=s(\ )''+(\ )'-\frac{(\ )}{s}. \tag{78}$$

The solution of Eqs. (64) and (65) for a conical three-layer shell with transverse compressive deformation of the filler taken into account gives rise to serious mathematical difficulties. On the other hand, Eqs. (75) and (76) for a conical three-layer shell without allowance for transverse compressive deformation of the filler have a simpler mathematical appearance.

Cylindrical Shell

If it is assumed in Eqs. (67) and (68) that $E_c = \infty$ and $\lambda_2 = 0$, we obtain the equations for cylindrical three-layer shells without allowance for transverse compression of the filler

$$a\beta_{xx}-\frac{V}{D}=-\frac{ma}{D}; \tag{79}$$

$$aV_{xx}-\frac{V}{(h+t)aG_c}C^*+C^*\beta=-\left(q_x+\frac{\mu p}{a}\right)a^2. \tag{80}$$

Spherical Shell

In the case of a spherical shell we assume that in the general equations (71) and (72) $R_1 = R_2 = a$ (where a is the radius of the sphere). We obtain

$$L_2(\beta)-\mu\beta-\frac{a}{D}V=-\frac{a^2}{D}m; \tag{81}$$

$$L_2(V)-\left[\frac{C^*}{(h+t)G_c}-\mu\right]V+aC^*\beta=-a^2q'-(1+\mu)pa^2. \tag{82}$$

Here

$$L_1(\ )=(\ )''+(\ )'\cot\theta-(\ )\cot^2\theta,$$
$$D=\frac{(1+\lambda)D^0}{(1+\lambda)^2-(\mu-\lambda)^2}. \qquad (83)$$

Boundary Conditions

In order to find the magnitudes of all the stresses and displacements occurring during the bending of a three-layer axisymmetrical shell, regardless of the assumptions $E_c = \sim$ or $E_c \neq \sim$, it is necessary to assign three boundary conditions on each edge, as in the case of a homogeneous axisymmetrical shell:

a) for a rigidly embedded edge

$$u=\beta=0,$$
$$\varepsilon_2=\frac{1}{C^0}(N_2-\mu N_1)=\frac{u\cot\theta+w}{R_2}=0 \qquad (84)$$

or

$$w=u=\beta=0; \qquad (85)$$

b) for a shifting pinched edge

$$w=\beta=N_1=0; \qquad (86)$$

c) for an edge supported on hinges

$$w=u=M_1=0 \qquad (87)$$

or

$$u=\varepsilon_2=M_1=0; \qquad (88)$$

d) for a freely supported edge

$$w=M_1=N_1=0; \qquad (89)$$

e) for a free edge

$$M_1=N_1=Q=0. \qquad (90)$$

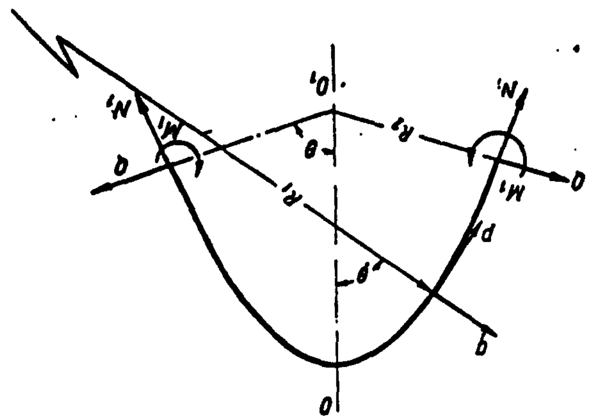

Fig. 7. For the determination of the load function.

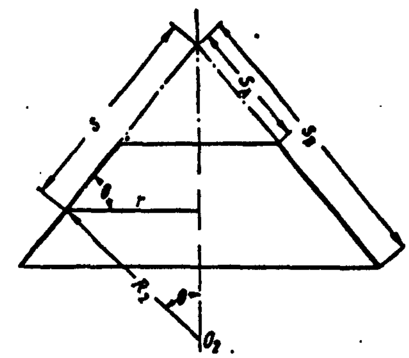

Fig. 8. Notation for a conical shell.

$ds_1 = R_1 d\,\theta;$

$ds_2 = r d\,\varphi;$

$R_2 = s \cot\theta.$

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