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RECURSION FORMULAE TO OBTAIN INTEGRAL ROOTS
OF REAL NUMBERS

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Abstract

Generalized recursion sequences to obtain the integral root of a number are developed. These sequences require, at most, the use of the square root operation in addition to the basic arithmetic operations; consequently, the sequences are suitable for use with desk calculators or manual computations.

For the \( n \)th root of a number, where \( n \) is an integer in the range

\[
2^{k-1} < n \leq 2^k, \quad (k \text{ integer}),
\]

the development in this paper provides for \( k + 1 \) recursion sequences, all of which exhibit second order convergence as the required root is approached.
INTRODUCTION

The problem of obtaining a recursion sequence to obtain any integral root of a number using a desk calculator or hand computation has been examined in the literature and a number of special forms of a more general recursion sequence have been derived. In this paper, the most general recursion sequence is obtained, and the criteria for the selection of the appropriate parameters in the recursion sequences are developed.

Although the procedure here is for the determination of a positive integral root, this method is the key to determination of a fractional root or a negative integral root. In the former case, where the root to be obtained is of the form, $p/q$, one merely obtains the $q^{th}$ root by the methods in this paper and raises the answer to the power, $p$. The determination of a negative integral root of a number, $N$, is the same as finding the positive integral root of the reciprocal of $N$.

Analysis:

In order to solve the problem, $L=A^{1/s}$, where $A$ and $s$ are given, recursion sequences of the form given in equation (1) are suggested in references 1 and 2:

$$A_{n+1} = \gamma A_n + \beta \left( A_n^{h} A \right)^{\frac{1}{s+h}}, \quad (1)$$

where $s = 2^k - h$, $1 \leq k \leq 2^{k-1}$, $h$ and $k$ are positive integers, and $\gamma$ and $\beta$ are constants to be chosen so as to make the convergence as rapid as possible. The form shown above can be generalized to:
where $s$, $h$, $Y$, and $\beta$ have the same meaning as in equation (1), and $n$ and $r$ are integers which are to be determined. If we define the fractional error $\varepsilon_n$, of the $n^{th}$ estimate to the desired root by the equation

$$A_n = L (1 + \varepsilon_n)$$

and noting that $L^s = A$, we may use (3) in (2) to obtain

$$1 + \varepsilon_{n+1} = Y (1 + \varepsilon_n) + \beta (1 + \varepsilon_n) \left( \frac{r}{s+h} \right) 2^m \left\{ \left( \frac{r+s}{s+h} \right) 2^m - 1 \right\}$$

If we require that the resulting error equation be independent of $L$ or $A$, then the exponent of the term involving $L$ must equal zero, i.e., we require that

$$\left( \frac{r+s}{s+h} \right) 2^m = 1$$

and under this condition, equation (4) becomes

$$1 + \varepsilon_{n+1} = Y (1 + \varepsilon_n) + \beta (1 + \varepsilon_n) \left( \frac{r}{s+h} \right) 2^m$$

Since $s+h = 2^k$, we obtain from (5):

$$r = 2^{k-m} - s$$

From (7), $r$ will be an integer as long as $m \leq k$.

We will require that $\varepsilon_{n+1} = 0$ when $\varepsilon_n = 0$,

which when used in equation (6) gives:

$$Y + \beta = 1$$
From (6), we also note that $\epsilon_{n+1} = -1$, when $\epsilon_n = -1$, i.e., an initial estimate greater than zero is always required. The slope of $\epsilon_{n+1}$ vs $\epsilon_n$ is given by

$$\frac{\partial \epsilon_{n+1}}{\partial \epsilon_n} = \gamma + \beta \left( \frac{r}{s+h} \right) 2^m \left( 1 + \epsilon_n \right)^{(2^{-n}-1)} \delta,$$

For convergence to the point, $\epsilon_{n+1} = \epsilon_n = 0$, it is sufficient to require that

$$\left| \frac{\partial \epsilon_{n+1}}{\partial \epsilon_n} \right| < 1,$$

(10)

Using this criterion applied at the origin, i.e., from (9) and (10), when $\epsilon_n = 0$, we obtain

$$\left| \gamma + \beta \left( \frac{r}{s+h} \right) 2^m \right| < 1,$$

(11)

or using (7) and (8) in (11):

$$\left| 1 - \beta \left( \frac{s}{s+h} \right) 2^m \right| < 1,$$

(12)

or

$$s(1-2^m) + h + s \gamma 2^m < s+h$$

The most rapidly converging trajectories in the vicinity of the origin will be those whose slope $= 0$ at $\epsilon_n = 0$, i.e., those whose trajectories satisfy (using equation (9)):

$$\gamma + \beta \left( \frac{r}{s+h} \right) 2^m = 0,$$

(14)
The simultaneous solution of (8) and (14) gives:

\[ \beta = \frac{1}{2^m} \left( 1 + \frac{4}{s} \right) \]

\[ \gamma = 1 - \frac{1}{2^m} \left( 1 + \frac{4}{s} \right) \]

Equations of the form given by equation (2) will always be convergent as long as \[ \left| \frac{\partial \varepsilon_n}{\partial \varepsilon_n} \right| \] for large \( \varepsilon_n \) is less or equal to 1.

From (9), this requires then \( \gamma \leq 1 \) or

\[ \frac{1}{2^m} \left( 1 + \frac{4}{s} \right) \geq 0 \]

which imposes no real restriction. Since for large \( \varepsilon_n \), \( \frac{\partial \varepsilon_{n+1}}{\partial \varepsilon_n} \approx \gamma, \varepsilon_n \)

\[ \frac{\partial \varepsilon_{n+1}}{\partial \varepsilon_n} \approx 1 - \frac{2}{s} \]

for \( \varepsilon_n \gg 1 \), (17)

Consequently, the larger the value \( m \), the slower the rate of convergence at the larger values of \( \varepsilon_n \). On the other hand, increasing the value of \( m \) reduces the effort required to obtain subsequent estimates. For \( m = k-1 \), only one square root in each cycle must be computed; for \( m = k \), each cycle requires no operations other than the four basic arithmetic operations. From equation (17), \( \frac{\partial \varepsilon_{n+1}}{\partial \varepsilon_n} > 0 \) for large \( \varepsilon_n \), for integer values of \( m \geq 1 \). For the case where \( m = 0 \), \( \frac{\partial \varepsilon_{n+1}}{\partial \varepsilon_n} < 0 \), for large \( \varepsilon_n \) and there exists the possibility of \( \varepsilon_{n+1} \) becoming negative for sufficiently large positive values of \( \varepsilon_n \).

The range of \( \varepsilon_n \) for which the \( m = 0 \) recursion sequence will converge can be obtained by setting \( m = 0 \) and \( \varepsilon_{n+1} = -1 \) in (6) and solving for \( \varepsilon_n \).
The range of $\xi_n$ for which the $n = 0$ sequence converges is

$$-1 < \xi_n < \xi_n^*,$$ (18)

where $\xi_n^*$ is given by

$$\xi_n^* = \left(1 + \frac{5}{h}\right)^{\frac{1}{1 + \frac{4e}{h}}} - 1,$$ (19)

Note that as long as $0 < \lambda < \delta$, then

$$\left(1 + \frac{5}{h}\right)^{\frac{1}{1 + \frac{4e}{h}}} - 1 > \left(1 + \frac{s}{h}\right) - 1 = \frac{s}{h}$$

i.e., $\xi_n^* > \frac{s}{h}$ which establishes a range of $\xi_n$ from $-1$ to $s/h$.

In this situation, however, the discussion of range of convergence is purely academic insofar as practical application is concerned; the divergence is readily recognized by the appearance of a negative value of $A_{n+1}$, and can be eliminated by the simple expedient of taking some fractional value of $A_n$ for the value of $A_{n+1}$. The appearance of a positive subsequent estimate generates a solution.

The use of recursion formulae of the form suggested above, (for $m=4$), yields particularly simple results for some of more common integral roots. In particular, the formula for the square root of $A$ is given by

$$A_{n+1} = \frac{1}{2} \left( A_n + \frac{A}{A_n} \right),$$ (20)

which is both simple and rapidly convergent.

Since useful recursion sequences may be obtained for each integer value of $m$ from $0$ to $k$, it follows that one has a choice of $k + 1$ sequences of the form in (2) for any given problem.
References
