POWER AND SIGNAL-TO-NOISE CALCULATIONS
OF A CERTAIN PSEUDO-RANDOM SIGNAL

25 January 1963
ASTIA Availability Notice

Qualified requesters may obtain copies of this report from Armed Services Technical Information Agency, Arlington Hall Station, Arlington 12, Virginia, Attn: TIPCR.

Destruction Notice

Destroy; do not return
POWER AND SIGNAL-TO-NOISE CALCULATIONS
OF A CERTAIN PSEUDO-RANDOM SIGNAL

By

Eugene J. McManus, Jr.

DA Project No. 1-A-0-13001-A-039
AMC Management Structure Code No. 5010-11-844

Radiation Branch
Electromagnetic Laboratory
Directorate of Research and Development
U. S. Army Missile Command
Redstone Arsenal, Alabama
ABSTRACT

This report presents calculations of power and spectral distribution for a particular pseudo-random waveform. Signal-to-noise ratios are derived, and it is shown that the amplifier bandwidth, which maximizes the output signal-to-noise ratio is given by $\frac{\xi}{\tau}$, when $\tau$ is the bit length in the pseudo-random sequence.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. DISCUSSION</td>
<td>1</td>
</tr>
<tr>
<td>III. SUMMARY AND CONCLUSIONS</td>
<td>7</td>
</tr>
</tbody>
</table>

# LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>AUTO-CORRELATION FUNCTION</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>GRAPHICAL FOURIER TRANSFORMATION</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>RESOLUTION OF AUTO-CORRELATION FUNCTION INTO TWO COMPONENTS</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>EXAMPLES OF CONTENT, VARIATION AND WIGGLINESS</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>BOUNDS ON THE SPECTRUM OF THE WAVEFORM</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>LOG-LOG PLOT OF BOUNDS ON THE SPECTRUM</td>
<td>13</td>
</tr>
<tr>
<td>7</td>
<td>RECEIVED POWER AS A FUNCTION OF AMPLIFIER BANDWIDTH</td>
<td>14</td>
</tr>
<tr>
<td>8</td>
<td>SIGNAL-TO-NOISE RATIO AND NORMALIZED RECEIVED POWER AS A FUNCTION OF</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>BANDWIDTH OF THE AMPLIFIER</td>
<td></td>
</tr>
</tbody>
</table>
POWER AND SIGNAL-TO-NOISE CALCULATIONS
OF A CERTAIN PSEUDO-RANDOM SIGNAL

I. INTRODUCTION

Recently in work with radar waveforms it has been found necessary to compute the amount of pseudo-random signal power passing through an amplifier of a certain specified bandwidth. 1 A quick solution to this problem would be to determine the percentage of the power spectrum which falls within this specified bandwidth. This report describes a quick, simple method of computing the power spectrum.

There are several methods of calculating the spectrum of a signal. 2 For nonperiodic pulse signals two distinct routes exist: One, through the signal spectrum to its squared magnitude; the second, through the auto-correlation function to its transform. Both routes yield the energy density spectrum, \( \Psi (w) \). Similarly for periodic signals, the same transformations exist, but to \( \Phi (w) \), the power density spectrum. However, for random signals, 3 the Fourier Transform diverges so that only one path exists from the signal to its power density spectrum - by way of the auto-correlation function. Fortunately, for the signal under consideration the auto-correlation is known and is shown in Figure 1.

II. DISCUSSION

The purpose of this report is to present a simplified method of calculating Fourier transforms of waveforms. 4 Essentially, this method is based on the fact that differentiation in the time domain corresponds to multiplication by \( jw \) in the frequency domain. The procedure is thus: given \( u(t) \), we can write \( \Psi_k (w) \), differentiate \( u(t) \), which multiplies \( \Psi_k (w) \) by \( jw \). Differentiate \( u(t) \) again, this

---

1 Todd, W. H. Private Communication
2 See for example; Lee, Y. W., Statistical Theory of Communication, John Wiley and Sons, New York, 1960
4 The material in this and following paragraphs borrows heavily from Footnote 3, above.
yields impulses, and also gives \((j w)^2 \Psi_k(w)\). Translation of an impulse a distance \(\delta\) corresponds to multiplying the impulse by \(\exp(\delta j w)\) (see Figure 2). Thus we can write, by inspection, the relation:

\[
(j w)^2 \Psi_k(w) = -2 \left( \frac{K + B}{\delta} \right) + \left( \frac{K + B}{\delta} \right) \left( e^{j w \delta} - e^{-j w \delta} \right)
\]

This equation can be rearranged slightly to give:

\[
(j w)^2 \Psi_k(w) = -2 \left( \frac{K + B}{\delta} \right) + 2 \left( \frac{K + B}{\delta} \right) \left( e^{j w \delta} + e^{-j w \delta} \right)
\]

Solving for \(\Psi_k(w)\), after some simplifying and substituting, yields:

\[
\Psi_k(w) = 2 \left( \frac{K + B}{\delta \omega^2} \right) \left( 1 - \cos \delta \omega \right)
\]

Equation 3 can further be written as:

\[
\Psi_k(w) = 4 \left( \frac{K + B}{\delta \omega^2} \right) \sin^2 \left( \frac{\delta \omega}{2} \right)
\]

Multiplying numerator and denominator of the right hand side by \(\delta\), and manipulating, the following result is obtained:

\[
\Psi_k(w) = \delta (K + B) \frac{\sin^2 \left( \frac{\delta \omega}{2} \right)}{\left( \frac{\delta \omega}{2} \right)^2}
\]

which in alternate form is:

\[
\Psi_k(w) = \delta (K + B) \left( \frac{\sin \delta \omega}{2} \right)^2 \left( \frac{\delta \omega}{2} \right)^2
\]

Equation 6 is the expression for the power spectrum of the original signal.

It is both interesting and instructive to examine the bounds on the power spectrum. These bounds are obtained from the form constants of the signal; namely, the content, variation, and wiggliness. These terms are defined as follows:
\[
\text{Content} = \int_{-\infty}^{+\infty} \left| u(t) \right| \, dt \equiv < \left| u(t) \right> \quad (7)
\]

\[
\text{Variation} = \int_{-\infty}^{+\infty} \left| \frac{d u(t)}{d t} \right| \, dt \equiv < \left| \frac{d u(t)}{d t} \right> \quad (8)
\]

\[
\text{Wiggliness} = \int_{-\infty}^{+\infty} \left| \frac{d^2 u(t)}{d t^2} \right| \, dt \equiv < \left| \frac{d^2 u(t)}{d t^2} \right> \quad (9)
\]

The content is the area under the curve (see Figure 3). Since the area under the auto-correlation curve in question is infinite, some difficulty may exist. However, this difficulty can be solved by breaking the curve into two parts: A triangular pulse and a uniform band. From the following discussion, it can be seen that Part b will effectively contribute a delta spike at the origin. No other contribution will be made by Part b.

For Part a, the content is obviously K\(\delta\). The variation, which is a change in the slope, is the total upward or downward excursion of the signal. Both the upward and downward excursions are counted as positive (see Figure 4). Thus, the variation is 2K. The wiggliness is the total variation in the slope of the signal in which all variations count as positive. This is \(\frac{4K}{\delta}\). Two frequencies, \(\omega_a\) and \(\omega_b\), are defined by the following:

\[
\omega_a = \frac{\text{variation}}{\text{content}} \quad \omega_b = \frac{\text{wiggliness}}{\text{variation}}
\]

For this wave:

\[
\omega_a = \frac{2K}{K\delta} = \frac{2}{\delta} \quad (10)
\]

\[
\omega_b = \frac{4K}{2K} = \frac{2}{\delta} \quad (11)
\]

We are now in a position to sketch the bounds on \(\psi_k(w)\). These bounds are the content, variation divided by \(\omega\), and wiggliness divided by \(\omega^2\). The calculation of \(\omega_a\) and \(\omega_b\) shows that both of these frequencies are the same. The bounds on \(|\psi_k(w)|\) are shown in Figure 5, and a log-log plot of the same figure is presented in Figure 6. In Figure 6, the actual curve will be 3 db below the corner frequency \(\omega_a\) since this is an asymptote plot.
We can now determine the amount of signal power passed by an amplifier as a function of the amplifier bandwidth. To do this, it is most convenient to have the band limited power normalized. The power passed by the amplifier is divided by the total signal power. To determine the total signal power, Figure 6 is integrated from 0 to \( \omega \), and multiplied by 2. The equation for Figure 6 is given by:

\[
\Psi(\omega) = \begin{cases} 
C & 0 \leq \omega \leq \omega_a \\
\frac{KV}{\omega} & \omega_a \leq \omega \leq \omega_b \\
\frac{MW}{\omega^2} & \omega \geq \omega_b 
\end{cases} \tag{12}
\]

(Where \( C \) stands for content, etc.) At the two frequencies, \( \omega_a \) and \( \omega_b \), the curves must agree. Thus at \( \omega_a \):

\[
C = \frac{KV}{\omega_a} \quad \text{or} \quad K = \frac{C\omega_a}{V}
\]

at \( \omega_b \):

\[
\frac{KV}{\omega_b} = \frac{MW}{\omega_b^2} \quad \text{or} \quad M = \frac{KV\omega_b}{W}
\]

Thus, the equation describing \( \Psi(\omega) \) can be written as follows (performing the above substitutions):

\[
\Psi(\omega) = \begin{cases} 
C & 0 \leq \omega \leq \omega_a \\
\frac{C\omega_a}{\omega} & \omega_a \leq \omega \leq \omega_b \\
\frac{C\omega_a\omega_b}{\omega^2} & \omega \geq \omega_b 
\end{cases} \tag{12a}
\]
Integrating $\Psi(w)$ over all values of $w$ from 0 to infinity,

$$\int_0^\infty \Psi(w) \, dw = \int_0^w a w \, dw + \int_w^b a w \, dw + \int_b^\infty a w b \, dw$$

(13)

or

$$\int_0^\infty \Psi(w) \, dw = C w_a + C w_a \ln \left( \frac{w_b}{w_a} \right) + C w_a - C w_a w_b$$

(14)

$w_r = 0$ if $w_q \neq w_a$

$w_s = 0$ if $w_b \neq w_b$

One-half the total power is given when $w_q = w_a$, $w_r = w_b$, and $w_s = \infty$.

Then:

$$\int_0^\infty \Psi(w) \, dw = 2C w_a + C w_a \ln \left( \frac{w_b}{w_a} \right)$$

(15)

Since only one side of the curve has been considered, it is necessary only to double Equation 15 to get the total power. Thus:

$$\int_0^\infty \Psi(w) \, dw = 2C w_a \left[ 2 + 1 \ln \left( \frac{w_b}{w_a} \right) \right]$$

(16)

is the expression for the total power. For the particular waveform under consideration, $w_a = w_b$ so that the expression for total power reduces to:

$$\int_0^\infty \Psi(w) \, dw = 4C w_a$$

(17)

To find the power passed by an amplifier of specified bandwidth, Equation 14 is used. Equation 14, rewritten, is (for the waveform under consideration):

$$\int_0^w \Psi(w) \, dw = \begin{cases} 2C w_q \quad w_q < w_a \\ 2C w_a \left( 2 - \frac{w_a}{w_s} \right) \quad w_s > w_a \end{cases}$$

(18)

A plot of this curve is shown in Figure 7.
The percentage of power received, as a function of bandwidth, can now be written. Letting \( \epsilon(\omega) \) be the percentage of power received, the expression for \( \epsilon(\omega) \) is:

\[
\epsilon(\omega) = \frac{\int_{-\infty}^{\omega} \Psi(\omega) \, d\omega}{\int_{-\infty}^{\infty} \Psi(\omega) \, d\omega} \times 100
\]

(19)

For the waveform under consideration,

\[
\frac{2C \omega_q}{4C \omega_a} \times 100 \quad \omega_q < \omega_a
\]

\[\epsilon(\omega) = \frac{2C \omega_a \left(2 - \frac{\omega_a}{\omega_s}\right) \times 100}{4C \omega_a}
\]

(20)

which becomes:

\[
\epsilon(\omega) = \begin{cases} 
50 \frac{\omega_q}{\omega_a} & \omega_q < \omega_a \\
50 \left(2 - \frac{\omega_a}{\omega_s}\right) & \omega_s > \omega_a
\end{cases}
\]

(21)

Finally, the expression for the signal-to-noise ratio as a function of bandwidth can be calculated. This is simply the signal power divided by the noise power, as a function of bandwidth. Thus,

\[
\left(\frac{S}{N}\right) = \frac{\int_{-\infty}^{\omega} \Psi(\omega) \, d\omega}{e_n^2(\omega)}
\]

(22)

Now, \( e_n^2 \) as a function of \( \omega \) is given by:

\[
e_n^2(\omega) = \int_{-\infty}^{\infty} 2K T \, d\omega
\]

(23)
The signal-to-noise ratio, as a function of bandwidth, is then:

\[
\frac{S}{N} = \begin{cases} 
\frac{2C}{4KT} & w < w_a \\
\frac{2C (2 - \frac{w_a}{w_s})}{4KT} & w_s > w_a
\end{cases}
\]  

(24)

By considering the spike contributed by Part b of Figure 3, the expression for the signal-to-noise ratio is:

\[
\frac{S}{N} = \begin{cases} 
\frac{2C}{4KT} + \frac{BL}{4KT w_r} & w_r < w_a \\
\frac{2C w_a (2 - \frac{w_a}{w_s})}{4KT w_s} + \frac{BL}{4KT w_s} & w_s > w_a
\end{cases}
\]  

(25)

where B is defined in Figure 3 and L is the total word length. Figure 8 shows a composite plot of signal-to-noise ratio and ε versus w.

III. SUMMARY AND CONCLUSIONS

The power spectrum of the given pseudo-random wave was derived and is given by Equation 6. By treating the pseudo-random wave as a random wave, expressions for the power received as a function of bandwidth are given by Equation 21. The expression for the signal-to-noise ratio is given by Equation 25. From Equation 25 it is seen that the best signal-to-noise ratio occurs when the amplifier bandwidth is zero; however, by Equations 20 and 24 it is seen that the signal-to-noise ratio is not materially degraded when the amplifier bandwidth is w_a or \(\frac{2}{T}\).
Figure 1. AUTO-CORRELATION FUNCTION
Figure 2. GRAPHICAL FOURIER TRANSFORMATION
Figure 3. RESOLUTION OF AUTO-CORRELATION FUNCTION INTO TWO COMPONENTS

\[ \phi(\tau) = \phi_1(\tau) - \xi(\tau) \]
Figure 4. EXAMPLES OF CONTENT, VARIATION, AND WIGGLINESS
Figure 5. **BOUNDS ON THE SPECTRUM OF THE WAVEFORM**
Figure 6. LOG-LOG PLOT OF BOUNDS ON THE SPECTRUM
Figure 7. RECEIVED POWER AS A FUNCTION OF AMPLIFIER BANDWIDTH
Figure 8. SIGNAL-TO-NOISE RATIO AND NORMALIZED RECEIVED POWER AS A FUNCTION OF BANDWIDTH OF THE AMPLIFIER
25 January 1963

APPROVED:

Waite H. Todd
WAITE H. TODD
Chief, Radiation Branch

Delman E. Rowe
DELMAN E. ROWE
Director, Electromagnetic Laboratory
### DISTRIBUTION

**Copy No.**

U. S. Army Missile Command Distribution List A for Technical Reports (2 January 1963) 1-88

Commanding General
U. S. Army Missile Command
Redstone Arsenal, Alabama
Attn: AMSMI -R, Mr. McDaniel

<table>
<thead>
<tr>
<th>89</th>
<th>-RE</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>-REC</td>
</tr>
<tr>
<td>91</td>
<td>-REE</td>
</tr>
<tr>
<td>92</td>
<td>-REN</td>
</tr>
<tr>
<td>93</td>
<td>-REN, Mr. W. E. Wood</td>
</tr>
<tr>
<td>94</td>
<td>Mr. E. E. Mills</td>
</tr>
<tr>
<td>95</td>
<td>Mr. H. Buie</td>
</tr>
<tr>
<td>96</td>
<td>Mr. E. J. McManus</td>
</tr>
<tr>
<td>97-106</td>
<td>-REO</td>
</tr>
<tr>
<td>107</td>
<td>.-REP</td>
</tr>
<tr>
<td>108</td>
<td>-RER</td>
</tr>
<tr>
<td>109</td>
<td>-RES</td>
</tr>
<tr>
<td>110</td>
<td>-RET</td>
</tr>
<tr>
<td>111</td>
<td>-REX</td>
</tr>
<tr>
<td>112</td>
<td>-RB</td>
</tr>
<tr>
<td>113-116</td>
<td>-RAP</td>
</tr>
<tr>
<td>117</td>
<td>.</td>
</tr>
</tbody>
</table>


This report presents calculations of power and spectral distribution for a particular pseudo-random waveform. Signal-to-noise ratios are derived, and it is shown that the amplifier bandwidth which maximizes the output signal-to-noise ratio is given by $\frac{2}{r}$ when $r$ is the bit length in the pseudo-random sequence.
<table>
<thead>
<tr>
<th>AD Accession No.</th>
<th>UNCLASSIFIED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Army Missile Command, Directorate of Research and Development, Electromagnetic Laboratory, Redstone Arsenal, Alabama</td>
<td></td>
</tr>
<tr>
<td>POWER AND SIGNAL-TO-NOISE CALCULATIONS OF A CERTAIN PSEUDO-RANDOM SIGNAL - Eugene J. McManus, Jr.</td>
<td></td>
</tr>
<tr>
<td>This report presents calculations of power and spectral distribution for a particular pseudo-random waveform. Signal-to-noise ratios are derived, and it is shown that the amplifier bandwidth which maximizes the output signal-to-noise ratio is given by $\frac{2}{T}$, when $T$ is the bit length in the pseudo-random sequence.</td>
<td></td>
</tr>
</tbody>
</table>

**DISTRIBUTION:** Copies obtainable from ASTIA, Arlington Hall Station, Arlington 12, Virginia

<table>
<thead>
<tr>
<th>AD Accession No.</th>
<th>UNCLASSIFIED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Army Missile Command, Directorate of Research and Development, Electromagnetic Laboratory, Redstone Arsenal, Alabama</td>
<td></td>
</tr>
<tr>
<td>POWER AND SIGNAL-TO-NOISE CALCULATIONS OF A CERTAIN PSEUDO-RANDOM SIGNAL - Eugene J. McManus, Jr.</td>
<td></td>
</tr>
<tr>
<td>This report presents calculations of power and spectral distribution for a particular pseudo-random waveform. Signal-to-noise ratios are derived, and it is shown that the amplifier bandwidth which maximizes the output signal-to-noise ratio is given by $\frac{2}{T}$, when $T$ is the bit length in the pseudo-random sequence.</td>
<td></td>
</tr>
</tbody>
</table>

**DISTRIBUTION:** Copies obtainable from ASTIA, Arlington Hall Station, Arlington 12, Virginia

<table>
<thead>
<tr>
<th>AD Accession No.</th>
<th>UNCLASSIFIED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Army Missile Command, Directorate of Research and Development, Electromagnetic Laboratory, Redstone Arsenal, Alabama</td>
<td></td>
</tr>
<tr>
<td>POWER AND SIGNAL-TO-NOISE CALCULATIONS OF A CERTAIN PSEUDO-RANDOM SIGNAL - Eugene J. McManus, Jr.</td>
<td></td>
</tr>
<tr>
<td>This report presents calculations of power and spectral distribution for a particular pseudo-random waveform. Signal-to-noise ratios are derived, and it is shown that the amplifier bandwidth which maximizes the output signal-to-noise ratio is given by $\frac{2}{T}$, when $T$ is the bit length in the pseudo-random sequence.</td>
<td></td>
</tr>
</tbody>
</table>

**DISTRIBUTION:** Copies obtainable from ASTIA, Arlington Hall Station, Arlington 12, Virginia