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FINAL TECHNICAL REPORT.

The Royal Institute of Technology
Division: Geodesy
Stockholm 70.

ON AN EXPLICIT SOLUTION OF THE GRAVIMETRIC BOUNDARY VALUE PROBLEM FOR AN ELLIPSOIDAL SURFACE OF REFERENCE

Contract No. DA-91-591-EUC-2033

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I N D E X.

ON THE DETERMINATION OF THE SHAPE OF THE GEOID AND THE SHAPE OF THE EARTH FROM AN ELLIPSOIDAL SURFACE OF REFERENCE.

ABSTRACT:

By introducing gravity data into geodesy it has been possible to solve some of the geodetic problems not only in a purely geometrical way but also with full consideration of the dynamic parts of the problem. The first approach for the determination of the shape of the earth by the aid of gravimetric data is based on the famous formula of Stokes.

According to this formula it is possible to determine the shape of an equipotential surface if the gravity is defined in all parts of the surface. The method has been extensively used up to now for a determination of the so-called geoid of the earth, in spite of the fact that the formula is only correct for a spherical surface and not for a spheroid such as our own earth. Another objection to using this formula is the fact that one has to know the gravity values on the so-called geoid and not on the surface of the earth. However, this is in practice an important limitation because measurements can normally not be made at the surface of the geoid in other places than on the Oceans.

Evidently we have in total two fundamental objections against using the method of Stokes for our geodetic problems. During the last 15 years there has been extensive work in this field in order to improve the present technique. The most important contributions have been made by Russian scientists as for example Molodensky and Sagrebin. Molodensky showed in his contribution that it was possible to solve the integral equation which defines the shape of the earth directly from gravity measurements at the physical surface of the earth. Sagrebin made a similar solution of the integral equation when there is a change of the reference surface from a sphere to an ellipsoid.

Although the two methods have not hitherto been used to any wider extent for practical applications, it is still evident that they have given a new view of some of the fundamental problems in geodesy. For a special study of the ellipsoidal surface of reference, the method of Sagrebin has been chosen as the theoretical background. However, a study of the deduction of the sum of certain Legendre polynomials used by Sagrebin shows that most of Sagrebins final formulas are incorrect. Therefore it was considered important to resolve the problem by

the aid of new derivations. This study is based on the same formulas as Sagrebin used for the conversion of the Lamé functions to Legendre polynomials.

For a final computation of the resolvent equation, all functions have been recomputed in order to get a correct resolvent. The new expressions are somewhat more complicated than those of Sagrebin and an electronic computer was required for the final study.

According to the method used by Sagrebin, it is necessary to make not less than nine complete integrations over the earth in order to obtain the final value of the geoidal height of one point. This means that the method is too tedious for practical use. Another way to approach the problem is to make use of an iterative method such as Molodensky has suggested. There is no objection to such a method, except that in most cases it is desired to obtain the final answer after just one integration. It is evident from a purely abstract point of view that repeated integrations will increase the accuracy of the computations.

Preliminary studies made it clear, however, that little increase of the final accuracy is obtained by using such a high number of integrations. In the method used here for computing the resolvent, all integration steps are taken into consideration, but all this information is compiled in such a way that two integrations are sufficient.

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Stockholm, 28th November, 1962.

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THE FIGURE OF A SURFACE DETERMINED BY THE AID OF GRAVITY DATA.

There are a few fundamental relations which are of utmost importance for any geodetic use of gravity data. The Newtonian potential of a body is determined by the function

$$W = f\iiint \frac{\rho}{r}\, dV$$

where

W = the potential

ρ = the density of mass

V = the volume

r = the distance between the volume-element and the actual point.

The potentional W is said to be harmonic if it satisfies the Laplace equation:

$$\Delta W = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0$$

The potential, due to the gravitation effect of mass, is harmonic in all points not occupied by this mass. In a continuous massdistribution the potential satisfies the so-called equation of Poisson:

$$\Delta W = -4\pi\rho$$

For a rotating body the potential is no longer harmonic as the total potential is defined by the function

$$W = f\iiint \frac{\rho}{r}\, dV + 0.5\,\omega^2 p^2$$

where p is the distance of the point to the axis of rotation.

The gradient is now given by

$$\operatorname{grad} W = -f\iiint \frac{\rho}{r^2} \begin{bmatrix} \cos(r,x) \\ \cos(r,y) \\ \cos(r,z) \end{bmatrix} dV + \omega^2 p \begin{bmatrix} \cos(p,x) \\ \cos(p,y) \\ \cos(p,z) \end{bmatrix}$$

For a simple solution of geodetic problems it is necessary to work with an harmonic potential-function. Therefore the geodetic potential is normally replaced by an harmonic parameter which is defined as the difference between the geodetic potential and the potential of a theoretical earth. If these two potentials are identical with respect to the centrifugal part, then our new parameter, the so-called disturbance potential, is harmonic, and the well-known Green theorem is valid

$$W_P - U_P = T_P = \frac{1}{2\pi}\iint \left[(W-U) \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r}\left(\frac{\partial W}{\partial n} - \frac{\partial U}{\partial n} \right) \right] dS$$

where

W_P = the true potential at the actual point
U_P = the potential of the theoretical earth at the actual point
n = the normal of the surface
r = the distance between the actual point and the running point
S = the surface.

In order to obtain an harmonic disturbance potential it is required that the theoretical earth (the reference-surface) and the actual earth coincide with respect to their axes of rotation.

This is the first important condition for a solution of conventional type. In order to facilitate the solution, another important condition will be added later. For a full understanding of this approach we note that in case we have a given surface, it is possible to compute the potentials directly by aid of Greens theorem. This problem is the so-called Neumann problem.

If the surface is not known, then there is no simple straight-forward solution. The solution according to Stokes has, however, been a useful approximation. (See equation 28 "The Earth Form from Potentials and Gravity".) It has however to be remembered that any solution of this type is based on the approximation of performing the integration over a sphere instead of over the true surface of the earth. Of much greater importance is another limitation which is caused by using Stokes' formula. From our formula 26 we find that the Legendre polynomial of order one is omitted from our solution. This means that the gravity centers of our theoretical earth and the actual earth have to coincide. This is the second important condition for a solution of conventional type.

We have to note that the solution from the Stokes formula is only valid for an equipotential surface. It can be proved that for a number of applications the solution gives a useful approximation but when it becomes necessary to make a detailed study of the shape of the earth, then Stokes' formula is insufficient.

The following study is made in order to obtain a resolvent which can replace Stokes' formula for an ellipsoidal reference surface.

STATEMENT OF THE PROBLEM.

The following quantities are assumed to be given at the outset:

a) Some reference surface: In the general case this may be an ellipsoid with three unequal axes, but because of the very small eccentricity of the equator this will later be taken as an ellipsoid of revolution.

b) Gravity anomalies $\Delta g = g - \gamma$ where g is the acceleration of gravity on the geoid and γ is the acceleration of the theoretical gravity on the reference surface.

The quantities sought are the height-differences N between the geoid and the reference surface, measured along the normal of the reference surface. The height-differences N are related to the disturbance potential T according to the following formula:

$$N = \frac{T}{\gamma} \tag{1.1}$$

The geoid can be determined as soon as an equation is obtained to solve the T. If the reference surface is a sphere, the required relation is given by the so-called "fundamental equation of gravimetry":

$$\frac{2T}{R} + \frac{\partial T}{\partial n} = -\Delta g \tag{1.2}$$

which is satisfied by Stokes' solution:

$$T = \frac{1}{4\pi R} \int \Delta g \; S(\psi) \; dS \tag{1.3}$$

Where n is the outer normal of the reference surface and dS is the surface element of a sphere with radius R. The goal is now to obtain a similar boundary condition, valid for an ellipsoidal reference surface.

THE FUNDAMENTAL EQUATION.

In dealing with problems concerning ellipsoids, it is advantageous to use the so-called "ellipsoidal coordinates". The orthogonal system of surfaces is a set of confocal quadrics, represented by the equation

$$\frac{x^2}{\lambda^2 - h^2} + \frac{y^2}{\lambda^2 - k^2} + \frac{z^2}{\lambda^2} = 1 \tag{2.1}$$

Considering (2.1) as an equation in λ one can distinguish three sets of roots, depending on the sign of the denominators in equation (2.1). Calling these sets ρ, μ, ν we have the, relation

$$\rho^2 > h^2 > \mu^2 > k^2 > \nu^2 \tag{2.2}$$

From (2.1) and (2.2) it follows that the ρ defines a set of ellipsoids. It must be noticed that the three axes of these ellipsoids, in order of increasing magnitude, are located respectively along the x, y and z-axes. μ and ν define two sets of hyperboloids, orthogonal to the ellipsoids and to each other.

Equation (2.1) must be transformed in order to make the quantity h equal to unity.

$$\frac{\left(\frac{x}{h}\right)^2}{\left(\frac{\lambda}{h}\right)^2 - 1} + \frac{\left(\frac{y}{h}\right)^2}{\left(\frac{\lambda}{h}\right)^2 - \left(\frac{k}{h}\right)^2} + \frac{\left(\frac{z}{h}\right)^2}{\left(\frac{\lambda}{h}\right)^2} = 1 \tag{2.3}$$

This can be rewritten as follows:

$$\frac{x^2}{\lambda^2 - 1} + \frac{y^2}{\lambda^2 - \eta^2} + \frac{z^2}{\lambda^2} = 1 \tag{2.4}$$

Therefore relation (2.2) becomes

$$\rho^2 > 1 > \mu^2 > \eta^2 > \nu^2 \tag{2.5}$$

Every point in space is determined by giving the ellipsoidal coordinates ρ, μ, ν.

If a_1, a_2 and b are the half-axes of the reference ellipsoid in question, in order of decreasing magnitude, this ellipsoid can be characterized by the relations

$$\rho = \rho_0 = a_1 \tag{2.6}$$

and

$$\eta^2 = \frac{a_1^2 - a_2^2}{a_1^2 - b^2} \tag{2.6}$$

Let U be the gravity potential of the reference ellipsoid which at the surface $\rho = \rho_o$ has the value U_o

$$(U)_{\rho=\rho_o} = U_o \tag{2.7}$$

The acceleration of gravity on this surface is determined by the equation

$$\gamma = -\frac{\partial U}{\partial n} = \left(-\frac{\partial U}{\partial \rho}\frac{d\rho}{dn}\right)_{\rho=\rho_o} \tag{2.8}$$

The gravity potential of the geoid is denoted by W. W is considered to be composed of the potential of the reference ellipsoid U, and an additional part, the disturbance potential T. On the surface of the geoid

$$W = U + T = W_o \tag{2.9}$$

The acceleration of gravity, g, is obtained by the formula:

$$g = -\frac{\partial W}{\partial n} = -\frac{\partial U}{\partial n} - \frac{\partial T}{\partial n} = -\frac{\partial U}{\partial \rho}\frac{d\rho}{dn} - \frac{\partial T}{\partial n} \tag{2.10}$$

If $\Delta\rho$ is the increment in the ellipsoidal coordinate ρ, corresponding to an increment N along the normal of the reference surface, the following relation is valid

$$\Delta\rho = N\frac{d\rho}{dn} \tag{2.11}$$

In (2.10) we replace U by the first two terms of Taylor's expansion:

$$U = (U)_{\rho=\rho_o} + \Delta\rho\left(\frac{\partial U}{\partial \rho}\right)_{\rho=\rho_o} \tag{2.12}$$

or

$$U = (U)_{\rho=\rho_o} + N\left(\frac{\partial U}{\partial \rho}\frac{\partial \rho}{dn}\right)_{\rho=\rho_o} \tag{2.13}$$

The next term can be shown to be of the order of α^2 (α is the flattening of the earth). Terms of this order will be neglected in all subsequent formulas. ($\alpha^2 \approx 10^{-5}$).

(2.13) is reduced with the aid of (2.7) and (2.8) to

$$U = U_o - N\gamma \tag{2.14}$$

Substituting (2.14) into (2.9) gives

$$U_o - N\gamma + T = W_o \tag{2.15}$$

or

$$T = W_o - U_o + N\gamma \tag{2.16}$$

If we assume $W_o = U_o$ we have found equation (1.1). This relation is sometimes called "the lemma of Bruns". We now proceed to derive the fundamental equation for T. By Taylor's expansion of (2.10) we obtain

$$g = -\left\{\left(\frac{\partial U}{\partial \rho}\right)_{\rho=\rho_o} + \Delta\rho\left(\frac{\partial^2 U}{\partial \rho^2}\right)_{\rho=\rho_o}\right\}\left\{\left(\frac{d\rho}{dn}\right)_{\rho=\rho_o} + \Delta\rho\left(\frac{d}{d\rho}\frac{d\rho}{dn}\right)_{\rho=\rho_o}\right\} - \frac{\partial T}{\partial n}$$

As in (2.12) we neglect here the terms with $\Delta\rho^2$ so

$$g = -\left(\frac{\partial U}{\partial \rho}\frac{d\rho}{dn}\right)_{\rho=\rho_o} - \Delta\rho\left\{\left(\frac{\partial^2 U}{\partial \rho^2}\frac{d\rho}{dn}\right)_{\rho=\rho_o} + \left(\frac{\partial U}{\partial \rho}\frac{d}{d\rho}\left(\frac{d\rho}{dn}\right)\right)_{\rho=\rho_o}\right\} - \frac{\partial T}{\partial n}$$

using (2.8) and (2.11) we get

$$g - \gamma = N\frac{\partial \gamma}{\partial n} - \frac{\partial T}{\partial n} \tag{2.17}$$

and taking (1.1) into account

$$g - \gamma = \frac{T}{\gamma}\frac{\partial \gamma}{\partial n} - \frac{\partial T}{\partial n} \tag{2.18}$$

In expression (2.18) the derivatives of γ and T are both taken along the normal to the ellipsoid. This is not exact in as much as T must be differentiated along the normal to the geoid. In neglecting the angle ϵ between the two normals (called plumbline deflection) we introduce only a small error, namely the difference between $\cos\epsilon$ and 1. Since ϵ is always smaller than one minute of arc the error is of the order of α^3. We can safely use formula (2.18) or in its final form:

$$-\frac{1}{\gamma}\frac{\partial \gamma}{\partial n} T + \frac{\partial T}{\partial n} = -\Delta g \tag{2.19}$$

This important formula can be derived from simple geometric reasoning.

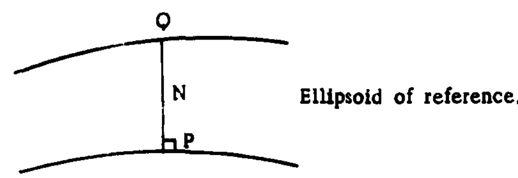


Fig. (2.1)

Omitting terms of the order of N^2, we obtain for the theoretical gravity at the point Q:

$$\gamma (Q) = \gamma(P) + N \frac{\partial \gamma}{\partial n}$$

or

$$\gamma (Q) = \gamma (P) + \frac{T}{\gamma}\frac{\partial \gamma}{\partial n} \tag{2.20}$$

On the other hand

$$g(Q) = -\frac{\partial W}{\partial n} = -\left(\frac{\partial U}{\partial n}\right)_Q - \frac{\partial T}{\partial n} = \gamma (Q) - \frac{\partial T}{\partial n}$$

which with the aid of (2.20) results in equation (2.19).

DETERMINATION OF THE UNDULATIONS OF THE GEOID WITH RESPECT TO THE ELLIPSOIDAL REFERENCE SURFACE.

We now turn to the solution of the boundary value problem, defined by (2.19). In this solution we are going to use a rotational ellipsoid as the reference surface, although some of the intermediate formulas are referred to the more general case of a tri-axial ellipsoid. Hence

$$a_1 = a_2 = a.$$

According to a formula of Bruns (1) the coefficient of T in (2.19) is

$$-\frac{1}{\gamma}\frac{\partial \gamma}{\partial n} = \left(\frac{1}{\rho_m} + \frac{1}{\rho_n}\right) + \frac{2\omega^2}{\gamma} \tag{3.1}$$

Here ω is the angular velocity of the earth. ρ_m and ρ_n are the principal radii of curvature at the point of the ellipsoid.

$$\rho_m = \frac{a\,(1 + e_1^2 \sin^2\beta)^{\frac{3}{2}}}{1 + e_1^2} \tag{3.2}$$

$$\rho_n = a\,(1 + e_1^2 \sin^2\beta)^{\frac{1}{2}} \tag{3.3}$$

$$e_1^2 = \frac{a^2 - b^2}{b^2}$$

where e_1 is called the second eccentricity of the ellipsoid. β is the reduced latitude of the point in question.

The boundary relation for T can be written in the form

$$-\frac{1}{\gamma}\frac{\partial \gamma}{\partial \rho}T + \frac{\partial T}{\partial \rho} = -\Delta g^* \tag{3.4}$$

(1) Bruns, H. Die Figur der Erde. Publik. Preusz. Geodät. Inst. Berlin 1878.

Here

$$\Delta g^{*} = \frac{dn}{d\rho}\,\Delta g = \sqrt{1 + e_1^2 \sin^2\beta}\cdot\Delta g \tag{3.5}$$

Neglecting terms of the order of e_1^4 and since $\alpha = \frac{a - b}{a}$ or $b = (1 - \alpha)\,a$, we have

$$e_1^2 = \frac{\alpha\,(2 - \alpha)}{(1 - \alpha)^2} = 2\alpha + \theta\,(\alpha^2)$$

and

$$-\frac{1}{\gamma}\,\frac{\partial\gamma}{\partial\rho} = \frac{2}{\rho}\,(1 + q + \alpha\cos^2\beta) \tag{3.6}$$

where

$$q = \frac{\omega^2 a}{\gamma}$$

Hence the final boundary condition is

$$(1 + q + \alpha\cos^2\beta)\,\frac{2T}{\rho} + \frac{\partial T}{\partial\rho} = -\Delta g^{*} \tag{3.7}$$

This formula is an extension of the fundamental formula of gravimetry for the case of a slightly flattened ellipsoid of revolution. As is easily seen, the relation (3.7) reduces to (1.2) for the case of a sphere $(q = o;\ \alpha = o;\ \frac{dn}{d\rho} = 1;\ \rho = R;)$.

Since the quantities of the order of α^2 are neglected, we can write for q the value it has at the equator

$$q = \frac{\omega^2 a}{\gamma_e}$$

Writing C for $(1 + q)$ and $\theta(\beta)$ for $\cos^2\beta$, which is a function of the position on the ellipsoid, the coefficient between parentheses in (3.7) becomes $C + \theta(\beta)$. T is expanded in a series according to powers of α, for which, as previously explained, only the first two terms are used

$$T = T^{(o)} + \alpha\,T^{(1)} \tag{3.8}$$

Combining (3.8) and (3.7), and comparing the coefficients of equal powers of α, we get the two equations

$$C\,\frac{2T^{(o)}}{\rho} + \frac{\partial T^{(o)}}{\partial\rho} = -\Delta g^{*} \tag{3.9}$$

and

$$C\,\frac{2T^{(1)}}{\rho}+\frac{\partial T^{(1)}}{\partial\rho}=-\,\theta\,(\beta)\,\frac{2T^{(o)}}{\rho} \qquad (3.10)$$

If the quantity $T^{(o)}$ is found from (3.9) the right hand expression in (3.10) is known and (3.10) can be solved in the same manner as (3.9). The complete disturbance potential is then obtained from (3.8).

The height-differences N are, according to the lemma of Bruns, also split up into two terms.

$$N = N^{(o)} + \alpha\, N^{(1)} \qquad (3.11)$$

In all series developments in the preceding formulas, as well as in the following ones, the terms of the order of magnitude of α^2 and higher orders are neglected. The error in the height-differences N may thus be of the order of $\alpha^2 N$, or of the order of 1 cm. Errors of this order can always be neglected. In Stokes' solution the flattening of the earth is entirely neglected.

The solution resulting from the following method shows an improvement by a factor of α over the classical Stokes' approach.

EVALUATION OF THE DISTURBANCE POTENTIAL.

In the following derivations, use is made of the previously mentioned ellipsoidal coordinates ρ, μ and ν, and of some special harmonic functions of ρ, μ and ν, denoted by $R(\rho)$, $M(\mu)$ and $N(\nu)$, the so-called Lamé functions or ellipsoidal harmonics. (The reader who is unfamiliar with Lamé's functions may consult any convenient treatise about this topic.)

The problem is to find a function $T^{(o)}$ which is harmonic outside of the ellipsoid, $\rho = \rho_o$, and which on the ellipsoid still satisfies the boundary condition (3.9). A bar is added to the quantities on the ellipsoid to distinguish them from those in space. The right hand member of (3.9) must be regarded as a predetermined function, $-f(\mu, \nu)$, of the two ellipsoidal coordinates μ and ν.

$$f(\mu, \nu) = \Delta g^{*} = \frac{dn}{d\rho} \Delta g \tag{4.1}$$

but $\frac{d\rho}{dn}$ can be written

$$\frac{d\rho}{dn} = l_o \sqrt{(\rho_o^2 - \eta^2)(\rho_o^2 - 1)} = l_o \; R_1(\rho_o) \; R_2(\rho_o) \tag{4.2}$$

where, according to Liouville

$$l^2 = \frac{1}{(\rho^2 - \mu^2)(\rho^2 - \nu^2)}$$

R_1 and R_2 are the first two functions of Lamé (see Poincaré "Figures d'équilibre d'une masse fluide" 1902).

The boundary condition (3.9) becomes

$$C \frac{2\overline{T}^{(o)}}{\rho} + \frac{\partial \overline{T}^{(o)}}{\partial \rho} = - f(\mu, \nu) \tag{4.3}$$

where

$$f(\mu, \nu) = \frac{1}{l_o R_1(\rho_o) \, R_2(\rho_o)} \Delta g \tag{4.4}$$

The function $f(\mu, \nu)$ can be expressed as a sum of products of the form $M(\mu)$; $N(\nu)$ as $\frac{1}{l_o}$ is a limited and continuous function on the surface. The Lamé functions S, R, M and N, will be given two indices, n and m, the first one to denote the order of the function and the second for the number within the order. For the functions of the first order R_1^o, R_1^1, R_1^2, we keep the notation used by Poincaré: R_1, R_2 and R_3.

Although the starting formula (3.9) has been obtained for an ellipsoid of revolution, in this section the general notation defining Lamé's functions for a tri-axial ellipsoid will be used. It can be noticed that for an ellipsoid of revolution $R_2 = R_3$.

Since T satisfies the equation of Laplace outside the ellipsoid, we can express $T^{(o)}$ as follows:

$$T^{(o)} = \sum_{n=2}^{\infty} \sum_{m=o}^{2n} A_n^m R_n^m(\rho_o) S_n^m(\rho) M_n^m(\mu) N_n^m(\nu) \tag{4.5}$$

The disturbance potential T is not uniquely defined without knowledge about the position of the reference ellipsoid and its relation to the earth. Therefore we add the assumption $U_o = W_o$, which implies that the term of order $n = o$ is equal to zero, and the assumption that the centers of gravity of the earth and the reference ellipsoid coincide, which implies that also the term of order $n = 1$ is equal to zero. As can be seen in (4.5) the summation over n is taken from two to infinity. The function $f(\mu, \nu)$ is thought to be expressed in the form

$$f(\mu, \nu) = \sum_{n=2}^{\infty} \sum_{m=o}^{2n} B_n^m M_n^m(\mu) N_n^m(\nu) \tag{4.6}$$

For the sake of brevity the following notation is used.

$$A_n^m M_n^m(\mu) N_n^m(\nu) = P_n^m$$

$$B_n^m M_n^m(\mu) N_n^m(\nu) = g_n^{*m}$$

Expressions (4.5) and (4.6) become now

$$T^{(o)} = \sum_{n=2}^{\infty} \sum_{m=o}^{2n} R_n^m(\rho_o) S_n^m(\rho) P_n^m \tag{4.7}$$

and

$$\Delta g^{*} = \sum_{n=2}^{\infty} \sum_{m=o}^{2n} g^{*m}_{n} \tag{4.8}$$

At the surface of the ellipsoid $\rho = \rho_o$ we have

$$\frac{2C}{\rho_o} \overline{T}^{(o)} = \sum_{n=2}^{\infty} \sum_{m=o}^{2n} \frac{2C}{\rho_o} S^m_n(\rho_o) R^m_n(\rho_o) P^m_n \tag{4.9}$$

$$\frac{\partial \overline{T}^{(o)}}{\partial \rho} = \sum_{n=2}^{\infty} \sum_{m=o}^{2n} R^m_n(\rho_o) \left(\frac{\partial S^m_n(\rho)}{\partial \rho}\right)_{\rho=\rho_o} P^m_n \tag{4.10}$$

From (4.3), (4.8), (4.9) and (4.10) we obtain

$$\sum_{n=2}^{\infty} \sum_{m=o}^{2n} R^m_n(\rho_o) S^m_n(\rho_o) \left\{\frac{2C}{\rho_o} + \frac{1}{S^m_n(\rho_o)} \left(\frac{\partial S^m_n(\rho)}{\partial \rho}\right)_{\rho=\rho_o}\right\} P^m_n =$$

$$= - \sum_{n=2}^{\infty} \sum_{m=o}^{2n} g^{*m}_{n} \tag{4.11}$$

Thus,

$$P^m_n = - \frac{g^{*m}_{n}}{R^m_n(\rho_o) S^m_n(\rho_o) \left\{\frac{2C}{\rho_o} + \frac{1}{S^m_n(\rho_o)} \left(\frac{\partial S^m_n(\rho)}{\partial \rho}\right)_{\rho=\rho_o}\right\}} \tag{4.12}$$

Substituting (4.12) into (4.7) we get for the disturbance potential the following expression

$$T^{(o)} = \sum_{n=2}^{\infty} \sum_{m=o}^{2n} - \frac{S^m_n(\rho)\, g^{*m}_n}{S^m_n(\rho_o) \left\{\frac{2C}{\rho_o} + \frac{1}{S^m_n(\rho_o)} \left(\frac{\partial S^m_n(\rho)}{\partial \rho}\right)_{\rho=\rho_o}\right\}} \tag{4.13}$$

Replacing $R(\rho_o)$ and $S(\rho_o)$ by R and S we get for $\rho = \rho_o$

$$\overline{T}^{(o)} = \sum_{n=2}^{\infty} \sum_{m=o}^{2n} \frac{g^{*m}_n}{-\frac{1}{S^m_n} \cdot \frac{\partial S^m_n}{\partial \rho} - \frac{2C}{\rho}}$$

or

$$T^{(o)} = \rho \sum_{n=2}^{\infty} \sum_{m=o}^{2n} \frac{g_n^m}{-\frac{d \ln S_n^m}{d \ln \rho} - 2C} \qquad (4.14)$$

In the special case of a non-rotating sphere this yields Stokes' solution.

THE HEIGHT-DIFFERENCE N EXPRESSED AS AN INTEGRAL.

We try to get now an expression for the height difference N analogous to the expression of Stokes. Taking into account the orthogonality of the Lamé functions, we can write:

$$g^{*m}_{n} = \frac{1}{4\pi\, I^{m}_{n}} \int l_{o}\ \Delta g^{*} \left[M^{m}_{n}(\mu)\ N^{m}_{n}(\nu)\ M^{m}_{n}(\mu')\ N^{m}_{n}(\nu') \right] d\omega \qquad (5.1)$$

where

$$I^{m}_{n} = \frac{1}{4\pi} \int l_{o} (M^{m}_{n}\ N^{m}_{n})^{2}\ d\omega \qquad (5.2)$$

$d\omega$ represents here the surface element of the ellipsoid. The series (4.14) is written in a shorter form as

$$\overline{\overline{T}}^{(o)} = \rho \sum_{n=2}^{\infty} \sum_{m=o}^{2n} T^{m}_{n} \qquad (5.3)$$

$$T^{m}_{n} = \frac{g^{*m}_{n}}{E^{m}_{n}} \qquad (5.4)$$

$$E^{m}_{n} = -\frac{d\ \ln\ S^{m}_{n}}{d\ \ln\ \rho} - 2C \qquad (5.5)$$

Thus,

$$T^{m}_{n} = \frac{1}{4\pi\, I^{m}_{n}\, E^{m}_{n}} \int l_{o}\ \Delta g^{*} M^{m}_{n}(\mu)\ N^{m}_{n}(\nu)\ M^{m}_{n}(\mu')\ N^{m}_{n}(\nu')\ d\omega \qquad (5.6)$$

Using (5.3) we have

$$T^{(o)} = \frac{\rho}{4\pi} \int l_{o} \Delta g^{*} \left[\sum_{n=2}^{\infty} \sum_{m=o}^{2n} \frac{M^{m}_{n}(\mu)\ N^{m}_{n}(\nu)\ M^{m}_{n}(\mu')\ N^{m}_{n}(\nu')}{I^{m}_{n}\ E^{m}_{n}} \right] d\omega \qquad (5.7)$$

By (4.4) we obtain

$$\Delta g^{*} = \frac{1}{l_o \ R_1 R_2} \ \Delta g$$

Noting that $R_3 = \rho$, we have

$$T^{(o)} = \frac{R_3}{4\pi \ R_1 R_2} \int \Delta g \ G\ (\mu, \nu, \mu', \ \nu') \ d\omega \tag{5.8}$$

where

$$G(\mu, \nu, \mu', \nu') = \sum_{n=2}^{\infty} \sum_{m=o}^{2n} \frac{M_n^m(\mu) N_n^m(\nu) M_n^m(\mu') N_n^m(\nu')}{I_n^m E_n^m} \tag{5.9}$$

For the case of an ellipsoidal reference surface we have a formula analogous to that of Stokes:

$$N^{(o)} = \frac{R_3}{4\pi \ R_1 R_2 \ \gamma} \int \Delta g \ G\ (\mu, \nu, \mu', \nu') \ d\omega \tag{5.10}$$

or

$$N^{(o)} = \frac{\rho}{4\pi\gamma} \int l_o \ \Delta g^{*} G\ (\mu, \nu, \mu', \nu') \ d\omega \tag{5.11}$$

We have the relation $l_o d\omega = d\sigma$, where $d\sigma$ is the surface element of the unit sphere, and $d\omega$ is the element of surface on the ellipsoid.

THE CASE OF AN ELLIPSOID OF REVOLUTION.

In the course of deriving the expression for the height difference N, the deviations of the geoid from ellipsoidal shape, we have not required the ellipsoid to possess any rotational symmetry. As pointed out before, the eccentricity of the equator of the earth is not very pronounced. As a reference figure, which does not necessarily claim to be the final solution, an ellipsoid of revolution serves excellently.

In this case, the products M N degenerate into ordinary spherical harmonics, and the functions R and S are transformed into Legendre polynomials of the first and second kind respectively and of imaginary argument.

Thus we have for the case of rotational symmetry:

$$R_2 = R_3; \qquad \mu = \sin\theta$$

$$M_n^m(\mu) = X_n^m(\cos\theta)$$

$$N_n^m(\nu) = \begin{cases} \sin m\lambda \\ \cos m\lambda \end{cases}$$

$\theta = 90^\circ - \beta$, where β is the reduced latitude.

$$\sqrt{1-\rho^2} = is; \qquad \rho = \sqrt{1+s^2}; \qquad s = \sqrt{\rho^2 - 1};$$

$$R_n^m(is) = X_n^m(is) \qquad S_n^m(is) = Q_n^m(is)$$

$$\frac{d}{d\rho} = \frac{d}{ds}\frac{ds}{d\rho} = \frac{\sqrt{1+s^2}}{s}\frac{d}{ds}$$

Thus we get

$$I_n^m = \frac{1}{4\pi}\int \left[P_n^m(\cos\theta)\begin{Bmatrix}\cos m\lambda \\ \sin m\lambda\end{Bmatrix}\right]^2 d\sigma = \frac{(n+m)!}{(2n+1)(n-m)!}\,\frac{1}{c_m} \tag{6.1}$$

($c_m = 1$ for $m = o$; $c_m = 2$ for $m \neq o$)

$$E_n^m = -\frac{1+s^2}{s}\frac{d}{ds}\ln Q(is) - 2C \tag{6.2}$$

$$C = 1 + q$$

That is

$$G(\mu, \nu, \mu', \nu') = Z(\theta_o, \lambda_o, \theta, \lambda) \tag{6.3}$$

where θ_o, λ_o now denote the fixed point and θ, λ the running point, and

$$Z(\theta_o, \lambda_o, \theta, \lambda) = \sum_{n=2}^{\infty}\sum_{m=o}^{n}\frac{(2n+1)(n-m)!}{(n+m)!}\frac{c_m P_n^m(\cos\theta)\cos m(\lambda-\lambda_o)}{E_n^m} \tag{6.4}$$

Note: we sum over from 0 to n (not 2n) because the rotational symmetry introduces a degeneration. For a given n, there now exist only(n+1) linearly independent functions. Quite naturally, the product M_n^m N_n^m must differ from the elementary spherical harmonic $P_n^m(\cos\theta)\begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix}$ by a constant factor, but this factor is cancelled, as its square occurs both in the numerator and the denominator of the formula (5.9) for $G(\mu, \nu, \mu', \nu')$ since I_n^m is in the denominator.
Thus (5.11) takes the form

$$N^{(o)} = \frac{1}{4\pi R_1 \gamma}\int_{E_o} \Delta g Z(\theta_o, \lambda_o, \theta, \lambda)\, d\omega \tag{6.5}$$

where Z is given by (6.4).

Now we transform the domain of integration from that of the ellipsoidal surface to that of the unit sphere. Remembering that $l_o\, d\omega = d\sigma$ and taking (5.11) into account, we obtain the formula:

$$N^{(o)} = \frac{\rho}{4\pi\gamma} \int \Delta g^{*} Z\ (\theta_o, \lambda_o, \theta, \lambda,)\ d\sigma \tag{6.6}$$

When finally we introduce the facts that $\Delta g^{*} = \sqrt{1+e_1^2 \sin^2\beta}\ \Delta g$ and $\rho = a$ we obtain for the ellipsoidal surface of reference:

$$N^{(o)'} = \frac{a}{4\pi\gamma} \int \sqrt{1+e_1^2 \sin^2\beta}\ \Delta g\ Z\ (\theta_o, \lambda_o, \theta, \lambda)\ d\sigma \tag{6.7}$$

where Z is given by (6.4), E_n^m is determined by (6.2). It can be shown that (6.6) contains Stokes' formula as a special case, and thus (6.6) can be regarded as an extension of this formula.

The function Z given by (6.4) must now be studied more closely. Doing this in accordance with Sagrebin, we find the following expression after a series expansion:

$$Z(\theta_o, \lambda_o, \theta, \lambda) = S(\psi) + e_1^2\ \phi\ (\psi; \theta_o, \lambda_o, \theta, \lambda) + 2q\ \Psi(\psi) \tag{6.8}$$

where ϕ and Ψ stand for the following expressions :

$$\phi(\psi; \theta_o, \lambda_o, \theta, \lambda) = - \sum_{n=2}^{\infty} \frac{n^2(2n+1)}{(n-1)^2(2n+3)} P_n(\cos\psi) +$$

$$+ 2 \sum_{n=2}^{\infty} \frac{2n+1}{(n-1)^2(2n+3)} P_n(\cos\psi) - \sum_{n=2}^{\infty} \frac{2n+1}{(n-1)^2(2n+3)} \frac{\partial^2}{\partial\lambda^2} P_n(\cos\psi) \tag{6.9}$$

$$\Psi(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{(n-1)^2} P_n \cos(\psi)\ . \tag{6.10}$$

INVESTIGATIONS OF THE FUNCTIONS Φ AND Ψ.

We begin studying the function

$$\Phi(\psi; \theta_o, \lambda_o, \theta, \lambda) = -\Phi_1(\psi) + 2\Phi_2(\psi) - \Phi_3(\psi; \theta_o, \lambda_o, \theta, \lambda) \qquad (7.1)$$

The first two terms of (7.1) are functions of the angular distance ψ between the two points (θ_o, λ_o) and (θ, λ) where θ is the complement of the reduced latitude. These functions are expressed by the equations (cf 6.9)

$$\Phi_1(\psi) = \sum_{n=2}^{\infty} \frac{n^2(2n+1)}{(n-1)^2(2n+3)} P_n(\cos\psi) \qquad (7.2)$$

$$\Phi_2(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{(n-1)^2(2n+3)} P_n(\cos\psi) \qquad (7.3)$$

The third term in (7.1) depends not only upon the relative position of the two points but also upon their absolute position on the ellipsoid. Thus the third term is a function of all four coordinates $\theta_o, \lambda_o, \theta, \lambda$.

If we change the order of summation and differentiation in the last term of (6.9) we have the formula

$$\Phi_3(\psi; \theta_o, \lambda_o, \theta, \lambda) = \frac{\partial^2}{\partial \lambda_o^2} \Phi_2(\psi) \qquad (7.4)$$

or

$$\Phi_3(\psi; \theta_o, \lambda_o, \theta, \lambda) = \frac{\partial^2 \Phi_2(\psi)}{\partial \psi^2} \left(\frac{\partial \psi}{\partial \lambda_o}\right)^2 + \frac{\partial \Phi_2(\psi)}{\partial \psi} \frac{\partial^2 \psi}{\partial \lambda_o^2} \qquad (7.5)$$

It is clear that the derivatives $\frac{\partial \psi}{\partial \lambda_o}$ and $\frac{\partial^2 \psi}{\partial \lambda_o^2}$ must be functions of both sets of coordinates $\theta_o, \lambda_o, \theta, \lambda$.

The derivatives can be obtained from the formula (the spherical cosine theorem):

$$\cos \psi = \cos \theta_o \cos \theta + \sin \theta_o \sin \theta \cos (\lambda - \lambda_o) \tag{7.6}$$

Thus for the evaluation of the function ϕ it is sufficient to obtain the functions $\phi_1(\psi)$ and $\phi_2(\psi)$ whereupon ϕ_3 can be found by ordinary differentiation. To obtain ϕ_1 and ϕ_2, we decompose the fractions preceding $P_n(\cos \psi)$ in (7.2) and (7.3) into partial fractions. We obtain

$$\frac{n^2(2n+1)}{(n-1)^2(2n+3)} = 1 + \frac{34}{25}\frac{1}{n-1} + \frac{3}{5}\frac{1}{(n-1)^2} - \frac{18}{25}\frac{1}{2n+3}$$

and

$$\frac{2n+1}{(n-1)^2(2n+3)} = \frac{4}{25}\frac{1}{n-1} + \frac{3}{5}\frac{1}{(n-1)^2} - \frac{8}{25}\frac{1}{2n+3}$$

Thus

$$-\phi_1(\psi) + 2\phi_2(\psi) = -\sum_{n=2}^{\infty} P_n(\cos \psi) - \frac{26}{25}\sum_{n=2}^{\infty}\frac{1}{n-1}P_n(\cos \psi)$$

$$+\frac{3}{5}\sum_{n=2}^{\infty}\frac{1}{(n-1)^2}P_n(\cos \psi) + \frac{2}{25}\sum_{n=2}^{\infty}\frac{1}{2n+3}P_n(\cos \psi)$$

The function $\phi(\psi)$ (cf 6.10) takes the form

$$\phi(\psi) = 2\sum_{n=2}^{\infty}\frac{1}{n-1}P_n(\cos \psi) + 3\sum_{n=2}^{\infty}\frac{1}{(n-1)^2}P_n(\cos \psi)$$

Writing the above expressions in a shorter form, we have

$$-\phi_1(\psi) + 2\phi_2(\psi) = -\varphi_o(\psi) - \frac{26}{25}\varphi_1(\psi) + \frac{3}{5}\varphi_2(\psi) + \frac{2}{25}\varphi_3(\psi) \tag{7.6}$$

$$\phi_2(\psi) = \frac{4}{25}\varphi_1(\psi) + \frac{3}{5}\varphi_2(\psi) - \frac{8}{25}\varphi_3(\psi) \tag{7.7}$$

$$\Psi(\psi) = 2\ \varphi_1\ (\psi) + 3\ \varphi_2\ (\psi) \tag{7.8}$$

where for convenience we have introduced the expressions:

$$\varphi_o\ (\psi) = \Sigma\ P_n \tag{7.9}$$

$$\varphi_1\ (\psi) = \Sigma\ \frac{1}{n-1}\ P_n \tag{7.10}$$

$$\varphi_2\ (\psi) = \Sigma\ \frac{1}{(n-1)^2}\ P_n \tag{7.11}$$

$$\varphi_3\ (\psi) = \Sigma\ \frac{1}{2n+3}\ P_n \tag{7.12}$$

All summations are taken from $n = 2$ to $n = \infty$, and P_n stands for P_n (cos ψ).
The sums (7.9 - 12) can be obtained by utilizing the generating function for the Legendre polynomials:

$$\sum_{n=o}^{\infty} P_n\ (\cos\ \psi\ )\ x^n = \frac{1}{\sqrt{1\ -\ 2x\ \cos\ \psi\ +\ x^2}}$$

or

$$\sum_{n=2}^{\infty} P_n\ x^n = \frac{1}{\sqrt{1\ -2x\ \cos\ \psi\ +\ x^2}} - 1 - x\ \cos\ \psi \tag{7.13}$$

Putting $x = 1$ in (7.13), we have

$$\varphi_o\ (\psi) = \frac{1}{\sqrt{2-2\ \cos\ \psi}} - 1\ -\ \cos\ \psi = \frac{1}{2}\ \text{cosec}\ \frac{\psi}{2} - 1\ -\ \cos\ \psi \tag{7.14}$$

The sums $\varphi_1\ (\psi)$ and $\varphi_3\ (\psi)$ are of the general form: $\Sigma\ \frac{1}{n-k}\ P_n$.
We introduce the expression

$$\Sigma\ \frac{1}{n-k}\ P_n\ x^{n-k} = S_k\ (\psi,\ x) \tag{7.15}$$

By differentiation we find the relation

$$\frac{\partial}{\partial x}\ S_k\ (\psi,\ x) = \Sigma\ P_n\ x^{n-k-1}$$

That is

$$S_k(\Psi, x) = \int (\Sigma P_n x^{n-k-1})\, dx \tag{7.16}$$

But we have, by (7.13)

$$\Sigma P_n x^{n-k-1} = \frac{1}{x^{k+1}} \left[\frac{1}{\sqrt{1-2x\cos\Psi + x^2}} - 1 - x\cos\Psi \right] \tag{7.17}$$

Thus

$$S_k(\Psi, x) = \int_0^x \frac{1}{x^{k+1}} \left[\frac{1}{r(x)} - 1 - x\omega \right] dx \tag{7.18}$$

where $r(x) = \sqrt{1-2x\,\omega + x^2}$ and $\omega = \cos\Psi$ to achieve shortness of notation

To obtain φ_1 we put $k = 1$:

$$S_1(\Psi, x) = \int_0^x \frac{1}{x^2} \left(\frac{1}{r(x)} - 1 - x\omega \right) dx \tag{7.19}$$

and then

$$\varphi_1(\Psi) = S_1(\Psi, 1) \tag{7.20}$$

$S_1(\Psi, x)$ is evaluated as follows:

$$S_1(\Psi, x) = \int_0^x \left(\frac{1}{x^2 r(x)} - \frac{1}{x^2} - \frac{\omega}{x} \right) dx = \int_0^x \frac{1}{x^2 r(x)}\, dx + \left(\frac{1}{x} - \omega \ln x \right)_0^x$$

(Singularities which arise when $x \to o$ must cancel in the end result.)

$$\int \frac{dx}{x^2 r(x)} = (\text{substituting } y = \frac{1}{x}) = -\int \frac{y\, dy}{r(y)}$$

but

$$\frac{\partial}{\partial y} r(y) = \frac{y - \omega}{r(y)}$$

Thus

$$-\int \frac{y\,dy}{r(y)} = -\int \left(\frac{\partial r(y)}{\partial y} + \frac{\omega}{r(y)}\right) dy = -r(y) - \omega \ln \left| y - \omega + r(y) \right|$$

$$= -\frac{r(x)}{x} - \omega \ln \left| 1 - x\omega + r(x) \right| + \omega \ln x$$

It follows then that:

$$S_1\ (\psi, x) = \left\{ \frac{1 - r(x)}{x} - \omega \ln \left| 1 - \omega x + r(x) \right| \right\}_0^x$$

As x approaches zero, $\frac{1 - r(x)}{x}$ approaches the constant ω.

Thus

$$S_1(\psi, x) = \frac{1 - r(x)}{x} - \omega \ln \frac{1}{2} \left| 1 - \omega x + r(x) \right| - \omega \qquad (7.21)$$

Putting $x = 1$ we obtain

$$\varphi_1(\psi) = S_1(\psi, 1) = 1 - \sqrt{2 - 2\omega} - \omega - \omega \ln \frac{1}{2} (1 - \omega + \sqrt{2 - 2\omega})$$

But $\omega = \cos \psi$, thus $\frac{1 - \omega}{2} = \sin^2 \frac{\psi}{2}$

which yields

$$\varphi_1\ (\psi) = 1 - 2 \sin \frac{\psi}{2} - \cos \psi - \cos \psi \ln (\sin^2 \frac{\psi}{2} + \sin \frac{\psi}{2}) \qquad (7.22)$$

To obtain $\varphi_2(\psi)$, we must study the more general sum

$$H\ (\psi, x) = \Sigma \frac{1}{(n-1)^2}\ P_n\ x^{n-1} \qquad (7.23)$$

Differentiating, we obtain

$$\frac{\partial}{\partial x}\ H\ (\psi, x) = \Sigma \frac{1}{n-1}\ P_n x^{n-2} = \frac{1}{x} S_1(\psi, x) \qquad (7.24)$$

by the use of (7.15).

Thus

$$H(\Psi, x) = \int_0^x \frac{1}{x} S_1(\Psi, x)\, dx$$ [1]

That is

$$H(\Psi, x) = \int_0^x \left(\frac{1-r(x)}{x^2} - \frac{\omega}{x} \ln \frac{1}{2} \left|1-\omega x + r(x)\right| - \frac{\omega}{x} \right) dx \qquad (7.25)$$

Putting $x = 1$,

$$H(\Psi, 1) = \varphi_2(\Psi) = \int_0^1 \left(\frac{1-r(x)}{x^2} - \frac{\omega}{x} - \frac{\omega}{x} \ln \frac{1}{2} \left|1-\omega x + r(x)\right| \right) dx$$

$$(7.26)$$

We consider the expression

$$J_{1\epsilon}(\Psi) = \int_\epsilon^1 \frac{1}{x} S_1(\Psi, x)\, dx = J_{1\epsilon}^{(1)} + \omega \ln \epsilon - \omega J_{1\epsilon}^{(2)} \qquad (7.27)$$

where ϵ is a small positive quantity which approaches zero , and

$$J_{1\epsilon}^{(1)} = \int_\epsilon^1 \frac{1-r(x)}{x^2}\, dx \qquad (7.28)$$

$$J_{1\epsilon}^{(2)} = \int_\epsilon^1 \frac{1}{x} \ln \frac{1}{2} \left| 1 - \omega x + r(x) \right| dx \qquad (7.29)$$

We now proceed to the evaluation of $J_{1\epsilon}^{(1)}$ and $J_{1\epsilon}^{(2)}$.

$$J_{1\epsilon}^{(1)} = \frac{1-r(x)}{x^2}\, dx = \left[- \frac{1-r(x)}{x} \right]_\epsilon^1 - \int_\epsilon^1 \frac{x-\omega}{x\, r(x)}\, dx =$$

$$= -1 + \sqrt{2-2\omega} + \omega + O(\epsilon) - \int_\epsilon^1 \frac{dx}{r(x)} + \omega \int_1^{1/\epsilon} \frac{dy}{r(y)}$$

1) Sagrebin did not include the, "x" in the denominator of the integrand.

where we have used partial integration, the substitution $y = \frac{1}{x}$, and the fact that

$\frac{1 - r(\epsilon)}{\epsilon} = \omega + O(\epsilon)$

Remembering that $\int \frac{dx}{r(x)} = \ln \left| x - \omega + r(x) \right| +$ a constant and disregarding terms $O(\epsilon)$ (of the order of ϵ), we have

$$J_{1\epsilon}^{(1)} = -1 + \omega + \sqrt{2 - 2\omega} - \ln \frac{1 - \omega + \sqrt{2 - 2\omega}}{1 - \omega} +$$

$$+ \omega \ln \frac{2}{1 - \omega + \sqrt{2 - 2\omega}} - \omega \ln \epsilon = -1 + \cos \psi + 2 \sin \frac{\psi}{2} -$$

$$- (1 + \cos \psi) \ln (1 + \sin \frac{\psi}{2}) + (1 - \cos \psi) \ln \sin \frac{\psi}{2} - \omega \ln \epsilon$$

Thus letting ϵ approach zero:

$$\varphi_2(\psi) = -1 + \cos \psi + 2 \sin \frac{\psi}{2} + (1 - \cos \psi) \ln \sin \frac{\psi}{2} -$$

$$- (1 + \cos \psi) \ln (1 + \sin \frac{\psi}{2}) - \cos \psi \, J_1^{(2)} \qquad (7.30)$$

It remains to evaluate $J_1^{(2)}$. It is not possible to express this function in terms of elementary functions, but it can be transformed by the substitution $x = \frac{1}{y}$ and by partial integration:

$$J_1^{(2)} = \int_0^1 \frac{1}{x} \ln \left| \frac{1}{2} (1 - \omega x + r(x)) \right| dx =$$

$$(x = \frac{1}{y} ; \quad \frac{dx}{x} = - \frac{dy}{y}) =$$

$$= \int_1^\infty \frac{1}{y} (\ln \left| \frac{1}{2} (y - \omega + r(y) \right| - \ln y) dy =$$

$$= \left[- \ln^2 y + \ln y \ln \frac{1}{2} (y - \omega + r(y)) \right]_1^\infty$$

$$+ \int_1^\infty (\frac{1}{y} - \frac{1}{r(y)}) \ln y \, dy$$

Thus

$$J_1^{(2)} = - L(\Psi) \tag{7.31}$$

where $L(\Psi)$ is given by the equation:

$$L(\Psi) = \int_1^{\infty} \left(\frac{1}{r(x)} - \frac{1}{x} \right) \ln x \, dx \tag{7.32}$$

The term $\frac{\ln x}{x}$ is included to make the integral converge at the upper limit. Thus the expression for $\varphi_2(\Psi)$ now reads:

$$\varphi_2(\Psi) = -1 + \cos\Psi + 2\sin\frac{\Psi}{2} + (1 - \cos\Psi)\ln\sin\frac{\Psi}{2} -$$

$$-(1 + \cos\Psi)\ln\left(1 + \sin\frac{\Psi}{2}\right) + \cos\Psi \, L(\Psi) \tag{7.33}$$

We now make a closer study of the function $L(\Psi)$. In the limits for $\omega = \pm 1$, that is, $\Psi = 0^{\circ}$ or 180°, we may compute the exact values of $L(\Psi)$ as follows:

1. $\Psi = 0$; $\omega = 1$

$$L(0) = - J_1^{(2)}(0) = - \int_0^1 \frac{\ln(1-x)}{x} \, dx$$

By series expansion we have

$$L_{(0)} = \int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^{n-1}}{n} \right) dx = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{n-1} \, dx =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

2. $\Psi = 180^{o}$, $\omega = -1$

$$L(180^{o}) = -\int_{o}^{1} \frac{\ln(1+x)}{x}\, dx$$

$$L(o) + L(180^{o}) = -\int_{o}^{1} \frac{\ln(1-x^{2})}{x}\, dx$$

Making the substitution $x^{2} = t$; we have $2 \ln x = \ln t$ and

$$\frac{dx}{x} = \frac{1}{2}\frac{dt}{t}$$

Thus

$$L(o) + L(180^{o}) = \frac{1}{2} L(o)$$

or

$$L(180^{o}) = -\frac{1}{2} L(o) = -\frac{\pi^{2}}{12}$$

The derivatives of $L(\Psi)$ can be obtained explicitly by differentiating the terms under the integral sign with respect to Ψ (see 7.32):

$$\frac{dL(\Psi)}{d\Psi} = \frac{dL(\Psi)}{d\omega} \cdot \frac{d\omega}{d\Psi} = -\sin\Psi\, L' \tag{7.34}$$

$$L' = \frac{dL}{d\omega} = \int_{1}^{\infty} \frac{x \ln x}{r^{3}(x)}\, dx \qquad \text{(see 7.32)} \tag{7.35}$$

But

$$\frac{\partial}{\partial x}\frac{1}{r(x)} = -\frac{x}{r(x)^{3}} + \frac{\omega}{r(x)^{3}}$$

thus,

$$\frac{x}{r(x)^{3}} = \frac{\omega}{r(x)^{3}} - \frac{\partial}{\partial x}\frac{1}{r(x)}$$

$$\frac{1}{r(x)^{3}} = \frac{r(x)^{2}}{r(x)^{3}} + \frac{2\omega x - x^{2}}{r(x)^{3}} = \frac{1}{r(x)} + \frac{\omega x}{r(x)^{3}} + \frac{x(\omega - x)}{r(x)^{3}}$$

$$= \frac{1}{r(x)} + \frac{\omega x}{r(x)^{3}} + x\frac{\partial}{\partial x}\frac{1}{r(x)}$$

This gives

$$L' = \omega \int_1^\infty \frac{\ln x}{r(x)}\, dx + \omega^2 \int_1^\infty \frac{x \ln x}{r(x)^3}\, dx + \omega \int_1^\infty x \ln x \frac{\partial}{\partial x} \frac{1}{r(x)}\, dx$$

$$- \int_1^\infty \ln x \frac{\partial}{\partial x} \frac{1}{r(x)}\, dx$$

$$L' = \omega \int_1^\infty \frac{\ln x}{r(x)}\, dx + \omega^2 L' + \omega \left[\frac{x \ln x}{r(x)}\right]_1^\infty - \omega \int_1^\infty \frac{\ln x}{r(x)}\, dx$$

$$- \omega \int_1^\infty \frac{dx}{r(x)} - \left[\frac{\ln x}{r(x)}\right]_1^\infty + \int_1 \frac{dx}{r(x)x}$$

$$= \omega^2 L' + \omega \left[\frac{x \ln x}{r(x)} - \ln \left| x - \omega + r(x) \right|\right]_1^\infty - \left[\ln \left| \frac{1}{x} - \omega + \frac{1}{x} r(x) \right|\right]_1^\infty$$

Thus

$$(1-\omega^2)\, L' = \omega \ln \frac{1 - \omega + \sqrt{2 - 2\omega}}{2} + \ln \frac{1 - \omega + \sqrt{2 - 2\omega}}{1 - \omega}$$

$$= \omega \ln (s + s^2) + \ln (s + s^2) - \ln s^2 = (1 + \omega) \ln (1 + s)$$

$$- (1 - \omega) \ln s$$

where

$$s = \sin \frac{\psi}{2}$$

We have now

$$L' = \frac{1}{1 - \omega} \ln (1 + s) - \frac{1}{1 + \omega} \ln s$$

and

$$\frac{dL}{d\psi} = - 2\, s\, c\, L' = \frac{s}{c} \ln s - \frac{c}{s} \ln (1 + s)$$

where

$$c = \cos \frac{\psi}{2}$$

Thus the formula for $\frac{dL(\psi)}{d\psi}$ is:

$$\frac{dL(\psi)}{d\psi} = tg\frac{\psi}{2}\ \ln\sin\frac{\psi}{2} - \cot\frac{\psi}{2}\ \ln\left(1 + \sin\frac{\psi}{2}\right) \tag{7.36}$$

This formula is differentiated once again:

$$\frac{d^2L(\psi)}{d\psi^2} = 1 - \frac{1}{2}\operatorname{cosec}\frac{\psi}{2} + \frac{1}{2}\operatorname{cosec}^2\frac{\psi}{2}\ \ln\left(1+\sin\frac{\psi}{2}\right) + \frac{1}{2}\sec^2\frac{\psi}{2}\cdot$$

$$\cdot\ln\sin\frac{\psi}{2} \tag{7.37}$$

We now turn to the evaluation of the function $\varphi_3(\psi)$ given by (7.12).

$$\varphi_3(\psi) = \Sigma\frac{1}{2n+3}P_n = \frac{1}{2}\Sigma\frac{1}{n+\frac{3}{2}}P_n$$

Putting $k = -\frac{3}{2}$ and $x = 1$ in the general formula (7.18) we have

$$\varphi_3(\psi) = \frac{1}{2}S_{-\frac{3}{2}}(\psi, 1) = \frac{1}{2}\int_0^1\frac{1}{x^{-\frac{1}{2}}}\left(\frac{1}{r(x)} - \omega x - 1\right)dx$$

That is,

$$\varphi_3(\psi) = \frac{1}{2}\int_0^1\frac{\sqrt{x}\,dx}{\sqrt{1-2x\omega+x^2}} - \frac{1}{5}\omega - \frac{1}{3} \tag{7.38}$$

Thus it is necessary to compute the integral

$$J_2(\psi) = \frac{1}{2}\int_0^1\frac{\sqrt{x}\,dx}{\sqrt{1-2x\omega+x^2}} \tag{7.39}$$

Substituting $\sqrt{x} = tg\frac{\varphi}{2}$ we obtain

$$J_2(\psi) = \frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{tg^2\frac{\varphi}{2}\,d\varphi}{\cos^2\frac{\varphi}{2}\sqrt{1-2\omega\,tg^2\frac{\varphi}{2}+tg^4\frac{\varphi}{2}}}$$

from which we have

$$J_2(\psi) = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{tg^2\frac{\varphi}{2}\, d\varphi}{\sqrt{1 - k^2\sin^2\varphi}}$$

where

$$k^2 = \frac{1}{2}(1 + \omega) = \frac{1}{2}(1 + \cos\psi) \text{ or } k = \cos\frac{\psi}{2} \tag{7.40}$$

However this last integral can be expressed as follows:

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{tg^2\frac{\varphi}{2}\, d\varphi}{\sqrt{1 - k^2\sin^2\varphi}} = \left[tg\frac{\varphi}{2}\sqrt{1 - k^2\sin^2\varphi}\right]_0^{\frac{\pi}{2}} +$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} - \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2\varphi}\, d\varphi$$

which can be expressed in snorter notation as

$$J_2(\psi) = \sin\frac{\psi}{2} + \frac{1}{2}K(\psi) - E(\psi) \tag{7.41}$$

where $K(\psi)$ and $E(\psi)$ are complete elliptic integrals of modulus $k = \cos\frac{\psi}{2}$.
Thus, finally

$$\varphi_3(\psi) = \sin\frac{\psi}{2} + \frac{1}{2}K(\psi) - E(\psi) - \frac{1}{5}\cos\psi - \frac{1}{3} \tag{7.42}$$

As we are going to require the first and second derivatives of $\varphi_3(x)$ in the following, we state here the corresponding derivatives of $K(\psi)$ and $E(\psi)$:

$$\frac{dE}{dk} = \frac{1}{k}(E - K) \tag{7.43}$$

$$\frac{dK}{dk} = \frac{1}{k}\left(\frac{E}{1 - k^2} - K\right) \tag{7.44}$$

Formulas (7.43) and (7.44) can be found in many standard tables (see Dwight: "Tables of Integrals and other Mathematical Data". 4th Ed," 1961, Macmillan Company, New York.)

Now

$$\frac{dE}{d\Psi} = \frac{dk}{d\Psi}\frac{dE}{dk} = -\frac{1}{2}\sin\frac{\Psi}{2}\frac{dE}{dk}$$

Thus,

$$\frac{dE}{d\Psi} = +\frac{1}{2}\,\mathrm{tg}\,\frac{\Psi}{2}(K - E) \tag{7.45}$$

and similarly

$$\frac{dK}{d\Psi} = \frac{1}{2}\,\mathrm{tg}\,\frac{\Psi}{2}K - \mathrm{cosec}\,\Psi\cdot E \tag{7.46}$$

Differentiating once more, we obtain in the same way:

$$\frac{d^2E}{d\Psi^2} = \frac{1}{4}\sec^2\frac{\Psi}{2}K - \frac{1}{4}\left(1 + \sec^2\frac{\Psi}{2}\right)E \tag{7.47}$$

and

$$\frac{d^2K}{d\Psi^2} = \frac{1}{4}\,\mathrm{tg}^2\frac{\Psi}{2}K + \frac{\cos\Psi}{\sin^2\Psi}E \tag{7.48}$$

EXPLICIT EXPRESSION FOR THE FUNCTION $Z(\theta_o, \lambda_o, \theta, \lambda)$.

The dominant function in the expression for Z is the well-known function of Stokes:

$$S(\psi) = \operatorname{cosec}\frac{\psi}{2} + 1 - 6\sin\frac{\psi}{2} - 5\cos\psi - 3\cos\psi\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) \tag{8.1}$$

Considering the equations (6.8) and (7.1) we can express Z as follows:

$$Z(\theta_o, \lambda_o, \theta, \lambda) = S(\psi) + e_1^2 f(\psi) - e_1^2 \Phi_3(\psi; \theta_o, \lambda_o, \theta, \lambda) + 2q\Psi(\psi) \tag{8.2}$$

where we have

$$f(\psi) = -\Phi_1(\psi) + 2\Phi_2(\psi) \tag{8.3}$$

The explicit expressions for $f(\psi)$ and $\Psi(\psi)$ can be found immediately by the aid of formulas: 7.6, 7.8, 7.14, 7.22, 7.33, 7.42.

$$\begin{aligned} f(\psi) = & -\frac{2}{3} - \frac{1}{2}\operatorname{cosec}\frac{\psi}{2} + \frac{84}{25}\sin\frac{\psi}{2} + \frac{328}{125}\cos\psi + \\ & + \left(\frac{11}{25}\cos\psi - \frac{3}{5}\right)\ln\left(1 + \sin\frac{\psi}{2}\right) + \\ & + \left(\frac{11}{25}\cos\psi + \frac{3}{5}\right)\ln\sin\frac{\psi}{2} + \frac{3}{5}\cos\psi\, L(\psi) + \\ & + \frac{1}{25}K(\psi) - \frac{2}{25}E(\psi) \end{aligned} \tag{8.4}$$

$$\begin{aligned} \Psi(\psi) = & -1 + 2\sin\frac{\psi}{2} + \cos\psi - (3 + 5\cos\psi)\ln\left(1 + \sin\frac{\psi}{2}\right) \\ & + (3 - 5\cos\psi)\ln\sin\frac{\psi}{2} + 3\cos\psi\, L(\psi) \end{aligned} \tag{8.5}$$

To obtain the function Φ_3, we need $\Phi_2(\psi)$, which can also be found easily with the aid of the formulas 7.7, 7.22, 7.33 and 7.42.

$$\Phi_2(\Psi) = -\frac{1}{3} + \frac{14}{25}\sin\frac{\Psi}{2} + \frac{63}{125}\cos\Psi -$$

$$-\left(\frac{3}{5} + \frac{19}{25}\cos\Psi\right)\ln\left(1+\sin\frac{\Psi}{2}\right) +$$

$$+\left(\frac{3}{5} - \frac{19}{25}\cos\Psi\right)\ln\sin\frac{\Psi}{2} + \frac{3}{5}\cos\Psi\, L(\Psi)$$

$$-\frac{4}{25}K(\Psi) + \frac{8}{25}E(\Psi) \tag{8.6}$$

The derivatives of $\Phi_2(\Psi)$ can now be obtained from the formulas 8.6, 7.36, 7.37, 7.45 - 7.48.

$$\frac{d\Phi_2(\Psi)}{d\Psi} = \frac{32}{125}\sin\Psi - \frac{12}{25}\cos\frac{\Psi}{2} - \frac{2}{25}\,\frac{\cot\frac{\Psi}{2}}{1+\sin\frac{\Psi}{2}} +$$

$$+\left(\frac{34}{25}\sin\Psi - \frac{3}{5}\cot\frac{\Psi}{2}\right)\ln\left(1+\sin\frac{\Psi}{2}\right)$$

$$+\left(\frac{34}{25}\sin\Psi - \frac{3}{5}\operatorname{tg}\frac{\Psi}{2}\right)\ln\sin\frac{\Psi}{2}$$

$$-\frac{3}{5}\sin\Psi\, L(\Psi) + \frac{2}{25}\operatorname{tg}\frac{\Psi}{2}K(\Psi) + \frac{4}{25}\cot\Psi\, E(\Psi) \tag{8.7}$$

$$\frac{d^2\Phi_2(\Psi)}{d\Psi^2} = \frac{8}{5}\sin\frac{\Psi}{2} + \frac{202}{125}\cos\Psi + \frac{1}{25}\operatorname{cosec}^2\frac{\Psi}{2} -$$

$$-\frac{3}{10}\operatorname{cosec}\frac{\Psi}{2} - \frac{1}{25}\,\frac{1}{1+\sin\frac{\Psi}{2}} +$$

$$+\left[\frac{49}{25}\cos\Psi - \frac{3}{5} - \frac{3}{10}\sec^2\frac{\Psi}{2}\right]\ln\sin\frac{\Psi}{2}$$

$$+\left[\frac{49}{25}\cos\Psi + \frac{3}{5} + \frac{3}{10}\operatorname{cosec}^2\frac{\Psi}{2}\right]\ln\left(1+\sin\frac{\Psi}{2}\right)$$

$$-\frac{3}{5}\cos\Psi\, L(\Psi) + \frac{1}{25}\left(\sec^2\frac{\Psi}{2} + 1\right)K(\Psi) - \frac{2}{25}\left[2\operatorname{cosec}^2\Psi + 1\right]E(\Psi)$$

$$\tag{8.8}$$

We still have to evaluate the derivatives of the angular distance Ψ with respect to λ_o. These derivatives can in principle be obtained from the spherical cosine theorem but are more easily found from three fundamental formulas of spherical trigonometry.

First, we make the change of variables $\beta_o = 90^o - \theta_o$; $\beta = 90^o - \theta$, where β is the reduced latitude of the point in question. The three fundamental formulas are (cf. fig. 8.1).

$$\sin \Psi \sin A_o = \cos \beta \sin (\lambda - \lambda_o) \tag{8.9}$$

$$\sin \Psi \cos A_o = \sin \beta \cos \beta_o - \cos \beta \sin \beta_o \cos(\lambda-\lambda_o) \tag{8.10}$$

$$\cos \Psi = \sin \beta_o \sin \beta + \cos \beta_o \cos \beta \cos (\lambda - \lambda_o) \tag{8.11}$$

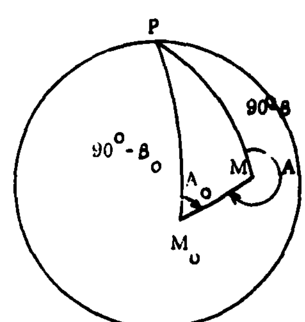


Fig. 8.1

Fig. 8.1

P = the north pole

M_o = the fixed point

M = the running point

A_o. = the azimuth

The angle M_oPM is equal to λ-λ_o.

Differentiating (8.11) with respect to λ_o and taking (8.9) into account we obtain

$$\frac{\partial \Psi}{\partial \lambda_o} = - \sin A_o \cos \beta_o \tag{8.12}$$

Differentiating once more:

$$\frac{\partial^2 \Psi}{\partial \lambda_o^2} = - \cos \beta_o \cos A_o \frac{\partial A_o}{\partial \lambda_o} \tag{8.13}$$

The derivative $\frac{\partial A_o}{\partial \lambda_o}$ can be obtained from (8.9) and (8.10) after differentiation:

$$\cos\Psi \sin A_o \frac{\partial \Psi}{\partial \lambda_o} + \cos A_o \sin\Psi \frac{\partial A_o}{\partial \lambda_o} = -\cos\beta \cos(\lambda - \lambda_o)$$

$$\cos\Psi \cos A_o \frac{\partial \Psi}{\partial \lambda_o} - \sin A_o \sin\Psi \frac{\partial A_o}{\partial \lambda_o} = -\cos\beta \sin\beta_o \sin(\lambda - \lambda_o)$$

From this we have

$$\sin\Psi \frac{\partial A_o}{\partial \lambda_o} = \cos\beta \left[\sin\beta_o \sin A_o \sin(\lambda - \lambda_o) - \cos A_o \cos(\lambda - \lambda_o)\right]$$

But we also have:

$$\cos(360^o - A) = -\cos A_o \cos(\lambda - \lambda_o) + \sin A_o \cos(\lambda - \lambda_o) \sin\beta_o$$

Thus

$$\frac{\partial A_o}{\partial \lambda_o} = \frac{\cos\beta \cos A}{\sin\Psi}$$

When we substitute this expression into (8.13), we obtain:

$$\frac{\partial^2 \Psi}{\partial \lambda_o^2} = -\frac{\cos\beta_o \cos\beta \cos A_o \cos A}{\sin\Psi} \tag{8.14}$$

Regarding the function Φ_3 as a function of the four variables β_o, β, A_o, A we obtain

$$\Phi_3(\Psi;\ \beta_o, \beta, A_o, A) = f_1(\Psi) \sin^2 A_o \cos^2\beta_o -$$

$$- f_2(\Psi) \cos A_o \cos A \cos\beta_o \cos\beta \tag{8.15}$$

where

$$f_1(\Psi) = \frac{d^2 \Phi_2(\Psi)}{d\Psi^2} \tag{8.16}$$

$$f_2(\Psi) = \frac{1}{\sin\Psi} \frac{d\Phi_2(\Psi)}{d\Psi} \tag{8.17}$$

The expressions for f_1 and f_2 are determined further by the formulas (8.7) and (8.8). Thus we can write for the function Z:

$$Z\ (\psi;\ \beta_o, \beta, A_o, A) = S\ (\psi) + 2\ \alpha\ f(\psi) + 2q\, \Psi(\psi)$$

$$- 2\alpha \left[f_2(\psi)\ \sin^2 A_o \cos^2 \beta_o - f_2(\psi)\ \cos\ A_o \cos A \cos \beta_o \cos \beta \right] \qquad (8.18)$$

where $S(\psi)$ is Stokes' function and the functions $\Psi(\psi)$, $f(\psi)$, $f_1(\psi)$, $f_2(\psi)$ are given by (8.4), (8.5), (8.16), (8.17). The quantity e_1^2 is replaced by the approximation $2\ \alpha$. The function Z, as given by (8.18), is still apparently a function of the variables A and β of the running point. In reality, Z is a function of the angles ψ and A_o only if we assume β_o to be given.

From the spherical triangle PM_oM we have:

$$\cos\ \beta\ \cos\ A = \sin\ \psi\ \sin\ \beta_o - \cos\ \beta_o\ \cos\ \psi\ \cos\ A_o \qquad (8.19)$$

Introducing (8.19) into (8.18) we obtain

$$Z(\ \beta_o;\ \psi,\ A_o) = S\ (\psi) + 2\alpha\ f(\psi) + 2q\, \Psi(\psi) -$$

$$- 2\alpha \left[f_1(\psi)\ \sin^2\ A_o\ \cos^2 \beta_o - f_2(\psi)\ \cos\ \beta_o\ \cos\ A \right.$$

$$\left. (\ \sin\ \psi\ \sin\ \beta_o - \cos\ \beta_o\ \cos\ \psi\ \cos\ A_o) \right] \qquad (8.20)$$

Thus we can write the expression for the height difference $N^{(o)}$ at the fixed point, determined by (β_o, λ_o), in the form:

$$N^{(o)} = \frac{a}{4\pi\gamma} \int_0^{\pi} \int_0^{2\pi} \Delta g^{*} Z\ (\beta_o;\ \psi,\ A_o)\ \sin\psi\, d\psi\, d\, A_o \qquad (8.21)$$

Using the variables ψ, A_o the surface element $d\sigma = \sin\psi\, d\psi\ dA_o$.

The quantity $\Delta\ g^{*}$ is connected to the gravity anomaly Δg by the equation

$$\Delta g^{*} = \sqrt{1 + e_1^2\ \sin^2 \beta}\ \Delta g = \Delta g\ (1 + \frac{e_1^2}{2} \sin^2 \beta + \ldots) =$$

$$= \Delta g\ (1 + \alpha\ \sin^2\ \beta) \qquad (8.22)$$

where $\sin^2\beta$ is given in terms of the coordinates of the fixed point and the angles Ψ and A_o by the relation

$$\sin^2\beta = \sin^2\beta_o \cos^2\Psi + 2 \sin\beta_o \cos\beta_o \sin\Psi \cos\Psi \cos A_o \tag{8.23}$$

The integration in (8.21) is performed on the circumscribed sphere with the greater half-axis as radius. Our problem is now solved.

CONCLUSIONS.

We may now write equation (8.21) in the form

$$N^{(o)} = \frac{a}{2\pi\gamma} \int_o^{\pi} \int_o^{2\pi} \Delta g^{*} Z^{(o)}(\psi;\ \beta_o, A_o, \beta, A\ d\psi d\ A_o) \tag{9.1}$$

The function $Z^{(o)}$ is given by the expression

$$Z^{(o)}(\beta_o, \psi, A_o) = F(\psi) + 2a \Big[F_o(\psi) - F_1(\psi) \sin^2 A_o \cos^2 \beta_o$$

$$+ F_2 \psi \cos \beta_o \cos A_o (\sin \psi \sin \beta_o - \cos \beta_o \cos \psi \cos A_o) \Big]$$

$$+ 2q\ F_3(\psi) \tag{9.2}$$

There we have

$$F(\psi) = \frac{1}{2} \sin \psi\ S(\psi) \tag{9.3}$$

$$F_o(\psi) = \frac{1}{2} \sin \psi\ f(\psi) \tag{9.4}$$

$$F_1(\psi) = \frac{1}{2} \sin \psi\ f_1(\psi) \tag{9.5}$$

$$F_2(\psi) = \frac{1}{2} \sin \psi\ f_2(\psi) \tag{9.6}$$

$$F_3(\psi) = \frac{1}{2} \sin \psi\ \ddot{\psi}(\psi) \tag{9.7}$$

$F(\psi)$ is sometimes called Helmert's function. Thus we have as our final formula for $N^{(o)}$:

$$N^{(o)} = \frac{a}{2\pi\gamma} \int_o^{\pi} \int_o^{2\pi} \Delta g \left[1 + a \sin^2 \beta \right] \Big[F(\psi) + 2a \Big\{ F_o(\psi) -$$

$$F_1(\psi) \sin^2 A_o \cos^2 \beta_o + F_2(\psi) \cos A_o \cos \beta_o (\sin \psi \sin \beta_o -$$

$$- \cos A_o \cos \beta_o \cos \psi) \Big\} + 2q\ F_3(\psi) \Big] d\psi dA \tag{9.8}$$

or with the same degree of approximation:

$$N^{(o)} = \frac{a}{2\pi\gamma} \int_{o}^{\pi} \int_{o}^{2\pi} \Delta g \; Z^{(o)}(\beta_o ; \Psi, A_o) \; d\Psi dA_o \tag{9.9}$$

where

$$Z^{(o)}(\beta_o ; \Psi, A_o) = F(\Psi) \left[1 + a \sin^2 \beta \right] + 2a \Big[F_o(\Psi) - F_1(\Psi) \cos^2 \beta_o \sin^2 A_o + \frac{1}{2} F_2(\Psi) \sin 2\beta_o \sin \Psi \cos A_o - F_2(\Psi) \cos^2 \beta_o \cos \Psi \cos^2 A_o + 2q \; F_3(\Psi) \Big] \tag{9.10}$$

Here $\sin^2 \beta$ is given by (8.23) as a function of β_o, Ψ, A. The quantities in (9.9) and (9.10) are defined as follows:

$N^{(o)}$ = the height difference between the geoid and the ellipsoid
a = the greater half-axis of the ellipsoid
q = the quotient between centrifugal force and gravitational force at the equator
γ = the gravity of the theoretical earth
β_o = reduced latitude of the fixed point
Ψ = angular distance between fixed point and running points.

The anomaly Δg in (9.9) is regarded as a function of the reduced latitude and the longitude. The functions F, F_o, F_1, F_2, F_3 occurring in the expression for $Z^{(o)}(\beta_o ; \Psi, A_o)$ are given by the formulas (cf. 9.3 - 9.7, 8.4, 8.5, 8.16 and 8.17).

$$F(\Psi) = \sin \Psi \left\{ \frac{1}{2} \operatorname{cosec} \frac{\Psi}{2} + \frac{1}{2} - 3 \sin \frac{\Psi}{2} - \frac{5}{2} \cos \Psi - \frac{3}{2} \cos \Psi \ln \left(\sin \frac{\Psi}{2} + \sin^2 \frac{\Psi}{2} \right) \right\} \tag{9.11}$$

$$F_o(\Psi) = \sin\Psi\Big[-\frac{1}{3} - \frac{1}{4}\,\mathrm{cosec}\,\frac{\Psi}{2} + \frac{42}{25}\sin\frac{\Psi}{2} + \frac{164}{125}\cos\Psi\cdot$$

$$(\frac{11}{50}\cos\Psi - \frac{3}{10})\ \ln(1+\sin\frac{\Psi}{2}) + (\frac{11}{50}\cos\Psi + \frac{3}{10})\ \ln\sin\frac{\Psi}{2}$$

$$+\frac{3}{10}\cos\Psi\ L(\Psi) + \frac{1}{50}K(\Psi) - \frac{1}{25}\ E(\Psi)\Big] \tag{9.12}$$

$$F_1(\Psi) = \sin\Psi\Big[\Big(\frac{4}{5}\sin\frac{\Psi}{2} + \frac{101}{125}\cos\Psi - \frac{1}{50}\cdot\frac{1}{1+\sin\frac{\Psi}{2}} +$$

$$+\Big(\frac{49}{50}\cos\Psi + \frac{3}{10} + \frac{3}{20}\cdot\frac{1}{\sin^2\frac{\Psi}{2}}\Big)\cdot \mathrm{Ln}\,(1+\sin\frac{\Psi}{2}) - \frac{3}{10}\cos\Psi\ L(\Psi) -$$

$$-\frac{1}{25}\ E(\Psi)\Big] - \frac{3}{10}\cos\frac{\Psi}{2} + \frac{1}{25}\Big[\sin\frac{\Psi}{2}K(\Psi)\Big(\cos\frac{\Psi}{2} + \frac{1}{\cos\frac{\Psi}{2}}\Big) +$$

$$\cot\frac{\Psi}{2} - 2\cdot E(\Psi)\,\frac{1}{\sin\Psi}\Big] + \Big(\frac{49}{25}\cos\frac{\Psi}{2}\cos\Psi - \frac{3}{5}\cos\frac{\Psi}{2} - \frac{3}{10}\,\frac{1}{\cos\frac{\Psi}{2}}\Big)$$

$$\sin\frac{\Psi}{2}\ln\sin\frac{\Psi}{2} \tag{9.13}$$

$$F_2(\Psi) = \frac{16}{125}\sin\Psi - \frac{6}{25}\cos\frac{\Psi}{2} - \frac{1}{25}\ \frac{1}{1+\sin\frac{\Psi}{2}}\ \cot\frac{\Psi}{2} +$$

$$+\Big(\frac{17}{25}\sin\Psi - \frac{3}{10}\cot\frac{\Psi}{2}\Big)\ \ln(1+\sin\frac{\Psi}{2}) + \Big(\frac{17}{25}\sin\Psi -$$

$$-\frac{3}{10}\,\mathrm{tg}\,\frac{\Psi}{2}\Big)\ \ln\sin\frac{\Psi}{2} - \frac{3}{10}\sin\Psi\ L(\Psi) + \frac{1}{25}\,\mathrm{tg}\,\frac{\Psi}{2}K(\Psi) + \frac{2}{25}\cot\Psi\ E(\Psi) \tag{9.14}$$

$$F_3(\Psi) = \frac{1}{2}\sin\Psi\Big[-1 + 2\sin\frac{\Psi}{2} + \cos\Psi + (3-5\cos\Psi)\ \ln\sin\frac{\Psi}{2}$$

$$-(3+5\cos\Psi)\ \ln(1+\sin\frac{\Psi}{2}) + 3\cos\Psi\ L(\Psi)\Big] \tag{9.15}$$

In the last four formulas $K(\psi)$ and $E(\psi)$ are complete elliptic integrals of the first and second kind and of modulus $\cos\frac{\psi}{2}$.

Thus ,

$$K(\psi) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - \cos^2\frac{\psi}{2}\sin^2\varphi}} \tag{9.16}$$

$$E(\psi) = \int_0^{\frac{\pi}{2}} \sqrt{1 - \cos^2\frac{\psi}{2}\sin^2\varphi}\; d\varphi \tag{9.17}$$

$L(\psi)$ is a new function, given by the formula:

$$L(\psi) = \int_0^1 \left(\frac{1}{\sqrt{1 - 2x\cos\psi + x^2}} - \frac{1}{x}\right) dx \tag{9.18}$$

(cf. formulas 7.34 et seq.)

This function can be tabulated (see table below) for some values by the use of its derivative

$$\frac{dL(\psi)}{d\psi} = \operatorname{tg}\frac{\psi}{2}\ln\sin\frac{\psi}{2} - \cot\frac{\psi}{2}\ln\left(1 + \sin\frac{\psi}{2}\right) \tag{9.19}$$

and the fact that $L(o) = \frac{\pi^2}{6}$.

After evaluating $N^{(o)}$ according to formula (9.9) (in principle for every point of the earth) we must compute the correction $N^{(1)}$. This can be done by replacing Δg^* $(= \Delta g + \Delta g \sin^2\beta)$ in (9.8) or (9.9) by $\frac{2N^{(o)}\gamma}{a}\cos^2\beta$ (cf. eq. 3.9 - 3.11 and 1.1).

It will be sufficient to take into consideration just the term $F(\psi)$ from (9.8) or (9.9). As we neglect terms of order α^2 we do not need more than this term of our expression for $\alpha N^{(1)}$.

Thus,

$$N^{(1)} = \frac{a}{2\pi\gamma}\int_0^{\pi}\int_0^{2\pi} \Delta g^{(1)} F(\psi)\, d\psi\, dA \tag{9.20}$$

where

$$\Delta g^{(1)} = \frac{2N^{(o)}\gamma}{a}\cos^2\beta \tag{9.21}$$

We have now solved our main problem, and it remains only to test the results on a suitable model. The solution is given by the formulas (9.9) and (9.20).

$$N = N^{(o)} + a\ N^{(1)} \tag{9.22}$$

The new functions are compared to Sagrebin's functions in a diagram below. A special test of the function φ_2 has been performed by directly summing the series on an electronic computer

$$\varphi_2 = \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} P_n (\cos \psi)$$

for different values of ψ. The results obtained agree completely with those found from the explicit formula (7.33) for $\varphi_2(\psi)$.

THE TEST MODEL.

For a simple test of the resolvent function Z, the following "model earth" is designed (Fig. 10.1).

The theoretical gravity γ is obtained from the requisite massdistribution to make the ellipsoid an equipotential surface. The "formula of Bruns" then holds for γ:

$$-\frac{1}{\gamma}\frac{\partial\gamma}{\partial n} = \left(\frac{1}{\rho_m} + \frac{1}{\rho_n}\right) + \frac{2\omega^2}{\gamma} \tag{10.1}$$

To obtain a disturbance potential T with the gravity centres of the theoretical earth and the actual earth coinciding we put four unit masses into the ellipsoid in the plane $\lambda = 0, 180^o$. Furthermore, we put a negative mass of -4 units in the centre. The radius of the circumscribed sphere and the gravitational constant are taken to be the unity.

Thus

$$T = \sum_{i=1}^{4} \frac{1}{r_{ij}} - \frac{4}{r_{oj}} \tag{10.2}$$

$$\Delta g = -\frac{\partial T}{\partial n} + \frac{1}{\gamma}\frac{\partial\gamma}{\partial n}T \tag{10.3}$$

We know the disturbance potential a priori, and therefore we can simplify the computations to one single step. This means we have to use the boundary condition

$$(1 + q)\frac{2T}{\rho} + \frac{\partial T}{\partial\rho} = -\Delta g^{*} \tag{10.4}$$

where

T = the total disturbance potential

Δg^{*} = the gravity anomaly for our resolvent.

Furthermore we have the relation

$$\Delta g = \Delta g^{*}(1 + 2\alpha \sin^2\beta)^{-\frac{1}{2}} \tag{10.5}$$

Substituting this Δ g into the resolvent formula we obtain

$$T = \frac{1}{2\pi} \int_0^{\pi} \int_0^{2\pi} \Delta g Z \, d\Psi \, dA \tag{10.6}$$

This value should coincide with the value from the Newtonian computation.

The computation of Δ g.

Starting from (10.2) we have:

$$\frac{\partial T}{\partial n} = \sum_{i=1}^{4} \frac{\partial}{\partial n} \frac{1}{r_{ij}} - 4 \frac{\partial}{\partial n} \cdot \frac{1}{r_{oj}} = - \sum_{i=1}^{4} \frac{(\overline{n} \cdot \overline{r}_{ij})}{r_{ij}^3} + 4 \frac{(\overline{n} \cdot \overline{r}_{oj})}{r_{oj}^3} \tag{10.7}$$

Symbols

$\overline{r}_{oj}$	=	the vector from the origin to a point on the ellipsoid
$\overline{r}_{oi}$	=	the vector from the origin to a disturbing mass
r	=	the length of a vector
$\overline{r}_{ij}$	=	$-\overline{r}_{io} + \overline{r}_{jo}$
$\overline{\overline{n}}$	=	the outer normal of the ellipsoid (unit vector).
$\overline{\overline{r}}$	=	unit vector

Thus we have to investigate the quantities $(\overline{n} \cdot \overline{r}_{oi})$ and $(\overline{n} \cdot \overline{r}_{oj})$.

The normal of the ellipsoid is given by the formula

$$\overline{\overline{n}} = \overline{\overline{r}}_{oj} + \alpha \sin 2 \beta_j \overline{\overline{\beta}}_j \tag{10.8}$$

where $\overline{\overline{\beta}}_j$ is the unitvector of the tangent (towards the north pole) of the sphere at the reduced point corresponding to j.

Thus

$$(\overline{\overline{n}} \cdot \overline{r}_{oj}) = r_{oj} + \alpha \sin 2\beta_j (\overline{\overline{\beta}}_j \cdot \overline{r}_{oj}) = r_{oj} + O(\alpha^2) \tag{10.9}$$

as $(\overline{\overline{\beta}}_j \cdot \overline{r}_{oj}) = O(\alpha)$

$$(\overline{\overline{n}} \cdot \overline{r}_{oi}) = (\overline{\overline{r}}_{oj} + \alpha \sin 2\beta_j \overline{\overline{\beta}}_j) \cdot \overline{r}_{oi} = (\overline{\overline{r}}_{oj} \cdot \overline{r}_{oi}) +$$

$$\alpha \, r_{oi} \sin 2\beta_j (\overline{\overline{\beta}}_j \cdot \overline{\overline{r}}_{oi}) \tag{10.10}$$

$$(\bar{\bar{r}}_{oj} \cdot \bar{\bar{r}}_{oi}) = r_{oi} \cos v_{ij} \tag{10.11}$$

where v_{ij} is the angle between the vectors $\bar{r}_{oi}$ and $\bar{r}_{oj}$.

The quantity $(\bar{\bar{\beta}}_j \cdot \bar{\bar{r}}_{oi})$ remains to be found. In Cartesian coordinates (x, y, z) we have

$$\bar{\bar{r}}_{oi} = \bar{\bar{R}}_i \cos\beta_i + \bar{\bar{z}} \sin\beta_i + O(a) \tag{10.12}$$

$$\bar{\bar{\beta}}_j = -\bar{\bar{R}}_j \sin\beta_j + \bar{\bar{z}} \cos\beta_j + O(a) \tag{10.13}$$

Here

$$\bar{\bar{R}} = \bar{\bar{x}} \cos\lambda + \bar{\bar{y}} \sin\lambda . \tag{10.14}$$

$$(\bar{\bar{\beta}}_j \cdot \bar{\bar{r}}_{oi}) = -\cos\beta_i \sin\beta_j \cos(\lambda_j - \lambda_i) + \sin\beta_i \cos\beta_j + O(a)$$

We do not want the quantities β_j and λ_j to enter into the final result, since we integrate over Ψ and A $(= A_{jk})$. Here β_k, λ_k and β_i, λ_i are given beforehand and may enter explicitly into the formulas for Δg and T.

We have the formulas

$$\sin\Psi_{jk} \sin A_{jk} = \cos\beta_j (\sin\lambda_j \cos\lambda_k - \sin\lambda_k \cos\lambda_j) \tag{10.15}$$

$$\cos\Psi_{jk} = \sin\beta_j \sin\beta_k + \cos\beta_j \cos\beta_k (\cos\lambda_j \cos\lambda_k + \sin\lambda_j \sin\lambda_k) \tag{10.16}$$

(cf. eq. (0.9) and (0.11)).

From the equations (10.15) and (10.16) we can solve for the quantities

$$\cos\beta_j \cos\lambda_j \ (= c_j) \quad \text{and} \quad \cos\beta_j \sin\lambda_j \ (= s_j).$$

$$c_j = \frac{\cos\Psi_{jk} - \sin\beta_k \sin\beta_j}{\cos\beta_k} \cos\lambda_k - \sin\Psi_{jk} \sin A_{jk} \sin\lambda_k \tag{10.17}$$

$$s_j = \frac{\cos\Psi_{jk} - \sin\beta_k \sin\beta_j}{\cos\beta_k} \sin\lambda_k + \sin\Psi_{jk} \sin A_{jk} \cos\lambda_k \qquad (10.18)$$

$A_{jk} = A$ and $\Psi_{jk} = \Psi$ are the integration variables.

Furthermore

$$\frac{\cos\Psi - \sin\beta_k \sin\beta_j}{\cos\beta_k} \equiv \frac{\sin\Psi\cos A - \sin\beta_j \cos\beta_k}{\sin\beta_k}$$

(cf. (8.10) and (8.11))

This expression can be used when $\cos\beta_k$ approaches zero.

By permutating the angles

$$90^\circ - \beta_j \rightarrow \Psi \rightarrow 90^\circ - \beta_k$$

$\lambda - \lambda_k \rightarrow A$ in (10.16) we obtain

$$\sin\beta_j = \sin\beta_k \cos\Psi + \cos\beta_k \sin\Psi\cos A \qquad (10.19)$$

Since

$-90^\circ < \beta_j < 90^\circ$ we have simply

$$\cos\beta_j = \sqrt{1 - \sin^2\beta_j} \quad \text{(always positive)} \qquad (10.20)$$

The "reduced" angle Ψ_{ij} between the vectors $\bar{r}_{oi}$ and $\bar{r}_{oj}$ is given by

$$\cos\Psi_{ij} = \sin\beta_i \sin\beta_j + \cos\beta_i (c_j \cos\lambda_i + s_j \sin\lambda_i) \qquad (10.21)$$

To obtain the physical angle v_{ij} in (10.11) we have to replace β_i by

$(\beta_i - \frac{a}{2}\sin 2\beta_i)$ and analogously for β_j.

Thus by series expansion

$$\cos v_{ij} = \cos\Psi_{ij}(1 + a(\sin^2\beta_i + \sin^2\beta_j)) - 2a\sin\beta_i \sin\beta_j \qquad (10.22)$$

Expression (10.15) reduces to

$$(\bar{\bar{\beta}}_j \cdot \bar{\bar{r}}_{oi}) = -\frac{\cos\beta_i \sin\beta_j}{\cos\beta_j}(c_j\cos\lambda_i + s_j\sin\lambda_i) + \sin\beta_i\cos\beta_j$$

Thus

$$a\sin 2\beta_j(\bar{\bar{\beta}}_j \cdot \bar{\bar{r}}_{oi}) = 2a\sin\beta_j(\sin\beta_i\cos^2\beta_j - \cos\beta_i\sin\beta_j$$

$$(c_j\cos\lambda_i + s_j\sin\lambda_i)) \qquad (10.23)$$

We have also

$$r_y^2 = r_{oi}^2 + r_{oj}^2 - 2r_{oi}\, r_{oj}\cos v_{ij} \qquad (10.24)$$

From the equation of the ellipsoid we have

$$r_{oj}^2 = 1 - 2a\sin^2\beta_j \qquad (10.25)$$

$$x^2 + y^2 + \frac{z^2}{1-e^2} = 1$$

(z axis along the axis of rotation)

where e is the first eccentricity of the ellipsoid.

$$x^2 + y^2 + z^2 = 1 - 2a\, z^2 + O(a^2) \qquad z = \sin\beta_j + O(a)$$

Now we can write (cf. 10.7)

$$\frac{\partial T}{\partial n} = -\bar{n}\left(\sum_{i=1}^{4}\frac{\bar{r}_{ij}}{r_{ij}^3} - 4\frac{\bar{r}_{oj}}{r_{oj}^3}\right) = \frac{4}{r_{oj}^2} - \sum_{i=1}^{4}\frac{1}{r_{ij}^3}\Big(r_{oj} - r_{oi}\cos v_{ij} -$$

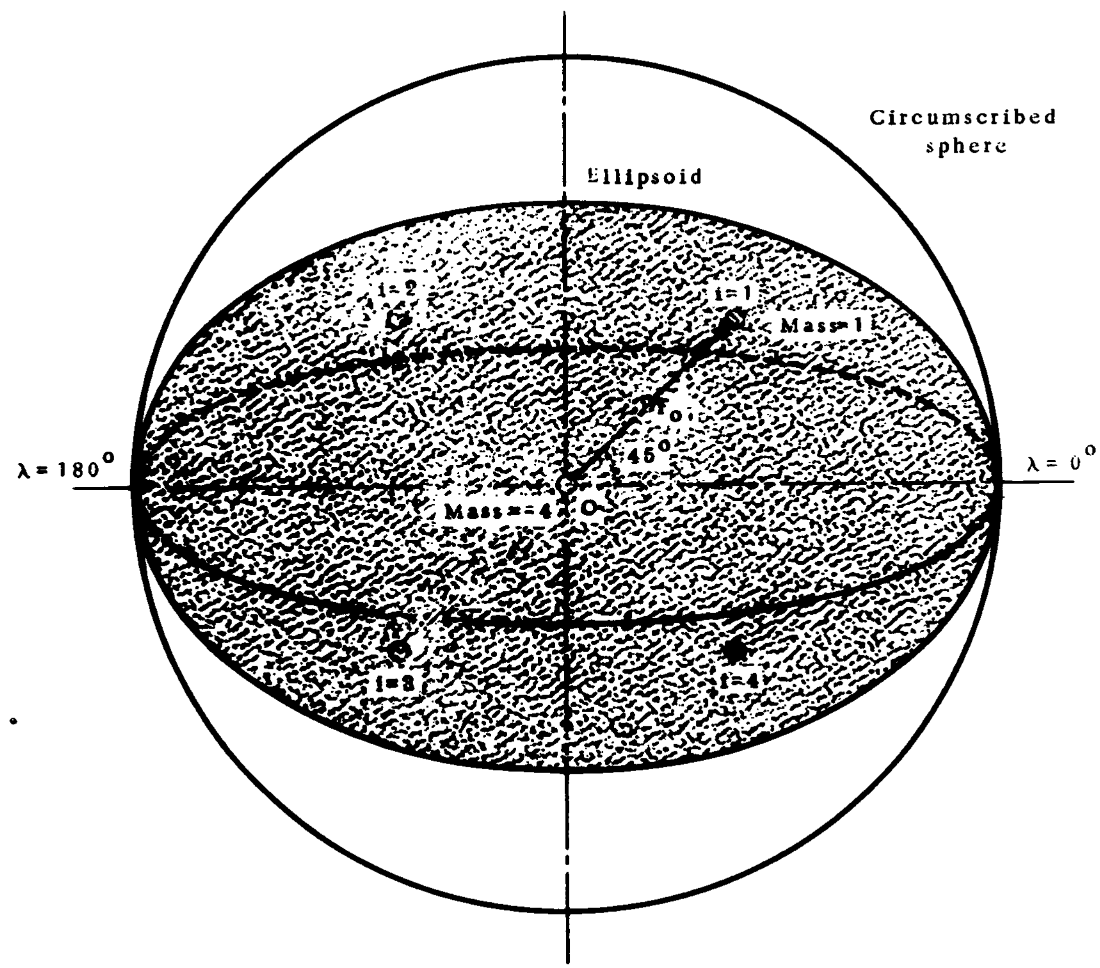
$$2a\sin\beta_j(\sin\beta_i\cos^2\beta_j - \cos\beta_i\sin\beta_j\, d_{ji})\Big) \qquad (10.26)$$

where

$$d_{ji} = c_j\cos\lambda_i + s_j\sin\lambda_i$$

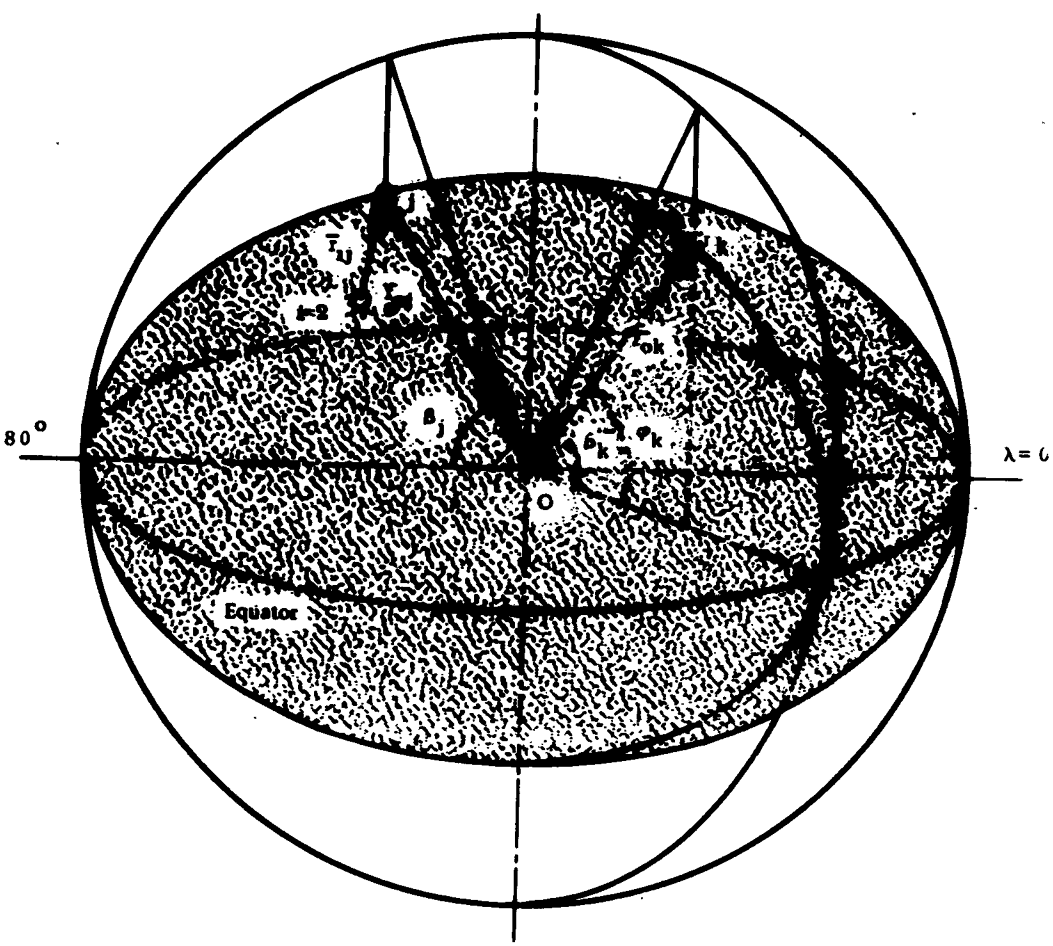
The formulas given here contain all the information necessary to test the modified Sagrebin resolvent by the aid of an electronic computer.

MASS ANOMALIES.



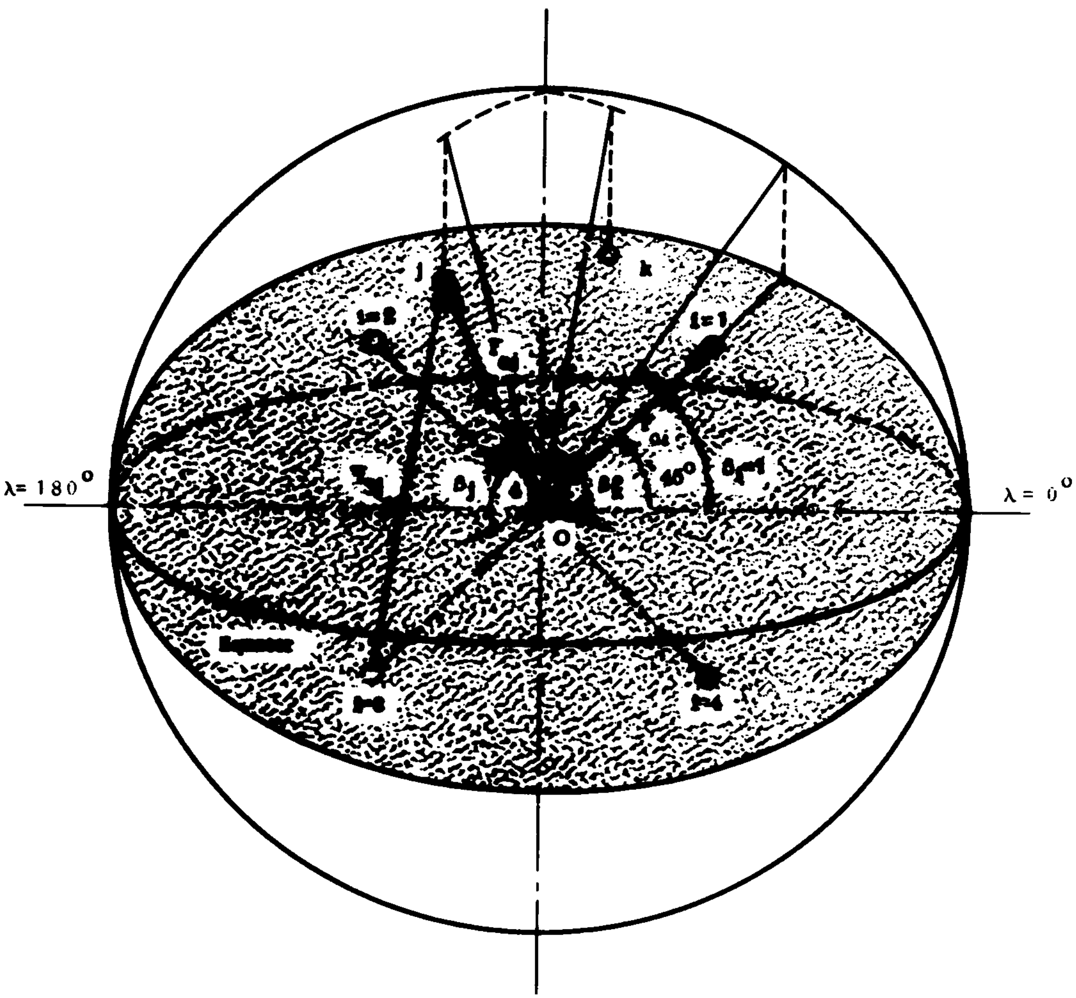
The position of the five pointmasses in the test model. The central mass is equal to -4 and the remaining masses are equal to +1.

RELATION BETWEEN POINTS ON THE ELLIPSOID AND THE CIRCUMSCRIBED SPHERE.



β_j, β_k are reduced latitudes.

THE PRINCIPAL VECTORS USED IN THE TEST MODEL.



k = The fixed point.

j = The running point.

The points i = 1, 2, 3, 4 are placed in the corners of a square in the plane $\lambda = 0^{\circ}$, 180°, and carry a mass of one unit each. The central point O carries mass equal to -4.

RESULTS FROM THE TEST MODEL.

Test N:o 1

$\alpha = 0$ $\beta = 60^{o}$ $\lambda = 15^{o}$

Compartment: $d\beta = 15^{o}$ $d\lambda = 15^{o}$

Newtonian T = 0.81870

Resolvent T = 0.84866 (Stokes)

Error +0.02996

Test N:o 2

$\alpha = 0$ $\beta = 60^{o}$ $\lambda = 15^{o}$

Compartment: $d\beta = 5^{o}$ $d\lambda = 5^{o}$

Newtonian T = 0.81870

Resolvent T = 0.81790 (Stokes)

Error -0.00080

Test N:o 3

$\alpha = 0.001$ $\beta = 60^{o}$ $\lambda = 15^{o}$

Compartment: $d\beta = 15^{o}$ $d\lambda = 15^{o}$

Newtonian T = 0.81962

Resolvent T = 0.85102

Error +0.03140

Test N:o 4

$\alpha = 0.001$ $\beta = 60^{o}$ $\lambda = 15^{o}$

Compartment: $d\beta = 5^{o}$ $d\lambda = 5^{o}$

Newtonian T = 0.81962

Resolvent T = 0.81998

Error +0.00036

Conclusions from numerical tests.

It has been possible to verify the principal formula $\varphi_2(x)$ by the aid of a numerical series expansion on an electronic computer. This calculation has proofed that the new derivation is correct. The final resolvents includes a number of functions which all are derived from this principal formula. The whole assembly of formulas is verified by the aid of the test model. The study of this model has shown that error in the computations is decreasing with approximately forth power of size of the integration compartments. Already with a compartment of $1^o \times 1^o$ the resolvent solution will fit better than 10^{-5} of T. The new resolvent for the ellipsoid has given an error which was only half of that corresponding to the same solution according to Stokes for a real sphere ($5^o \times 5^o$). One can expect both formulas to have approximately the same accuracy.

COMPARISON BETWEEN SAGREBIN'S FUNCTIONS AND THE NEW FUNCTIONS.

Sagrebin:

$$F_0(\Psi) = \sin\Psi\left(\frac{4}{15} - \frac{1}{4}\operatorname{cosec}\frac{\Psi}{2} + \frac{12}{25}\sin\frac{\Psi}{2} + \frac{253}{250}\cos\Psi + \right.$$

$$+ \frac{3}{10}\ln\left(\sin\frac{\Psi}{2} + \sin^2\frac{\Psi}{2}\right) + \frac{11}{50}\cos\Psi\ln\left(\sin\frac{\Psi}{2} + \sin^2\frac{\Psi}{2}\right)$$

$$+ \frac{1}{50}K(\Psi) - \frac{1}{25}E(\Psi)$$

Corrected $F_0(\Psi)$ function:

$$F_0(\Psi) = \sin\Psi\left(-\frac{1}{3} - \frac{1}{4}\operatorname{cosec}\frac{\Psi}{2} + \frac{42}{25}\sin\frac{\Psi}{2} + \right.$$

$$+ \frac{164}{125}\cos\Psi \cdot \left(\frac{11}{50}\cos\Psi - \frac{3}{10}\right)\ln\left(1 + \sin\frac{\Psi}{2}\right) +$$

$$+ \left(\frac{11}{50}\cos\Psi + \frac{3}{10}\right)\ln\sin\frac{\Psi}{2} + \frac{3}{10}\cos\Psi\, L(\Psi) + \frac{1}{50}K(\Psi) - \frac{1}{25}E(\Psi)$$

In a similar way the three remaining functions $F_1(\Psi)$ $F_2(\Psi)$ and $F_3(\Psi)$ are changed.

Sagrebin:

$$\varphi_2(\Psi) = 1 - 2\sin\frac{\Psi}{2} + 2\sin^2\frac{\Psi}{2}\ln\left(\sin\frac{\Psi}{2} + \sin^2\frac{\Psi}{2}\right)$$

Correct function:

$$\varphi_2(\Psi) = -1 + \cos\Psi + 2\sin\frac{\Psi}{2} + (1-\cos\Psi)\ln\sin\frac{\Psi}{2} -$$

$$- (1 + \cos\Psi)\ln\left(1 + \sin\frac{\Psi}{2}\right) - \cos\Psi\, J_1^{(2)}$$

where

$$J_1^{(2)} = \int_0^1 \frac{1}{x}\ln\left|\frac{1}{2}\left(1 - \omega x + r(x)\right)\right| dx = \int_1^\infty \left(\frac{1}{r(x)} - \frac{1}{x}\right)\ln x\, dx$$

etc.

From a theoretical point of view the new results seem to be of the greatest importance since hitherto modern geodesy is based on the fact that the errors in the generally used Stokes approach could be determined according to Pizetti and then finally according to Sagrebin. However, Sagrebin has shown that Pizetti's method is insufficient to give any real information. Sagrebin's own theoretical study is probably one of the most important in modern geodesy, in spite of the fact that it is incorrect in all new functions.

It is supposed that some of the fundamental questions in geodesy can be solved by the aid of this new study.

TABLE OF FUNCTIONS (15°)

X	F(X)	F0(X)	F1(X)	F2(X)	F3(X)
0°	1.0000	-.5000	.0000	-.5000	.0000
15	1.1121	-.3618	-.1031	-.7901	.8928
30	.4736	-.0683	.1625	-.7924	.7923
45	-.3070	.1793	.5495	-.6429	.1359
60	-.8957	.3103	.8677	-.4053	-.6297
75	-1.1049	.3137	.9841	-.1365	-1.1127
90	-.9142	.2197	.8598	.1128	-1.1150
105	-.4500	.0808	.5531	.3025	-.6825
120	.0773	-.0487	.1872	.4066	-.0528
135	.4578	-.1281	-.1034	.4152	.4705
150	.5590	-.1391	-.2275	.3347	.6590
165	.3702	-.0879	-.1713	.1859	.4574
180	.0000	.0000	.0000	.0000	.0000

$$F(\psi) = \frac{1}{2}\sin\psi\, S(\psi) = \frac{1}{2}\sin\psi\, \frac{1}{4\pi}\sum_{n=2}^{\infty}\frac{2n+1}{n-1}P_n(\cos\psi)$$

$$F_0(\psi) = \frac{1}{2}\sin\psi\, f(\psi) = \frac{1}{2}\sin\psi\,(-\phi_1(\psi) + 2\,\phi_2(\psi)) =$$

$$\frac{1}{2}\sin\psi\left[-\sum_{n=2}^{\infty}\frac{n^2(2n+1)}{(n-1)^2(2n+3)}P_n(\cos\psi) + 2\sum_{n=2}^{\infty}\frac{(2n+1)}{(n-1)^2(2n+3)}P_n(\cos\psi)\right]$$

$$F_1(\psi) = \frac{1}{2}\sin\psi\, f_1(\psi) = \frac{1}{2}\sin\psi\,\frac{d^2\phi_2(\psi)}{d\psi^2}$$

$$F_2(\psi) = \frac{1}{2}\sin\psi\, f_2(\psi) = \frac{1}{2}\frac{d\,\phi(\psi)}{d\psi}$$

$$F_3(\psi) = \frac{1}{2}\sin\psi\,\Psi(\psi) = \frac{1}{2}\sin\psi\sum_{n=2}^{\infty}\frac{(2n+1)}{(n-1)^2}P_n(\cos\psi).$$

TABLE OF FUNCTIONS (5°)

X	F(X)	F0(X)	F1(X)	F2(X)	F3(X)
0°	1.0000	-.5000	.0000	-.5000	.0000
5	1.2165	-.5055	-.1071	-.6567	.4613
10	1.2146	-.4456	-.1288	-.7409	.7364
15	1.1121	-.3618	-.1031	-.7901	.8928
20	.9410	-.2663	-.0407	-.8123	.9460
25	.7224	-.1666	.0504	-.8120	.9080
30	.4736	-.0683	.1625	-.7924	.7923
35	.2100	.0242	.2881	-.7561	.6140
40	-.0543	.1076	.4195	-.7055	.3894
45	-.3070	.1793	.5495	-.6429	.1359
50	-.5371	.2375	.6715	-.5706	-.1292
55	-.7357	.2813	.7792	-.4907	-.3895
60	-.8957	.3103	.8677	-.4053	-.6297
65	-1.0121	.3248	.9330	-.3165	-.8369
70	-1.0820	.3255	.9723	-.2262	-1.0004
75	-1.1049	.3137	.9841	-.1365	-1.1127
80	-1.0821	.2908	.9684	-.0490	-1.1696
85	-1.0169	.2588	.9262	.0346	-1.1696
90	-.9142	.2197	.8598	.1128	-1.1150
95	-.7804	.1756	.7726	.1843	-1.0103
100	-.6230	.1286	.6688	.2478	-.8630
105	-.4500	.0808	.5531	.3025	-.6825
110	-.2700	.0342	.4307	.3475	-.4794
115	-.0916	-.0095	.3070	.3823	-.2656
120	.0773	-.0487	.1872	.4066	-.0528
125	.2292	-.0822	.0761	.4201	.1475
130	.3577	-.1088	-.0219	.4228	.3249
135	.4578	-.1281	-.1034	.4152	.4705
140	.5258	-.1396	-.1657	.3975	.5773
145	.5596	-.1432	-.2073	.3704	.6409
150	.5590	-.1391	-.2275	.3347	.6590
155	.5251	-.1281	-.2271	.2913	.6320
160	.4608	-.1106	-.2075	.2413	.5631
165	.3702	-.0879	-.1713	.1859	.4574
170	.2589	-.0610	-.1219	.1263	.3222
175	.1331	-.0312	-.0633	.0638	.1664
180	.0000	.0000	.0000	.0000	.0000

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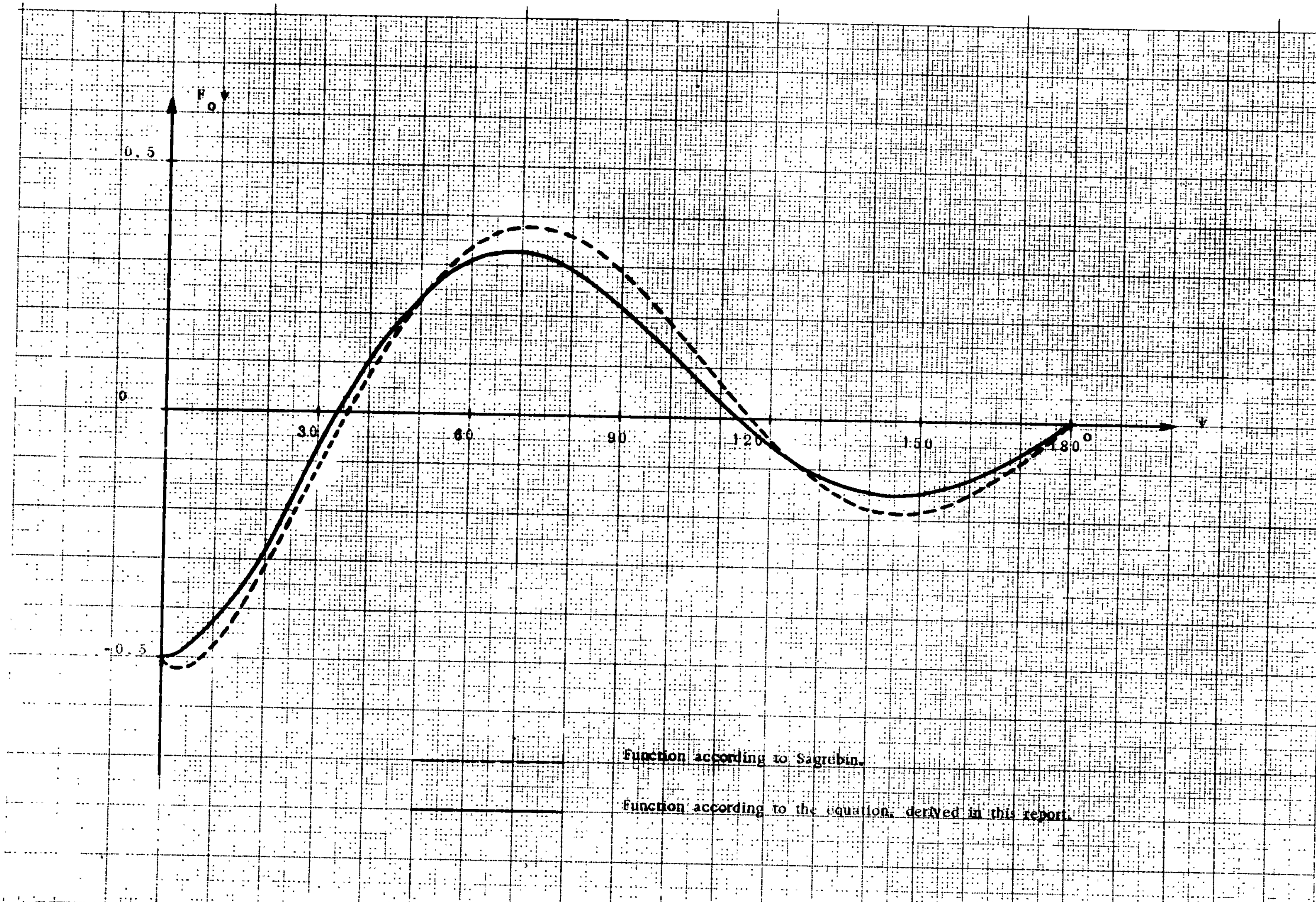
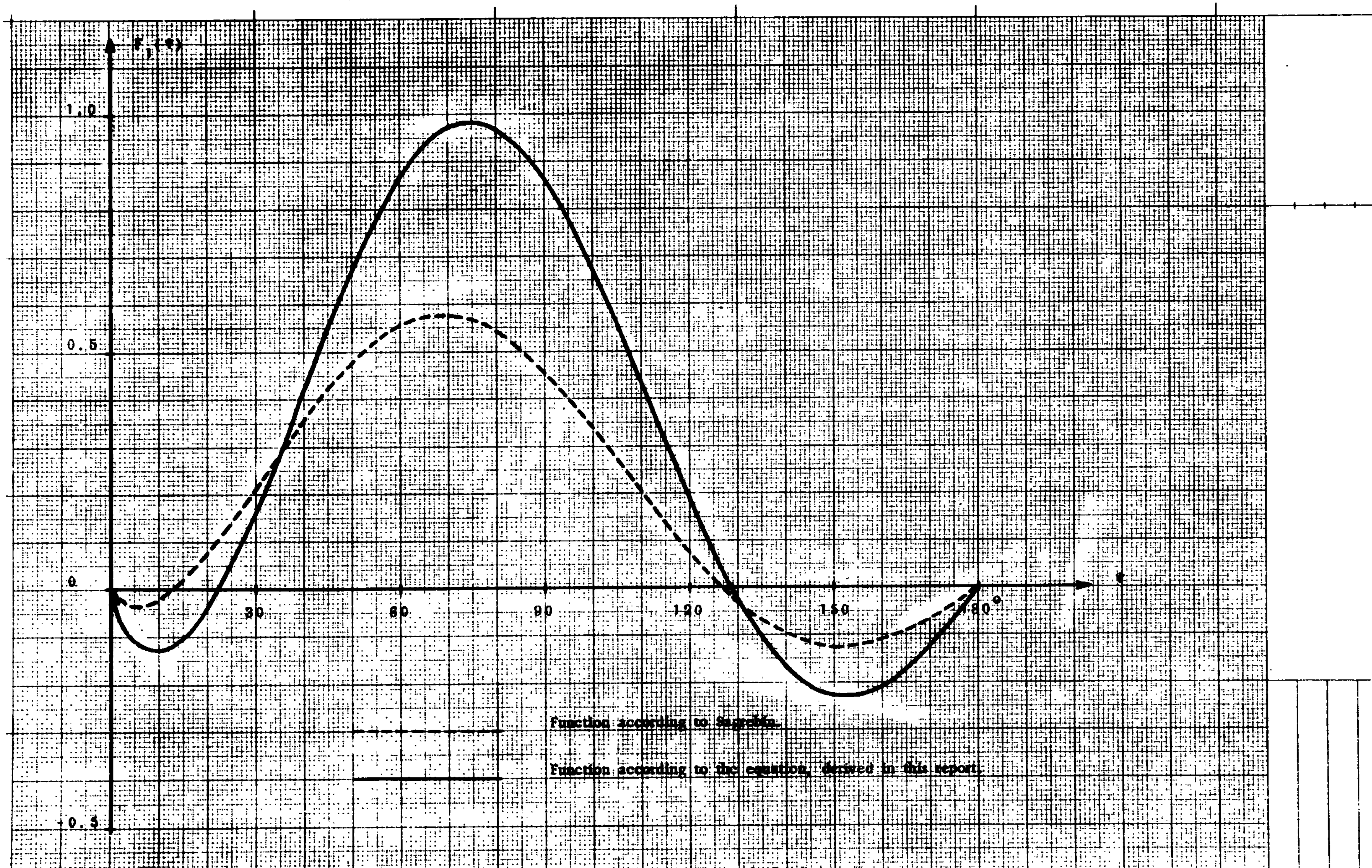
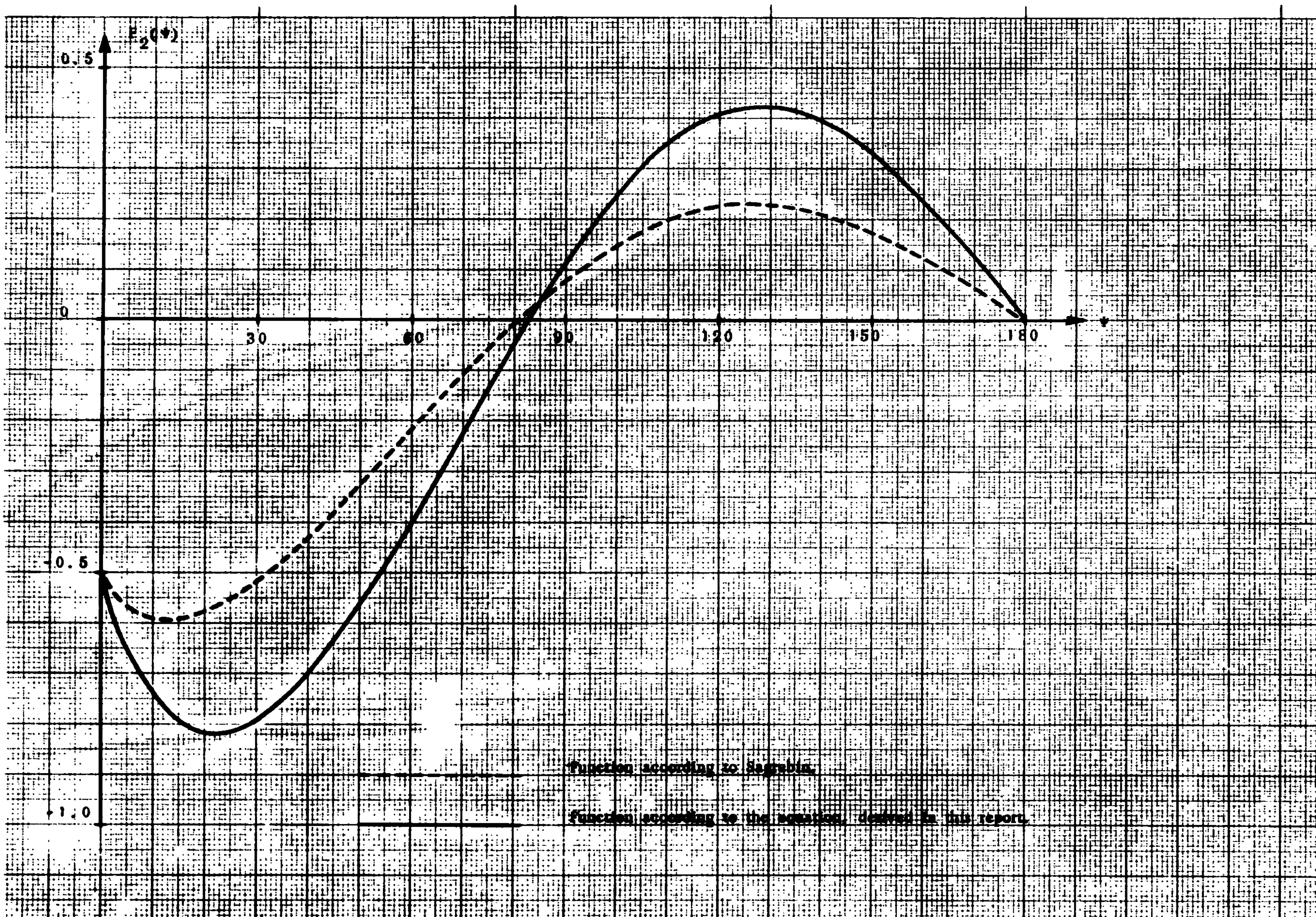
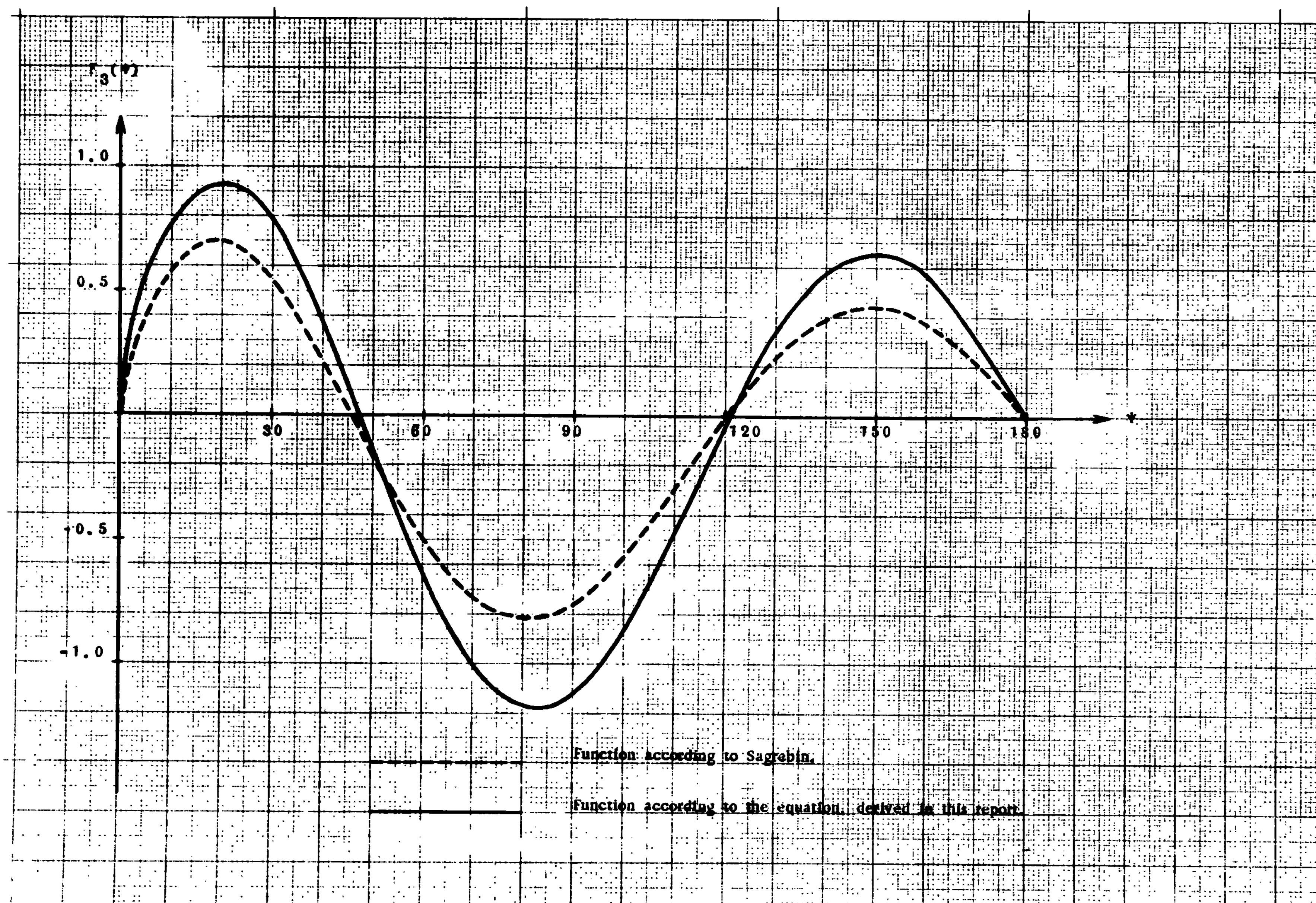


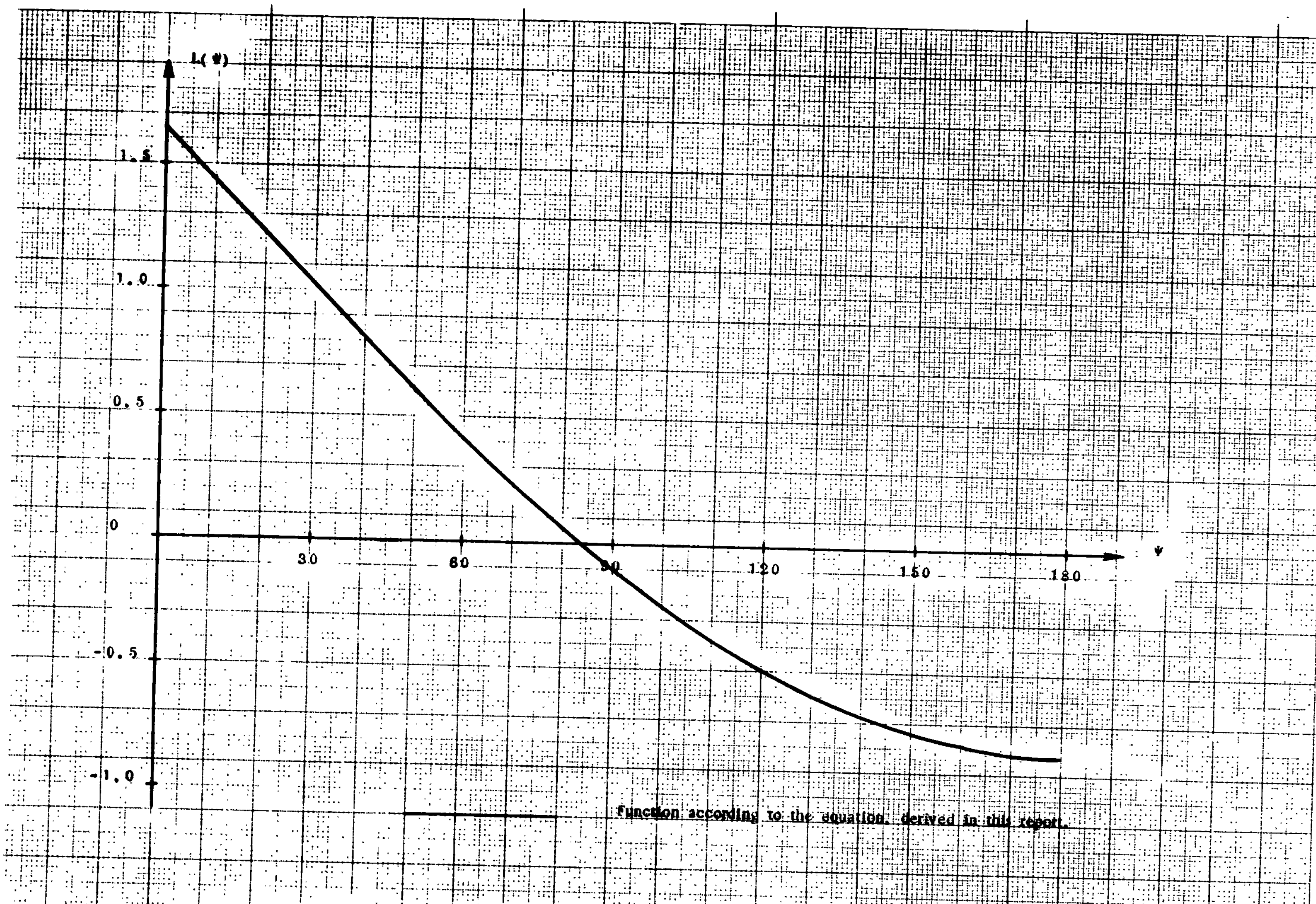


$F_1(\varphi)$
1.0
0.5
0
-0.5
30
60
90
120
150
180°
φ
Function according to Sagrebin.
Function according to the equation, derived in this report.


$F_2(\psi)$
0.5
0
-0.5
-1.0
30
60
90
120
150
180
ψ
Function according to Sagrebin.
Function according to the equation, derived in this report.



$F_3(\varphi)$
1.0
0.5
-0.5
-1.0
30
60
90
120
150
180
φ
Function according to Sagrebin.
Function according to the equation, derived in this report.



L(ψ)
1.5
1.0
0.5
0
-0.5
-1.0
30
60
90
120
150
180
ψ
Function according to the equation, derived in this report.

A P P E N D I X .

TABLE FOR THE RESOLVENT OF AN ELLIPSOIDAL SURFACE OF REFERENCE.

The table can be used for determining the height over the geoid in the following manner.

We start with formula

$$N^{(o)} = \frac{a}{2\pi\gamma} \int_0^{\pi} \int_0^{2\pi} \Delta g \left[1 + \frac{i^2}{2}(\sin^2\beta_o \cos^2\psi + 2\sin\beta_o \cos\beta_o \sin\psi\cos\psi \cos A + \cos^2\beta_o \sin^2\psi \cos^2 A)\right] (F(\psi) + i^2 F_o(\psi) - i^2 F_1(\psi)\sin^2 A\cos^2\beta_o + i^2 F_2(\psi)\cos A\cos\beta_o \sin\psi\sin\beta_o - i^2 F_2(\psi)\cos^2 A\cos^2\beta_o \cos\psi + 2q\, F_3(\psi))\, d\psi dA \qquad [13.0]$$

This formula is transcribed

$$N^{(o)} = C \int\int \Delta g \left[1 + \mathrm{corr}_1\right] \left(\sum_{v=-1}^{3} a_v F_v(\psi)\right) dS \qquad [13.1]$$

where

$$\mathrm{corr}_1 = O(i^2)$$

$$F_{-1}(\psi) = F(\psi) \quad \text{def.}$$

$$a_v = \begin{cases} O(i^2) & v = 0, 1, 2, 3 \\ 1 & v = -1 \end{cases}$$

Every a_v is a function of

β_o = reduced latitude for the fixed point,

ψ = angle between the fixed and the running point

A = azimuth for the running point (measured in the orthonormal system determined by the fixed point).

We substitute now for [13.1] the equivalent:

$$N^{(o)} = C \iint \Delta g \ F(\psi) \left[\frac{\Sigma\ a_v F_v}{F(\psi)}\right] (1 + corr_1)\ dS \qquad [13.2]$$

where

$F(\psi) = F_{-1}(\psi)$ is independant of the position of the fixed point. The factor $(1 + corr_1)\ \Sigma\,(a_v F_v) : F(\psi) = 1 + G(\beta_o;\ \psi,\ A) \cdot O\ (i^2)$ is near to 1 for any point on the ellipsoid and equals 1 for every point on a spherical surface. This fact enables us to use the following approximation formula for $N^{(o)}$

$$N^{(o)} = C \sum_i \sum_j \Delta g_{ij}\ F(\psi_i) \cdot K(\beta_o;\ \psi_i,\ A_{ij})\ \Delta\psi_i\,\Delta A_{ij} \qquad [13.3]$$

where the magnitude of K in accordance with our considerations above is near to 1.

Now an appropriate network for ψ_i and A_{ij} is defined and we have in our example chosen:

$$\psi_i = i \cdot 5^o \qquad i = 0, 1, \ldots .36$$
$$A_{ij} = j \cdot 10^o \qquad j = 0, 1, \ldots .36 \text{ for every } i$$

As reduced latitude for the fixed point we have taken $\beta_o = +30^o$. Now we have computed the values of $K(30;\ \psi_i, A_{ij})$ headed under ψ_i, furthermore the reduced latitude, β_{ij}, for $(\psi_i,\ A_{ij})$ in the orthonormal system determined by the axis of the ellipsoid and the difference in azimuth, $l_{ij} - l_o$, between the fixed and the running points in the same system.

The system of equations for the coordinate transformation is:

$$\cos\beta_{ij}\ \cos(l_{ij} - l_o) = \cos\psi_i \cos\beta_o + \sin\psi_i \cos A_{ij} \sin\beta_o$$

$$\cos\beta_{ij}\ \sin(l_{ij} - l_o) = \sin\psi_i \sin A_{ij}$$

$$\sin\beta_{ij} = \cos\psi_i \sin\beta_o - \sin\psi_i \cos A_{ij} \cos\beta_o$$

The table is constructed so that for any point where Δg is given one can get an approximate value of the K-function. We have then to multiply K with its corresponding value of $F(\psi)$ and with that element of surface, that the actual point is intended to represent, to get the actual partial sum of [13.3]. (Note that $F(\psi) = \frac{1}{2} \sin\psi\ S(\psi)$, where $S(\psi)$ is Stokes' function, and that therefore "the element of surface" lacks the $\sin\psi_i$- factor).

Example of tabulated resolvent values according to (13.3).

BETA = Reduced latitude.   L-L(O) = Longitude difference.

PSI = 5.0

BETA	L-L(O)	$Z(0)/F(0) = K(\beta_o, \psi_i, A_{ij})$
25.00°	.00°	1.0035
25.07	.96	1.0035
25.29	1.89	1.0032
25.64	2.77	1.0029
26.12	3.58	1.0025
26.72	4.29	1.0021
27.41	4.88	1.0017
28.18	5.33	1.0014
29.01	5.63	1.0012
29.87	5.77	1.0011
30.74	5.73	1.0012
31.60	5.52	1.0013
32.40	5.13	1.0016
33.14	4.57	1.0020
33.77	3.86	1.0024
34.30	3.02	1.0027
34.68	2.08	1.0030
34.92	1.06	1.0032
35.00	360.00	1.0033
34.92	358.94	1.0032
34.68	357.92	1.0030
34.30	356.98	1.0027
33.77	356.14	1.0024
33.14	355.43	1.0020
32.40	354.87	1.0016
31.60	354.48	1.0013
30.74	354.27	1.0012
29.87	354.23	1.0011
29.01	354.37	1.0012
28.18	354.67	1.0014
27.41	355.12	1.0017
26.72	355.71	1.0021
26.12	356.42	1.0025
25.64	357.23	1.0029
25.29	358.11	1.0032
25.07	359.04	1.0035
25.00	.00	1.0035

PSI= 10.0

BETA	L-L(0)	Z(0)/F(0)
20.00	.00	1.0059
20.14	1.84	1.0058
20.55	3.64	1.0056
21.23	5.34	1.0052
22.16	6.92	1.0047
23.31	8.33	1.0043
24.66	9.52	1.0038
26.17	10.48	1.0035
27.79	11.15	1.0032
29.50	11.51	1.0031
31.23	11.54	1.0031
32.95	11.21	1.0033
34.58	10.52	1.0036
36.09	9.47	1.0040
37.42	8.08	1.0045
38.51	6.37	1.0049
39.32	4.40	1.0052
39.83	2.25	1.0054
40.00	360.00	1.0055
39.83	357.75	1.0054
39.32	355.60	1.0052
38.51	353.63	1.0049
37.42	351.92	1.0045
36.09	350.53	1.0040
34.58	349.48	1.0036
32.95	348.79	1.0033
31.23	348.46	1.0031
29.50	348.49	1.0031
27.79	348.85	1.0032
26.17	349.52	1.0035
24.66	350.48	1.0038
23.31	351.67	1.0043
22.16	353.08	1.0047
21.23	354.66	1.0052
20.55	356.36	1.0056
20.14	358.16	1.0058
20.00	.00	1.0059

PSI= 15.0	BETA	L-L(O)	Z(O)/F(O)
	15.00	.00	1.0080
	15.20	2.67	1.0079
	15.80	5.28	1.0076
	16.79	7.77	1.0072
	18.14	10.08	1.0066
	19.81	12.17	1.0061
	21.77	13.97	1.0055
	23.97	15.44	1.0051
	26.36	16.53	1.0048
	28.88	17.19	1.0046
	31.46	17.39	1.0047
	34.03	17.07	1.0049
	36.52	16.19	1.0053
	38.83	14.75	1.0058
	40.89	12.71	1.0063
	42.62	10.13	1.0068
	43.91	7.06	1.0073
	44.72	3.63	1.0075
	45.00	360.00	1.0076
	44.72	356.37	1.0075
	43.91	352.94	1.0073
	42.62	349.87	1.0068
	40.89	347.29	1.0063
	38.83	345.25	1.0058
	36.52	343.81	1.0053
	34.03	342.93	1.0049
	31.46	342.61	1.0047
	28.88	342.81	1.0046
	26.36	343.47	1.0048
	23.97	344.56	1.0051
	21.77	346.03	1.0055
	19.81	347.83	1.0061
	18.14	349.92	1.0066
	16.79	352.23	1.0072
	15.80	354.72	1.0076
	15.20	357.33	1.0079
	15.00	.00	1.0080

GRAVITATION

Historical

The study of gravitation has attracted an exceptional amount of interest in recent years. The foremost reason for this is found in the rapid development which is occuring at present in the disciplines connected with geophysics. The problems connected with satellites and robot weapons have accented this interest further. The general manner of development is briefly as follows.

In Sweden and the other "western countries" developments have previously followed the classical representation of STOKES. In the application of this theory one has sought to obtain gravity material by direct measurement of gravity in various places over the whole earth which then was applied to determine the so called geoid. The auxilliary surface can be said to constitute the gravitations model earth. If the continents were cut through by a network of canals, the mean water surface thus obtained would define the geoid. One difficulty with the application of STOKES' formula for the determination of the geoid is that no masses can lie outside the actual geoid. In the practical application of STOKES' theories one is therefore forced to eliminate by some artifice the masses which lie outside the geoid. Western scientists have not been successful in solving this problem. At the same time, many of the works which have been carried out toward this goal will certainly be significant in various other connections even if they do not succeed in the solution of the main problem. To correctly understand the older manner of consideration it can be useful to give the historical development. In the triangle measurements in India, J.H. PRATT in 1855 found large triangle misclosures which were measured on the surface of the earth in the vicinity of the Himalaya mountains. From this he drew the conclusion that the large masses of mountains had deep "roots" with less density than the surrounding area so that the visible outer masses were completely compensated at a lower level. In such a manner he could assume that two equally large land areas always contain masses of equal size. The assumption for this is only that the two volumes compared include all the masses down to the earths inner surface of compensation. The hypotheses presented by PRATT have led to a special discipline which is called "isostasy". Of great significance for the application of isostatic hypotheses was the determination of the "depth of compensation" which is required to obtain equilibrium among the masses. On this point views differ. In his own theories

PRATT assumed that the depth of compensation was everywhere constant and many writers said that this depth should be measured from the unknown surface of the geoid. The American JOHN F. HAYFORD was the first to apply PRATT's theories to a large project and he chose to measure the depth of compensation from the physical surface of the earth which introduced the complication that the surface thus determined was not in hydrostatic equilibrium. Since, however, many scientists considered it to be a necessary condition that the surface of compensation should be in hydrostatic equilibrium, they chose to reckon the depth of compensation from the geoid. It is interesting, however, that to a large degree the schools obtained the same result: namely that the depth of compensation is approximately 120 km. At the same time as PRATT G. B. AIRY presented a somewhat more detailed hypothesis concerning the interior of the earth. AIRY also accepts in principle the isostatic concept, but makes it more precise by giving the outer body of the earth (sial), a density of 2.67, which floats on the inner layer (sima) which has a density of 3.30. The continents should then more or less float like icebergs on the interior masses of the earth. If these theories were correct one could find in them a good basis for the ideas which were later presented by the German, A. L. WEGENER. According to him the South American and African continents were at one time a single continent, but have since then separated. This hypothesis of the movements of continents has been much discussed, but no successful geodetic measurements have yet been carried out which could prove or disprove the hypothesis. We do not know the distance at the present time than to approximately several hundred meters and the movements which are supposed to occur probably are essentially less. Therefore it will be some time before this theory can be checked, if in fact this will ever be possible. The actual significance of isostasy is that it offers the possibility of reduction of the disturbance effects of masses lying outside the geoid. A disadvantage of the isostatic reduction method is that it is very tedious, but in spite of this isotatic methods have come into wide use. For example, in Sweden, Rikets Allmänna Kartverk has performed a computation of a suitable geoid for Sweden with the aid of isostatic reduction methods.

Among other theories for the elimination of masses outside the geoid one put forward by the Russian scientist M. P. RUDZKY in 1905 can be cited. He sought to eliminate the disturbing masses by an imaginary transportation of masses of all the material outside the geoid. In this original manner RUDZKY obtained a model earth which he considered to be free of disturbing effects. A number of other theories followed and the state had almost been reached in 1950 where each geodesist had his own theory for the computation of the geoid.

The cause for this dilemma can perhaps be sought for along various lines. First and foremost the situation was such that the various theories were often so complicated that it could be difficult for any critics to find the correct points of attack. Furthermore all the theories contained fundamental hypotheses concerning the structure of the interior of the earth which were impossible to check completely.

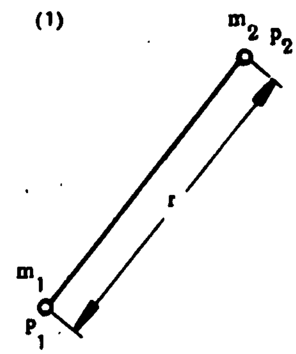
It was therefore a scientific sensation of the first order when at the General Assembly for Geodesy and Geophysics in Toronto in 1957, a paper written by the Russian scientist M. MOLODENSKY was presented in which it was shown that the classical geoid is not required and that a mathematically correct method, free of hypotheses, developed in the Russian language, had already been available for several years. Furthermore, this was generally applied in the Soviet Union. The situation is such that for a flat lowland the previous theories can be applied when the effects there are negligible. MOLODENSKY had computed the effect of disturbance masses on the plumb line for a mountain 4000 metres in height. A correct computation showed a plumb line deflection of 50", while a computation according to classical method gives a result of 15,4". It is seen that the older method does not even give the correct order of magnitude.

The greatest significance of MOLODENSKY's work certainly does not lie in the purely practical field. Above all, it is through the mathematical concepts themselves developed correctly and elegantly that assumptions are afforded for reasonable analyses of all the problems which are connected with the gravitational field of our earth.

GRAVITATION

According to NEWTON two point-mass bodies with masses m_1 and m_2 are attracted to each other by a **force** k defined by

$$k = \frac{f \cdot m_1 \cdot m_2}{r^2} \qquad (1)$$



f = gravitational constant

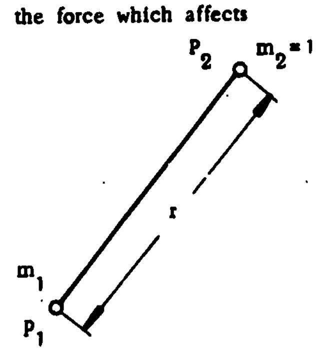
r = distance between masses

Quantities: 1 Newton = force which gives the mass of 1 kg an acceleration of 1 m/sec^2.

1 Kilopond = force which gives the mass of 1 kg an acceleration of 9.8067 m/sec^2.

When the body m_2 has the mass equal to unity ($m_2 = 1$), the force which affects the unit mass is called the **acceleration** g

$$g = \frac{f\ m_1}{r^2} \qquad (2)$$



Quantities: In geodesy the unit for acceleration is

1 gal = 1 cm/sec^2

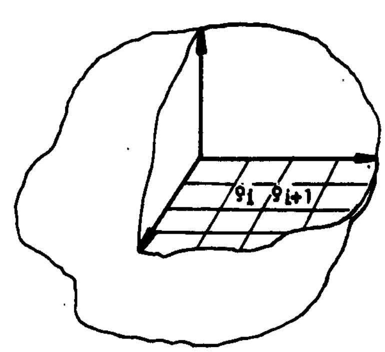
The mass M for an arbitrary body is determined by the formula

$$M = \iiint_V \varrho \, dV \qquad (3)$$

V = volume

ϱ = density

Quantity: The unit for the mass is 1 kg.

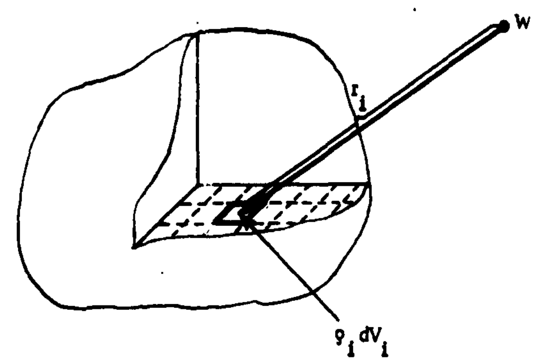


The mass M according to (2) also defines a so-called gravitational potential W according to the relation

$$W = f \iiint \frac{\varrho}{r} \, dV \qquad (4)$$

r = distance of the volume element from the actual point

Quantities: gal. cm. and kilogal-meters.



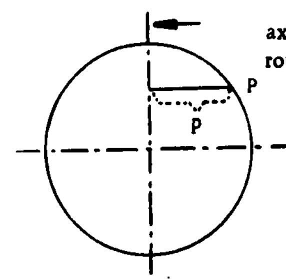
The gravitational potential is considered, unless otherwise stated, as a purely mathematical definition. According to (4) the gradient to the gravitational potential is

$$\text{grad } W = \nabla W = \begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix} = - f \iiint \frac{\varrho}{r^2} \begin{bmatrix} \cos (r, x) \\ \cos (r, y) \\ \cos (r, z) \end{bmatrix} dV \qquad (5)$$

As is seen from equations (2) - (5), grad W constitutes the acceleration which is caused by the actual mass. From this it also follows that the potential difference between two points represents the work which is required to move a unit mass from one point to the other. Generally, one can say that the potential according to equation (4) represents the work required to move a unit mass from the actual point to infinity. This "absolute potential" is often comparatively difficult to obtain and in general one must accept a certain constant error in the determination of W .

GRAVITY

A body at rest is not influenced by any gravitation other than that defined by NEWTON. A moving body is also affected by the so-called motional gravitation. For the earth this additional gravitation is generated on the first hand by the earth's rotation about its own axis. This centrifugal force (g) is computed as follows



$$q = \omega^2 p$$

where

ω = angular velocity in radians

p = distance between the actual point to the axis of rotation.

The work (Q) required to take the unit mass from the actual point to the rotational axis thus is

$$Q = \int_0^p \omega^2 x \, dx = 0.5 \, \omega^2 p^2$$

If the rotating body has a mass, the total potential is

$$W = f \iiint \frac{\varrho}{r} dV + 0.5 \, \omega^2 p^2$$

This composed potential is generally called the gravity potential. The gradient to this potential evidently constitutes the acceleration caused by gravity.

$$\nabla W = - f \iiint \frac{\varrho}{r^2} \begin{bmatrix} \cos(r, x) \\ \cos(r, y) \\ \cos(r, z) \end{bmatrix} dV + \omega^2 p \begin{bmatrix} \cos(p, x) \\ \cos(p, y) \\ \cos(p, z) \end{bmatrix}$$

where

$\cos(p, z) = 0$ for the co-ordinate system in which the z-axis coincides with the axis of rotation.

At each point the gradient is normal to the equipotential surface which includes the point. Furthermore the gradient is tangent to the "plumb line". It is also evident that from a rigorously mathematical point of view the "plumb line" cannot be considered as a straight line. However in most practical applications this should be justified.

THE CONCEPT OF THE GRAVIMETRIC POTENTIAL

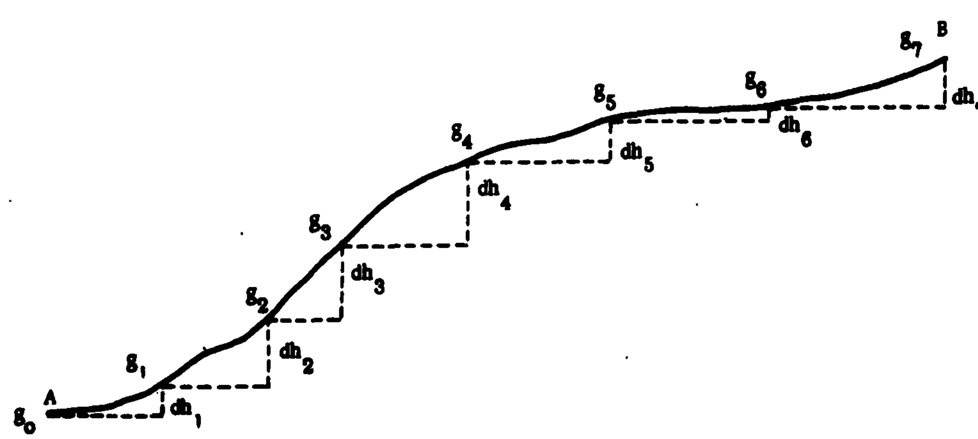
The concept of potential has been extensively applied in electrical theory for a long time. In modern geodesy the concept of potential also plays a significant role. To explain the geodetic concept of potential in greater detail it may be useful to begin with the better known definition of work. If two points A and B are given, it is possible to define uniquely the work required to transport the mass m from the lower point A to the higher point B .

$$\text{Work } A - B = \int_A^B mg\,dh$$

where

g = the acceleration of gravity along the chosen path

dh = height differences



Here <u>the potential difference</u> is identical with the work required to transport a unit mass along the same path. Consequently we get the potential difference

$$A - B = \int g\,dh = \sum_A^B g\,dh = g_1 dh_1 + g_2 dh_2 + g_3 dh_3 + g_4 dh_4 + g_5 dh_5 + g_6 dh_6 + g_7 dh_7$$

The potential difference is the only mathematical expression free of objections for "height differences" between two points. For practical purposes one of the following height concepts is often used.

Unreduced height

In levelling a direct measurement of the quantities dh is made. For simpler measurements the variations in gravity can be disregarded and the height differences be between two points be given as

$$\sum_{A}^{B} dh$$

It is to be noted however, that such a computation does not give a unique result since the result is directly dependent on the path chosen between the two given points.

Orthometric height

The international gravity formula reads

$$\gamma = 978.049 (1 + 0.0052884 \sin^2 \varphi - 0.0000059 \sin^2 2\varphi) \text{ cmsek}^{-2}$$

From a comparison of the gravity values at the equator and pole we get

$$\gamma_p = \gamma_E (1.005288)$$

Consequently, if two potential differences of equal magnitude are compared we obtain

$$\gamma_p \cdot h_p = \gamma_E \cdot h_E$$

$$h_E = \frac{\gamma_p}{\gamma_E} h_p = 1.0053 \text{ hp}$$

In a corresponding manner a theoretical correction can be applied to points situated at arbitrary latitudes. The correction is called the orthometric correction. The correction assumes that the actual gravity agrees with the theoretical gravity.

Dynamic height

The potential difference between two points A and B is defined as

$$\sum_{A}^{B} g_i \, dh_i$$

To decide "in which way the water will run" orthometric heights cannot be used. After division of the potential difference by a suitable mean value of gravity we get a height difference

$$\frac{\Sigma g_i h_i}{\gamma_m}$$

This height difference is called "the dynamic height difference". Only the dynamic height difference give the so-called "work heights".

THE THEORETICAL MODEL EARTH

For the geometric definition of the theoretical model earth an ellipsoid (1924) with the following data is used.

Semi-major axis: $a = 6\ 378\ 388.000$ m.

Semi-minor axis: $b = 6\ 356\ 911.946$ m.

Flattening: $\frac{a - b}{a} = \frac{1}{297} = \alpha$

Angular velocity: $\omega = 0.729\ 211\ 513 \cdot 10^{-4}$/sec. (radians)

For the geophysical definition of the model earth the following data is used.

Gravity on the ellipsoid's surface: $\gamma_o = 978049\ (1 + 0.0052884 \sin^2 \varphi - 0.0000059 \sin^2 2\varphi)$ (milligal)

where

φ = latitude.

Gravity at the elevation z outside the ellipsoid:

$$\gamma_z = \gamma_o + \frac{\partial \gamma}{\partial z} z + \frac{1}{2} \frac{\partial^2 \gamma}{\partial z^2} z^2 + \ldots \text{ or}$$

$$\gamma_z = \gamma_o - (308.78 - 0.45 \sin^2 \varphi)\, z + 0.0727\, z^2 \quad \text{(milligal and km)}$$

$$\gamma_z = \gamma_o - \frac{2\gamma_o \cdot z}{R} + \ldots \quad \text{(for the sphere)}$$

Theoretical height z from the potential difference $(W_o - W)$

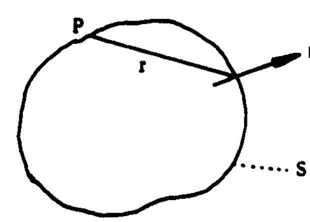
$$z = \frac{W_o - W}{\gamma_o} + \left(\frac{W_o - W}{\gamma_o}\right)^2 (0.000\ 157\ 854 - 0.000\ 001\ 034 \sin^2 \varphi) + \left(\frac{W_o - W}{\gamma_o}\right)^3 0.000000025$$

THE EARTH'S FORM FROM POTENTIALS AND GRAVITY

For the computation of the shape of the earth by means of gravity measurements, it is assumed that the acceleration of gravity is known at all points of the earth's surface. It is further assumed that the latitudes and longitudes of all points of measurement are known, while the elevation of the points above the international reference ellipsoid are unknown and sought quantities. The solution to the problem offers various difficulties and therefore, in general, a direct determination has not been carried out, but instead the choice has been to determine the potential of a theoretical auxilliary model which is assumed to have the same surface as the actual earth. Such a solution also assumes a knowledge of the potential differences between all the measured points concerned. In this case one can obtain from GREEN's second theorem two integral equations which define the relationship between the potentials and gravity of the two models.

For the actual earth we obtain

$$W_p = \frac{1}{2\pi} \iint \left(W \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r} \frac{\partial W}{\partial n} \right) dS \quad (1)$$



where

W_p = potential at the actual point
W = potential at the running point
r = distance between the actual point and the running point
n = normal to the surface (outer)
S = surface.

For the theoretical model earth we obtain

$$U_p = \frac{1}{2\pi} \iint \left(U \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r} \frac{\partial U}{\partial n} \right) dS \qquad (2)$$

where

U_p = theoretical potential at the actual point
U = theoretical potential at the running point.

Neither of the two integral equations are correct for the case represented here since GREEN's theorem requires that the potentials be harmonic functions, e.g. that the following LAPLACE equations should be satisfied

$$\Delta W = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0 \tag{3}$$

and

$$\Delta U = 0 \tag{4}$$

These LAPLACE equations are only satisfied for a body which does not rotate. However, in the further computations the differences between the two integral equations are considered and the potential differences are certainly harmonic functions.

Consequently we obtain

$$W_p - U_p = T_p = \frac{1}{2\pi} \iint \left[T \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r} \left(\frac{\partial W}{\partial n} - \frac{\partial U}{\partial n} \right) \right] dS \tag{5}$$

where

T_p = disturbance potential at the actual point

T = disturbance potential at the running point.

We now want to solve this integral equation with respect to T. We obtain

$$\frac{\partial W}{\partial n} = \frac{\partial W}{\partial \bar{g}} \cdot \frac{\partial \bar{g}}{\partial \bar{n}} = - g \cos (\bar{g}, \hat{n}) \qquad \bar{g} = \nabla W \tag{6}$$

$$\frac{\partial U}{\partial n} = \frac{\partial U}{\partial \bar{\gamma}} \cdot \frac{\partial \bar{\gamma}}{\partial \bar{n}} = - \gamma \cos (\bar{\gamma}, \hat{n}) \qquad \bar{\gamma} = \nabla U \tag{7}$$

where n = unit vector of the normal.

For an approximately spherical surface we consider the following permissible approximations

$$\frac{\partial W}{\partial n} = - g \tag{8}$$

$$\frac{\partial U}{\partial n} = - \gamma \tag{9}$$

Here g is the gravity on the surface of the earth while γ is the corresponding quantity for the model earth. For the computation of the theoretical gravity values we require first a knowledge of the gravity field of the model earth. From the geometric point of view the model earth is assumed to be a rotational ellipsoid with the semi-major axis of 6 378 388 m and a flattening of 1:297. Gravity on the surface of the ellipsoid is assumed to be defined according to the International Gravity Formula as follows

$$\gamma_o = 978049 (1 + 0.0052884 \sin^2 \varphi - 0.0000059 \sin^2 2\varphi) \text{ mgal} \tag{10}$$

where

φ = latitude.

Gravity at the elevation z above the ellipsoid can be expressed by means of a series development of the following type

$$\gamma_z = \gamma_o + \frac{\partial \gamma}{\partial z} z + \frac{1}{2} \frac{\partial^2 \gamma}{\partial z^2} z^2 + \frac{1}{6} \frac{\partial^3 \gamma}{\partial z^3} z^3 + \dots \tag{11}$$

For the theoretical model earth we get

$$\gamma_z = \gamma_o - (308.78 - 0.45 \sin^2 \varphi) z + 0.0727 z^2 + \dots \tag{12}$$

(Units: milligal and km.)

or for the sphere

$$\gamma_z = \gamma_o - \frac{2\gamma_o z}{R} + \dots \qquad (R = \text{radius of sphere}) \tag{12 a}$$

Since the height of the physical surface of the earth above the ellipsoid is an unknown quantity it is not possible to perform a direct computation of the theoretical gravity according to (10). However, it is assumed here that the actual potentials W are known for every point on the surface of the earth. Furthermore it is assumed that the theoretical model earth is an equipotential surface with a known potential U_o. The potential differences $W - U_o$ define a height difference in the potential field of the model earth as follows

$$W = U_o + \frac{\partial U_o}{\partial z} z + \frac{1}{2} \frac{\partial^2 U_o}{\partial z^2} z^2 + \frac{1}{6} \frac{\partial^3 U_o}{\partial z^3} z^3 + \dots \tag{13}$$

A known potential difference $W - U_o$ can be evaluated as a "theoretical height" according to the formula

$$z = \frac{U_o - W}{\gamma} + \left(\frac{U_o - W}{\gamma}\right)^2 k_2 + \left(\frac{U_o - W}{\gamma}\right)^3 k_3 + \dots \tag{14}$$

For the model earth, we obtain the corresponding "theoretical height"

$$z = \frac{U_o - W}{\gamma_o} + \left(\frac{U_o - W}{\gamma_o}\right)^2 (0.000\,157\,854 - 0.000\,001\,034 \sin^2 \varphi) + \left(\frac{U_o - W}{\gamma_o}\right)^3 0.000000025 + \dots \tag{15}$$

Units: U_o and W in kilogal and meters; γ_o in gal; z in km.

Here it is assumed that the z-axis at each point is orthogonal to the ellipsoid.

Computation of the corresponding gravity (γ_z) can now be performed simply according to equation (12)

Consequently the normal derivative $\frac{\partial U}{\partial n}$ can be denoted by

$$\frac{\partial U}{\partial n} = \cos(\bar{\gamma}, \bar{n}) \left[\gamma_z + \frac{\partial \gamma}{\partial z} \frac{T}{\gamma} + \dots\right] \tag{16}$$

For the sphere the corresponding expression can be denoted as

$$\frac{\partial U}{\partial n} = \left[\gamma_z - \frac{2T}{R}\right] \tag{16 a}$$

R = radius of the earth.

From equations (5) and (16) we obtain

$$T_p = \frac{1}{2\pi} \iint \left\{ - \frac{T}{r^2} \cos(\bar{r}, \bar{n}) + \frac{1}{r}\left[g \cos(\bar{g}, \bar{n}) - (\gamma_z + \frac{\partial \gamma}{\partial z} \cdot \frac{T}{\gamma}) \cos(\bar{\gamma}, \bar{n})\right]\right\} dS \tag{17}$$

Here we have an integral equation with T as the unknown quantity. After T is computed the corresponding height difference (Δz) can be determined according to equation (15). The final height above the ellipsoid is ($z + \Delta z$).

For a spherical surface

$$\frac{\partial \frac{1}{r}}{\partial n} = - \frac{1}{r^2} \cos(\bar{r}, \bar{n}) = - \frac{1}{2r\, R} \tag{18}$$

Consequently the integral equation for an approximately spherical surface is

$$T_p = \frac{1}{2\pi} \iint \left[\frac{3T}{2r\, R} + \frac{(g - \gamma_z)}{r}\right] dS \tag{19}$$

and

$$T_p - \frac{3}{4\pi} \iint \frac{T}{r} dS = \frac{1}{2\pi} \iint \frac{g - \gamma_z}{r} dS \tag{19 a}$$

This integral equations can be solved by means of spherical functions. The following parameter is introduced

$$T = \sum_{n=0}^{\infty} \sum_{|m| \leq 1} a_{nm} Y_{nm} (\varphi, \lambda) = \Sigma\Sigma\, a\, Y \tag{20}$$

Thus we get

$$\Sigma\Sigma\, a\, Y - \frac{3}{4\pi\, R} \iint (\Sigma\Sigma\, a\, Y\, \Sigma\Sigma \frac{4\pi}{2n+1}\, Y\, \bar{Y})\, dS = \frac{1}{2\pi} \iint \Sigma\Sigma \frac{4\pi}{2n+1}\, Y\, \bar{Y}\, (g - \gamma_z)\, dS \tag{21}$$

A further parameter is introduced.

$$b = \frac{1}{2\pi} \iint \frac{4\pi\, \bar{Y}}{2n+1} (g - \gamma_z)\, dS \tag{22}$$

and from integral equation (21) we obtain

$$\Sigma\Sigma\, a\, Y - 3\, \Sigma\Sigma \frac{a\, Y}{2n+1} = \Sigma\Sigma\, b\, Y \tag{23}$$

The relationship between a and b can now be determined

$$\frac{(2n - 2)\, a}{2n + 1} = b$$

or

$$a = \frac{2n + 1}{2(n - 1)}\, b$$

Finally we obtain

$$T = \Sigma\Sigma \frac{2n + 1}{2(n - 1)}\, b\, Y = \Sigma\Sigma \frac{(2n + 1)\, Y}{2\,(n - 1)} \iint \frac{2\, \bar{Y}}{2n + 1} (g - \gamma_z)\, dS \qquad (24)$$

or

$$T = \iint \Sigma\Sigma \frac{1}{n - 1}\, Y\, \bar{Y}\, (g - \gamma_z)\, dS \qquad (25)$$

In terms of LEGENDRE polynomials we obtain

$$T = \frac{1}{4\pi} \iint \Sigma \frac{2n + 1}{(n - 1)}\, P_n (\cos \omega)\, (g - \gamma_z)\, dS \qquad (26)$$

Consequently the disturbance potential can now be computed from

$$T = \iint k\, (g - \gamma_z)\, dS \qquad (27)$$

where

$$k = \frac{1}{4\pi} \left(\operatorname{cosec} \frac{\omega}{2} - 6 \sin \frac{\omega}{2} + 1 - \cos \omega \left[5 + 3 \ln (\sin^2 \frac{\omega}{2} + \sin \frac{\omega}{2})\right]\right) \qquad (28)$$

This is the well known STOKES' formula.

EQUIPOTENTIAL SURFACES

A surface on which the potential is everywhere the same is called an equipotential surface. According to GREEN's formula the following is valid on the surface of a body

$$W_p = \frac{1}{2\pi} \iint \left(W \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r} \frac{\partial W}{\partial n} \right) dS$$

Thus for an quipotential surface we obtain

$$W_p \left[1 + \frac{1}{2\pi} \iint \frac{1}{r^2} \cos (\bar{r}, \bar{n})\, dS \right] = -\frac{1}{2\pi} \iint \frac{1}{r} \frac{\partial W}{\partial n}\, dS$$

or

$$W_p \left[1 + \frac{1}{2\pi} \iint \frac{1}{r^2} \cos (\bar{r}, \bar{n})\, dS \right] = \frac{1}{2\pi} \iint \frac{g}{r} \cos (\bar{g}, \bar{n})\, dS$$

For the sphere

$$\cos (\bar{r}, \bar{n}) = \frac{r}{2R} \quad \text{and we get}$$

$$W_p = \frac{1}{4\pi} \iint \frac{g}{r}\, dS$$

Finally, if g is also constant we obtain

$$\boxed{W_p = gR}$$

From the geodetic point of view the most interesting equipotential surface is that equilibrium surface which coincides with the oceans of the world. In geodesy this equipotential surface is called the **g e o i d**.

MATHEMATICAL PROPERTIES OF AN EQUIPOTENTIAL SURFACE

A level surface is characterized by the fact that at each point on the surface the potential is of the same magnitude.

If we denote W as the scalar expression for the potential we have

$$W = \text{constant.}$$

In an arbitrary point P_o the potential is W_o.

If P_o is made the origin in a coordinate system we get the following expression for the potential in an arbitrary point after a TAYLOR series development

$$W(x, y, z) = W_o + xW_x + yW_y + zW_z + \frac{1}{2}\left\{x^2W_{xx} + y^2W_{yy} + z^2W_{zz} + 2xy\,W_{xy} + 2xz\,W_{xz} + 2yz\,W_{yz}\right\}$$

+ terms of higher order

where W_x = partial derivative with respect to x

W_y = - " - y

etc.

With the aid of matrices this expression becomes

$$W(x, y, z) = W_o + \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} W_{xx} & W_{xy} & W_{xz} \\ W_{yx} & W_{yy} & W_{yz} \\ W_{zx} & W_{zy} & W_{zz} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

+ terms of higher order.

Here the potential is expressed by means of the following parameters

1:o Potential scalar = W_o

2:o Potential vector = $\begin{bmatrix} W_x & W_y & W_z \end{bmatrix}$

3:o Potential matrix = $\begin{bmatrix} W_{xx} & W_{xy} & W_{xz} \\ W_{xy} & W_{yy} & W_{yz} \\ W_{xz} & W_{yz} & W_{zz} \end{bmatrix}$

The potential vector is also denoted by the gradient to the level surface. For this the following symbols are used

$$\nabla W = \text{grad } W = \begin{bmatrix} \frac{\partial W}{\partial x} \\ \frac{\partial W}{\partial y} \\ \frac{\partial W}{\partial z} \end{bmatrix} = \begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix} = \bar{g}$$

From this we obtain the relation

$$\bar{g} \cdot \bar{g} = W_x^2 + W_y^2 + W_z^2 = g^2 \cdot 1 = g^2$$

It is evident that the gradient is tangent to the plumb line at the actual point. We orient the coordinate axes so that the z-axis coincides with the normal to the level surface and obtain

$$W_x = 0, \; W_y = 0 \text{ and } W_z = g$$

The radius of curvature of the plumb line is computed from the principal normals as follows

$$\frac{1}{R} = \sqrt{\bar{n} \cdot \bar{n}}$$

In the actual case we get

$$\hat{g} = \frac{1}{W_z} \begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix} ; \qquad \frac{\partial \bar{g}}{\partial z} = \frac{1}{W_z^2} \begin{bmatrix} W_z \cdot W_{xz} - W_x \cdot W_{zz} \\ W_z \cdot W_{yz} - W_y \cdot W_{zz} \\ W_z \cdot W_{zz} - W_z \cdot W_{zz} \end{bmatrix} = \bar{n}$$

From this we obtain for the normal to the plumb line

$$\bar{n} = \frac{1}{g} \begin{bmatrix} W_{xz} \\ W_{yz} \\ 0 \end{bmatrix}$$

and

and

$$\frac{1}{R} = \frac{1}{g} \sqrt{W_{xz}^2 + W_{yz}^2 + 0^2}$$

or

$$\frac{1}{R} = \frac{1}{g} \sqrt{g_x^2 + g_y^2} = \frac{1}{g} \cdot g_n$$

where g_n = the derivative of the acceleration for the direction φ where $\mathrm{tg}\, \varphi = \frac{g_y}{g_x}$

POTENTIAL MATRICES AND THE LEVEL SURFACE

We have already oriented the coordinate system so that the z-axis coincides with the normal to the level surface. We complement this with a reorientation of the xy-plane so that the rectangular xy-terms are eliminated.

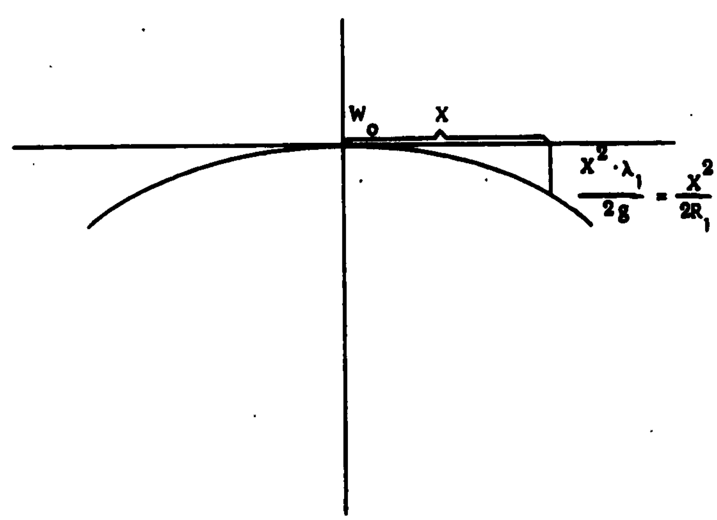
For this we determine the latent roots to the system

$$\begin{vmatrix} W_{xx} - \lambda & W_{yx} \\ W_{yx} & W_{yy} - \lambda \end{vmatrix} = 0$$

which gives

$$\lambda_1 = -\frac{W_{xx} + W_{yy}}{2} + \frac{1}{2}\sqrt{(W_{xx} - W_{yy})^2 + 4\,W_{xy}^2}$$

$$\lambda_2 = -\frac{W_{xx} + W_{yy}}{2} - \frac{1}{2}\sqrt{(W_{xx} - W_{yy})^2 + 4\,W_{xy}^2}$$



The potential equation can now be written

$$W(x\ y\ z) = W_o + Z\,W_Z + \frac{1}{2}(X^2\lambda_1 + Y^2\lambda_2 + Z^2 W_{ZZ}) + \dots +$$

$$+ \text{terms of higher order}$$

The quadratic potential term can now be easily evaluted in terms of height differences after division by g .

$$dZ_X = \frac{X^2\lambda_1}{2g} \qquad dZ_Y = \frac{Y^2\lambda_2}{2g}$$

where $$\lambda_1 = \frac{g}{R_1} \qquad \lambda_2 = \frac{g}{R_2}$$

R_1 and R_2 are here the two principal radii of curvature to the level surface.

(The X- and Y-axes are necessarily orthogonal in this system.)

The trace of the complete potential matrix is known as LAPLACE's equation

$$Sp\ W = W_{xx} + W_{yy} + W_{zz} = \Delta W$$

where $Sp\ W$ = trace of W .

For NEWTON's gravitation in empty space

$$Sp\ W = 0 \text{ is valid.}$$

A potential which satisfies this condition is said to be harmonic.

When the actual body rotates we get

$$Sp\ W = 2\omega^2 \qquad \omega = \text{angular velocity}$$

and where the density of the mass = ρ

$$Sp\ W = 2\omega^2 - 4\pi f \rho$$

f = NEWTON's gravitational constant.

SIMPLE LAYER POTENTIAL

The potential is a unique and continuous scalar quantity. Ignoring the choice of units, the potential according to NEWTON can be denoted as follows

$$W = \iiint_V \frac{\varrho \, dV}{r}$$

ϱ = density of mass (kg/m^3); V = volume

r = distance between volume element and the actual point

In the interior of the mass with continuous mass distribution

$$\nabla^2 W = -4\pi \varrho \qquad \text{(POISSON's equation)}$$

is valid and in empty space

$$\nabla^2 W = 0 \qquad \text{(LAPLACE's equation)}$$

In the latter case the function is said to be harmonic.

While the above "volume potential" is connected directly to known physical relations, parallel to this we use, for example, the potential from a hypothetical layer of mass on the surface.

$$W = \iint \frac{\sigma}{r} dS \qquad \nabla^2 W = -4\pi \sigma$$

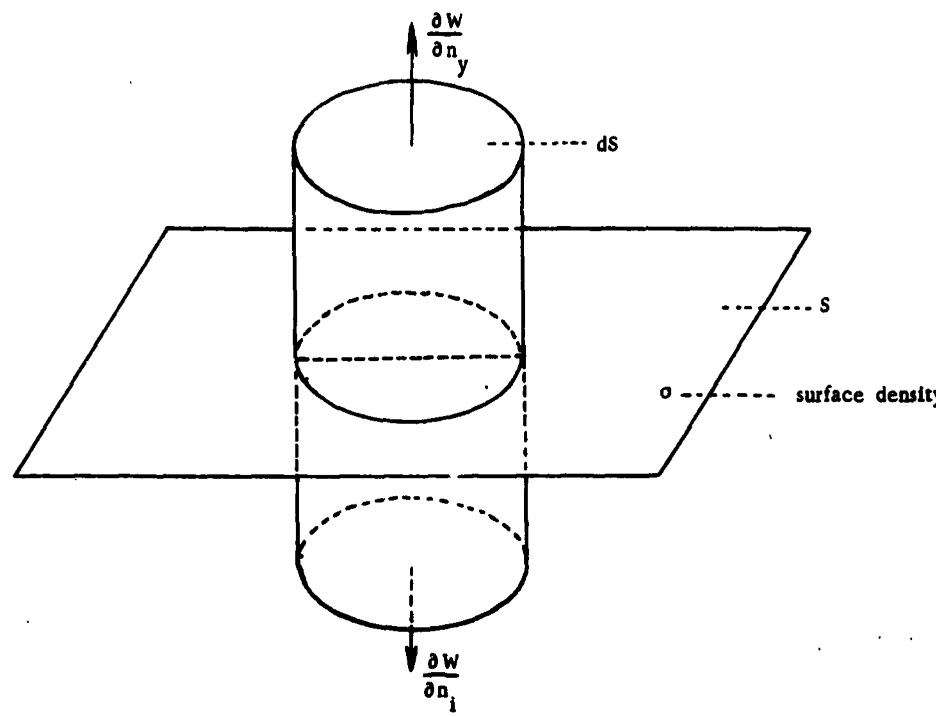
σ = layer density (kg/m^2); S = surface

r = distance between the surface element and the actual point.

In this case we employ a so called single layer potential. The procedure lacks physical background in geodesy but nevertheless constitutes a valuable mathematical auxilliary tool.

While the gradients to the volume potential are both unique and continuous, the single layer potential derivatives exhibit a "pimp" (discontinuity) at the surface. This can be shown in the following manner.

Consider an infinitesimal cylinder which cuts through the surface S and is parallel to the surface normal



If we start from POISSON's equation we obtain, after integration

$$\int_V \nabla^2 W \, dV = - \int_V 4\pi \sigma \, dV$$

From GAUSS' theorem and GAUSS' law we further obtain

$$\int_V \nabla^2 W \, dV = \int_S \nabla V \, \overline{dS} = - 4\pi \int_S \sigma \, dS$$

If the gradients are computed at right angles to the surface, they are parallel to the vector $\overline{dS}$ and we get

$$\left(\frac{\partial W}{\partial n_y} - \frac{\partial W}{\partial n_i}\right) = - 4\pi \sigma$$

From this we get

$$\frac{\partial W}{\partial n_y} = \frac{\partial W}{\partial n_i} - 4\pi \sigma$$

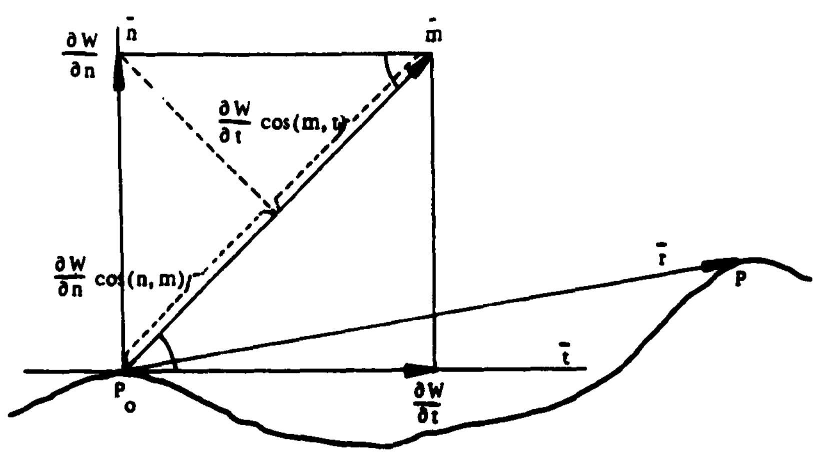
To deduce an expression for the continuous part of the derivatives we differentiate in the direction of the tangent

$$\frac{\partial W}{\partial t} = - \iint_S \frac{\sigma}{r^2} \frac{\partial r}{\partial t} \, dS$$

When the continuous part is known we obtain the derivatives in the direction of the normal after addition of the discontinuities above

$$\boxed{\begin{aligned} \frac{\partial W}{\partial n_y} &= - \iint_S \frac{\sigma}{r^2} \frac{\partial r}{\partial n} \, dS - 2\pi\sigma_0 \\ \frac{\partial W}{\partial n_i} &= - \iint_S \frac{\sigma}{r^2} \frac{\partial r}{\partial n} \, dS + 2\pi\sigma_0 \end{aligned}}$$

For the computation of the derivative in an arbitrary direction we perform a vectorial addition according to the figure below



$$\frac{\partial W}{\partial m} = \frac{\partial W}{\partial n} \cos(n, m) + \frac{\partial W}{\partial t} \cos(m, t)$$

or

$$\frac{\partial W}{\partial m} = - \int_S \frac{\sigma}{r^2} \left[\frac{\partial r}{\partial n} \cos(n, m) + \frac{\partial r}{\partial t} \cos(m, t) \right] dS - 2\pi\sigma_0 \cos(n, m)$$

where

$$\frac{\partial r}{\partial n} = \cos (r, n)$$

$$\frac{\partial r}{\partial t} = \cos (r, t)$$

From these we finally obtain

$$\boxed{\frac{\partial W}{\partial m} = - \int \frac{\sigma}{r^2} \cos (r, m)\, dS - 2\pi \sigma_o \cos (n, m)}$$

where

(r, m) = angle between the vector $\bar{m}$ and the vector from P_o to the running point

(n, m) = angle between the surface normal and the vector $\bar{m}$

σ_o = layer density at the point P_o .

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