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Benton F. Massell

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In a recent and highly provocative article,[1] Evsey Domar poses the following problem:[2] Consider a sector of the economy composed of several fully integrated industries, producing final products only.[3] Write the sectoral production function

$$(1) \quad Q(t) = A(t)K(t)^{\beta} L(t)^{1-\beta},$$

where Q = output, K = capital input, L = labor input, t = time, and A is a technology parameter. Further, let the production function for industry i be written

$$(2) \quad Q_i(t) = A_i(t)K_i(t)^{\beta_i} L_i(t)^{1-\beta_i}.$$

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[1] E. D. Domar, "On the Measurement of Technological Change," The Economic Journal, Vol. LXXI (December 1961), pp. 709-29.

[2] The problem posed by Professor Domar is actually broader than these comments suggest; however, I am concerned with only one part of his argument.

[3] Domar considers two industries only, but the extension to n industries is straightforward.

Now, the percentage rate of technical change -- what Domar terms
the "Residual" -- for the sector can be expressed[4]

$$(3) \quad \frac{\dot{A}}{A} = \frac{\dot{Q}}{Q} - \beta \frac{\dot{K}}{K} - (1-\beta) \frac{\dot{l}}{L} ,$$

and similarly for an industry. The problem, then, is to find a
method for weighting and aggregating the industry production functions
which leaves the rate of technical change invariant with respect to
aggregation.[5] Professor Domar's solution is to raise both sides of
(2) to the v_i power, where $v_i = \frac{Q_i}{Q}$, and to multiply industry
production functions together to obtain the sector function.

Professor Domar is troubled by the fact that a Cobb-Douglas
function, in which factors enter multiplicatively, is often used
to describe an aggregate production process, where aggregation of
output and inputs over industries is additive. He suggests that
aggregation over industries, in order to yield consistent results
(by which he means results which leave the rate of technical change
invariant with respect to the aggregation process) should also be
multiplicative. "It is clear that arithmetic aggregation (addition)
of the production equations would not give a consistent result."[6]

The issue raised by Domar brings to mind a similar problem which
I have always found disturbing: in the case of a Cobb-Douglas -- or
any multiplicative -- production function, whether factors are added

[4]See Robert M. Solow, "Technical Change and the Aggregate
Production Function," The Review of Economics and Statistics,
Vol. XXXIX (August 1957), pp. 312-20; also Domar, op. cit., p. 711.

[5]More precisely, a method such that the aggregate rate of
technical change will be a weighted arithmetic mean of the industry
rates of technical change.

[6] Domar, op. cit., p. 718, note. 2.

or multiplied together depends on what system of classification is used. Thus, in equation (1), machines are added to other machines -- and to structures and land -- and different workers are added together, regardless of their particular skills and job assignments, but a worker and a machine are multiplied. However, if we rewrite (1) as[7]

$$(4) \quad Q(t) = B(t)K_1(t)^{\beta_1} K_2(t)^{\beta_2} L(t)^{1-(\beta_1 + B_2)},$$

where K_1 and K_2 represent different machines, then the two types of capital, which in (1) were added together, are now multiplied.

It is of some interest to ask under what conditions the rate of technical change is invariant with respect to aggregation of the capital stock,[8] assuming that aggregation is always by simple addition. In other words, if we distinguish between equipment and structures, will this result in an estimate of $\dot{B}/B$ in equation (4) which differs from that of A/A in equation (1), where both types of capital are lumped together? Or, what if we distinguish between red machines and green machines -- will this affect our estimates? If so, then one must not attribute too much significance to estimates of technical change which are not invariant with respect to such arbitrary classificatory devices.

The rate of technical change can be written from equation (4) as

$$(5) \quad \frac{\dot{B}}{B} = \frac{\dot{Q}}{Q} - \beta_1 \frac{\dot{K_1}}{K_1} - \beta_2 \frac{\dot{K_2}}{K_2} - \left[1-(\beta_1 + \beta_2) \right] \frac{\dot{L}}{L} .$$

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[7] If the exponents equal the ratio of the value of the corresponding input to the value of output (as they do in Domar's study), then labor's exponent in (4) will be the same as that in (1).

[8] We could as well disaggregate the labor force.

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It is readily seen that

$$(6) \quad \frac{\dot{A}}{A} = \frac{\dot{B}}{B} \text{ if } \beta \frac{\dot{K}}{K} = \beta_1 \frac{\dot{K}_1}{K_1} + \beta_2 \frac{\dot{K}_2}{K_2} \ ,$$

of it

$$(7) \quad \beta_1 \left(\frac{\dot{K}}{K} - \frac{\dot{K}_1}{K_1} \right) + \beta_2 \left(\frac{\dot{K}}{K} - \frac{\dot{K}_2}{K_2} \right) = 0.$$

Condition (7) will hold either if (i) K_1 and K_2 change at the same relative rates or if (ii) the marginal productivity of the two kinds of capital is the same. This latter follows from the fact that

$$(8) \quad \beta_1 = \frac{f^1 K_1}{Q}$$

where $f^1 =$ the marginal product of K_1, so that

$$(9) \quad \beta_1 \frac{\dot{K}_1}{K_1} = \frac{f^1}{Q} \dot{K}_1.$$

Thus, it can be seen that arbitrary division of the stock into red machines and green ones will have no affect on our estimate of the Residual, assuming, of course, that the color of a machine has no bearing on its marginal productivity. With regard to land vs. man-made assets, or equipment vs. structures, however, it may turn out in some cases that condition (7) is not satisfied. If, in a particular sector, equipment earns a higher return for a certain period of time than structures, and if the proportions in which the two are used changes over time, then our estimate of technical change in that sector will depend on the level of disaggregation employed.

Returning to Professor Domar's problem, it seems appropriate to ask whether comparable conditions can be established for aggregation

over industries. That is, under what conditions will additive aggregation provide invariance?

We can write the rate of technical change for an industry,

$$(10) \quad \frac{\dot{A}_1}{A_1} = \frac{\dot{Q}_1}{Q_1} - \beta_1^* \; \frac{\dot{K}_1}{K_1} - (1 - \beta_1^*) \frac{\dot{L}_1}{L_1} \; .$$

Now, I have shown elsewhere[9] that, on the basis of simple additive aggregation of output and inputs, technical change is given by

$$(11) \quad \frac{\dot{A}}{A} = \gamma_1 + \gamma_2 + \gamma_3,$$

where

$$(12) \begin{cases} \gamma_1 = \Sigma \; \frac{\dot{Q}_1}{Q} \\[2mm] \gamma_2 = \beta \; \Sigma \; \frac{f_1^k}{f^k} \; \dot{w}_1^k \\[2mm] \gamma_3 = (1-\beta) \; \Sigma \; \frac{f_1^L}{f^L} \; \dot{w}_1^L \end{cases}$$

and where

$$(13) \begin{cases} w_1^k = \frac{K_1}{K} \\[2mm] w_1^L = \frac{L_1}{L} \end{cases}$$

It follows that aggregate technical change is equal to a simple weighted average of the industrial rates of technical change (where an industry's weight is the proportion of aggregate output produced by the industry) if

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[9] B. F. Massell, "A Disaggregated View of Technical Change," The Journal of Political Economy, Vol. LXIX (December 1961), pp. 547-57.

$$(14) \quad \gamma_2 = \gamma_3 = 0.$$

It is clear from inspection of (12) that $\gamma_2 = 0$ either if [10]

$$(15) \quad f_i^k = f^k$$

or if

$$(16) \quad \dot{v}_i^k = 0$$

all i,

and similar conditions hold with respect to γ_3. In other words, aggregation leaves technical change invariant if the marginal productivities of capital and labor are the same in all industries, or if each industry's share of both inputs remains constant over time. In general, these conditions are unlikely to be satisfied and, as shown in the paper referred to above,[11] technical change in aggregate U. S. manufacturing has exceeded a weighted average of the individual industrial rates of technical change by a considerable margin.

But the final point I should like to raise is that there is no reason why, if neither (15) nor (16) holds, we should wish aggregation to leave the Residual invariant. For, if some industries are experiencing a more rapid rate of technical change than others, and if the return to factors is higher in the more progressive industries, then there is a net gain to society from transferring

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[10] These are sufficient but not necessary conditions.

[11] Massell, op. cit., p. 555.

resources into these industries. It is this interindustry transfer
of resources which accounts for the discrepancy between aggregate
technical change and a weighted sum of technical change in the
industries -- what I have termed "interindustry technical change."
According to Professor Domar's method, there is no discrepancy; but
I would argue that there should be, and that it is of interest to
be able to measure this factor, which can be regarded as an index
of the gain from interindustry factor mobility within the sector.
Forcing the $\mathring{A}_i/A_i$ to average out to $\mathring{A}/A$ amounts to disregarding the
differences among industries in rates of growth and in returns to
factors.

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On page 5, the top line of equation (12) should read:

$$Y_1 = \frac{Q_1}{Q} \cdot \frac{\dot{A}_1}{A_1}$$