63-2-4

FILTERS FOR DETECTING MOVING UNDERWATER OBJECTS

by

Arthur E. Laemmel

Research Report No. PIBMRI-1037-62

Contract No. Nonr-839(28)

for

Office of Naval Research

Washington 25, D. C.

21 June 1962

MRI

POLYTECHNIC INSTITUTE OF BROOKLYN

MICROWAVE RESEARCH INSTITUTE

ELECTROPHYSICS DEPARTMENT

Report PIBMRI-1037-62
Contract No. Nonr-839(28)

FILTERS FOR DETECTING MOVING UNDERWATER OBJECTS

by

Arthur E. Laemmel

Polytechnic Institute of Brooklyn
Microwave Research Institute
55 Johnson Street
Brooklyn 1, New York

Report PIBMRI-1037-62
Contract No. Nonr-839(28)

21 June 1962

Title Page
Acknowledgment
Abstract
Table of Contents
12 Pages of Text
References (1 Page)
Appendices (6 Pages)

Arthur E. Laemmel

Arthur E. Laemmel
Research Associate Professor

Prepared for
Office of Naval Research
Washington 25, D. C.

ACKNOWLEDGMENT

The work reported in this memorandum was performed in connection with Contract No. Nonr-839(28) with the Office of Naval Research.

ABSTRACT

~~A solid object moving horizontally beneath the surface of a fluid will cause a wake to be formed on the surface.~~ The detection of ~~the~~ a wake is usually very difficult because of its small amplitude, and because of the natural waves which are usually present. It is shown that the three dimensional spectra of the wake and of the natural waves are of different forms, and that this difference might be used to design a filter to enhance the wake and attenuate the natural waves. Some of the practical limitations on such a filter are also discussed.

TABLE OF CONTENTS

FILTERS FOR DETECTING MOVING UNDERWATER OBJECTS

by

Arthur E. Laemmel

1. Introduction The problem of detecting the surface wake of an object moving beneath a fluid surface can be considered as analogous to the familiar problem of detecting an electrical signal. In both cases the desired effect is partially hidden by a random disturbance: surface gravity waves in the case of the wake, and antenna or tube noise in the case of the electrical signal. Several types of filter for separating the desired effect or signal from the random disturbance or noise have been extensively studied in the electrical case, for example the filters of North[1] and Dwork,[2] and Wiener[3]. Some extensions have been made to more than one dimension in the case of optics, infra red, and radar; but these have generally been applied to several spatial dimensions without considering the time dimension simultaneously. The filters analyzed below are similar to the MTI (moving target indicator) radars, but the special properties of water waves introduce several novel and promising opportunities for improvement.

2. Conditions of the problem Let a rectangular coordinate system be established with the origin at the undisturbed surface of the fluid with z going upward and x and y going horizontally. Let $A(x, y, t)$ be the wake of the moving object and $B(x, y, t)$ be the random disturbances on the surface of the fluid. The variables A and B can represent the value of z at the surface, radar reflectivity etc.; the only restriction being that the noise B is linearly added to the signal A before being received by the detector. A three dimensional spectrum will be defined for A and called α, and according to the usual Fourier transform theory, A can then in turn be represented in terms of α:

$$\alpha(u, w, f) = \int\int\int_{-\infty}^{\infty} A(x, y, t)\, e^{-2\pi i(xu + yw + tf)}\, dx\,dy\,dt \tag{1}$$

$$A(x, y, t) = \int\int\int_{-\infty}^{\infty} \alpha(u, w, f)\, e^{2\pi i(xu + yw + tf)}\, du\,dw\,df \tag{2}$$

Analogous expressions can be written representing B in terms of its spectrum β. Note that f is the frequency familiar to electrical engineers and is measured in cycles per second, while u and w are the spatial frequencies which are beginning to be common in optical engineering and are measured in cycles per foot or other unit of length. Let $\gamma(u, w, f)$ be the frequency function of a linear filter which acts on the signal as follows:

$$\overline{\alpha}(u, w, f) = \gamma(u, w, f)\, \alpha(u, w, f) \tag{3}$$

The representation of the filtering in the space-time domain can be shown to be the following convolution:

$$\overline{A}(\xi, \zeta, \tau) = \iiint_{-\infty}^{\infty} \Gamma(\xi - x, \zeta - y, \tau - t)\, A(x, y, t)\, dx\, dy\, dt \tag{4}$$

where Γ is defined analogously to A in Eq 2. The function Γ can be regarded as the filter response to a unit impulse at the origin $x = 0$, $y = 0$ and at time $t = 0$. Note that a bar is used to denote that the corresponding function has been filtered, and $\overline{B}$ and $\overline{\beta}$ will be similarly defined. The object of the filter can be very generally stated as this: to reduce the function $\overline{B}$ to as small a value as possible while keeping $\overline{A}$ large at somewhere near the target's position. The filter of North and Dwork maximizes the peak value of $\overline{A}$ without regard to whether or not $\overline{A}$ corresponds to A in shape. Wiener's filter attempts to make $\overline{A} + \overline{B}$ as much like A in shape as possible. The former specifies the phase and amplitude of the filter spectrum, the later only the power spectrum. These and several other types of filter have one simple feature in common, i.e. they pass regions of the spectral plane where the signal is large and attenuated regions where the noise is large. This effect is intuitively obvious; the more formal theories are only necessary, where the spectra of signal and noise overlap, to give the relative weighting in such overlap regions. The greater the overlap (i.e. the similarity in shape of signal and noise spectra) the more one must rely on the formulas, but also the less likely that a successful separation of signal and noise can take place.

3. <u>Spectrum of gravity wave noise</u> Some simple examples of noise and signal will be examined to give a feeling for the relative disjointness of the spectra. The physical processes are most completely understood for the case where the variables A and B represent the vertical displacement of the surface of the fluid. If these displacements are small, the wake of a moving submerged object will be linearly super imposed on

the waves normally present on the surface of the fluid. An elementary wave on the surface of a fluid of constant depth h is [4, 5]

$$b(x, y, t) = \cos(\omega t - kx \cos\theta - ky \sin\theta + \psi) \tag{5}$$

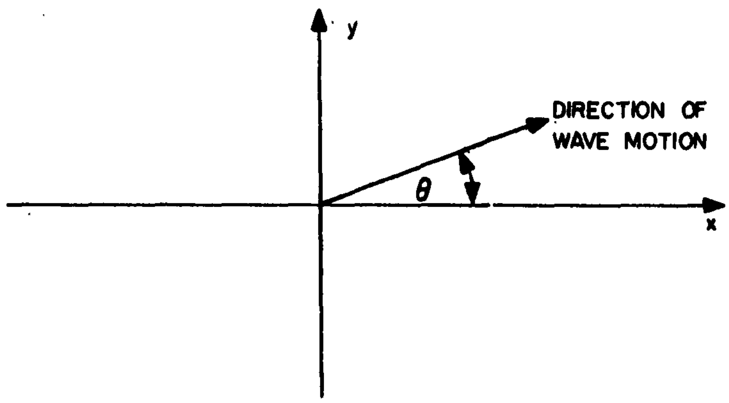
where

$$\omega = +\sqrt{gk \tanh kh} \tag{6}$$

ψ = a phase constant

θ = angle of propagation from x axis:

g = acceleration of the Earth's gravity



The wavelength is seen to be $\frac{2\pi}{k}$ $(=\lambda)$. The frequency of oscillation at a fixed point is $\frac{\omega}{2\pi}$ cycles per second. The phase velocity of the wave is

$$c = \frac{\omega}{k} = +\sqrt{\frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}}$$

It will be noted that although ω and k can assume any positive value between 0 and ∞, the value of c ranges between 0 and $\sqrt{gh}$. If the wavelength is long enough for c to be near its upper bound the wave is called a tidal wave. In the present problem the opposite case is of more interest: If the wavelength is small compared to the depth the velocity is approximately $\sqrt{\frac{g\lambda}{2\pi}}$.

The most general function describing the free vibrations of the fluid surface is obtained by combining many components of the type represented by Eq. 5.

$$B(x,y,t) = \sum_{\nu} \mu_{\nu} \quad \cos(\omega \; t - k_{\nu} x \cos\theta_{\nu} - k \quad y \sin\theta_{\nu} + \psi_{\nu}.) \tag{8}$$

$$\text{where } \omega_{\nu}^{2} = g k_{\nu} \tanh h\, k_{\nu} \tag{9}$$

A more general representatiôn is obtained if this summation is replaced by an integral and if the substitutions

$$\begin{aligned} 2\pi u &= -k_{\nu} \cos\theta_{\nu} \\ 2\pi w &= -k_{\nu} \sin\theta_{\nu} \end{aligned} \tag{10}$$

are made:

$$B(x,y,t) = \int_{-\infty}^{\infty} \int_{+\infty}^{\infty} \mu(u,w) \cos\left[\omega(u,w)t + 2\pi u x + 2\pi w y + \psi(u,w)\right] \; i u d w \tag{11}$$

$$\text{where } \omega(u,w) = \sqrt{2\pi\, g \sqrt{(u^{2}+w^{2})} \tanh 2\pi h \sqrt{u^{2}+w^{2}})} \tag{12}$$

This can be rewritten in the following exponential forms:

$$B(x,y,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \beta(u,w,f)\, e^{2\pi i (ux + wy + ft)} \; du\, dw\, df \tag{13}$$

$$\text{where } \beta(u,w,f) = \frac{1}{2} \mu(u,w)\, e^{i\psi(u,w)} \; \delta\left[f - \frac{\omega(u,w)}{2\pi}\right] + \frac{1}{2} \mu(-u,-w) e^{-i\psi(-u,-w)} \; \delta\left[f + \frac{\omega(u,w)}{2\pi}\right] \tag{14}$$

and $\delta(f) = 0 \ (f \neq 0), \int_{-\infty}^{\infty} \delta(f)\, df = 1$ (impulse functions)

and where $\omega(\ )$ is defined in Eq. 12.

Several very interesting properties of the spectrum of the wave "noise" are at once apparent:

1) $\beta(-u, -w, -f) = \beta^{*}(u, w, f)$

2) $\beta(u, w, f) = 0$ except on the surface $f = \pm \frac{\omega(u, w)}{2\pi}$ (see Eq. 12)

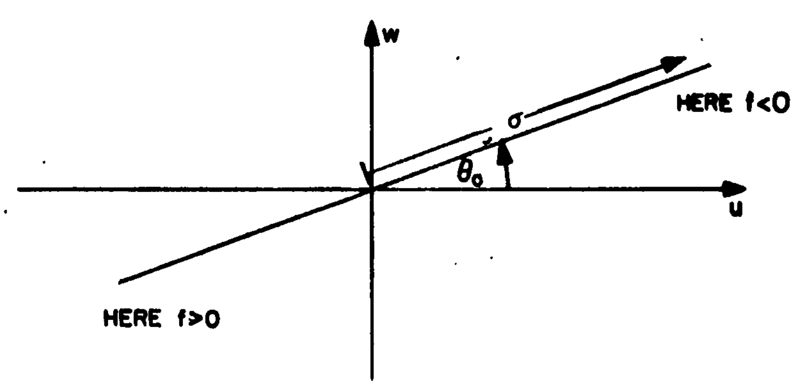
3) If all of the original wave components are travelling in the in the same direction, say at an angle θ_o with the x axis, then $\beta(u, w, f) = 0$ except along the line defined by the equations:

$$u = \sigma \cos \theta_o$$

$$w = \sigma \sin \theta_o$$

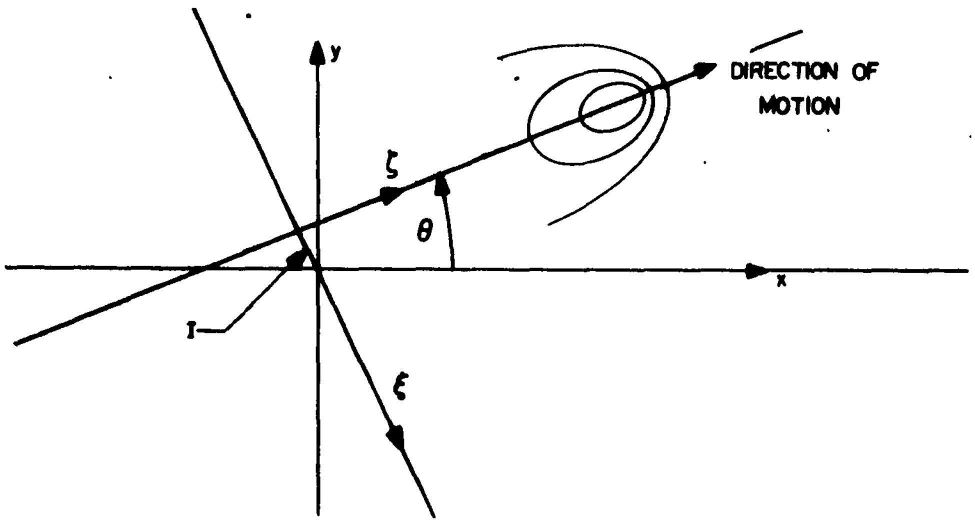
$$f = \left\{ \begin{array}{ll} \frac{1}{2\pi}\, \omega(\sigma \cos \theta_o, \sigma \sin \theta_o) & 0 \leq \sigma \\ -\frac{1}{2\pi}\, \omega(\sigma \cos \theta_o, \sigma \sin \theta_o) & 0 \leq 0 \end{array} \right\} \text{ see Eq 12}$$

The projection of this line on the u, w plane is sketched below:



4. Signal spectrum: wake of a moving submerged object The simplest case to consider will be that in which the object sets up a surface distrubance which travels with it without changing shape. If the object is travelling with a velocity c in a direction making an angle θ with the positive x axis, then the signal function can be written

$$A(\xi, \zeta - ct) \quad \text{where} \begin{cases} \xi = x \sin\theta - y\cos\theta + I \\ \zeta = x\cos\theta + y\sin\theta \end{cases}$$



The spectrum can now be calculated by Eq. (1):

$$a(u,w,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\xi, \zeta - ct)\, e^{-2\pi i(ux + wy + ft)}\, dx\, dy\, dt \tag{15}$$

$$a(u,w,f) = \mathcal{A}(u \sin\theta - w\cos\theta,\ u\cos\theta + w\sin\theta)\ \delta(f + uc\cos\theta + wc\sin\theta)\, e^{-2\pi i I(w\cos\theta - u\sin\theta)} \tag{16}$$

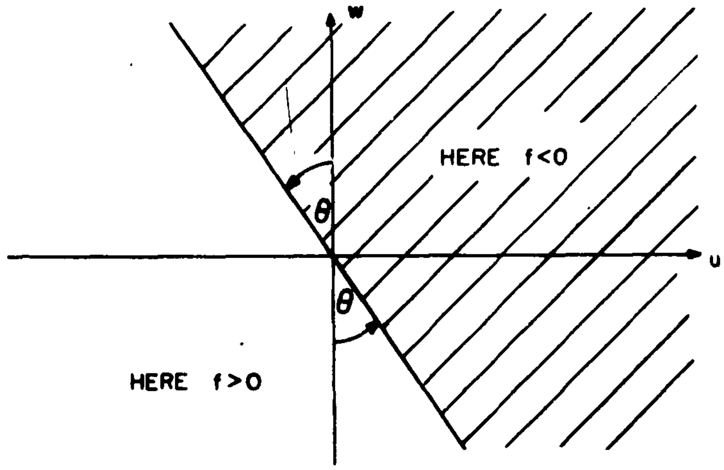
$$\text{where } \mathcal{A}(u,w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\xi,\zeta)\, e^{2\pi i(\xi u + \zeta w)}\, ds\, d\zeta$$

The following interesting properites may now be deduced:

1) $\alpha(-u, -w, -f) = \alpha^*(u, w, f)$

2) $\alpha(u, w, f) = 0$, except on the (plane) surface $f = -uc \cos\theta - wc \sin\theta$

3) The intercept of the above plane with the (u, w) plane is the line $u = -w \tan\theta$



Note particularly that this line does not depend on the distance of the object's path from the origin (I). The angle between the two planes depends only on the magnitude of the velocity and is fact $\tan^{-1} c$.

Thus, the first order theory shows that the signal spectrum exists only on plane in the spectral domain and the noise spectrum exists only on a paraboloid of revolution (for infinite depth). Offhand, this sounds almost ideal from the viewpoint of filtering. Unfortunately, the plane and the paraboloid intersect and most of the signal is near this curve of intersection.

5. Filtering in two dimensions Most of the principles underlying the proposed methods of filtering can be illustrated in two dimensions, thus avoiding temporarily some of the mathematical complexity of the three dimensional case. It is not suggested that the two dimensional case indicates anything about the feasibility of detection in three dimensions.

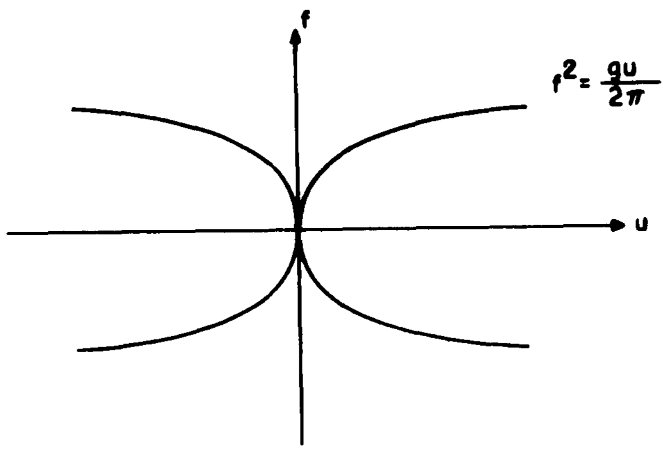
Assume, as before, a coordinate system with its origin at the undisturbed level of the fluid, z going upward, and with all boundaries and variables independent of y. If the depth is infinite, the "noise" will be free gravity waves of the form (see Eq. 8)

$$B(x,t) = \sum_{\upsilon} \mu_{\upsilon} \cos(\omega_{\upsilon} t + k_{\upsilon} x + \psi_{\upsilon}) \tag{17}$$

where $$\omega_{\upsilon} = +\sqrt{g|k_{\upsilon}|} \tag{18}$$

$$|k_{\upsilon}| = \frac{2\pi}{\lambda_{\upsilon}} \qquad C_{\upsilon} = \frac{\omega_{\upsilon}}{|k_{\upsilon}|} = \frac{\lambda_{\upsilon}}{t_{\upsilon}} = +\sqrt{\frac{g\lambda_{\upsilon}}{2\pi}}$$

The convention will be adapted that the wavelength λ_{υ} will always be positive, but that k_{υ} can be positive (waves going in the - x direction) or k_{υ} can be negative (waves going in the + x direction). The spectrum of a general function of x and t can be represented quite generally by drawing " contour" lines of constant amplitude and constant phase in the plane of one spatial frequency (cycles per foot) and are temporal frequency (cycles per second). This spectral plane should not be confused with the more usual complex frequency plane where the temporal frequancy is given an imaginary part (makes per second). In the present case the spectrum vanishes almost everywhere in the spectral plane except along the lines specified by Eq. 18. Along these lines Dirac delta functions are distributed, as in Eq. 14.



The signal to be detected in the present example will be taken as the wake of a circular cylinder of radius b moving with a constant velocity c in the - x direction at a constant depth z = - D. A modification of the method used by Lamb is given in Appendix 1 and results in:

$$A(x,t) = \frac{2b^2 c^2}{g} \frac{(x+ct)^2 - D^2}{\left[(x+ct)^2 + D^2\right]^2} - \frac{4\pi g b^2}{c^2} e^{\frac{-gD}{c^2}} \sigma(x+ct) \sin\left(\frac{gx}{c^2} + \frac{gt}{c}\right) \tag{19}$$

$$\text{where } \sigma(\xi) = \begin{cases} 0 & \xi < 0 \\ 1 & 0 < \xi \end{cases}$$

The spectrum can be obtained from an intermediate result in Lamb's method, since he actually satisfied the surface boundary condition in terms of the spectral function. From Eq. < 12 in appendix 1:

$$A(x,t) = 2b^2 \int_0^\infty \frac{k e^{-kd}}{k - \frac{g}{c^2}} \cos(kx + kct)\, dk \tag{20}$$

$$A(x,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b^2 \delta(\omega - kc) \frac{|k| e^{-|k|D}}{|k| - \frac{g}{c^2}} e^{i(kx + \omega t)} d\omega\, dk \tag{21}$$

Thus

$$\alpha(u,f) = 4\pi^2 b^2 \delta(f - uc) \frac{|u| e^{-2\pi |u| D}}{|u| - \frac{g}{2\pi c^2}} \tag{22}$$

This vanishes everywhere in the (u, f) plane except along the line f = u c.

The filter which is to separate signal from noise can be designed in several ways: reject frequencies which represent free surface waves, pass only those frequencies representing a moving object, or to pass or attenuate any particular region according to its effect on the signal to noise ratio or probability of detection. The first is perhaps simplest and does not require knowledge of the object's velocity. To see the effect of such a filter on the desired wake, Eq. 21 must be evaluated over the whole plane except for a band

$$\pm\sqrt{g|k|} - \Delta < \omega < \pm\sqrt{g|k|} + \Delta \tag{23}$$

where the $\pm$ is to be interpreted as the same sign as k.
This is most easily done by subtracting the contribution from this band:

$$A_o(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_o(k,\omega)\, b^2\, \delta(\omega - kc)\, \frac{|k|\, e^{-|k|D}}{|k| - \frac{g}{c^2}}\, e^{i(kx+\omega t)} \tag{24}$$

$$\text{where } \gamma_o(k,\omega) = \begin{cases} 1 & \sqrt{gk} - \Delta < \omega < \sqrt{gk} + \Delta \qquad 0 < k \\ 1- & \sqrt{g|k|} - \Delta < \omega < -\sqrt{g|k|} + \Delta \qquad k < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$A_o(x,t) = \int_{-\infty}^{\infty} \gamma_o(k,kc)\, b^2\, \frac{|k|\, e^{-|k|D}}{|k| - \frac{g}{c^2}}\, e^{ik(x+ct)}\, dk \tag{25}$$

This can be approximated as follows for small Δ*:

$$A_o(x,t) \approx 2b^2 \int_0^{\Delta_1} \frac{ke^{-kD}}{k - \frac{g}{c^2}} \cos k(x+ct)dk + 2b^2 \int_g^{\frac{g}{c^2}+\Delta_2} \frac{ke^{-kD}}{k - \frac{g}{c^2}} \cos k(x+ct)dk$$

*The bandwidths Δ_1 and Δ_2 are functions of Δ and other parameters. They approach zero. Their calculation is discussed below.

Neglecting terms involving powers of Δ_1 and Δ_2 higher than the first:

$$A_o(x,t) \approx -\frac{4b^2 g}{c^2} e^{-\frac{gD}{c^2}} \; Si\left[\Delta_2 |x+ct|\right] \sin\frac{g}{c^2}|x+ct| \tag{26}$$

The value of the integral over all space is

$$A_\infty(x,t) = \frac{2b^2c^2}{g}\frac{(x+ct)^2 - D^2}{\left[(x+ct)^2+D^2\right]^2} - \frac{2\pi g b^2}{c^2} e^{-\frac{gD}{c^2}} \sin\frac{g}{c^2}|x+ct| \tag{27}$$

The wake function after filtering by the band elimination filter is therefore the difference between these two:

$$\bar{A}(x,t) = \frac{2b^2c^2}{g}\frac{(x+ct)^2 - D^2}{\left[(x+ct)^2+D^2\right]^2} - \frac{2\pi g b^2}{c^2} e^{-\frac{gD}{c^2}}\left\{1 - \frac{2}{\pi} Si\left[\Delta_2|x+ct|\right]\right\} \sin\frac{g}{c^2}|x+ct| \tag{28}$$

The disturbance represented by Eq. 28 is seen to be composed of two parts: 1) a symmetrical depression of amplitude $\frac{2b^2c^2}{gD^2}$ ft. centered over the cylinder and of width 2D ft. or $\frac{2D}{c}$ sec, and 2) a packet of sine waves of amplitude $\frac{2\pi g b^2}{c^2} e^{-\frac{gD}{c^2}}$, of period $2\pi\frac{c^2}{g}$ ft. or $2\pi\frac{c}{g}$ sec., and of duration $\frac{2}{\Delta_2}$ ft. or $\frac{2}{\Delta_2 c}$ sec. The later component is also symmetrically centered over the cylinder, but for certain types of filter it may be behind the cylinder because of physical realizability conditions on the filter. For various reasons, some of which are analyzed in Appendix 2, the noise spectrum is not confined to a line $\omega = \sqrt{gk}$. If the noise (surface gravity waves) is to be eliminates, then a band of frequencies, or several bands of frequencies, must be rejected by the filter. One of the principal reasons for the zero spectral width is the deviation from simple linear waves caused by finite wave amplitude. Thus, the higher the waves. the wider the spectral rejection band, and the shorter the wave packet in Eq. 28 will be. A limit is reached when the wave packet is reduced to a single loop.

Some idea of the size of the various parameters can be obtained from the following table:

c knots	λ feet	t sec.	2a feet	$2a_2$ feet	$^2/_{\Delta 2}$ feet	cycles
2	2.23	.66	0.2	0.028	17.8	8
5	13.9	1.65	1.25	0.18	111	8
10	55.7	3.30	5	0.71	446	8
20	223	6.60	10	0.71	7140	32
50	1390	16.5	10	0.11	1,730,000	1240

The second and third columns are the wavelength and period respectively of the oscillitory part of the wake of an object moving at the indicated speed. These two columns also characterize the gravity waves which will cause the most trouble in detection of the moving object. The fourth column shows an assumed height for the gravity waves naturally present. The height is double the amplitude of fundamental component, but this is less than the peak height usually measured. The values shown in the fourth column were arrived at as follows: The maximum peak height of a wave is known from hydrodynamic theory to be about 1/7 of its length[4, 5]. For shorter wavelengths, the double amplitude of the fundamental component is taken as 1/5.57 of its length. For larger wavelengths, common experience shows that the amplitude does not increase indefinitely with wavelength. Hence, for λ = 55.7 ft. and above fundamental double amplitude is kept at 10 ft. The fifth column shows the double amplitude of the second harmonic of the gravity waves, computed according to Eq. A 3 of Appendix 2. The sixth column shows the number of cycles in the filtered oscillitory wake when the gravity wave rejection filter is adjusted to have a bandwidth given by Eq. A14.

REFERENCES

1. D. O. North, "Analysis of factors which determine signal-noise discrimination in pulsed carrier systems, RCA Tech. Rpt. PTR-6C, June 1943.

2. B. M. Dwork, "Detection of a pulse superimposed on fluctuation noise", Proc. IRE, Vol. 38, pp 771-774, July 1950.

3. N. Wiener, "Extrapolation, interpolation, and smoothing of stationary time series", John Wiley and Sons, New York, 1949 (also as a report in 1942)

4. H. Lamb, "Hydrodynamics", 6th Ed., Dover, New York, 1945.

5) J. J. Stoker, "Water Waves, the Mathematical Theory with its Applications", Interscience, New York, 1957.

Appendix 1 - Wake of Submerged Cylinder

Refer to Lamb[4], Section 247, p. 410. Everything checks up to Eq. 12, and this is correct formally, but it does not remove the dominant contribution of the integral. Retain the first form of Eq. 12

$$\zeta = 2b^2 \int_0^\infty \frac{e^{-kf} \cos kx}{k - \kappa} dk \tag{L12}$$

and transform the variable in this just as Lamb does. Instead of his Eq. 14, this gives:

$$\zeta = -2\pi\kappa b^2 e^{-\kappa f} \sin\kappa x + 2b^2 \int_0^\infty \frac{\kappa \cos mf - m \sin mf}{\kappa^2 + m^2} me^{-mx} dm$$

Expanding the denominator in a power series:

$$\frac{1}{\kappa^2 + m^2} = \frac{1}{\kappa^2} - m^2 + \frac{m^4}{\kappa^2} - \ldots$$

Integrating term by term, arranged according to ascending powers of m:

$$\zeta = -2\pi\kappa b^2 e^{-\kappa f} \sin\kappa x + \frac{2b^2}{\kappa} \frac{x^2 - f^2}{(x^2 + f^2)^2} - \ldots$$

This is for $x > 0$, for $x < 0$ note that ζ is an even function of x in Eq. 12. Add a free gravity wave, just as Lamb does, to satisfy the upstream boundary condition. Instead of his Eq. 17, there results:

$$\left.\begin{array}{ll} \zeta(x) \approx \dfrac{2b^2c^2}{g} \dfrac{x^2 - f^2}{(x^2+f^2)^2} - \dfrac{4\pi g b^2}{c^2} e^{-\frac{gf}{c^2}} \sin \dfrac{gx}{c^2} & 0 < x \\[2ex] \zeta(x) \approx \dfrac{2b^2c^2}{g} \dfrac{x^2 - f^2}{(x^2+f^2)^2} & x < 0 \end{array}\right\} \quad \text{(L17}$$

Appendix 2 - Width of the Spectrum of Gravity Waves

If the spectrum of the gravity wave "noise" is really confined to a line (one space, one time dimension) or to a surface (2 space one time dimensions) in the spectral space, then the "noise" can be reduced indefinitely allowing extremely small wakes to be detected. As was pointed out in Section 5 above, the sensitivity of a detector using the filtering method proposed in this report depends directly on how narrow the rejection band can be made. Therefor, some estimates will be made here on how wide a spectral region the actual waves occuppy. Note that in most cases the spectrum is still a line (or surface) but shifted from the normal position. This allows the possibility that a filter could be adopted to the new situation, provided that the new conditions apply consistantly. For example, a wave having an amplitude equal to 1/5 its wavelength would merely have its infinitesmal-amplitude spectrum displaced*, but it is then unreasonable to assume that smaller waves would not also be present similtaneously.

<u>Finite depth</u> From Eq. 6, the frequency of a single component plane gravity wave can be seen to be

$$\omega = \sqrt{gk \tanh hk} \approx \sqrt{gk} \quad (1-e^{-2hk}) \tag{A1}$$

<u>Capillarity</u> From Lamb, p. 459 Eq. 3, a rearrangement gives

$$\omega = \sqrt{gk + \frac{k^3 T_1}{\rho}} \approx \sqrt{gk} \left(1 + \frac{k^2 T_1}{2 g \rho}\right) = \sqrt{gk} \left(1 + \frac{1}{653 \lambda^2}\right) \tag{A2}$$

where T_1 = surface tension = $74 \frac{\text{dynes}}{\text{cm}} = 0.163 \frac{\text{lb.}}{\text{sec}^2}$

ρ = density = $64 \ {}^{\text{lb}}/{}_{\text{ft}^2}$

<u>Finite amplitude</u> Refer to Lamb, p 417. The vertical displacement of the surface, disregarding the "DC" level, is given by his Eq. 3

$$\zeta(x,t) = a \cos k(x+ct) + \frac{ka^2}{2} \cos 2k(x+ct) + \frac{3k^2a^3}{8} \cos 3k(x+ct) \tag{A3}$$

---

*Harmonics could also be added, these also along lines or surfaces.

Note that the wave is considered to be moving here instead of the whole body of water. His Eq. 6 gives

$$\omega = \sqrt{gk\,(1+k^2 a^2)} \approx \sqrt{gk}\;\left[1 + 19.7\left(\frac{a}{\lambda}\right)^2\right] \tag{A4}$$

$$\Delta \approx gk\;\frac{k^2 a^2}{4}$$

In this case another effect is also observed, namely that not only is the fundamental frequency shifted slightly upward, but also distinct harmonic terms are added. The time and space frequencies of these harmonics can be expressed in terms of the fundamental k as follows:

$$\left.\begin{aligned} \omega_2 &= 2\sqrt{gk}\left(1+\frac{k^2a^2}{2}\right) & k_2 &= 2k \\ \omega_3 &= 3\sqrt{gk}\left(1+\frac{k^2a^2}{2}\right) & k_3 &= 3k \end{aligned}\right\} \tag{A5}$$

$$\cdots$$

Also, the harmonic time frequencies can be expressed as functions of the harmonic space frequencies:

$$\left.\begin{aligned} \omega_2 &= 2\sqrt{g\,k_2}\left(1+\frac{k_2^2\;a^2}{8}\right) \\ \omega_3 &= 3\sqrt{g\,k_3}\left(1+\frac{k_3^2\;a^2}{36}\right) \end{aligned}\right\} \tag{A6}$$

$$\cdots$$

Nonlinear superposition The simple theory of gravity wave noise breaks down unless the waves are of infinitesimal amplitude. One such breakdown is considered above under "finite amplitude," and this is analogous to harmonic distortion in electrical systems. Another breakdown which occurs in analogous to intermodulation distortion. Under finite amplitude above only waves which are periodic were considered. If two waves of different periods are superinposed they act independently of each other if they are both of infinitesimal amplitude, i.e. the displacement of the surface with both waves present is the sum of the displacements when the two waves exist separately. The spectra also add linearly, and therefor the gravity wave noise spectrum is confined to the parabolic line mentioned above.

Consider now the case where two waves are present at the same time, one of much longer wavelength than the other. If x is the distance along the surface and ζ the elevation of the surface the component with the longer wavelength can be represented as

$$\zeta_1(x,t) \approx a_1 \cos\left(\frac{2\pi}{\lambda_1} x + \sqrt{\frac{2\pi g}{\lambda_1}}\, t\right) \qquad \text{(A7}$$

Now if another component is present with a wavelength λ_2 ($<< \lambda_1$), then it is subject to downward body forces from not only gravity, but also from the vertical acceleration of the water's surface due to the first component. The effective acceleration of gravity for the second component is thus:

$$\tilde{g}(x,t) = g + \frac{\partial^2}{\partial t^2} \zeta_1(x,t) \qquad \text{(A8}$$

$$\tilde{g}(x,t) = g - 2\pi \frac{a_1}{\lambda_1} g \cos\left(\frac{2\pi}{\lambda_1} x + \sqrt{\frac{2\pi g}{\lambda_1}}\, t\right) \qquad \text{(A9}$$

It is assumed that the second component will have little effect on the first. The problem of finding a free surface wave under the influence of a sinusoidally varying gravity force is analogous to that of obtaining frequency modulation by varying the capacitance in an oscillator circuit.. The solution cannot be obtained by merely substituting a variable g function in Eq. A8.

The method of solution for the smaller wave will be a simple modification of that given by Lamb[4] and Stoker[5]. First, solve

$$\frac{\partial^2 \phi}{2x^2} + \frac{\partial^2 \phi}{2z^2} = 0 \qquad \text{(A10}$$

in the interior of the liquid subject to the following boundary conditions:

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{for } z = -\infty \qquad \text{(A11}$$

$$\frac{\partial^2 \phi}{2t^2} + \tilde{g}\frac{\partial \phi}{\partial z} = 0 \qquad \text{for } z = 0 \qquad \text{(A12}$$

Finally, the surface displacement caused by the second component is given in terms of ϕ by:

$$\zeta_2 = -\frac{1}{g}\left.\frac{\partial\phi}{\partial t}\right|_{z=0} \tag{A13}$$

Assume a solution of the boundary value problem of the form:

$$\phi = e^{k_2 z}\cos\left[k_2 x + h_2 t - j\sin(k_1 x + h_1 t)\right] \tag{a14}$$

where $k_1 = \frac{2\pi}{\lambda_1}$ and $h_1 = \sqrt{\frac{2\pi g}{\lambda_1}}$. The above expression satisfies Eqs. A10, A11, and A12 to good approximations if the following conditions hold:

$$\frac{a_1}{\lambda_1} << 1 \qquad \frac{\lambda_2}{\lambda_1} << 1 \qquad h_2 = \sqrt{gk_2} \tag{A15}$$

$$j = \pi\frac{a_1}{\lambda_1}\sqrt{\frac{\lambda_1}{\lambda_2}} \tag{A16}$$

Combining Eqs. A9 and A13, these results

$$\zeta_2 = \frac{-\frac{1}{g}\frac{\partial\phi}{\partial t}\Big|_{z=0}}{1-2\pi\frac{a_1}{\lambda_1}\cos(k_1 x + h_1 t)}$$

and upon substituting from Eq. 14:

$$\zeta_2 \approx \sqrt{\frac{\pi}{g\lambda_2}}\;\frac{1-\pi\frac{a_1}{\lambda_1}\cos(k_1 x + h_1 t)}{1-2\pi\frac{a_1}{\lambda_1}\cos(k_1 x + h_1 t)}\sin\left[k_2 x + h_2 t - j\sin(k_1 x + h_1 t)\right]$$

$$\zeta_2 \approx \sqrt{\frac{2\pi}{g\lambda_2}}\left[1+\pi\frac{a_1}{\lambda_1}\cos(k_1 x + h_1 t)\right]\sin\left[k_2 x + h_2 t - j\sin(k_1 x + h_1 t)\right]$$

$$\zeta_2 \approx \sqrt{\frac{2\pi}{g\lambda_2}} \left\{ \sin\left[k_2 x + h_2 t - j \sin(k_1 x + h_1 t)\right] + \frac{\pi}{2} \frac{a_1}{\lambda_1} \sin\left[(k_2 + k_1)x + (h_2 + h_1)t \right.\right.$$

$$\left.\left. - j \sin(k_1 x + h_1 t)\right] + \frac{\pi}{2} \frac{a_1}{\lambda_1} \sin\left[(k_2 - k_1)x + (h_2 - h_1)t - j \sin(k_1 x + h_1 t)\right] \right\} \quad \text{(A 17}$$

Note that these terms are similar to the frequency modulated waves of radio systems, and that the added terms are linear in $\frac{a_1}{\lambda_1}$. The deviation from the first order theory is thus more serious when several components are present in the wave structure, since the perturbations of a single component due to finite amplitude are only proportional to $\left(\frac{a}{\lambda}\right)^2$. This large " intermodulation distortion" may not be entirely disadvantageous as far as the detection of an submerged object is concerned, since the moving object may cause the long waves which modify the natural waves of much shorter wavelength.