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The Dynamic Stability of Elastic Systems
Volume II

(V. V. Bolotin)

Translated by V. I. WEINGARTEN,
K. N. TRIROGOFF and K. D. GALLEGOS

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THE DYNAMIC STABILITY OF ELASTIC SYSTEMS, VOLUME II
(V. V. Bolotin)

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TRANSLATORS' PREFACE TO THE ENGLISH EDITION

Within the past few years, interest in the dynamic stability of elastic systems has increased. This interest is reflected in the appearance of a large number of papers on this subject. Unfortunately, few are in English, the majority of these papers being written either in Russian or German. In addition, there is no English text available which presents the mathematical theory of the subject with its applications. However, such a text is available in Russian. This work, "Dynamic Stability of Elastic Systems," (Gostekhizdat, Moscow, 1956) by V. V. Bolotin, together with its German translation (Veb Deutscher Verlag Der Wissenschaften, Berlin, 1961), is the only comprehensive book on the subject of dynamic stability. To fill this need in American scientific literature, it was decided to undertake the translation of this unique book.

At the time the translation was initiated, the German edition was not yet available, and the book was translated from the Russian. With the appearance of the German edition, the translation was checked and all changes in the German edition were incorporated into the English translation. Footnotes in the Russian edition were eliminated and the German referencing system was adapted for our use. A number of notes were made by the translators in order to bring the American edition of the text up to date. References to new works published in the United States and an additional Russian reference were added.

The translators would also like to thank Dr. Paul Seide, Dr. Robert Cooper, and Dr. John Yao of Aerospace Corporation for reading the translation.

El Segundo, California, November 1962 V.I.W., K.N.T., K.D.G.

PREFACE TO THE GERMAN EDITION

Four years have passed since the appearance of the Russian edition of the present book. During this time a series of works on dynamic stability were published, containing interesting results. In addition, the general theory was applied to a new class of problems whose analysis appeared to be of a mathematical nature which fell within the narrow limits of the theory of dynamic stability. All of the above are considered in the German edition of the book. References are included of some works published thru 1956, which were unknown to me when the book was first published.

G. Schmidt, from the Institute for Applied Mathematics and Mechanics, German Academy of Sciences, Berlin, prepared the excellent translation and contributed corrections in some places. The translation was also reviewed at the Institute for Vibration Technology, Karlsruhe Technical University; F. Weidenhammer and G. Benz especially gave worthwhile advice. C. W. Mishenkov, from the Moscow Power Institute, assisted me with the preparation of complete references. To all these people, I would like to express my deep thanks.

December 1960
Moscow

Author - V. V. Bolotin

PREFACE TO THE RUSSIAN EDITION

This book is an attempt to present systematically the general theory of dynamic stability of elastic systems and its numerous applications. Investigations of the author are used as the basis for the book, part of which was published previously in the form of separate articles. The author's method of presentation is retained where the problems treated have been analyzed by other authors.

The book is devoted to the solution of technical problems. As in every other engineering (or physics) investigation, the presentation consists of first choosing an initial scheme or pattern, and then using the approximate mathematical methods to obtain readily understood results. This intent, and the desire to make the book easily understood by a large number of readers, is reflected in the arrangement and structure of the book.

The book consists of three parts. PART I is concerned with the simplest problems of dynamic stability which do not require complicated mathematical methods for their solutions. By using these problems, the author wishes to acquaint the reader with previously investigated problems. At the same time, certain peculiarities of the phenomena of instability are clarified, which previously have been only sketchily mentioned. PART I also contains methods of solution of the general problem.

PART II begins with two chapters containing the minimum necessary mathematical information; a conversant reader can disregard these chapters. The properties of the general equations of dynamic stability are then examined; methods are presented for the determination of the boundaries of the regions of instability and the amplitudes of parametrically excited vibrations for the general case.

PART III is concerned with applications. Various problems of the dynamic instability of straight rods, arches, beams, statically indeterminant rod systems, plates, and shells are examined. The choice of examples was dictated by the desire to illustrate the general methods and present solutions to practical problems. The number of examples was limited by the size of the book.

I would like to take this opportunity to express my sincere thanks to A. S. Vol'mir for having read the manuscript and for having given valuable advice.

January 1956 Author - V. V. Bolotin
Moscow

ABSTRACT

This is Volume II of four proposed volumes of the translation of V. V. Bolotin's book, "The Dynamic Stability of Elastic Systems." Volume II contains the translation of Chapter Three, Chapter Four, and Chapter Five. Nonlinear effects are discussed in Chapter Three. Free and forced vibrations of a nonlinear system are discussed in Chapter Four.

Amplitudes of vibrations at the principal parametric resonance obtained by using nonlinear theory are discussed in Chapter Five.

CONTENTS

CHAPTER THREE

DETERMINATION OF NONLINEAR TERMS

•11. PRELIMINARY REMARKS

From linear theory one expects the vibration amplitudes in the regions of dynamic instability to increase unboundedly with time, and indeed very rapidly, i.e., they increase exponentially. However, this conclusion contradicts experimental results, which show that vibrations with steady-state amplitudes exist in the instability region. The first portion of the oscillogram (Fig. 8) shows that the amplitude of the vibrations increases approximately exponentially. As the amplitudes increase, the character of the vibrations changes; the speed of the growth gradually decreases until vibrations of constant (or almost constant) amplitude are finally established.

The forces acting upon a rod can be considered to be linear functions of displacements, velocities, and accelerations only for sufficiently small deflections. With increasing amplitudes, the influence of nonlinear factors becomes more and more apparent, i.e., these terms limit the infinite increase of amplitudes predicted by linear theory. Therefore, linear theory cannot determine either the magnitude of the steady-state amplitudes or whether or not the vibrations become stationary. Determination of the above is possible only on the basis of nonlinear theory.

The question arises whether the presence of nonlinear factors introduces some changes in the distribution of regions of instability given by linear theory. This is not the case. The boundaries of the regions of dynamic instability can be given accurately by the linear differential equations.

While rigorous investigation of this question will be postponed until Chapter XVI, in the meantime let us refer to its analogy with the problems of static stability, for example, to the problem of the buckling of a straight rod subjected to a static longitudinal force. If one linearizes the equations of the problem, it is well-known that one obtains the correct magnitude of the critical force (Euler buckling force); however, the deflections of the rod in the critical and postcritical stages cannot be determined. For the determination of these deflections, one must take an exact (nonlinear) expression of the curvature.

The idea that the linear approximation is sufficient for determining the dynamic stability of a rod was first clearly formulated by Gol'denblat (Ref. 55). Assertions to the contrary can frequently be found. Beliaev (Ref. 4) assumed that the consideration of the nonlinear curvature must in some manner be reflected on the regions of dynamic instability.

For a more precise determination of the boundaries of the regions of instability at large values of P_t, Beliaev proposed to make use of "differential equations for a slightly curved rod". Analogous assumptions have been made repeatedly up to the present.

An investigation of the relationship between linear and nonlinear theory is of basic importance to the problems under consideration. We shall return to this question in the second part of the book, where we shall relate it to the results of Liapunov's theory of the stability of motion.

12. BUCKLING IN THE POSTCRITICAL REGION

In the present chapter we shall analyze the nonlinearities on which the amplitudes of parametrically excited vibrations depend. We will begin with the simplest problem of strength of materials in which it is necessary to consider the effect of a nonlinearity, i.e., we shall begin with the problem of the bending of a straight rod by a longitudinal force that exceeds the Euler buckling value (Fig. 15). The equation for the buckling (elastic line) has the form

$$\frac{1}{\rho} + \frac{Pv}{EJ} = 0. \tag{3.1}$$

where $\frac{1}{\rho}$ is the exact expression for the curvature. Take the distance measured along the arc of the deformed rod as an independent variable. If the axial deformation along the rod axis is disregarded, then the arc length coincides with the x-coordinate for the undeformed rod.

Let s be the arc length and ϕ the angle between the ox-axis and the tangent of the elastic line (Fig. 15). Taking into account that $\sin \phi = dv/ds$ and differentiating this expression along the arc length s, we obtain

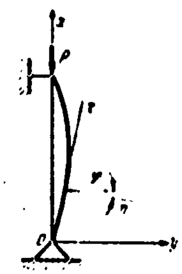

Figure 15

$$\frac{d^2 v}{ds^2} = \cos\varphi \cdot \frac{d\varphi}{ds}$$

which gives

$$\frac{1}{\varrho} = \frac{d\varphi}{ds} = \frac{d^2 v}{ds^2} \cdot \frac{1}{\cos\varphi}.$$

Since

$$\cos\varphi = \sqrt{1 - \sin^2\varphi} = \sqrt{1 - \left(\frac{dv}{ds}\right)^2}$$

we obtain the following expression for the curvature

$$\frac{1}{\varrho} = \frac{\frac{d^2 v}{ds^2}}{\sqrt{1 - \left(\frac{dv}{ds}\right)^2}}.$$

Equation (3.1) for the buckling of a rod takes the form

$$\frac{\frac{d^2 v}{ds^2}}{\sqrt{1-\left(\frac{dv}{ds}\right)^2}}+\frac{Pv}{EJ}=0. \tag{3.2}$$

The nonlinear differential equation obtained can be reduced to a form more convenient for solution. By expanding the radical in the binominal expansion

$$\left[1-\left(\frac{dv}{ds}\right)^2\right]^{-\frac{1}{2}}=1+\frac{1}{2}\left(\frac{dv}{ds}\right)^2+\frac{3}{8}\left(\frac{dv}{ds}\right)^4+\cdots, \tag{3.3}$$

Eq. (3.2) can be rewritten in the following form:

$$\frac{d^2 v}{ds^2}\left[1+\frac{1}{2}\left(\frac{dv}{ds}\right)^2+\frac{3}{8}\left(\frac{dv}{ds}\right)^4+\cdots\right]+\frac{Pv}{EJ}=0. \tag{3.4}$$

The first term in Eq. (3.3) corresponds to the usual linear approximation in strength of materials. We obtain the first nonlinear approximation by retaining two terms in Eq. (3.3); such an approximation will be suitable for small deflections.

If the longitudinal force does not greatly exceed the Euler buckling value, then the elastic curve differs only slightly from the form of the first characteristic function of the linear problem,

$$v(s)=f\sin\frac{\pi s}{l}. \tag{3.5}$$

We shall make use of the Galerkin variational method for finding the unknown deflection f. Variational methods recently have found greater and more effective application in different divisions of the applied theory of elasticity and structural mechanics; we shall assume that the reader is familiar with these methods.[1]

Following the method of Galerkin, we substitute Eq. (3.5) into the left-hand side of Eq.˚ (3.4) and require the resulting equation to the selected function $\sin \frac{\pi s}{l}$ to be orthogonal:

$$\int_0^l \left\{ \frac{d^2 v}{ds^2} \left[1 + \frac{1}{2} \left(\frac{dv}{ds} \right)^2 + \frac{3}{8} \left(\frac{dv}{ds} \right)^4 + \cdots \right] + \frac{Pv}{KJ} \right\} \sin \frac{\pi s}{l} \, ds = 0.$$

With the known integrals

$$\int_0^l \sin \frac{\pi s}{l} \, ds = \frac{l}{2}, \qquad \int_0^l \cos^2 \frac{\pi s}{l} \sin^2 \frac{\pi s}{l} \, ds = \frac{l}{8},$$
$$\int_0^l \cos^4 \frac{\pi s}{l} \sin^2 \frac{\pi s}{l} \, ds = \frac{l}{16}$$

we obtain

$$\left(1 + \frac{\pi^2 f^2}{8 l^2} + \frac{3}{32} \frac{\pi^4 f^4}{l^4} + \cdots \right) f - \frac{P}{P_e} f = 0. \tag{3.6}$$

for the determination of f. Here, as before

$$P_e = \frac{\pi^2 KJ}{l^2}.$$

One of the possible solutions of Eq. (3.6) is f = 0; this solution obviously corresponds to the initial (uncurved) shape of the rod. The non-zero solutions can be found from the condition

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[1] We recommend Ref. 81 as a first reading. See Refs. 82 and 83 for readings in English.

$$1 + \frac{\pi^2 f^2}{8 l^2} + \frac{3}{32} \frac{\pi^4 f^4}{l^4} + \cdots - \frac{P}{P_*} = 0. \tag{3.7}$$

If higher powers of f are neglected in Eq. (3.7), we obtain the well-known approximate equation[2]

$$f = \frac{2\sqrt{2}}{\pi} l \sqrt{\frac{P}{P_*} - 1}. \tag{3.8}$$

A comparison with the exact solution of the same problem, using elliptical integrals, shows that Eq. (3.8) gives good results for $f < 0.2 l$; consequently, $P < 1.045 P_*$. In order to obtain the second approximation, one should retain the f^4 terms, and so on.

Note that this equation and also the equations for the higher approximations can be obtained as a special case from the results of Euler, (Ref. 85). In some later works a correction to the second approximation was calculated incorrectly as indicated by Nikolai (Ref. 86).

§13. NONLINEAR ELASTICITY

1. We return to the problem of the dynamic stability of a hinged straight rod that is compressed by a periodic longitudinal force. We shall proceed from the equation

$$f'' + 2\varepsilon f' + \Omega^2 (1 - 2\mu \cos \theta t) f + \psi(f, f', f'') = 0 \tag{3.9}$$

that differs from Eq. (2.1) by the presence of certain nonlinear functions of displacements, velocities, and accelerations, $\psi(f, f', f'')$. The determination of this function constitutes our next problem.

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[2] See Ref. 84, Volume I for example.

Among the terms entering into the nonlinear function $\psi\,(f,\ f',\ f'')$, it is always possible to single out such terms that do not contain derivatives of displacements with respect to time. These nonlinear terms characterize the nonlinear elasticity of the system. Taking this into account, we will group all the static nonlinear factors under the general designation of nonlinear elasticity, regardless of whether they are of a geometrical or physical origin.

Nonlinear elasticity is the only nonlinear factor in the problem of the bending of a rod by a force exceeding the Euler buckling load. The governing equation can be obtained from Eq. (3. 9), if one assumes f = const. and eliminates terms explicitly depending on time. As a result, we obtain

$$\Omega^2 f + \psi(f) = 0$$

or

$$\left(1 - \frac{P}{P_*}\right) f + \frac{1}{\omega^2}\,\psi(f) = 0. \tag{3. 10}$$

Comparing Eq. (3. 10) with Eq. (3. 6), we can conclude that the nonlinear function corresponding to the complete expression for the curvature has the form

$$\psi(f) = \omega^2 f\left(\frac{\pi^2 f^2}{8 l^2} + \frac{3}{32}\,\frac{\pi^4 f^4}{l^4} + \cdots\right)$$

This means that the function $\psi(t)$, which takes into account the influence of the nonlinear curvature, can be represented in the form of a series that contains odd powers of the deflection

$$\psi(f) = \sum_{k=1}^{\infty} \gamma_k f^{2k+1}; \qquad (3.11)$$

where

$$\gamma_1 = \frac{\pi^2 m^2}{8 l^2}, \quad \gamma_2 = \frac{3}{32}\frac{\pi^4 m^4}{l^4}, \quad \ldots .$$

and so on.

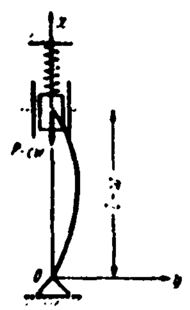

Figure 16

It can be shown that consideration of certain other nonlinearities leads also to nonlinear functions of the type in Eq. 3.11. Let us examine, for example, a rod with a longitudinal elastic spring (Fig. 16). During vibration, an additional longitudinal force arises from the reaction of the spring:

$$\Delta P = -c\,w. \qquad (3.12)$$

The displacement of the moving end of the rod is denoted by w, the stiffness of the spring is denoted by c, and the axial deformation of the rod is not considered.

Essentially the longitudinal displacement w is nonlinearly related to the transverse deflection of the rod. The longitudinal displacement of the moving end can be found as the difference between the initial length l and the projection of its deformed center line

$$w = l - \int_0^l \cos\varphi\, ds = l - \int_0^l \sqrt{1 - \left(\frac{dv}{ds}\right)^2}\, ds.$$

Expanding the radical in a series

$$\sqrt{1-\left(\frac{dv}{ds}\right)^2} = 1 - \frac{1}{2}\left(\frac{dv}{ds}\right)^2 - \frac{1}{8}\left(\frac{dv}{ds}\right)^4 + \cdots,$$

and integrating by terms, we obtain

$$w = \frac{1}{2}\int_0^l \left(\frac{dv}{ds}\right)^2 ds + \frac{1}{8}\int_0^l \left(\frac{dv}{ds}\right)^4 ds + \cdots. \tag{3.13}$$

The first term of this series is frequently encountered in the bending calculation as an expression for the deflection of the moving end of a rod.

If we substitute into Eq. (3.13) the expression

$$v(s, t) = f(t) \sin \frac{\pi s}{l}.$$

and use the defined integrals

$$\int_0^l \cos^2 \frac{\pi s}{l}\, ds = \frac{l}{2}, \quad \int_0^l \cos^4 \frac{\pi s}{l}\, ds = \frac{3}{8} l, \quad \ldots$$

etc., we find

$$w = \frac{\pi^2 f^2}{4l} + \frac{3}{64}\frac{\pi^4 f^4}{l^3} + \cdots. \tag{3.14}$$

Taking into account the additional longitudinal force in Eq. (3.12), the equation of the vibrations of a rod, Eq. (2.1), takes the form

$$f'' + 2\varepsilon f' + \omega^2\left(1 - \frac{P_* + P_t \cos\theta t + \Delta P}{P_*}\right) f = 0$$

(Compare it with Eq. (1.4)). After substituting Eq. (3.14), we obtain

$$f'' + 2\varepsilon f' + \omega^2\left(1 - \frac{P_* + P_t \cos\theta t}{P_*}\right) f + \psi(f) = 0$$

where

$$\psi(f) = \frac{\pi^2 \omega^2 c}{4 l P_*} f^3 + \frac{3}{64} \frac{\pi^4 \omega^2 c}{l^3 P_*} f^5 + \cdots .$$

Thus a consideration of a longitudinal elastic spring also results in the nonlinear function of the type in Eq. (3.11). We will limit outselves to non-linear terms less than third order in further considerations and will write the nonlinear function in the form

$$\psi(f) = \gamma f^3 \tag{3.15}$$

The coefficient of nonlinear elasticity[3] is

$$\gamma = \frac{\pi^2 \omega^2}{8 l^2}\left(\frac{2 c l}{P_*} + 1\right); \tag{3.16}$$

where the second term accounts for the effect of nonlinear curvature. It is seen from this formula that, with accuracy up to quantities of third order magnitude, the influence of the nonlinear curvature is equivalent to the influence of a longitudinal spring with a spring stiffness

$$c = \frac{\pi^2 E J}{2 l^3} .$$

[3] More exactly, it is the coefficient of the cubic term in the developed series.

2. Nonlinearities of the above type often arise in applications. This nonlinearity is contained, for example, in rods of statically indeterminate systems which remain geometrically unchanged after the rods are unloaded (so-called redundant rods).

In fact, during the bending of such rods, the ends approach one another, causing additional elastic forces to appear in the remaining part of the system tending to hinder the approach of the ends. Since the approach of the ends described by Eq. (3. 14) is nonlinearly related to the deflections of the rod, the elastic forces also have a nonlinear character.

Nonlinear elastic forces can arise also in systems that are usually treated as being statically determinate. For example, riveted or welded girders are statically determinate when they are assumed to be hinge-connected and statically indeterminate if the stiffness of the hinges is considered. Still more complicated relations arise in spatial constructions, e.g., in the frames of metal bridges. In determining the nonlinear characteristics of any rod entering into the composition of a frame structure, the total influence of the spatial construction must be considered.

The presence of nonlinear elasticity reflects on the carrying capacity of the compressed rods, which lose their stability in the elastic range. As is well-known, for single rods in which the nonlinear elasticity is dependent on the nonlinear curvature, even an insignificant increase of critical force results in dangerous deformations. Therefore, the critical load for such rods can, in practice, be considered to be equal to the limit load.[4] If a rod is a part of a statically indeterminate system, its deflection in the post-critical stage can be considerably smaller. To illustrate this concept, let us consider a simple example (Fig. 17).

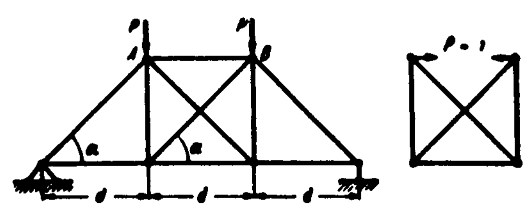

Figure 17

[4] The case of very flexible rods where large deflections occur without yielding and where creep of the material is possible will not be considered.

Under the action of an external loading, it is possible for element AB to become unstable. To determine the coefficient of resistance for the remaining part of the construction, we compute the approach of end points A and B under the action of unit forces. (Fig. 17). According to the well-known equation of structural mechanics,

$$\delta_{11} = \sum \frac{\bar{N}_1^2 s}{EF}.$$

For simplicity, the areas of all the rods will be assumed to be equal. Computations give

$$\delta_{11} = \frac{d}{EF}\left(1 + 2\tan^3\alpha + \frac{2}{\cos^3\alpha}\right),$$

hence the coefficient of resistance is

$$c = \frac{1}{\delta_{11}} = \frac{EF}{d\left(1 + 2\tan^3\alpha + \frac{2}{\cos^3\alpha}\right)}.$$

In accordance with Eq. (3. 16) the coefficient of nonlinear elasticity is

$$\gamma = \frac{\pi^4 \omega^2}{8d^3}\left(\frac{2cd}{P_e} + 1\right).$$

The expression in parentheses can be written in the form

$$\frac{2cd}{P_e} + 1 = \frac{2\lambda^2}{\pi^2\left(1 + 2\tan^3\alpha + \frac{2}{\cos^3\alpha}\right)} + 1$$

where λ is the fineness ratio of the rod. Following from Eqs. (3. 10) and (3. 15), the deflection of the middle of the rod is determined from the equation

$$f = \frac{\omega}{\sqrt{\gamma}} \sqrt{\frac{N}{N_*} - 1}$$

or

$$f = \frac{2d\sqrt{2}}{\pi k(\alpha, \lambda)} \sqrt{\frac{N}{N_*} - 1}$$

where N is the longitudinal force in the rod, N_* is its critical value, and k is the nondimensional coefficient

$$k(\alpha, \lambda) = \sqrt{\frac{2\lambda^2}{\pi^2\left(1 + 2\tan^2\alpha + \frac{2}{\cos^2\alpha}\right)} + 1}.$$

Since $k(\alpha, \lambda)$ is of the order of magnitude of the fineness ratio λ (at $\alpha = 45^{\circ}$ we have $k \gtrsim 0.15\lambda$) then the deflection of rod AB can be one-tenth the displacement of a single rod.

Further consideration of problems dealing with frame constructions in the postcritical state falls outside of the framework of the present book.

3. Hitherto, we have restricted our treatment to the case of an elastic material with linear properties. Nonlinear elastic properties of a material can also be considered with the aid of Eq. (3.11). Since for the majority of known materials, the stress σ and the strain e satisfy the inequality $(d^2\sigma)/(de^2) \leq 0$ (Fig. 18), nonlinearity of this kind in general will be "soft" in contrast to "hard" nonlinearity of cases considered earlier.[5]

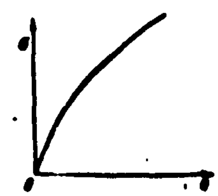

Figure 18

[5]The nonlinearity is called "soft" if the quasi-elastic coefficient decreases with displacement and "hard" if the quasi-elastic coefficient increases with displacement.

For example, let the properties of the material be described by the relation

$$\sigma = E(e - \beta e^3)$$

where β is a constant. Since plane sections remain plane, we can assume

$$e = y\frac{d^2v}{dx^2}$$

and we find

$$M = \int_F \sigma y\, dF = EJ\frac{d^2v}{dx^2} - B\left(\frac{d^2v}{dx^2}\right)^3$$

for the bending moment in the beam cross-section, where B is a new constant. After substitution in the ordinary differential equation, the last term on the right-hand side yields a nonlinear function for $\lambda < 0$ of the form in Eq. (3.15).

The question arises whether or not the nonlinear function in Eq. (3.11) can include all the possible cases of nonlinear elasticity. Even in the case where the magnitude of elastic forces depends on the sign of the deflections, we must introduce in Eq. (3.11) terms containing even powers of the deflection (nonsymmetrical power characteristic). Furthermore, nonlinear elastic forces exist whose dependence on the deflections cannot be represented in the form of converging power series. The simplest example of this sort is the rod with deflection restrictors which are springs (Fig. 19).

Figure 19

Assume that the stiffness of the restrictors is sufficiently small so that one can neglect the influence of the additional support on the form of an elastic curve (only in this case can the nonlinearity be condisered small). The nonlinear function then takes the form

$$\psi(f) = \begin{cases} 0 & \text{for } |f| \leq \Delta, \\ \frac{c\,l}{2m} f & \text{for } |f| > \Delta, \end{cases}$$

where c is the spring constant of the restrictor. An expression of this kind obviously cannot be represented in the form of a power series.

However, problems of this type are seldom encountered, while at the same time, functions such as Eq. (3.11) encompass the majority of problems. In all following considerations, we will confine ourselves to Eq. (3.11).

•14. NONLINEAR INERTIA[6]

Previously we have considered nonlinear factors of static origin. In dynamic problems one must also consider nonlinear inertia forces and nonlinear damping.

As an example of nonlinear inertia we will investigate a simple problem and assume that on the moving end of the rod there is a concentrated mass M_L (Fig. 20). In this case the additional longitudinal force

$$\Delta P = -M_L w''.$$

arises during vibration of the rod. As before, w(t) denotes the longitudinal displacement of the moving end

$$w = \frac{\pi^2}{4l} f^2 + \frac{3}{64} \frac{\pi^4}{l^3} f^4 + \cdots.$$

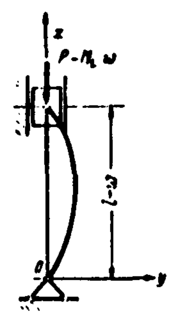

Figure 20

[6] This designates the nonlinear (more exactly, cubic) part of the expression for the inertia effect that enters mechanical problems through the choice of variables. In Lagrangian mechanics, on the other hand, one speaks only generally of inertia forces even if these are described by a nonlinear expression relating the chosen generalized coordinates.

By considering this longitudinal force, the differential equation of the tranverse vibrations takes the form

$$f'' + 2\varepsilon f' + \omega^2\left(1 - \frac{P_0 + P_t \cos\theta t - M_L w''}{P_*}\right) f = 0$$

where

$$w'' = \frac{\pi^2}{2l}\left[f f'' + (f')^2\right] + \frac{3}{16}\cdot\frac{\pi^4}{l^3} f^2\left[f f'' + 3(f')^2\right] + \cdots.$$

If we substitute the frequency Ω of the natural vibrations of a loaded rod and the exciting parameter μ, we obtain

$$f'' + 2\varepsilon f' + \Omega^2(1 - 2\mu\cos\theta t) f + 2\varkappa f\left[f f'' + (f')^2\right] = 0.$$

Terms which are higher than third order have been neglected. The influence of the nonlinear inertia forces is considered through the function

$$\varphi(f, f', f'') = 2\varkappa f\left[f f'' + (f')^2\right] \tag{3.17}$$

The coefficient

$$\varkappa = \frac{\pi^4 M_L}{4 m l^3} \tag{3.18}$$

will be called the coefficient of nonlinear inertia.

2. A nonlinear expression of the type in Eq. (3.17) was first found in an article by Krylov and Bogoliubov (Ref. 5) in connection with the investigation of the natural vibrations of struts. The necessity to account for this nonlinearity in the case of parametric resonce was indicated in an article by Gol'denblat (Ref. 55).

The presence of a concentrated mass on the moving end is not the only condition which gives rise to nonlinear inertia forces. During transverse vibrations, every section of the rod goes through some longitudinal displacement. This displacement is of second order in comparison to the transverse deflection

$$w(x) \approx \frac{1}{2}\int_0^x \left(\frac{\partial v}{\partial \xi}\right)^2 d\xi .$$

Therefore a distributed loading due to the inertia forces

$$n(x, t) = -m\,\frac{\partial^2 w}{\partial t^2}.$$

acts on the rod. Assume that this longitudinal loading influences the shape of the vibration only slightly and again take

$$v(x, t) = f(t)\sin\frac{\pi x}{l}. \tag{3.19}$$

After substitution and integration, we find

$$n(x, t) = -\frac{\pi^2 m}{2l^2}\left(x + \frac{l}{2\pi}\sin\frac{2\pi x}{l}\right)[f f'' + (f')^2].$$

Obviously the additional longitudinal force in each section is

$$\Delta N(x, t) = \int_l^{l-x} n(\xi, t)\, d\xi$$

or, after integration is

$$\Delta N(x, t) = -\frac{\pi^2 m}{4}\left(1 - \frac{x^2}{l^2} - \frac{1}{\pi^2}\sin^2\frac{\pi x}{l}\right)[f f'' + (f')^2].$$

We will now formulate the differential equation of the vibrations of a rod. In contrast to Eq. (1.2), it must contain an additional term that considers the influence of the longitudinal force $\Delta N(x, t)$ dependent on x. By neglecting damping, the equation will have the form

$$EJ\frac{\partial^4 v}{\partial x^4} + (P_0 + P_t \cos\theta t)\frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x}\left(\Delta N \frac{\partial v}{\partial x}\right) + m\frac{\partial^2 v}{\partial t^2} = 0.$$

With the aid of Galerkin's variational method, it can be reduced to an ordinary differential equation. In fact, if we substitute Eq. (3.19), multiply by $\sin\frac{\pi x}{l}$, and integrate, we obtain

$$f'' + \omega^2\left(1 - \frac{P_0 + P_t\cos\theta t}{P_*}\right)f + 2\varkappa f[f f'' + (f')^2] = 0$$

where

$$\varkappa = -\frac{\pi^2}{4l^2}\int_0^l \sin\frac{\pi x}{l}\,\frac{d}{dx}\left[\left(1 - \frac{x^2}{l^2} - \frac{1}{\pi^2}\sin^2\frac{\pi x}{l}\right)\cos\frac{\pi x}{l}\right]dx.$$

The calculation of the integral gives

$$\varkappa = \frac{\pi^4}{4l^2}\left(\frac{1}{3} - \frac{3}{8\pi^2}\right). \tag{3.20}$$

The coefficient χ, determined according to Eq. (3.20), takes into account the influence of the inertia forces of the rod itself. Comparing Eqs. (3.18) and (3.20), we see that the influence of the inertia forces of the rod has the same influence as an equivalent mass

$$M_L = \left(\frac{1}{3} - \frac{3}{8\pi^2}\right)ml$$

concentrated on the moving end.

3. We shall show how to estimate the magnitude of the coefficient of nonlinear inertia for elements of complicated rod systems. Consider, for example, element AB of the upper boom of a multiple span truss (Fig. 21). Under the action of the external loading

$$q = q_0 + q_t \cos \theta t$$

periodic stresses arise in the rods of the truss. The magnitude of these stresses can be calculated by ordinary structural dynamics methods. The truss experiences the usual forced vibrations in which additional terms of second order amplitude arise if one of the rods becomes unstable.

If rod AB becomes dynamically unstable, its vibrations due to the approach of the ends (expressed by a quantity w) will be accompanied by vibrations of the entire truss. Assuming that the truss is statically determinate (ideal hinges), we obtain the vibration form shown in Fig. 21b. The amplitudes obviously are quantities of second order in comparison to the deflection of rod AB. It is important to emphasize that we are concerned only with the additional deformation of the truss arising from the approach of the ends of element AB.

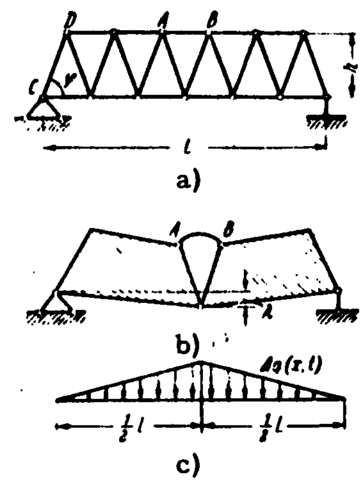

Figure 21

We will now calculate the coefficients of nonlinear elasticity. The deflection of the middle of a span can be found from simple geometric considerations and is

$$\lambda = \frac{wL}{4h},$$

where L is the span of the truss and h is its height. Additional inertia loading on the truss as seen from Fig. 21c is

$$\Delta q(x, t) = \begin{cases} -\dfrac{m_q x}{2h}\dfrac{d^2 w}{dt^2} & \text{for } x \leq \dfrac{L}{2}, \\ -\dfrac{m_q (L - x)}{2h}\dfrac{d^2 w}{dt^2} & \text{for } x > \dfrac{L}{2}; \end{cases}$$

where m_q denotes the sum of the mass of the truss per unit length and the mass of the previously given distributed load. Hence, one easily finds the additional stress[7] in rod AB.

$$\Delta N = -\frac{m_q L^3}{48h^2}\frac{d^2 w}{dt^2}. \tag{3.21}$$

As seen from Eq. (3.21), the influence of the inertia forces of the truss on the magnitude of the longitudinal force in the rod is equivalent to the influence of the concentrated mass

$$M_L = \frac{m_q L^3}{48h^2}$$

placed on the moving end of the rod. The coefficient of nonlinear inertia of the rod AB is

$$\varkappa \approx \frac{m_q n^2}{2m h^2}; \tag{3.22}$$

where m is the mass per unit length of the rod and n is the number of panels of the bottom ginder. The coefficient of the nonlinear inertia for other rods

[7] Our calculation is only approximate; we actually referred all the mass of the truss and the loading to the lower boom. This type of procedure is often used in "linear" structural dynamics. It appears that this approximate method gives sufficient accuracy for practical purposes (see Ref. 87).

is determined in a similar manner. Thus, for the supporting cross strut CD (Fig. 21), we obtain

$$\varkappa = \frac{\pi^4(n-1)^2}{6 n s^2 \cos\varphi} \frac{m_e}{m}, \tag{3.23}$$

where s is the length of the strut and ϕ is the angle it makes with the horizontal. In general, if one considers only the vertical inertia forces, the coefficient χ can be computed by the following approximate formula for beam elements (statically determinate and also statically indeterminate) of the truss:

$$\varkappa = \frac{\pi^4}{4 m l^3} \int\limits_L m_e(s)\, \psi^2(s)\, ds \tag{3.24}$$

In Eq. (3.24), $\psi(s)$ are ordinates of the influence line of a force which results from a unit vertical force in the corresponding element. To derive this formula, it must be assumed that all of the mass of the rod system can be referred to a definite line, e.g., to the line of action. Then the additional inertia forces can be determined by the formula

$$\Delta q(s, t) = -m_e(s)\, \psi(s) \frac{d^2 w}{dt^2}$$

from which, according to the loading of the influence line, we find

$$\Delta N(t) = -\frac{d^2 w}{dt^2} \int\limits_0^L m_e(s)\, \psi^2(s)\, ds$$

for the force in the rod. The equivalent mass obviously is

$$M_L = \int\limits_0^L m_e(s)\, \psi^2(s)\, ds.$$

from which Eq. (3.24) follows. Equations (3.22) and (3.23) are its special cases.

Equation (3.24) can also be easily extended to any rod system; for this purpose the concept of the "influence line" must be interpreted in a broader sense. Other examples for the determination of the coefficient of nonlinear inertia will occur.

•15. NONLINEAR DAMPING

1. The damping of free and forced vibrations has not yet been fully considered. The complex and diverse processes accompanying energy dissipation during vibrations and the influence of a large number of factors that are difficult to take into account by theoretical means have resulted, up to this point, in the reduction of the effect of damping to the addition of certain "suitable" terms in the equation of the conservative problem. These terms are selected so that the theoretical results are in satisfactory agreement with experimental data. A large number of "hypotheses" of every kind that do not agree with the experimental data are characteristic of the present state of the art on damping (Refs. 88 and 89).

We will investigate the equation of damped vibrations

$$f'' + \omega^2 f = R(f, f'). \tag{3.25}$$

where R(f, f') is an additional term which considers the resistance forces. The simplest and most widely used expression for the calculation of damping is

$$R(f') = -2\varepsilon f' \tag{3.26}$$

where ε is a constant, leads to the result briefly summarized in the following.

The amplitudes of free vibrations decrease in a geometric series so that the logarithmic decrement of damping

$$\ln \frac{f(t)}{f(t+T)} = \varepsilon T = \delta \tag{3.27}$$

is a quantity that is independent of the amplitude. The period of the free vibrations will be denoted by T:

$$T = \frac{2\pi}{\sqrt{\omega^2 - \varepsilon^2}} \approx \frac{2\pi}{\omega_0}.$$

The dissipation of energy

$$\Delta W = -\int_T R(f')\,df \tag{3.28}$$

for one period is proportional to the square of the amplitude, and the relative dissipation

$$\psi = \frac{\Delta W}{W} \tag{3.29}$$

is independent of the amplitude.[8]

Numerous experiments show, however, that energy loss during vibrations is only slightly dependent on the velocity. (For example, the area of loop of elastic hysteresis is practically independent of the duration of the vibration cycle, i.e., of the speed of deformation.) Meanwhile, according to Eq. (3.26), it would seem that the resistance forces are greatly dependent on the velocity. For many years "the hypothesis of viscous resistance" has

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[8]Computations give $\Psi = 2\delta$.

often been criticized for this.[9] It is proposed here to consider the forces of resistance proportional to the amplitude and oriented in the direction opposite to that of the velocity

$$R(f, f') = -\gamma |f| \operatorname{sign} f'; \tag{3.30}$$

where one can write

$$\operatorname{sign} f' = \begin{vmatrix} 1 & \text{for } f' > 0, \\ -1 & \text{for } f' < 0, \end{vmatrix}$$

or make use of "the complex modulus," borrowed from acoustics and electrical engineering, etc.

The quantities that the resistance forces depend upon are not available from Eq. (3.26). It is simply an indirect method of considering damping. Furthermore, there is no basis at hand for assigning the interpretation of a material constant to the damping coefficient. On the contrary, it is more logical (and this is confirmed by experimental data) to consider the corresponding energy dissipation as a material constant or (almost the same thing) the logarithmic decrement of damping. In this case, the "hypothesis of viscous resistance" and also the methods that have been proposed give analogous, and in the case of a suitable selection of the constants, coinciding results.

We will call the damping <u>linear</u> if the relative dissipation of energy ψ is independent of the amplitude. One can easily see that this definition is somewhat broader than the one that is generally used. For example, it also includes the damping described by Eq. (3.30) which is generally nonlinear. For the explicit calculation we are especially interested not in the change of the damping during the period but in its overall effect during a period. Its measurement describes the relative dissipation of energy. From this viewpoint, the expressions for the resistance forces given by Eqs. (3.26) and (3.30) are equivalent.

[9] For a survey of the literature, see Ref. 89.

The most detailed investigations on nonlinear damping consider effects of the internal damping in the material. As is known, the energy losses due to internal friction form the most significant part of the general energy dissipation. At the same time, this part lends itself to theoretical considerations.[10]

Among the expressions for the consideration of internal damping, we will indicate only a generalization of the "classical" damping expression Eq. (3.26) by

$$R(f') = -h|f'|^k \operatorname{sign} f'$$

For a determination of constants h and k, numerous experiments were carried out on bending and torsional vibrations. Thus, according to the data of Lunts (Ref. 91), k = 2.17 for steel; according to other data, the index k takes on values from 2 to 3.

Davidenkov (Ref. 88) related the magnitude of internal friction to the phenomenon of elastic hysteresis. In order to describe the upper and lower branches of the hysteresis curve $\sigma = \sigma(e)$, he proposed the equation

$$\sigma = E\left\{e \mp \frac{\eta}{n}\left[(e_0 \pm e)^n - 2^{n-1} e_0^n\right]\right\} \tag{3.31}$$

where η and n are material constants and e_0 is the deformation amplitude (Fig. 22). On the basis of Eq. (3.31), a series of special problems were investigated by Pisarenko (Ref. 92).

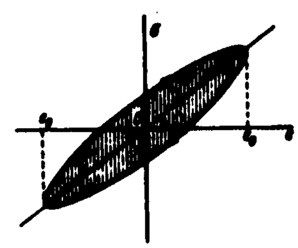


Figure 22

2. As is seen from the above, the usual methods for considering damping are rarely used for the solution of the indicated problem. In particular, resulting nonlinear "hypotheses" are inadequate for two reasons. First, even if one

[10] The physical aspect of the problem of internal friction is discussed, for example, in Ref. 90.

assumes that they are correct, they take into account only a comparatively small part of the total dissipation of energy. We have indicated this already (•9). Second, the application of these "hypotheses" creates serious mathematical difficulties which have not yet been applied to even the simplest problems of free and forced oscillations. All this stimulates the search for new methods which are more suitable and adaptable to the calculation of damping.

It appears to be more logical to consider the coefficient of damping in Eq. (3.26) not as a constant, but as a function of displacement. Considering that this function must be even, and approximating it by a power series, we obtain the following expression

$$R(f, f') = -2(\varepsilon + \varepsilon_1 f^2 + \varepsilon_2 f^4 + \cdots) f' \tag{3.32}$$

(here ϵ, ϵ_1, ϵ_2, . . . are constants to be determined from experiments).

The expression of the type in Eq. (3.32) is encountered in radio engineering in the theory of vacuum tube generators. This corresponds to the so called "soft" condition of the generators when the series of this expression contains the first two terms, and to the "hard" condition (see Ref. 93) when it contains three terms. It is well-known that problems with nonlinearities of the type in Eq. (3.32) are readily adaptable to mathematical treatment. An investigation of free vibrations produces an amplitude dependent decrement in the final calculation.

In this connection, we shall consider a recent proposal of Panovko (Ref. 89). We will not go into the basic theory of this proposal (the elliptic form of the loop of the hysteresis and the independence of its area on the frequency), but note only that it is equivalent to the expression for the resistance force

$$R(f, f') = -bA^n \sqrt{1 - \frac{f^2}{A^2}} \operatorname{sign} f'$$

where A corresponds to the amplitude of the vibrations, and b and n are constants. We shall now take

$$f = A\cos(\omega t + \lambda);$$

where A is a constant for the forced vibration problem and is a "slowly changing" function of time in the free vibration problem with a small amount of damping. Then

$$R(f, f') = -bA^n \sin(\omega t + \lambda)$$

or

$$R(f, f') = -\frac{b}{\omega} A^{n-1} f'. \tag{3.33}$$

For damped vibrations the last equation is approximate.

In a certain sense, Eqs. (3.32) and (3.33) are equivalent. If in Eq. (3.32) the damping coefficient is a function of the instantaneous value of displacement, then it is also dependent on the amplitude. Equation (3.33) is simpler, but it is too "quasi-linear". It becomes unsuitable, for example, for vibrations with two different frequencies. Independently from this, we are inclined to favor Eq. (3.32) since it can be reduced to the following mechanical model.

Let us return to our model (Fig. 16). Together with the usual resistance force we will also consider a friction force that arises during the guidance of the moving support. We will assume that it is proportional to the velocity of the displacement of the support

$$\Delta P = -k_L w'.$$

The equation for the free vibrations of a damped rod takes the form

$$f'' + 2\varepsilon f' + \omega^2\left(1 + \frac{k_L w'}{P}\right) f = 0 \qquad (3.34)$$

(Compare with •13 and •14.) By a termwise differentiation of Eq. (3.14), we find

$$w' = \frac{\pi^2}{2l} f f' + \frac{3}{16} \frac{\pi^4}{l^3} f^3 f' + \cdots.$$

Substitution into Eq. (3.34) yields

$$f'' + 2\varepsilon f' + \omega^2 f + \frac{\pi^2 k_L \omega^2}{2l P_*} f^2 f' + \frac{3}{16} \frac{\pi^4 k_L \omega^2}{l^3 P_*} f^4 f' + \cdots = 0;$$

which reduces to a nonlinear function of the type in Eq. (3.32).

If the displacements are small, then the nonlinear function can be confined to terms less than third order. This function assumes the form

$$\psi(f, f') = 2\varepsilon_L f^2 f' \qquad (3.35)$$

where ϵ_L is the coefficient of nonlinear damping

$$\varepsilon_L = \frac{\pi^4 k_L}{4 m l^3}. \qquad (3.36)$$

We shall show that if a rod is part of a rod system, the nonlinearity of the damping can be explained by the energy dissipation in the remaining part of the system. This follows from considerations similar to those expressed in •14.

As long as the amplitudes of the vibrations of the rod are sufficiently small, in practice the energy dissipation occurs in the rod itself. Because of the approach of the ends of the rod, for larger amplitudes additional displacements of the entire truss that are nonlinearly related to the displacements of the rod arise. The energy loss due to these displacements also produces damping which is nonlinear. Under certain additional assumptions, the nonlinear damping for this case can be determined by analytical means.

Let us investigate the truss shown in Fig. 21. As is often done, we will represent the resistance forces directed opposite to the motion in the form of a distributed loading that is proportional at every point to the velocity of the motion. Considering that

$$\lambda = \frac{wL}{4h}$$

we obtain:

$$\Delta q(x, t) = \begin{cases} -\frac{k_q x}{2h} w' & \text{for} \quad x \leq \frac{L}{2}, \\ -\frac{k_q (L-x)}{2h} w' & \text{for} \quad x > \frac{L}{2}; \end{cases}$$

where k_q is the resistance coefficient for the truss. An additional longitudinal force

$$\Delta P = -\frac{k_q L^3}{48h^2} w'.$$

is formed in the element AB. Thus, the damping in a truss is equivalent to the damping caused by the resistance developed in the moving support that has a coefficient

$$k_L = \frac{k_q L^3}{48h^2}$$

The coefficient of nonlinear damping will be

$$\varepsilon_L = \frac{\pi^4 k_q L^3}{192 m h^2 l^3}$$

or approximately

$$\varepsilon_L \approx \frac{k_0 n^2}{2 m \lambda^2}, \tag{3.37}$$

where n is the number of panels of the lower girder. It is seen from Eq. (3.37) that the nonlinear part of the damping increases rapidly as the number of panels of the truss increases.

3. The nonlinear resistances described so far are characterized by the fact that, depending upon the amplitude, its work in a period increases faster than the work of viscous (velocity proportional) friction. Somewhat apart from this, we have so-called dry friction, which usually is considered constant in magnitude with its direction opposite to that of the velocity:

$$R(f') = -k_0 \operatorname{sign} f'.$$

It is readily seen that the work of dry friction during a period

$$\Delta W = \int_T k_0 (\operatorname{sign} f')\, df$$

is proportional to the first power of the amplitude, i. e., it increases slower than the work of linear viscous friction. Therefore, dry friction, taken separately, cannot explain the presence of steady-state vibrations at parametric resonance.

Dry friction can arise in support attachments where it acts with viscous resistance (combined friction). It is interesting that the presence of dry friction in the moving support produces damping of the transverse vibrations of the rod according to a law that is characteristic for a linear resistance. In other words, dry friction in a moving support and viscous resistance of transverse vibration are in a certain sense equivalent. In fact, the frictional force in the moving support is

$$\Delta P = -k_0 \operatorname{sign} w'$$

or[11]

$$\Delta P = -k_0 \operatorname{sign}(f f').$$

A substitution in the equation of free vibrations yields

$$f'' + \omega^2 \left[1 + \frac{k_0 \operatorname{sign}(f f')}{P_*} \right] f = 0.$$

But since

$$f \operatorname{sign}(f f') = |f| \operatorname{sign} f'$$

this equation takes the form

$$f'' + \omega^2 f + \frac{k_0 \omega^2}{P_*} |f| \operatorname{sign} f' = 0.$$

Thus, the presence of dry friction in the moving support is considered in the transverse vibration equation by the component of the type in Eq. (3. 30), i. e., by a resistance equivalent to the viscous resistance.

[11] Terms of higher order obviously do not influence the sign of w'.

CHAPTER FOUR

FREE AND FORCED VIBRATIONS OF A NONLINEAR SYSTEM

•16. METHOD OF SLOWLY CHANGING AMPLITUDES

1. The modern methods for the investigation of nonlinear vibrations are based on the well-known works of Liapunov (Ref. 94) and Pioncaré (Ref. 95; see also Ref. 96).

A rigorous examination of nonlinear differential equations in the general case leads to serious mathematical difficulties. However, there exists a broad class of differential equations which can be solved by effective approximate methods. To this class of equations belong, in particular, the equations describing the vibrations of systems with small nonlinearities.

Let us consider the nonlinear differential equation

$$f'' + 2\varepsilon f' + \omega^2 f + \psi(f, f', f'') = 0. \tag{4.1}$$

By introducing a nondimensional time and a nondimensional displacement, we can write this equation in the form

$$\eta'' + \frac{2\varepsilon}{\omega}\eta' + \eta + \psi(\eta, \eta', \eta'') = 0.$$

The prime denotes differentiations with respect to the nondimensional time. We will call the nonlinearity small if the condition

$$\left|\frac{\psi(\eta, \eta', \eta'')}{\eta}\right| \ll 1 \tag{4.2}$$

is fulfilled. In addition, we will assume that the damping is small:

$$\frac{2\varepsilon}{\omega} = \frac{\delta}{\pi} \ll 1. \qquad (4.3)$$

As we will see, the problem under consideration will satisfy these conditions.

2. One of the simplest methods for the solution of differential equations with small nonlinearities is the method of slowly changing amplitudes (van-der-Pol method). Although it is not sufficiently rigorous, it possesses physical clearness. This method is widely applied in radio engineering where it has helped to obtain a whole series of important results (Ref. 93).

The basic idea of the method of slowly changing amplitudes is as follows. If the nonlinearities and the damping of the system is sufficiently small, the solution of the nonlinear equation, Eq. (4.1), at least during a period, differs only slightly from the solution of the linear differential equation

$$f'' + \omega^2 f = 0,$$

i.e., from the harmonic vibration

$$f(t) = a \sin\omega t.$$

It is said, for such a case, that the solution of the differential equation, Eq. (4.1), is of an almost periodic character. Accordingly, we will seek an approximate solution of Eq. (4.1) in the form

$$f(t) = a(t) \sin\tilde{\omega} t; \qquad (4.4)$$

where $\bar{\omega}$ is a frequency which in general differs from the "linear frequency" ω, and $a(t)$ is the "slowly changing amplitude." The last expression can be used only when the increase in amplitude for the period is sufficiently small in comparison with its mean value, i.e.,

$$\left.\begin{aligned} \left|\frac{a'}{a}\right| \frac{2\pi}{\omega} \ll 1, \\ \left|\frac{a''}{a'}\right| \frac{2\pi}{\omega} \ll 1. \end{aligned}\right\} \tag{4.5}$$

Since the initial instant of time is arbitrary, we will consider the initial phase angle equal to zero.

We now substitute Eq. (4.4) into the nonlinear function $\psi(f, f', f'')$ and expand it in a Fourier series

$$\psi(f, f', f'') = \Phi(a, \bar{\omega}) \sin\bar{\omega} t + \psi(a, \bar{\omega}) \cos\bar{\omega} t + \cdots. \tag{4.6}$$

Terms containing the higher harmonics are not written out. In determining the coefficients

$$\Phi(a, \bar{\omega}) = \frac{\bar{\omega}}{\pi} \int_0^{\frac{2\pi}{\bar{\omega}}} \psi(f, f', f'') \sin\bar{\omega} t \, dt,$$

$$\Psi(a, \bar{\omega}) = \frac{\bar{\omega}}{\pi} \int_0^{\frac{2\pi}{\bar{\omega}}} \psi(f, f', f'') \cos\bar{\omega} t \, dt$$

we make use of the inequality, Eq. (4.5), which is approximately written as

$$\left.\begin{aligned} f' &= a \bar{\omega} \cos\bar{\omega} t + a' \sin\bar{\omega} t \approx a \bar{\omega} \cos\bar{\omega} t, \\ f'' &= -a \bar{\omega}^2 \sin\bar{\omega} t + 2a' \bar{\omega} \cos\bar{\omega} t + a'' \sin\bar{\omega} t \approx -a \bar{\omega}^2 \sin\bar{\omega} t \end{aligned}\right\} \tag{4.7}$$

A substitution of Eqs. (4. 4) and (4. 6) into Eq. (4. 1) yields

$$-a\bar{\omega}^2 \sin\bar{\omega}t + 2a'\bar{\omega}\cos\bar{\omega}t + \underline{a'' \sin\bar{\omega}t + 2\varepsilon a' \sin\bar{\omega}t} + 2\varepsilon a\bar{\omega}\cos\bar{\omega}t$$
$$+ \omega^2 a \sin\bar{\omega}t + \Phi(a,\bar{\omega})\sin\bar{\omega}t + \Psi(a,\bar{\omega})\cos\bar{\omega}t + \cdots = 0.$$

On the basis of Eqs. (4. 3) and (4. 5), the underlined terms can be neglected. By equating the coefficients of sin $\bar{\omega}$t and cos $\bar{\omega}$t to zero, we obtain the equations

$$\left.\begin{aligned} &(\omega^2 - \bar{\omega}^2)a + \Phi(a,\bar{\omega}) = 0, \\ &\frac{da}{dt} = -\varepsilon a - \frac{1}{2\bar{\omega}}\Psi(a,\bar{\omega}). \end{aligned}\right\} \tag{4. 8}$$

The first equation of Eq. (4. 8) establishes the dependence of frequency on the vibration amplitude; the second equation determines the amplitude change with respect to time. In particular, for the linear system with damping, we obtain:[1]

$$\bar{\omega} = \omega = \text{const}, \quad a = a_0 e^{-\varepsilon t}.$$

3. Another basis for this method given by K. F. Teodorchik (Ref. 93) will now be discussed. The differential equation, Eq. (4. 1), will be written in the form

$$f'' + \bar{\omega}^2 f = (\bar{\omega}^2 - \omega^2)f - 2\varepsilon f' - \varphi(f, f', f''). \tag{4. 9}$$

[1]This solution differs from the exact solution only in the addition of the damping correction, Eq. (2. 3), to the frequencies.

The vibrations of this nonlinear system will be considered as forced vibrations of a linear conservative system with a frequency $\bar{\omega}$ subjected to the external force

$$\sum F = (\bar{\omega}^2 - \omega^2) f - 2\varepsilon f' - \varphi(f, f', f'')$$

Expanding the right-hand term of this expression in a Fourier series and using Eq. (4.7) gives

$$\sum F = F(a, \bar{\omega}) \sin \bar{\omega} t + G(a, \bar{\omega}) \cos \bar{\omega} t$$

where

$$F(a, \bar{\omega}) = (\bar{\omega}^2 - \omega^2) a - \Phi(a, \bar{\omega}),$$
$$G(a, \bar{\omega}) = -2\varepsilon \bar{\omega} a - \Psi(a, \bar{\omega}).$$

From the beginning we have assumed that the solution does not contain a cosine part

$$F(a, \bar{\omega}) = 0.$$

This equation is identical to the first equation in Eqs. (4.8). In the following, we will consider the equation:

$$f'' + \bar{\omega}^2 f = G(a, \bar{\omega}) \cos \bar{\omega} t. \tag{4.10}$$

If $G(a, \bar{\omega})$ is a slowly changing function, then

$$f(t) = a(t) \sin \bar{\omega} t$$

where the following equation is an approximate (asymptotic) solution of Eq. (4.10):

$$a(t) = \frac{1}{2\bar{\omega}} \int_0^t G(\tau)\, d\tau$$

Differentiation reduces the second of Eqs. (4.8) to

$$\frac{da}{dt} = \frac{1}{2\bar{\omega}} G(a, \bar{\omega}).$$

In spite of all their physical clarity, both justifications cited above are not mathematically rigorous. The rigorous proof of the method of slowly changing amplitudes was given by L. I. Mandel'shtam and N. D. Papaleksi (Ref. 56). Applying the method of the small parameter, they showed that the first approximations correspond in both methods. The first approximation according to the method of Krylov and Bogoliubov (Ref. 97; see also Ref. 98) and to the method of Andronov and Khackin (Ref. 99) lead also to analogous results.

If higher harmonics are introduced

$$f(t) = a_1(t) \sin \bar{\omega} t + a_2(t) \sin 2\bar{\omega} t + a_3(t) \sin 3\bar{\omega} t + \cdots,$$

we can increase the accuracy of the results. The smaller the nonlinearity the greater is the accuracy given by the first harmonic approximation.

4. The method of slowly changing amplitudes can also be applied to the problem of forced vibrations. For example, let us consider the differential equation

$$f'' + 2\varepsilon f' + \omega^2 f + \varphi(f, f', f'') = S \sin\theta t. \tag{4.11}$$

We will seek its solution in the form

$$f(t) = a(t)\sin\theta t + b(t)\cos\theta t; \tag{4.12}$$

where a(t) and b(t) are slowly changing amplitudes. Writing Eq. (4.11) in the form

$$f'' + \theta^2 f = S\sin\theta t + (\theta^2 - \omega^2) f - 2\varepsilon f' - \varphi(f, f' f''), \tag{4.13}$$

substituting Eq. (4.12) on the right-hand side of Eq. (4.13) and separating out the sin θt and cos θt terms, we obtain

$$f'' + \theta^2 f = F(a, b)\sin\theta t + G(a, b)\cos\theta t + \cdots. \tag{4.14}$$

This yields

$$F(a, b) = S + (\theta^2 - \omega^2) a - 2\varepsilon\,\theta\, b - \Phi(a, b),$$
$$G(a, b) = -(\theta^2 - \omega^2) b - 2\varepsilon\,\theta\, a - \Psi(a, b)$$

where

$$\Phi(a, b) = \frac{\theta}{\pi}\int_0^{\frac{2\pi}{\theta}} \varphi(f, f', f'')\sin\theta t\, dt, \qquad \Psi(a, b) = \frac{\theta}{\pi}\int_0^{\frac{2\pi}{\theta}} \varphi(f, f', f'')\cos\theta t\, dt.$$

An approximate solution of Eq. (4.14) is

$$a(t) = \frac{1}{2\theta}\int_0^t G(\tau)\,dt, \qquad , \qquad b(t) = \frac{1}{2\theta}\int_0^t F(\tau)\,dt.$$

Through differentiation, we obtain the van-der-Pol equations

$$\frac{da}{dt} = \frac{1}{2\theta}G(a, b) \quad , \quad \frac{db}{dt} = \frac{1}{2\theta}F(a, b)$$

or, more explicitly,

$$\begin{aligned} \frac{da}{dt} &= \frac{1}{2\theta}[-(\theta^2 - \omega^2)b - 2\varepsilon\theta a - \Psi(a, b)], \\ \frac{db}{dt} &= \frac{1}{2\theta}[S + (\theta^2 - \omega^2)a - 2\varepsilon\theta b - \Phi(a, b)]. \end{aligned}$$

Thus, instead of a differential equation of second order, Eq. (4.11), we have a simpler system of two equations of first order that are not explicitly dependent on time. In the case of stationary vibrations

$$\frac{da}{dt} = \frac{db}{dt} = 0,$$

For determining the steady-state amplitudes, we obtain the following system of algebraic equations:

$$\left.\begin{aligned} -(\theta^2 - \omega^2)b - 2\varepsilon\theta a - \Psi(a, b) &= 0, \\ S + (\theta^2 - \omega^2)a - 2\varepsilon\theta b - \Phi(a, b) &= 0. \end{aligned}\right\} \qquad (4.15)$$

This equation system can also be derived by other means. For example, one can substitute Eq. (4. 12) directly into Eq. (4. 11), which will be written in the form L(f, f',f'') = 0. One then requires that the result of the substitution (in accordance with the variational method of Galerkin) be orthogonal to the "functional coordinates" sin θt and cos θt:

$$\int_0^{\frac{2\pi}{\theta}} L(f, f', f'') \sin\theta t\, dt = 0, \qquad \int_0^{\frac{2\pi}{\theta}} L(f, f', f'') \cos\theta t\, dt = 0.$$

This method directly leads to Eqs. (4. 15).

•17. FREE VIBRATIONS OF A NONLINEAR SYSTEM

1. Let us consider the problem of the free vibrations of a rod, taking into account the nonlinear terms introduced in the previous chapter.[2] The differential equation of this problem is

$$f'' + 2\varepsilon f' + \omega^2 f + \psi(f, f', f'') = 0; \tag{4. 16}$$

where $\psi(f, f', f'')$ is a nonlinear function of displacements, velocities and accelerations. Limiting ourselves to quantities of less than third order, the nonlinear function can be represented in the form

$$\psi(f, f', f'') = \gamma f^3 + 2\varepsilon_L f^2 f' + 2\varkappa f[f f'' + (f')^2] \tag{4. 17}$$

The first term considers the influence of nonlinear terms of static origin, the second term considers the nonlinear character of damping, and

[2] A comprehensive theoretical and experimental investigation of the free vibrations and forced vibrations of a nonlinear system can be found in Refs. 64, 65, 66, and 67.

the third term considers the influence of the inertia forces that arise from longitudinal displacements. In the following paragraphs we will discuss nonlinear elasticity, nonlinear damping, and nonlinear inertia.

It is easy to show that the differential equation of the problem belongs to a class of differential equations with small nonlinearities. This characteristic follows from the problem itself. As is well-known, the linear treatment of problems on the free vibrations of rods gives results that are in good agreement with experimental data. The additional nonlinear terms in Eq. (4.16) are a correction through which the linear approximation is made more precise.

To obtain a quantitative estimate, one must reduce Eq. (4.16) to a nondimensional form and apply the condition in Eq. (4.2). The calculation will be left to the reader. During stable equilibrium the influence of nonlinearities depends upon the amplitude of the vibrations (its growth is proportional to the square of the amplitude). This means that the vibration amplitudes must be sufficiently small. In all practical problems this condition is assumed to be satisfied.

2. We will now consider calculations. Substituting the expression $f = a \sin \bar{\omega}t$ into Eq. (4.17) and introducing

$$\begin{aligned}\sin^3\bar{\omega}t &= \frac{1}{4}(3\sin\bar{\omega}t - \sin 3\bar{\omega}t),\\ \sin\bar{\omega}t\cos^2\bar{\omega}t &= \frac{1}{4}(\sin\bar{\omega}t + \sin 3\bar{\omega}t),\\ \sin^2\bar{\omega}t\cos\bar{\omega}t &= \frac{1}{4}(\cos\bar{\omega}t - \cos 3\bar{\omega}t)\end{aligned}$$

we obtain

$$\varphi(f, f', f'') = \left(\frac{3}{4}\gamma - \varkappa\bar{\omega}^2\right)a^3\sin\bar{\omega}t + \frac{\varepsilon_L\bar{\omega}}{4}a^3\cos\bar{\omega}t + \cdots. \tag{4.18}$$

In accordance with the method of slowly changing amplitudes, the derivatives a' and a", and also terms containing higher harmonics, are neglected. From Eq. (4.18) it follows that

$$\Phi(a, \bar{\omega}) = \left(\frac{3}{4}\gamma - \varkappa\bar{\omega}^2\right)a^2,$$
$$\Psi(a, \bar{\omega}) = \frac{\varepsilon_L \bar{\omega}}{4}a^2,$$

so that Eq. (4.8) for our case takes the form

$$\omega^2 - \bar{\omega}^2 + \left(\frac{3}{4}\gamma - \varkappa\bar{\omega}^2\right)a^2 = 0 \tag{4.19}$$

and

$$\frac{da}{dt} = -\left(\varepsilon + \frac{\varepsilon_L}{4}a^2\right)a \tag{4.20}$$

Equation (4.19) permits one to determine the frequency of the free vibrations of the nonlinear system

$$\bar{\omega} = \omega\sqrt{\frac{1 + \frac{3}{4}\frac{\gamma}{\omega^2}a^2}{1 + \varkappa a^2}}. \tag{4.21}$$

As one can see from Eq. (4.19), the natural frequency of the nonlinear system depends upon the vibration amplitude. Therefore, the existence of nonlinear elasticity leads to an increase of the frequency with amplitude; conversely, nonlinear inertia causes a decrease of the natural frequency. If we consider that the nonlinearity is small, we can write Eq. (4.21) in the form

$$\bar{\omega} \approx \omega\left[1 + \frac{a^2}{2}\left(\frac{3}{4}\frac{\gamma}{\omega^2} - \varkappa\right)\right] \tag{4.22}$$

For $\chi < (3/4)(\gamma/\omega^2)$ the frequency of free vibrations will increase with a growth of amplitude; for $\chi > (3/4)(\gamma/\omega^2)$ the frequency will decrease with an increase of amplitude. For

$$\varkappa = \frac{3}{4}\frac{\gamma}{\omega^2} \tag{4.23}$$

the effect of nonlinear elasticity and nonlinear inertia is the same; the free vibrations remain isochronous. In this sense a nonlinear inertia with a coefficient χ and a nonlinear elasticity with a coefficient $\gamma = (4/3)\chi\omega^2$ are equivalent. We are reminded that an analogous correspondence arises for the linear system: the influence of transverse forces of inertia is equivalent to the influence of a continuous elastic foundation with a resistance coefficient of $k = -m\omega^2$. The factor 4/3 in our case can be explained by the nonlinear character of the system.

Let us consider the case when the longitudinal elastic connection is absent. Then, as we will show, the nonlinear inertia term always turns out to be dominant in this case. The coefficient of nonlinear elasticity according to Eq. (3.16) is

$$\gamma = \frac{\pi^2\omega^2}{8l^2}. \tag{4.24}$$

Since the imperfect elastic material usually gives a "soft" nonlinearity, one must consider Eq. (4.24) as the maximum value of the coefficient of nonlinear elasticity in the absence of nonlinear connections. On the other hand, the smallest value of the coefficient χ given in Eq. (3.20) is

$$\varkappa = \frac{\pi^4}{4l^4}\left(\frac{1}{3} - \frac{3}{8\pi^2}\right)$$

If we consider that for this value

$$\frac{4}{3}\frac{\varkappa\omega^2}{\gamma} = \frac{8\pi^2}{3}\left(\frac{1}{3} - \frac{3}{8\pi^2}\right) \approx 8$$

we arrive at a conclusion which is important for all future considerations. In the absence of longitudinal couplings the predominant nonlinear factor is nonlinear inertia. This conclusion is valid for all rods which constitute statically determinate systems and also to those rods of statically indeterminate systems, for which $\chi > (3/4)(\gamma/\omega^2)$.

Let us now estimate the order of magnitude of the nonlinear correction to the frequency. If we substitute the values of coefficients γ and χ into Eq. (4.22), we can write this equation in the form

$$\bar{\omega} = \omega\left(1 - k\frac{a^2}{l^2}\right)$$

where

$$k \approx 3.2 + 12\frac{M_L}{ml}.$$

M_L denotes the "longitudinal mass" which is fixed to the moving end of the rod. The mass of the rod is not considered during computations. (It is accounted for in the term 3.2.)

Let $M_L = 0$. In this case the change in the free vibration does not exceed 3 percent even at such a large amplitude as $a/l = 0.1$. If the rod is part of a rod system, the influence of the nonlinear terms can then increase by ten or one hundred times. This can be shown by examining the example in •14.

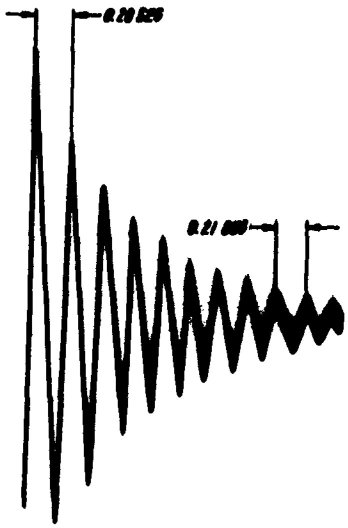


Fig. 23

An oscillogram obtained experimentally of the vibrations of a rod with a large inertia non-linearity is shown in Fig. 23. The change of period of the free vibration can easily be seen.

3. For an explanation of the law of damping one must return to Eq. (4.20). If we write it in the form

$$\varepsilon dt = -\frac{da}{a + \frac{\varepsilon_L}{4\varepsilon} a^3}$$

we obtain upon integration

$$-\varepsilon t = \ln a - \frac{1}{2}\ln\left(\frac{4\varepsilon}{\varepsilon_L} + a^2\right) + \ln C$$

or

$$\frac{Ca}{\sqrt{\frac{4\varepsilon}{\varepsilon_L} + a^2}} = e^{-\varepsilon t}.$$

The constant integration is

$$C = \frac{\sqrt{\frac{4\varepsilon}{\varepsilon_L} + a_0^2}}{a_0}.$$

where a_0 is the initial amplitude. Therefore

$$\frac{a_0}{a}\sqrt{\frac{\frac{4\varepsilon}{\varepsilon_L} + a_0^2}{\frac{4\varepsilon}{\varepsilon_L} + a^2}} = e^{-\varepsilon t}.$$

Solving this equation with respect to the amplitude a, we finally obtain

$$a = \frac{a_0 e^{-\varepsilon t}}{\sqrt{1 + (1 - e^{-2\varepsilon t}) \frac{\varepsilon_L}{4\varepsilon} a_0^2}}. \tag{4.25}$$

For the special case of linear damping,

$$a = a_0 e^{-\varepsilon t}. \tag{4.26}$$

Nonlinear damping gives a more rapid decrease of the amplitudes, as one might expect.

For practical purposes the following interpretation of Eq. (4.20) might be useful. If one divides the whole damping period into sufficiently small intervals of time, one can average the nonlinear part of the damping by assuming it is constant during every interval:

$$\bar{\varepsilon} = \varepsilon + \frac{\varepsilon_L}{4} \bar{a}^2. \tag{4.27}$$

Here $\bar{a}$ is an average value of the amplitude in the given time interval. For every interval Eq. (4.20) takes the form

$$\frac{da}{dt} = -\bar{\varepsilon} a$$

from which

$$a = a_0 e^{-\bar{\varepsilon} t}$$

Thus Eq. (4.20) describes damped vibrations that occur with a variable damping constant. Such a "quasi-linear" treatment is all the more suitable since essentially it gives the basis for a generally accepted method of reducing experimental diagrams of damped vibrations. As is well-known, the experimental damping constant is determined from the condition

$$\bar{\varepsilon} = \frac{1}{\Delta t} \ln \frac{a(t)}{a(t+\Delta t)} \tag{4.28}$$

whether it is constant during the entire process of damping or changes with amplitude. In the last case the damping is nonlinear, or more precisely, it does not follow the exponential law of Eq. (4.26). The application of Eq. (4.28) assumes that the damping is "linearized" within the "limits" of each interval Δt.

•18. FORCED VIBRATIONS OF A NONLINEAR SYSTEM

1. We will discuss briefly the problem of forced vibrations of a nonlinear system.[3] The steady-state amplitudes of these vibrations can be determined from Eq. (4.15). The amplitude of the "generalized force" is denoted by S, which will be

$$S = \frac{2P}{ml}.$$

for example, in the case of a rod loaded at the center (Fig. 24).

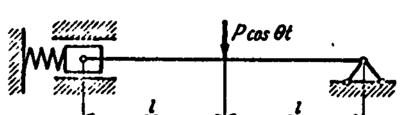


Fig. 24

Substituting the approximate solution

$$f(t) = a \sin\theta t - b \cos\theta t$$

[3] See footnote 2 on page 41.

into the expression for the nonlinear function, Eq. (4.17), and performing the operations indicated in the preceding paragraph, we obtain

$$\Phi(a, b) = A^2\left[\left(\frac{3}{4}\gamma - \varkappa\theta^2\right)a + \frac{\varepsilon_L}{4}\theta b\right],$$
$$\Psi(a, b) = A^2\left[-\left(\frac{3}{4}\gamma - \varkappa\theta^2\right)b + \frac{\varepsilon_L}{4}\theta a\right].$$

Here A is the amplitude of the vibrations, i.e., $A^2 = a^2 + b^2$. Therefore Eqs. (4.15) take the form

$$\left.\begin{aligned} &-(\theta^2 - \omega^2)b - 2\varepsilon\theta a - A^2\left[-\left(\frac{3}{4}\gamma - \varkappa\theta^2\right)b + \frac{\varepsilon_L}{4}\theta a\right] = 0, \\ &S + (\theta^2 - \omega^2)a - 2\varepsilon\theta b - A^2\left[\left(\frac{3}{4}\gamma - \varkappa\theta^2\right)a + \frac{\varepsilon_L}{4}\theta b\right] = 0 \end{aligned}\right\} \qquad (4.29)$$

The equation system, Eq. (4.29) is unsolvable in the general form; therefore we restrict ourselves to the case in which the damping, either linear or nonlinear, can be neglected. The equation system for this case is simplified to

$$-(\theta^2 - \omega^2)b + A^2 b\left(\frac{3}{4}\gamma - \varkappa\theta^2\right) = 0,$$
$$S + (\theta^2 - \omega^2)a - A^2 a\left(\frac{3}{4}\gamma - \varkappa\theta^2\right) = 0$$

and obviously can be satisfied by $b = 0$, $A = a$. The amplitude of the steady-state vibrations is determined from the cubic equation

$$S + (\theta^2 - \omega^2)A - A^3\left(\frac{3}{4}\gamma - \varkappa\theta^2\right) = 0. \qquad (4.30)$$

2. The roots of Eq. (4.30) can be graphically determined as the coordinates of intersection points of a straight line

$$y = \frac{S + (\theta^2 - \omega^2)A}{\frac{3}{4}\gamma - \chi\theta^2} \tag{4.31}$$

with the cubic parabola $y = A^3$. Figure 25 corresponds to the case in which the nonlinear elasticity is dominant, i.e., $(3/4)\gamma - \chi\omega^2 > 0$.

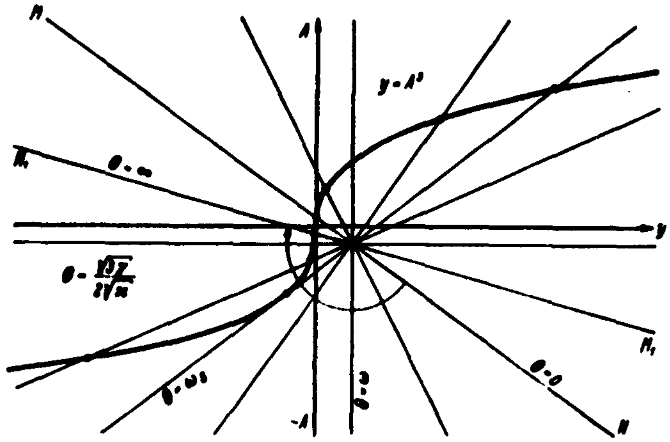


Fig. 25

During changes of the exciting frequencies from zero to infinity, the straight line, Eq. (4.31), rotates clockwise from the initial position MN to the end position M_1N_1. As long as the frequency is sufficiently small (at least for $\theta < \omega$), the straight line intersects the cubic parabola once, i.e., Eq. (4.30) has one real root. For larger frequencies, three real roots occur and, consequently, three possible amplitude values also occur. At

$$\theta = \frac{\sqrt{3}}{2}\sqrt{\frac{1}{\chi}} \tag{4.32}$$

the straight line takes a horizontal position so that Eq. (4.30), in addition to having the one finite root, has also the roots $A = \pm\infty$.

The resonance curve is shown in Fig. 26. Its character is different not only from the resonance curve of the linear problem but also from the resonance curve for the case in which only nonlinear elasticity is present.[4] As in the much investigated case of nonlinear elasticity, from the three correct solutions which exist in the region CD, only two will be stable. The unstable, i.e., the physically impossible solution, is denoted by a dotted line.

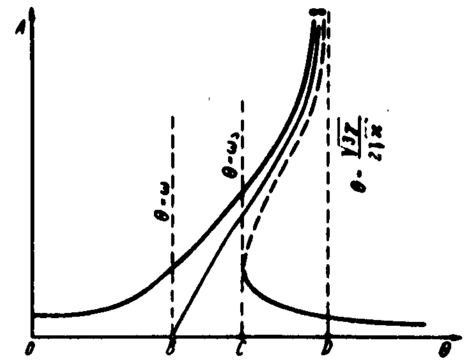
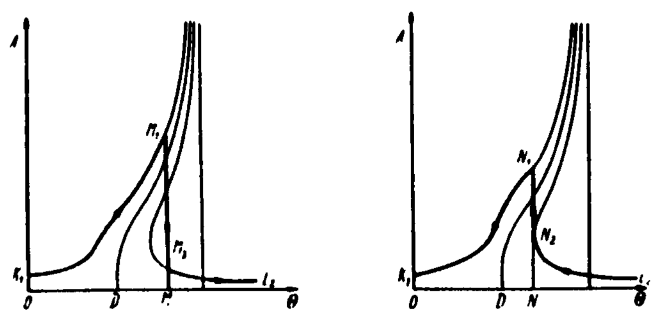


Fig. 26

The existence of two stable solutions leads to a result which is called the "overhang." By a gradual increase in the excitation frequency we can bring the system into the region where two stable solutions exist (Fig. 27a). The growth of the amplitudes will proceed initially along the curve K_1M_1, until vibrations occur at a certain point M_1. The amplitude (not calculated with respect to time) decreases suddenly to the magnitude MM_2, and then decreases along the curve M_2L_2. By reversing this frequency process, the amplitude of the vibrations will initially increase smoothly along the curve L_2N_2 (Fig. 27b). The sudden change



a) b)

Fig. 27

[4]This last case examined in many textbooks on the theory of vibrations (for example, Ref. 98) can be obtained from our equations, if in them one sets $X = 0$. The values of the amplitude increase for increasing frequencies.

now occurs at point N_2, after which the amplitudes follow the curve N_1K_1. This shows that the amplitude of the forced vibrations depends not only on excitation frequency in the considered instant of time but also on the "history" of the nonlinear system.

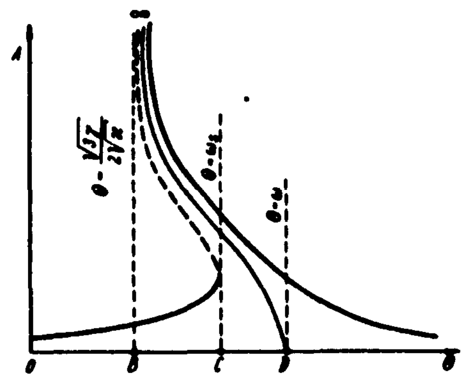


Fig. 28

The question of the "overhang" will be examined in connection with parametrically excited vibrations (•23).

We have examined the case in which the nonlinearities are due essentially to elasticity. If the influence of nonlinear inertia is predominant, the resonance curve bends toward the side of decreasing frequencies; the "overhang" of the resonance curve also bends towards the side of decreasing frequencies (Fig. 28).

CHAPTER FIVE

AMPLITUDES OF VIBRATIONS AT THE PRINCIPAL PARAMETRIC RESONANCE

•19. BASIC EQUATIONS

1. Let us investigate the differential equation

$$f'' + 2\varepsilon f' + \Omega^2(1 - 2\mu \cos\theta t) f + \psi(f, f', f'') = 0, \tag{5.1}$$

that describes parametrically excited vibrations including the effects of nonlinear factors, and let us formulate the problem of finding the solutions that correspond to steady-state vibrations.

It was established in •8 that the solutions of the linear problem have the form

$$f(t) = \sum_{k=1,3,5,\ldots}^{\infty} \left(a_k \sin\frac{k\theta t}{2} + b_k \cos\frac{k\theta t}{2}\right) \tag{5.2}$$

on the boundaries of the first, third, and generally, the odd regions of dynamic instability where a_k and b_k are constant coefficients. It would be natural to seek the solution of the nonlinear problem in such a form that is valid within the limits of the odd regions of instability. The following considerations lead to this form of solution.

For a specific choice of its coefficients, Eq. (5.2) can satisfy Eq. (5.1). In fact, the result of the substitution of the series into Eq. (5.1) will not contain other periodic terms except $\sin (k\theta t)/2$ and $\cos (k\theta t)/2$ for odd values of k. However, this applies only if the nonlinear function $\psi(f, f', f'')$ does not contain terms of even powers. The function

$$\psi(f, f', f'') = \gamma f^3 + 2\varepsilon_L f^2 f' + 2\varkappa f[f f'' + (f')^2] \tag{5.3}$$

will satisfy this requirement.

The third and deciding reason for the use of Eq. (5.2) is the experimental results, which indicate that the steady-state vibrations within the limits of the odd regions of instability have exactly such a form. We will postpone a discussion of the experimental data until •22.

2. Let us seek the solution of Eq. (5.1) in the form of Eq. (5.2). Substitution gives

$$
\begin{aligned}
&\sum_{k=1,3,5}^{\infty}\left(\Omega^2-\frac{k^2\theta^2}{4}\right)\left(a_k \sin\frac{k\theta t}{2}+b_k\cos\frac{k\theta t}{2}\right)\\
&\quad+\varepsilon\theta\sum_{k=1,3,5}^{\infty}k\left(a_k\cos\frac{k\theta t}{2}-b_k\sin\frac{k\theta t}{2}\right)\\
&\quad-\Omega^2\mu\sum_{k=1,3,5}^{\infty}a_k\left[\sin(k+2)\frac{\theta t}{2}+\sin(k-2)\frac{\theta t}{2}\right]\\
&\quad-\Omega^2\mu\sum_{k=1,3,5}^{\infty}b_k\left[\cos(k+2)\frac{\theta t}{2}+\cos(k-2)\frac{\theta t}{2}\right]+\varphi^*(f, f', f'')=0.
\end{aligned}
\tag{5.4}
$$

The nonlinear function in Eq. (5.4) is expanded in a Fourier series

$$
\varphi^*(f, f', f'')=\sum_{k=1,3,5}^{\infty}\left(\Phi_k\sin\frac{k\theta t}{2}+\Psi_k\cos\frac{k\theta t}{2}\right). \tag{5.5}
$$

Its coefficients

$$
\left.
\begin{aligned}
\Phi_k(a_1, a_3, \ldots, b_1, b_3, \ldots) &= \frac{\theta}{2\pi}\int_0^{\frac{4\pi}{\theta}}\varphi^*(f, f', f'')\sin\frac{k\theta t}{2}\,dt,\\
\Psi_k(a_1, a_3, \ldots, b_1, b_3, \ldots) &= \frac{\theta}{2\pi}\int_0^{\frac{4\pi}{\theta}}\varphi^*(f, f', f'')\cos\frac{k\theta t}{2}\,dt
\end{aligned}
\right\}
\tag{5.6}
$$

are obviously nonlinear functions of the coefficients of Eq. (5.2) and in the case of Eq. (5.3) are homogeneous functions of third order. Substituting

Eq. (5.5) into Eq. (5.4) and equating the coefficients of sin $(k\theta t)/2$ and cos $(k\theta t)/2$, we obtain the following system of equations

$$\left.\begin{aligned}
\left[\Omega^2(1+\mu)-\frac{\theta^2}{4}\right]a_1-\varepsilon\theta b_1-\Omega^2\mu a_3+\Phi_1(a_i,b_i)&=0,\\
\left[\Omega^2(1-\mu)-\frac{\theta^2}{4}\right]b_1+\varepsilon\theta a_1-\Omega^2\mu b_3+\Psi_1(a_i,b_i)&=0,\\
\left(\Omega^2-\frac{k^2\theta^2}{4}\right)a_k-k\varepsilon\theta b_k-\Omega^2\mu(a_{k-2}+a_{k+2})+\Phi_k(a_i,b_i)&=0,\\
\left(\Omega^2-\frac{k^2\theta^2}{4}\right)b_k+k\varepsilon\theta a_k-\Omega^2\mu(b_{k-2}+b_{k+2})+\Psi_k(a_i,b_i)&=0,\\
(k=3,5,\ldots).&
\end{aligned}\right\} \tag{5.7}$$

For brevity we write

$$\begin{gathered}
\Phi_k(a_1,a_3,\ldots,b_1,b_3,\ldots)=\Phi_k(a_i,b_i),\\
\Psi_k(a_1,a_3,\ldots,b_1,b_3,\ldots)=\Psi_k(a_i,b_i).
\end{gathered}$$

3. In order to determine the amplitudes of the steady-state vibrations within the limits of the even-numbered regions of excitation, we will seek solutions in the form

$$f(t)=b_0+\sum_{k=2,4,6}^{\infty}\left(a_k\sin\frac{k\theta t}{2}+b_k\cos\frac{k\theta t}{2}\right). \tag{5.8}$$

This form is suggested by considerations analogous to those stated in No. 1.

Substituting Eq. (5.8) into Eq. (5.1) and expanding the nonlinear function in the series

$$\psi^*(f,f',f'')=\Psi_0+\sum_{k=2,4,6}^{\infty}\left(\Phi_k\sin\frac{k\theta t}{2}+\Psi_k\cos\frac{k\theta t}{2}\right),$$

we obtain the following system of equations:

$$
\left.\begin{aligned}
\Omega^2(b_0 - \mu b_2) + \Psi_0(a_i, b_i) &= 0, \\
(\Omega^2 - \theta^2) a_2 - \mu \Omega^2 a_4 - 2\varepsilon\theta b_2 + \Phi_2(a_i, b_i) &= 0, \\
(\Omega^2 - \theta^2) b_2 - \mu \Omega^2 (2b_0 + b_4) + 2\varepsilon\theta a_2 + \Psi_2(a_i, b_i) &= 0, \\
\left(\Omega^2 - \frac{k^2\theta^2}{4}\right) a_k - \mu \Omega^2 (a_{k-2} + a_{k+2}) - k\varepsilon\theta b_k + \Phi_k(a_i, b_i) &= 0, \\
\left(\Omega^2 - \frac{k^2\theta^2}{4}\right) b_k - \mu \Omega^2 (b_{k-2} + b_{k+2}) + k\varepsilon\theta a_k + \Psi_k(a_i, b_i) &= 0 \\
(k = 4, 6, \ldots). &
\end{aligned}\right| \qquad (5.9)
$$

Coefficients Φ_k and Ψ_k for $k \geq 2$ are calculated according to Eq. (5.6), and the coefficient Ψ_0, according to the formula

$$
\Psi_0(a_2, a_4, \ldots, b_0, b_2, b_4, \ldots) = \frac{\theta}{4\pi} \int_0^{4\pi/\theta} \psi^*(f, f', f'')\, dt
$$

We note in conclusion, that Eqs. (5.7) and (5.9) could be obtained differently, i.e., by means of the Galerkin variational method. If one substitutes, for example, Eq. (5.2) into Eq. (5.1), one can require that the corresponding expression of L(f, f', f") be orthogonal to each of the functional coordinates sin $(k\theta t)/2$ and cos $(k\theta t)/2$:

$$
\int_0^{4\pi/\theta} L(f, f', f'') \sin\frac{k\theta t}{2}\, dt = 0, \qquad \int_0^{4\pi/\theta} L(f, f', f'') \cos\frac{k\theta t}{2}\, dt = 0
$$

$$
(k = 1, 3, 5, \ldots).
$$

These expressions reduce to Eq. (5.7).

20. DETERMINATION OF STEADY-STATE AMPLITUDES[1]

1. If we investigate the vibrations for the principal resonance $\theta \approx 2\Omega$, we can neglect the influence of higher harmonics in Eq. (5.2) and can assume

$$f(t) = a \sin\frac{\theta t}{2} + b \cos\frac{\theta t}{2} \tag{5.10}$$

as an approximation.[2] Therefore, Eqs. (5.7) are essentially simplified, and a system of two equations for the coefficients a and b remains:

$$\left.\begin{aligned} &\left[\Omega^2(1+\mu) - \frac{\theta^2}{4}\right] a - \varepsilon\theta b + \Phi(a,b) = 0, \\ &\left[\Omega^2(1-\mu) - \frac{\theta^2}{4}\right] b + \varepsilon\theta a + \Psi(a,b) = 0. \end{aligned}\right\} \tag{5.11}$$

For the determination of the quantities $\Phi(a, b)$ and $\Psi(a, b)$, we substitute Eq. (5.10) into Eq. (5.3). Computations give

$$\begin{aligned} \psi^*(f, f', f'') = {} & \frac{A^2}{4}(3\gamma a - \varepsilon_L \theta b - \varkappa \theta^2 a) \sin\frac{\theta t}{2} \\ & + \frac{A^2}{4}(3\gamma b + \varepsilon_L \theta a - \varkappa \theta^2 b) \cos\frac{\theta t}{2} + \cdots . \end{aligned}$$

Terms containing higher harmonics are not written out; the amplitude of steady-state vibrations is denoted by A where

$$A^2 = a^2 + b^2 .$$

Thus, the first coefficients of the expansion of the function $\psi^*(f, f', f'')$ in a Fourier series are

---

[1] See Ref. 57.

[2] The method of the slowly changing amplitude also leads to this approximation.

$$
\begin{aligned}
\Phi(a, b) &= \frac{A^2}{4}(3\gamma a - \varepsilon_L \theta b - \varkappa \theta^2 a), \\
\Psi(a, b) &= \frac{A^2}{4}(3\gamma b + \varepsilon_L \theta a - \varkappa \theta^2 b).
\end{aligned}
\qquad (5.12)
$$

2. For further investigation, we will write the system of Eqs. (5.11) in the following form:

$$
\begin{aligned}
(1 + \mu - n^2)a - \frac{n\Delta}{\pi} b + A^2\left(\frac{3\gamma}{4\Omega^2} a - \frac{n\Delta_L}{\pi} b - \varkappa n^2 a\right) = 0, \\
(1 - \mu - n^2)b + \frac{n\Delta}{\pi} a + A^2\left(\frac{3\gamma}{4\Omega^2} b + \frac{n\Delta_L}{\pi} a - \varkappa n^2 b\right) = 0
\end{aligned}
\qquad (5.13)
$$

where

$$
n = \frac{\theta}{2\Omega}, \quad \Delta = \frac{2\pi\varepsilon}{\Omega}, \quad \Delta_L = \frac{\pi\varepsilon_L}{2\Omega}. \qquad (5.14)
$$

It is obvious that Eqs. (5.13) will be satisfied for a = b = A = 0. This solution corresponds to the case when transverse vibrations of the rod are absent.

One can find the non-zero solutions in the following manner. We will consider Eq. (5.13) as a system of homogeneous linear equations with respect to a and b. This system has solutions that differ from zero only in the case when the coefficients of the determinant disappear:

$$
\begin{vmatrix}
1 + \mu - n^2 - A^2\left(\varkappa n^2 - \frac{3\gamma}{4\Omega^2}\right) & -\frac{n}{\pi}(\Delta + \Delta_L A^2) \\
\frac{n}{\pi}(\Delta + \Delta_L A^2) & 1 - \mu - n^2 - A^2\left(\varkappa n^2 - \frac{3\gamma}{4\Omega^2}\right)
\end{vmatrix} = 0. \qquad (5.15)
$$

Expanding the determinant and solving the equation obtained with respect to the amplitude A of the steady-state vibrations, we find

$$
A = \sqrt{\frac{p(1 - n^2) - \frac{n^2}{\pi^2}\Delta\Delta_L \pm \sqrt{\mu^2\left(p^2 + \frac{n^2}{\pi^2}\Delta_L^2\right) - \frac{n^2}{\pi^2}[p\Delta + \Delta_L(1 - n^2)]^2}}{p^2 + \frac{n^2}{\pi^2}\Delta_L^2}} \qquad (5.16)
$$

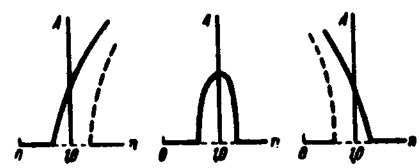
where

$$p = \varkappa n^2 - \frac{3\gamma}{4\Omega^2}. \tag{5.17}$$

3. Equation (5. 16) is too cumbersome for further investigation. Therefore we shall first clarify how the character of the resonance curves depends upon the form of the nonlinear function. For the case $\psi(f) = \gamma f^3$ (the case of nonlinear elasticity), Eq. (5. 16) gives

$$A = \frac{2\Omega}{\sqrt{3\gamma}} \sqrt{n^2 - 1 \pm \sqrt{\mu^2 - \frac{n^2 \Delta^2}{\pi^2}}}. \tag{5.18}$$

A diagram of the dependence of the amplitude on the frequency is shown in Fig. 29a, where the two plotted solutions correspond to the two signs in Eq. (5. 18). One of these solutions (represented by the dotted line) is obviously unstable. It is characteristic for the case of nonlinear elasticity that resonance curves are bent towards the side of the increasing exciting frequencies. We have already encountered analogous properties during the investigation of forced vibrations (•18).



a) b) c)

Fig. 29

Let us consider the case of nonlinear damping

$$\psi(f, f') = 2\varepsilon_L f^2 f'.$$

Equation (5. 16) assumes the form

$$A = \frac{\sqrt{\pi}}{\sqrt{n \Delta_L}} \sqrt{\sqrt{\mu^2 - (1 - n^2)^2} - \frac{n \Delta}{\pi}}$$

The corresponding resonance curve is represented in Fig. 29b. As is seen from the figure, the resonance curve in this case is approximately symmetrical with respect to the maximum.

Finally, for the case of the nonlinear inertia

$$\psi(f, f', f'') = 2\kappa f[f f'' + (f')^2],$$

we obtain the amplitude equation

$$A = \frac{1}{n\sqrt{\kappa}} \sqrt{1 - n^2 \pm \sqrt{\mu^2 - \frac{n^2 \Delta^2}{\pi^2}}}. \tag{5.19}$$

Here we also have two solutions, one of which is unstable (Fig. 29c). Contrary to the case of nonlinear elasticity, the resonance curves are bent toward the side of decreasing exciting frequencies.

We note that the base of the resonance curves in all three cases (as in the general case) does not depend on the magnitude and the character of the nonlinearity but coincides with the interval of the instability determined by the methods of linear theory. In fact, if we set A = 0 in Eq. (5.15), we obtain

$$\begin{vmatrix} 1 + \mu - n^2 & -\frac{n\Delta}{\pi} \\ \frac{n\Delta}{\pi} & 1 - \mu - n^2 \end{vmatrix} = 0,$$

coinciding with the equation of the critical frequencies, Eq. (2.12).

It is evident that in the case of nonlinear elasticity or nonlinear inertia, it is possible to have an "overhang" of the resonance curve beyond the limits of the region of excitation. (Compare with §18.) In the first case, the "overhang" will occur towards the side of increasing frequencies; in the second case, it will occur towards the side of decreasing frequencies. This phenomenon will be investigated in detail in §23.

4. The form of the resonance curves for each of the three fundamental cases is so typical, that according to the character of the curves, it is possible to judge which nonlinear factors predominate in each case. It can be seen from Eq. (5.16), that the manner in which the resonance curve is bent depends on the sign of the quantity

$$p = \varkappa n^2 - \frac{3\gamma}{4\Omega^2}$$

For

$$\frac{4}{3}\frac{\varkappa\Omega^2}{\gamma} > 1 \qquad (n \approx 1)$$

the resonance curves bend toward the side of decreasing frequencies, i.e., the amplitudes will increase as the exciting frequencies decrease. If, however,

$$\frac{4}{3}\frac{\varkappa\Omega^2}{\gamma} < 1 \qquad (n \approx 1),$$

the resonance curves will then bend toward the side of increasing frequencies.

As in the case of free vibrations (§17), it can be shown that if the rod does not have longitudinal elastic springs, then the nonlinear inertia is the decisive nonlinear factor. From the relation derived in §17

$$\frac{4}{3}\frac{\varkappa\omega^2}{\gamma} \simeq \frac{8\pi^2}{3}\left(\frac{1}{3} - \frac{3}{8\pi^2}\right)$$

it follows that

$$\frac{4}{3}\,\frac{\varkappa n^2 \Omega^2}{\gamma} \geq \frac{8\pi^2}{3}\left(\frac{1}{3} - \frac{3}{8\pi^2}\right)\frac{\theta^2}{4\omega^2}.$$

However, for the principal region of instability

$$\frac{8\pi^2}{3}\left(\frac{1}{3} - \frac{3}{8\pi^2}\right)\frac{\theta^2}{4\omega^2} > 1,$$

only if the longitudinal force is not in the vicinity of the Euler value.

The dominating influence of nonlinear inertia is confirmed by experimental data. Oscillograms of parametric excited vibrations taken near the principal resonance $\theta = 2\Omega$ are shown in Fig. 30. These oscillograms show

Fig. 30

the vibrations which arise for a decreasing exciting frequency. The first frame corresponds to the upper boundary of the region of parametric excitation (θ = 42.0 cps). Figure 11a is an enlargement of this frame. Along with the parametric excited vibrations, forced vibrations also occur with a doubled frequency (i.e., with a frequency of the external force). The additional frames correspond to a gradual decrease of the exciting frequency. It is characteristic that the accompanying vibrational amplitude increases. The sixth frame corresponds to an "overhang" of the resonance curve (θ = 36.5 cps), after which the "break" of the amplitude occurs. Only forced vibrations having a frequency of the external force remain.

The dependence of the amplitude on the frequency also can be seen in Fig. 31. In contrast to the oscillogram in Fig. 30, this oscillogram shows a continuous record. A decrease of the exciting frequency by 20 percent, which takes the system beyond the limits of the region of instability along the "overhang," increases the amplitudes by a factor greater than 3.

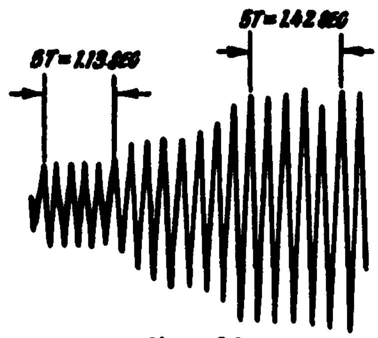


Fig. 31

§21. INVESTIGATION OF THE EQUATION FOR STEADY-STATE AMPLITUDES

1. We will first investigate the influence of linear damping on the magnitude of the steady-state amplitudes. For this purpose, we neglect in Eq. (5.16) the terms which take into account the nonlinear portion of the damping. This can be represented in the following form

$$A = \frac{1}{\sqrt{p}}\sqrt{1 - n^2 \pm \sqrt{\mu^2 - \frac{n^2\Delta^2}{\pi^2}}}, \tag{5.20}$$

where as before

$$p = \varkappa n^2 - \frac{3\gamma}{4\Omega^2}.$$

Equation (5.20) gives real values for the amplitude in the case $(n\Delta)/\pi < \mu$, or because $n \approx 1$, in the case $\Delta/\pi < \mu$. If however, $\Delta/\pi > \mu$, then steady-state vibrations do not arise. This result is found to be in complete agreement with the results of linear theory.

On the other hand, for sufficiently small damping, its influence on the magnitude of the amplitudes is imperceptible for practical purposes. For example, at n = 1, Eq. (5.20) assumes the form

$$A = \sqrt{\frac{\mu}{|p|}} \cdot \sqrt{1 - \left(\frac{\mu_*}{\mu}\right)^2}$$

where μ_* is the critical value of the excitation parameter (•9). Even at $\mu_* = 1/2\mu$, the influence of damping comprises an order of magnitude of 6 percent. With a further decrease of damping, the approximate values which were computed for the conservative case are still more exact. This is seen, for example, from Fig. 32, which represents the dependence of the dimensionless amplitude on the ratio of the excitation parameter. In general at

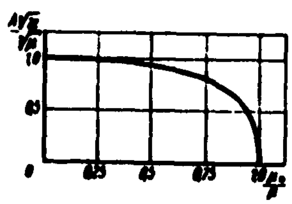


Fig. 32

$$\mu > 3\mu_* \tag{5.21}$$

the simple formula of the conservative problem

$$A = \sqrt{\frac{1 - n^2 \pm \mu}{p}} \tag{5.22}$$

gives sufficient accuracy for practical purposes. In particular, the largest amplitudes within the limits of the region of instability (i. e., without the consideration of the "delay") can be determined according to the formula

$$A = \sqrt{\frac{2\mu}{|p|}} \tag{5.23}$$

In the case of predominant nonlinear inertia ($p > 0$), the greatest amplitudes are attained on the lower boundary of the resonance region; in the case of predominant nonlinear elasticity ($p < 0$), the greatest amplitudes are attained on its upper boundaries. Equation (5.22) can also be written in the following form:

$$A^2 = \frac{1}{p}\left(\frac{\theta_*^2}{\theta^2} - 1\right) \tag{5.24}$$

Here, the upper or lower critical frequency is taken, depending on the sign of p instead of θ_*. For the proof, let us investigate Eq. (5.15) at $\Delta = \Delta_L = 0$:

$$\begin{vmatrix} 1 + \mu - n^2 - A^2 p & 0 \\ 0 & 1 - \mu - n^2 - A^2 p \end{vmatrix} = 0.$$

If we compare it with Eq. (2.12) for the frequency boundaries, we see that it is satisfied at $n^2 + A^2 p = n_*^2$. Consequently,

$$A^2 = \frac{1}{p}(n_*^2 - n^2) \qquad \left(n_* = \frac{\theta_*}{2\Omega}\right),$$

follows from Eq. (5.24).

In what follows, unless otherwise stipulated, we will assume that the condition in Eq. (5.21) is fulfilled, and we will make use of Eqs. (5.22), (5.23), and (5.24).

2. We shall now turn to the question concerning the influence of the magnitude of external loading on the steady-state amplitudes.

As seen from Eq. (5.23), the amplitude of the vibrations grows in proportion to the square root of the excitation parameter

$$A = \sqrt{\frac{2\mu}{|p|}}. \tag{5.25}$$

i.e., proportional to the square root of the amplitude of the periodic force. A nonlinear dependence between the loading and the vibration amplitude is usually characteristic for problems which are described by nonlinear differential equations. However, the amplitude of the longitudinal displacements at the point of application of the force also grows in proportion to the excitation parameter.

At first glance, the constant component of the longitudinal force P_0 must increase the amplitudes of the vibrations. Actually an increase of the force P_0 also increases the excitation parameters:

$$\mu = \frac{P_t}{2(P_* - P_0)}.$$

However, this dependence is more complicated. The force P_0 is usually of gravitational origin, i.e., one way or another it is related to the weight, and therefore an increase usually causes an increase in the nonlinear inertia of the system. This also can bring about a decrease of the amplitudes. We will illustrate this by a simple example (Fig. 20).

Let the simply-supported rod be loaded by the force $P_0 + P_t \cos \theta t$, where the force P_0 is associated with the rod mass P_0/g. Neglecting the influence of nonlinear elasticity and linear and nonlinear damping, we obtain

$$\varkappa = \frac{\pi^4 (P_0 + kG)}{4Gl^2};$$

where $G = mgl$ is the free weight of the rod, and k is the coefficient in Eq. (3.20) designated by χ. Eq. (5.22) at $\theta = 2\Omega$ gives $A^2 = \mu/\chi$, or

$$A^2 = \frac{2l^2}{\pi^4} \cdot \frac{P_t G}{(P_0 + kG)(P_* - P_0)}. \tag{5.26}$$

The dependence of the amplitude on the constant component P_0 is shown in Fig. 33. At $P_0 = 0$, the vibration amplitudes are relatively large:

$$A_0^2 = \frac{2l^2 P_1}{\pi^4 k P_*}$$

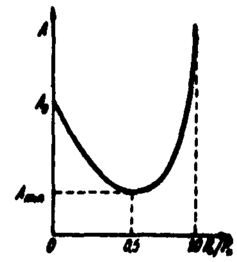


Fig. 33

An increase of the force P_0 decreases the amplitudes until a minimum is reached at $P_0 = 1/2(P + kG) \approx (1/2)P_*$. Subsequent amplitudes once again increase.

3. In the presence of two factors, i. e., the nonlinear elasticity and the nonlinear inertia, the resonance curves assume the form shown in Fig. 34. Figure 34a corresponds to the case $p > 0$ (predominant nonlinear inertia).

As seen from the diagrams, the "overhang" in this case is bounded by the frequency θ_∞. This frequency is determined by equating the expression for p to zero:

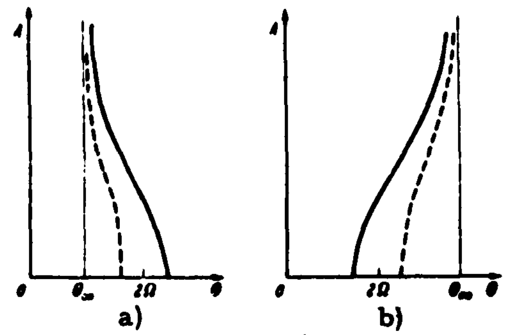


Fig. 34

$$\varkappa n^2 - \frac{3\gamma}{4\Omega^2} = 0.$$

Hence,

$$\theta_\infty = 2n_\infty \Omega = \sqrt{\frac{3\gamma}{\varkappa}}. \tag{5.27}$$

The magnitude $\omega_L = \sqrt{\gamma/\chi}$ is none other than the linearized frequency of the "longitudinal system" (the nonlinear elasticity plus the nonlinear inertia). Thus in the case of an end mass M_L and a connection with a spring constant, we have

$$\varkappa = \frac{\pi^4 M_L}{4 m l^3}, \quad \gamma = \frac{\pi^4 c}{4 m l^3}$$

which gives $\omega_L^2 = c/M_L$. Equation (5.27) can be interpreted as the condition for the occurrence of resonance in the "longitudinal" system although it differs from the condition of synchronization by the factor of $\sqrt{3}$ for the natural frequency.

If the nonlinear elasticity and the nonlinear inertia compensate one another, then the frequency θ_∞ lies in the region of dynamic instability. This is the most unfavorable case from the point of view of the amplitudes of the vibrations: the parametric resonance and the resonance in the "longitudinal" system are superimposed on one another (Fig. 35). The amplitudes are limited by nonlinear damping and by terms of higher order (in the expression for the nonlinear function).

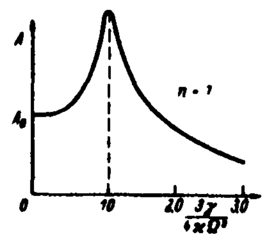


Fig. 35

The factor $\sqrt{3}$ in the resonance condition stems from the nonlinearities of the "longitudinal" system. In fact, the linear coefficient which replaces the elasticity of this system is very large during small displacements (infinitely large during displacements tending to zero, if the rod is considered incompressible) and approach a constant value c when the deflections are increasing. This process is shown in Fig. 36, where the reaction of the system to the longitudinal displacement is plotted along the vertical axis.

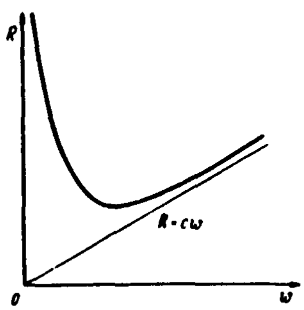


Fig. 36

4. We shall briefly dwell on methods for preventing parametrically excited vibrations. Besides the obvious prevention measure, i.e., the reduction or the total elimination of exciting forces and also the removal of the structure from the dangerous region by means of changing its parameters, the following methods are recommended.

1) Linear damping. This prevention measure is effective only when the damping is large, so that $\mu_* \approx \mu$. For $\mu_* > \mu$ parametric resonance is impossible. For a choice of suitable damping characteristics one should use the equation in •9.

2) Nonlinear damping. The introduction of a "longitudinal" damper reduces the amplitudes of the parametric vibrations approximately inversely proportional to $\sqrt{\Delta_L}$. This method should be considered only as a helping measure.

3) The introduction of nonlinearities by springs and inertia members. If the features of the construction allow its use, such a method may prove to be useful. One must remember, however, that the influence of the indicated nonlinearities is opposite, and that there exist such combinations which are unfavorable to the decrease of vibrations. Consequently, the increase of a nonlinear elasticity of the system does not always reduce the amplitudes. The same is true for nonlinear inertia. This method does not permit the total elimination of the vibrations.

•22. EXPERIMENTAL VERIFICATION OF THE THEORY

1. On the experimental set-up described in •6, a series of rods were investigated and the amplitudes of steady-state vibrations determined. The data of the experiment were compared with theoretical results. Some of these data are cited below, along with a description of methods for the determination of the nonlinear characteristics.

The coefficient of nonlinear elasticity γ was determined from the experiment in which the rod was loaded with a longitudinal force exceeding the critical value. In •13, the formula

$$f = \frac{\omega}{\sqrt{\gamma}} \sqrt{\frac{P}{P_*} - 1} \tag{5.28}$$

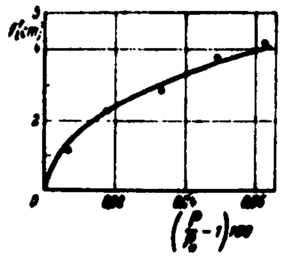


Fig. 37

was derived. Experiments have shown that Eq. (5.28) describes quite satisfactorily the dependence of the deflections on the longitudinal force if the latter is slightly greater than the Euler buckling force (Fig. 37). From Eq. (5.28) it follows that

$$\gamma = \frac{\omega^2}{l^2}\left(\frac{P}{P_*} - 1\right).$$

For a longitudinal force P within the limits $P_* \leq P \leq 1.00065\ P_*$ and deflections measured in the middle of the span, we obtained $\gamma/\omega^2 = 0.42 \times 10^{-4}\,\mathrm{cm}^2$ for one of the specimens.

The coefficient of nonlinear damping ϵ_L was determined from oscillograms of free damped vibrations.

These results indicate that the decrement of damping, computed according to the formula

$$\Delta = \frac{1}{T} \ln \frac{A(t)}{A(t+T)}$$

increases with amplitude. The experimental values for the above mentioned specimen are plotted in Fig. 38. The increase of the decrement is satisfactorily given by the parabola $\bar{\epsilon} = 0.10 + 0.02\,A^2$, from which it follows that $\epsilon_L = 0.08\ \mathrm{cm}^{-2}\mathrm{sec}^{-1}$.

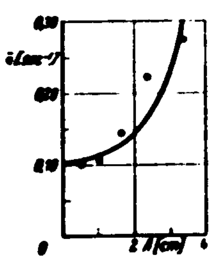


Fig. 38

The coefficient of nonlinear inertia was calculated from the formula

$$\varkappa = \frac{\pi^4 M_L}{4 m l^3}$$

where the reduced mass M_L is found from elementary considerations (Fig. 39).

We will introduce the following notations: let G_1 be the weight of the specimen, G_2 be the weight of loading arm CD that produces the constant component of axial force, P_1 be the weight of the moving support of the vibrator and other hardware, and P_2 be the weight of the additional loading on the end of the lever. If w is the vertical displacement of point B, then the displacement of the end of the lever arm will be $w(a + b)/a$. The axial inertia force in the rod is then

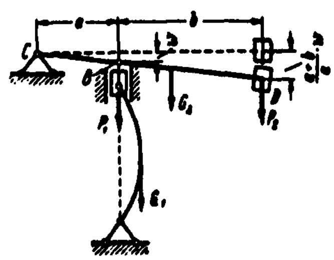


Fig. 39

$$\Delta N = \left[P_1 + kG_1 + \frac{(a+b)^2}{a^2}\left(P_2 + \frac{1}{3}G_2\right)\right]\frac{w''}{g},$$

from which

$$M_L = \frac{1}{g}\left[P_1 + kG_1 + \frac{(a+b)^2}{a^2}\left(P_2 + \frac{1}{3}G_2\right)\right]$$

For the case where $G_1 = 6.0$ kg, $G_2 = 31.0$ kg, $P_1 = 30.0$ kg, $P_2 = 12.0$ kg, $a = 40$ cm, $b = 130$ cm, $l = 170$ cm, we obtain $M_L g = 429.0$ kg. Consequently

$$\varkappa = \frac{3.14^4 \cdot 429}{4 \cdot 6.0 \cdot 170^2} = 0.061 \text{ cm}^{-2}.$$

The coefficient of nonlinear inertia (more accurately, the characteristic magnitude)

$$p = \varkappa n^2 - \frac{3\gamma}{4\Omega^2},$$

can also be determined from the experiment on the damping of free vibrations. According to the approximate formula, Eq. (4.22), the frequency of the nonlinear system is

$$\bar{\Omega} = \Omega\left[1 + \frac{1}{2}a^2\left(\frac{3\gamma}{4\Omega^2} - \varkappa\right)\right].$$

Near the principal resonance $n \approx 1$, the formula takes the form

$$\bar{\Omega} = \Omega\left(1 - \frac{1}{2}a^2 p\right)$$

If the natural frequencies are determined for various amplitudes, one can compute the coefficient

$$p = \frac{2}{a^2}\left(\frac{\Omega}{\bar{\Omega}} - 1\right).$$

This method, however, is unreliable; it can be recommended only for systems with pronounced nonlinearities.

2. The resonance curves were determined in such a manner that at least five-sixths of the experimental points are within the limits of the principal instability region (of the linear approximation). The recording was made stepwise for increasing and decreasing frequencies; this permitted the determination of "overhanging" vibrations.

The resonance curve for one of the specimens is given in Fig. 40. If one applies the equations derived from theory, one must consider that the amplitude of a periodic force produced by the vibrator grows proportionally to the square of the exciting frequency. The value of the coefficient at

$n = 1$ is denoted by $\bar{\mu}$. Then

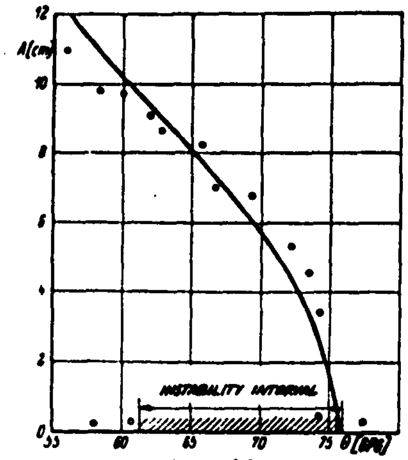


Fig. 40

$$\mu = \bar{\mu} n^2. \tag{5.29}$$

This expression for μ should be substituted in all previously derived theoretical formulas.

We will give the results of the calculations for the case corresponding to Fig. 40 for the following values:

Frequency of free vibrations	Ω	$= 33.8$ cps
Euler buckling force	P_*	$= 183$ kg
Constant component of the axial force	P_0	$= 55$ kg
Amplitude of the variable component (at $n = 1$)	$\bar{P}_t$	$= 54.2$ kg

Excitation parameter $\bar{\mu} = \dfrac{\bar{P}_t}{2(P_* - P_0)} = \dfrac{54.2}{2(183-55)} = 0.212$

Reduced longitudinal mass $M_L g = 316$ kg

Coefficient of nonlinear inertia $\chi = \dfrac{(3.14^4)(316)}{(4)(6.0)(170)^2} = 0.044\ \text{cm}^{-2}$

The nonlinearities of the elasticity and also the linear and nonlinear damping are so small that they can be neglected in the calculation of amplitudes. The experimentally determined coefficient of linear damping obtained from experiment, for example, is $\epsilon = 0.08\ \text{sec}^{-1}$; therefore, the critical coefficient of excitation is

$$\mu_* = \frac{\Delta}{\pi} = \frac{2\epsilon}{\Omega} = 0.0047 \ll \bar{\mu}.$$

The coefficient of nonlinear elasticity γ is small compared to $\chi\Omega^2$, and the influence of nonlinear elasticity can therefore also be neglected.

The boundaries of the principal region of excitation are determined from Eq. (1.43).

$$\theta_* = \frac{2\Omega}{\sqrt{1 \mp \bar{\mu}}}.$$

Substitution gives 61.3 cps and 75.9 cps for the lower and upper boundaries, respectively. Experimental results give 60.8 cps and 74.2 cps (Fig. 40). The deviation is within the limits of the accuracy of the experimental recording of the oscillogram.

The vibration amplitudes are determined according to Eq. (5.22) in conjunction with Eq. (5.29):

$$A^2 = \frac{1 - n^2(1 - \bar{\mu})}{\varkappa n^2}.$$

On the lower boundary of the region of excitation $n_*^2 = 1/(1 + \bar{\mu})$, from which $A^2 = 2\bar{\mu}/\chi$ follows. In our case

$$A = \sqrt{\frac{2 \cdot 0.212}{0.044}} = 9.84\ \text{cm}.$$

The theoretical results are plotted in Fig. 40 as a solid line. The somewhat higher values for the experimental amplitudes near the upper boundaries of the region of excitation may be explained by the fact that the vibrations are always accompanied by more or less intensive beats. However, in the figure the maximum amplitudes are given without any correction for the error caused by the beats.

Analogous experiments were conducted with a number of other specimens, where the constant and periodic component of the axial force were varied for each of them. All the results were in satisfactory agreement with the theory. The experimental data are shown in Figs. 41 and 42.

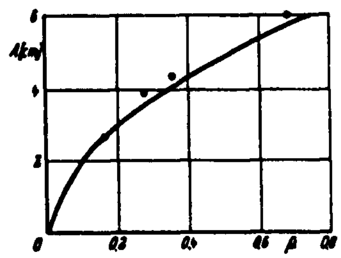


Fig. 41

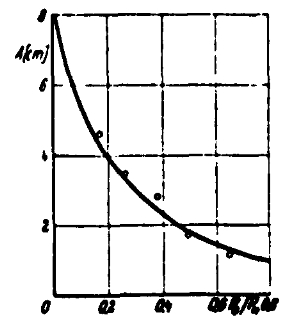


Fig. 42

Figure 42 shows the variation of the vibration amplitudes at $\theta = 2\Omega$ for a changing P_0. In the case of increasing P_0, the natural frequency Ω decreases and the variable loading P_t drops accordingly.

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