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RM-3502-PR
JANUARY 1963



# ANTENNA PATTERN DISTRIBUTIONS FROM RANDOM ARRAYS

Thomas M. Cover

PREPARED FOR:

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# PREFACE

This Memorandum presents the results of a theoretical study of the behavior of certain types of transmitting antennas. The material has possible eventual application to space communications and to problems in communications engineering and science.

## SUMMARY

This Memorandum deals with the determination of the probability distribution of the electric field resulting from an arbitrary random array of sources (or scatterers). The distribution is surprisingly simple, is easily calculated for most interesting array distributions, and has wide generality of application. Specifically, we find the antenna pattern distribution of a synthetic aperture antenna formed by a moving space vehicle emitting pulses randomly in time. However, our results apply not only to synthetic aperture antennas of arbitrary distribution but also to randomly deleted antennas and to chaff, meteor trail, and electron cloud diagnostics as well.

The problem is restricted to the study of the far field from N sources, the positions of which are independent identically distributed as $F(\underline{r})$. Markov's method is then used to analyze what is essentially a two-dimensional random walk induced by a three-dimensional distribution. It is shown that if the Fourier transform $\psi(\underline{k})$ of the distribution function $F(\underline{r})$ can be performed in closed form, then the limiting form of the probability density of the resultant electric field vector is immediately obvious for every frequency and direction of propagation. Finally, the probability density of the resultant power or envelope is determined in closed form, and the correlation between the resultant field at different angles and frequencies is exhibited.

## ACKNOWLEDGMENTS

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## I. INTRODUCTION

This Memorandum is concerned with the probability distribution of the far field resulting from an array of N sources, the positions of which are independent, identically distributed random variables. The importance or utility of this simply stated problem is essentially derived from the extreme simplicity of form of the answer.

Maffett[1] briefly discusses a specific random antenna--a linear array with N sources uniformly distributed. However, the analysis is incomplete in that he derives the distribution of only one of the two quadrature components of the field. Hence, nothing can be said of the distribution of the envelope or power. In this Memorandum an expression will be derived for the envelope distribution of the resultant field from arbitrary spatial distribution functions.

An interesting and very general study of the problem has been made by Kelly and Lerner[2] in which they exploit the mathematical analogy to the shot effect of the radar echo from a random collection of scatterers. However, their specializations are made to cases in which the expected antenna pattern is essentially zero.

The following examples will illustrate some of the array distributions which may be of interest. A gas diffusing from a point source might have a three-dimensional gaussian distribution. A synthetic aperture antenna formed by emitting pulses at Poisson increments in time from a linearly moving space vehicle can be considered, under certain assumptions on the data processing scheme, to have sources having a uniform probability density over a line segment. A short section of meteor trail may have its scatterers obeying a uniform distribution over a cylinder cut out by the meteor. Also, a phased array whose components are randomly deleted or damaged is a random array with a distribution corresponding to the related complete array. The expected changes in gain of randomly deleted arrays are discussed by Ogg.[3]

The far field assumption, made in order to give the answer its simplicity and utility, restricts some of the above examples to relatively uninteresting cases. In order for the far field assumption to hold, the lateral extent of the target must be less than the diameter of a Fresnel zone. For example, the lateral extent of a gas cloud, meteor trail segment, or synthetic aperture antenna--for distances on the order of 1000 km at wavelengths on the order of 1 m--must be less than 1 km. Then again, the qualitatively interesting cases occur when the extent of the array is a small number of wavelengths, because the field components will tend to add coherently rather than incoherently as the extent of the array decreases.

Figure 1 illustrates a typical sample random array where $\underline{k}$ is the direction of propagation vector with magnitude

$$|\underline{k}| = \frac{2\pi}{\lambda} \tag{1}$$

and points in the direction of propagation. Under the far field assumption, for identically polarized sources, the resultant electric field E at a point along the direction $\underline{k}$ can be written as the complex number

$$E = \sum_{j=1}^{N} e^{i\underline{k}\cdot\underline{r}_j} \tag{2}$$

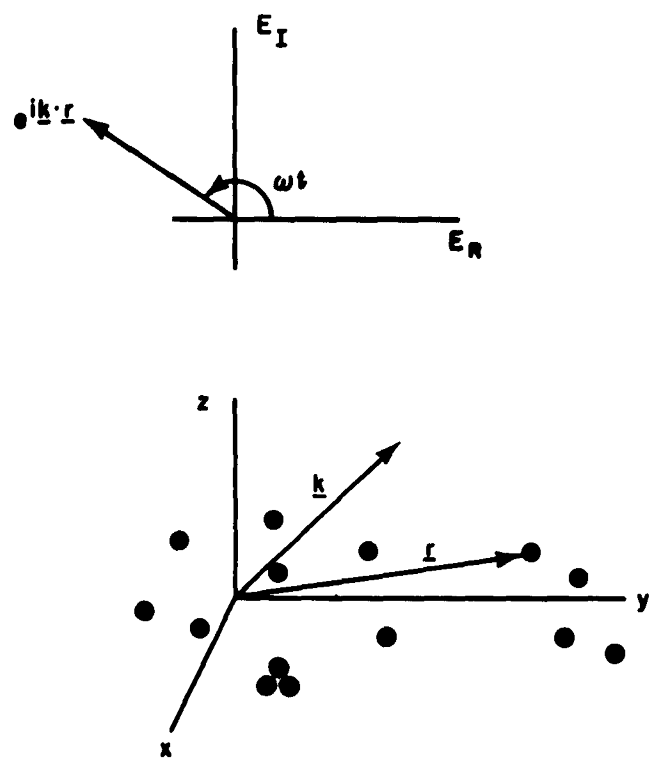


Fig. 1 — Sketch of array and electric field component arising from source at position $\underline{r}$

For radar echoes $\underline{k}$ becomes $2\underline{k}$ because of the doubled path. For a continuous distribution $g(\underline{r})$

$$E = \iiint_{-\infty}^{\infty} g(\underline{r})\, e^{i\underline{k}\cdot\underline{r}}\, d\underline{r} \tag{3}$$

## II. THE FIELD DISTRIBUTION FROM A RANDOM ARRAY

This section will provide the details of the derivation of the asymptotic probability distribution of the resultant electric field. For purposes of exposition we assume N isotropic radiators of equal amplitude, operating in phase. We may relax these assumptions later.

The very general method of characteristic functions will be used here, i.e., the characteristic function of the sum of independent (not necessarily identically distributed) random variables (in this case vector-valued random variables) is the product of the individual characteristic functions. The inversion formula[4] yields the desired distribution function of the vector-valued random variable. Chandrasekhar[5] ascribes to Markov the origin of this technique with respect to multidimensional random variables.

The essential simplifications occur when the N random variables are identically distributed and N is large (10 or more), thus allowing an asymptotic expansion of the characteristic function, which is then easily inverted.

Our problem is essentially that of solving a two-dimensional random walk (each step being an electric field vector) induced by the three-dimensional probability distribution of the positions of the radiators. We assume the use of isotropically radiating sources of equal intensity that are operating in phase. These sources are independently identically distributed in space according to the probability distribution $F(\underline{r})$ where $\underline{r} = (x,y,z)^t$, a three-dimensional column vector.*

$$F(\underline{r}) = P_r \left\{ X \le x,\ Y \le y,\ Z \le z \right\} \tag{4}$$

*The symbol "t" designates transpose.

To this distribution function $F(\underline{r})$ there corresponds the characteristic function[4,5] $\psi(\underline{k}) = \psi(k_x, k_y, k_z)$. If the radiators also have some randomness of phase $\delta$, we may define a new $\underline{k} = (k_x, k_y, k_z, 1)^t$ and a new $\underline{r} = (x, y, z, \delta)^t$ and treat this case along with the former. In either case we have the definition

$$\psi(\underline{k}) = E\left\{e^{i\underline{k}^t\underline{r}}\right\} = \iiint_{-\infty}^{\infty} e^{i\underline{k}^t\underline{r}}\, dF(\underline{r}) \tag{5}$$

The contribution to the electric radiation field in direction $\underline{k}$ due to a source at position $\underline{r}$ is $e^{i\underline{k}^t\underline{r}}$. The complex number $e^{i\underline{k}^t\underline{r}}$ will be written as the two-dimensional column vector $\underline{E} = (\cos \underline{k}^t\underline{r}, \sin \underline{k}^t\underline{r})^t$. Hence the total electric field will be the sum of $N$ two-dimensional vectors, each of unit length but having random phases, the distributions of which depend on $F(\underline{r})$ and $\underline{k}$. Our first task is to obtain the probability density $p(\underline{E})$ where

$$\underline{E} = \frac{1}{N} \sum_{i=1}^{N} \underline{E}_i \tag{6}$$

Note that $|\underline{E}| = 1$ if all the sources add coherently. Hence, we call $\underline{E}$ the normalized resultant vector.

It will be wise to consider first the distribution of the centered random variable

$$\underline{E}_{(N)} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\underline{E}_i - \underline{\mu}_i) \tag{7}$$

where

$$\underline{\mu}_i = E\left\{\underline{E}_i\right\}$$

is the expected value of the electric field in direction $\underline{k}$ due to the i-th source (averaged over the spatial distribution $F(\underline{r})$). Later, a change in variable will yield the desired distribution. The characteristic function $\Phi_{(N)}(\underline{w})$ of $\underline{E}_{(N)}$ is given by

$$\begin{aligned}\Phi_{(N)}(\underline{w}) &= \iiint_{-\infty}^{\infty} \exp\left[i\underline{w}^t\underline{E}_{(N)}\right] dF(\underline{r}_1,\underline{r}_2,\ldots,\underline{r}_N) \\ &= \iiint_{-\infty}^{\infty} \exp\left[\frac{i}{\sqrt{N}}\,\underline{w}^t \sum_{j=1}^{N} (\underline{E}_j-\underline{\mu}_j)\right] dF(\underline{r}_1)dF(\underline{r}_2)\ldots dF(\underline{r}_N)\end{aligned} \tag{8}$$

where the independence of the $\underline{r}_j$ has been used. Making use of the identical distribution of the $\underline{r}_j$, the following equation is true for any j:

$$\Phi_{(N)}(\underline{w}) = \left[\iiint_{-\infty}^{\infty} \exp\left[\frac{i}{\sqrt{N}}\,\underline{w}^t(\underline{E}_j-\underline{\mu}_j)\right] dF(\underline{r}_j)\right]^N \tag{9}$$

This integral can be evaluated only for very special forms of $F(\underline{r})$. See Section VI for such a case.

In order to proceed with the analysis for general $F(\underline{r})$, consider the expansion of Eq. (9)

$$\Phi_{(N)}(\underline{w}) = \left[\iiint_{-\infty}^{\infty} \left[1 + \frac{i}{\sqrt{N}}\,\underline{w}^t(\underline{E}_j-\underline{\mu}_j) - \frac{1}{2N}\,\underline{w}^t(\underline{E}_j-\underline{\mu}_j)(\underline{E}_j-\underline{\mu}_j)^t\underline{w} + o\left(N^{-3/2}\right)\right] dF(\underline{r}_j)\right]^N \tag{10}$$

$$= \left[1 - \frac{1}{2N}\,E\left\{\underline{w}^t(\underline{E}_j-\underline{\mu}_j)(\underline{E}_j-\underline{\mu}_j)^t\underline{w}\right\} + o\left(N^{-3/2}\right)\right]^N$$

because $E\left\{\underline{E}_j-\underline{\mu}_j\right\} = 0$. Then

$$\Phi_{(N)}(\underline{w}) = \exp\left[-\frac{\underline{w}^t Q\underline{w}}{2}\right] + 0\left(N^{-1/2}\right)$$

which tends in the limit for large N to the characteristic function

$$\Phi(\underline{w}) = \exp\left[-\frac{\underline{w}^t Q\underline{w}}{2}\right] \tag{11}$$

where Q is the 2 x 2 covariance matrix corresponding to a single radiator of unit amplitude:

$$Q = \begin{bmatrix} \mathrm{Var}(E_R) & \mathrm{Cov}(E_R E_I) \\ \mathrm{Cov}(E_I E_R) & \mathrm{Var}(E_I) \end{bmatrix} \tag{12}$$

(Incidentally, the asymptotic derivations of results similar to these in Ref. 5 are incorrect.*) The characteristic function (Eq. (11)) is easily inverted[6] to give the density

$$F\left(\underline{E}_{(N)}\right) = \frac{1}{2\pi\|Q\|^{1/2}} \exp\left[-\frac{1}{2}\,\underline{E}^t_{(N)}\, Q^{-1}\, \underline{E}_{(N)}\right]$$

A change of variable corresponding to definitions found in Eqs. (6) and (7) results in

$$F(\underline{E}) = \frac{N}{2\pi\|Q\|^{1/2}} \exp\left[-\frac{N}{2}\,(\underline{E}-\underline{\mu})^t\, Q^{-1}(\underline{E}-\underline{\mu})\right] \tag{13}$$

*The asymptotic expansions are incorrect in Eqs. (85), (91), and (96), although the answers exhibited in Eqs. (87) and (93) are correct. The general answer, Eq. (103), is incorrect because Eq. (100) is a moment matrix rather than a covariance matrix.

So, for large N, the probability density of the normalized resultant electric field from N independent identically distributed sources is bivariate normal with mean $\underline{\mu}$ and covariance matrix $\frac{1}{N}$ Q.

## III. EVALUATION OF MEAN AND COVARIANCE MATRIX

In this section the results are pleasing in their simplicity and interesting in their interpretation. We wish to find $\underline{\mu}$ and Q in terms of $F(\underline{r})$.

By the definition found in Eq. (8)

$$
\begin{aligned}
\underline{\mu} &= E\left\{\underline{E}_1\right\} \\
&= \left(E\left\{\cos \underline{k}\cdot\underline{r}\right\},\ E\left\{\sin \underline{k}\cdot\underline{r}\right\}\right)^t \\
&= \left(\iiint_{-\infty}^{\infty} \cos \underline{k}\cdot\underline{r}\ dF(\underline{r}),\ \iiint_{-\infty}^{\infty} \sin \underline{k}\cdot\underline{r}\ dF(\underline{r})\right)^t \\
&= \underline{\psi}(\underline{k}) = \left(\psi_R(\underline{k}),\ \psi_I(\underline{k})\right)^t
\end{aligned}
\qquad (14)
$$

where $\psi(\underline{k})$ is just the characteristic function of $F(\underline{r})$ (see Eq. (5)) and $\underline{\psi}(\underline{k})$ is the two-dimensional vector formed from the real and imaginary parts of $\psi$. We conclude, upon comparison with Eq. (3), that the <u>expected</u> antenna space factor from the random array with distribution $F(\underline{r})$ corresponds exactly to the <u>actual</u> space factor from a source <u>intensity</u> distribution $g(\underline{r})$ equal to the <u>probability density</u> $F'(\underline{r})$. This is intuitively obvious.

The covariance matrix Q also is easily expressed in terms of $\underline{\psi}(\underline{k})$. From Eq. (12) we have, for a single unit radiator, the matrix

$$
Q = \begin{bmatrix} \overline{E_R^2} - \overline{E}_R^2 & \overline{E_R E_I} - \overline{E}_R\overline{E}_I \\ \\ \overline{E_R E_I} - \overline{E}_R\overline{E}_I & \overline{E_I^2} - \overline{E}_I^2 \end{bmatrix} \qquad (15)
$$

By calculating $Q_{11}$, for example, we find

$$\overline{E_R^2} = \iiint_{-\infty}^{\infty} E_R^2 \, dF(\underline{r}) = \iiint_{-\infty}^{\infty} \cos^2 \underline{k}\cdot\underline{r} \, dF(\underline{r})$$

$$= \iiint_{-\infty}^{\infty} \frac{1}{2} (1 + \cos 2 \, \underline{k}\cdot\underline{r}) \, dF(\underline{r}) \tag{16}$$

$$= \frac{1}{2} + \psi_R(2\underline{k})$$

$$\overline{E_R^2} - \overline{E}_R^2 = \frac{1}{2} + \psi_R(2\underline{k}) - \psi_R^2(\underline{k}) \tag{17}$$

Proceeding in the same manner, we finally obtain

$$Q = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} \psi_R(2\underline{k}) - \psi_R^2(\underline{k}) & \frac{1}{2} \psi_I(2\underline{k}) - \psi_I(\underline{k}) \, \psi_R(\underline{k}) \\ \\ \frac{1}{2} \psi_I(2\underline{k}) - \psi_I(\underline{k}) \, \psi_R(\underline{k}) & \frac{1}{2} - \frac{1}{2} \psi_R(2\underline{k}) - \psi_I^2(\underline{k}) \end{bmatrix} \tag{18}$$

$$\underline{\mu} = \left(\psi_R(\underline{k}), \psi_I(\underline{k})\right)^t$$

and

$$P(\underline{E}) = N(\underline{\mu}, \frac{1}{N} Q)$$

where $N(\underline{\mu}, \frac{1}{N} Q)$ is the normal density function with mean $\underline{\mu}$ and covariance matrix Q. A comment on the simplicity of this result is in order. The characteristic function $\psi(\underline{k})$ is just the three-dimensional Fourier transform of $F(\underline{r})$ and is precisely the far field pattern for a nonrandom illumination $F(\underline{r})$. Here, we find that both $\underline{\mu}$ and Q, and therefore $P(\underline{E})$, are simple functions of $\psi$ evaluated at the argument $\underline{k}$ which "just happens" to be the direction of propagation vector. Thus, if the Fourier transform of the distribution function $F(\underline{r})$ can be performed in closed form, then the probability density of the normalized resultant electric field vector is immediately obvious for <u>every</u> frequency and direction of propagation.

# IV. EXAMPLES

## UNIFORM LINEAR ARRAY

We are concerned with the space factor of N isotropic sources operating in phase whose positions $\underline{r}_i$ are independent, identically distributed random variables uniformly distributed over an interval of length L. (See Fig. 2.)

$$F'(x,y,z) = \begin{cases} \frac{1}{L}\,\delta(y)\,\delta(z) & , \quad -\frac{L}{2} \le x \le \frac{L}{2} \\ 0 & , \quad \text{otherwise} \end{cases} \tag{19}$$

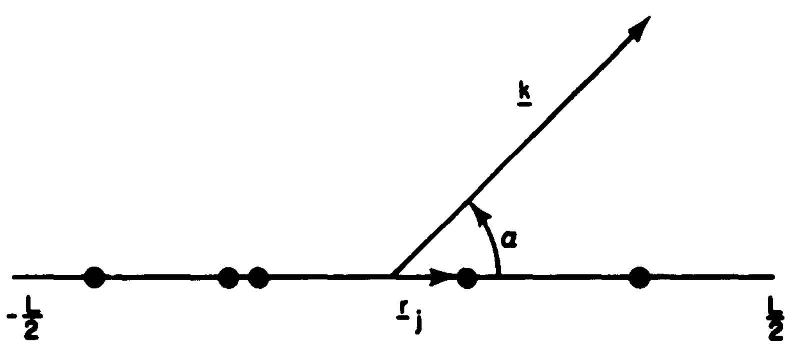


Fig. 2— Uniform linear array

Actually it was this problem (suggested by Irving S. Reed) which led to the investigations reported in this Memorandum. It would be of interest, for example, to know the radiation pattern of a synthetic aperture antenna realized by transmitting pulses at Poisson increments in time from a rapidly orbiting satellite, deep space vehicle, or meteor trail. The data are simply processed as if the pulse positions were unknown. Such a random antenna will have the property of suppressing the grating lobes arising from transmission of regularly spaced pulses.

To proceed, we find simply

$$\begin{aligned}\psi(\underline{k}) = \psi(k_x,k_y,k_z) &= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\underline{k}^t\underline{r}}\,\delta(y)\,\delta(z)\,d\underline{r} \\ &= \frac{\sin k_x \frac{L}{2}}{k_x \frac{L}{2}}\end{aligned} \tag{20}$$

where $k_x = \frac{2\pi}{\lambda}\cos\alpha$ and $\alpha$ is the angle included between the direction of propagation vector and the line of sources. We are motivated now to define

$$\gamma = \frac{k_x L}{2\pi} = \frac{L\cos\alpha}{\lambda} \tag{21}$$

and call it the "effective" antenna length in wavelengths.

$$\begin{aligned} P(\underline{E}) &= N\left(\underline{\mu}, \frac{1}{N}Q\right) \\ \underline{\mu} &= \begin{pmatrix} \dfrac{\sin\pi\gamma}{\pi\gamma} \\ 0 \end{pmatrix} \\ Q &= \begin{bmatrix} \frac{1}{2}+\frac{1}{2}\frac{\sin 2\pi\gamma}{2\pi\gamma}-\frac{\sin^2\pi\gamma}{(\pi\gamma)^2} & 0 \\ 0 & \frac{1}{2}-\frac{1}{2}\frac{\sin 2\pi\gamma}{2\pi\gamma} \end{bmatrix}\end{aligned} \tag{22}$$

Observe that for $\gamma = 1, 2, 3, \ldots$

$$\underline{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \; Q = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \tag{23}$$

Figure 3 shows the sketch of the expected value of E, along with the one standard deviation lines of the real and imaginary part of E.

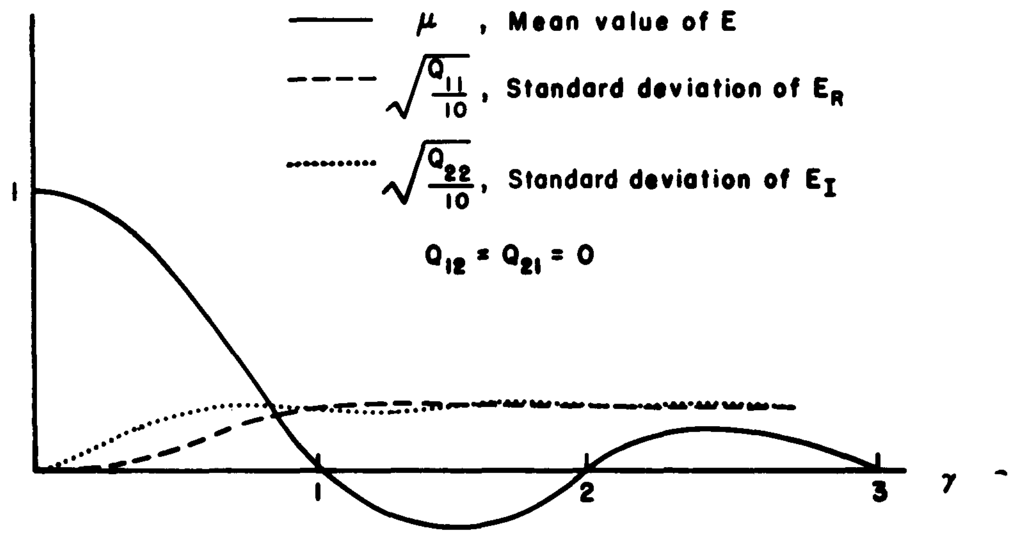


Fig. 3 — Uniform linear distribution (N=10)

Remark 1: Here P(E) depends on the direction of propagation $\underline{k}$ through $\gamma$.

Remark 2: A meteor entering the atmosphere and creating an ionization path will result in a $\gamma$ which increases linearly with time. Thus, a receiver will observe a return similar to Fig. 3 considered as a function of time.

## GAUSSIAN ARRAY IN THREE-DIMENSIONS

It is clear that echo statistics from an array of normally distributed sources (or scatterers) are of more than academic interest. Such a distribution arises naturally for a gas diffusing from some source point, dipoles subliming from an orbiting source, and in "nice" explosions (relatively collisionless expansion of a gas which was confined and in equilibrium at time zero).

Here we consider the distribution

$$F'(\underline{r}) = \frac{1}{\left(\frac{2\pi\sigma^2}{3}\right)^{3/2}} \exp\left(-3|\underline{r}|^2/2\sigma^2\right) \tag{24}$$

and easily calculate by Eq. (5) the corresponding characteristic function

$$\psi(\underline{k}) = \exp\left[-|\underline{k}|^2\,\sigma^2/6\right] \tag{25}$$

Since $|\underline{k}| = \frac{2\pi}{\lambda}$, we are led to introduce $\frac{\gamma}{2\pi}$, the cloud "radius" in wavelengths:

$$\frac{\gamma}{2\pi} = \frac{\sigma}{\lambda\sqrt{3}} \tag{26}$$

Thus $\quad \psi(\underline{k}) = e^{-\gamma^2/2}$

and $\quad \psi(2\underline{k}) = e^{-2\gamma^2}$

$$p(\underline{E}) = N(\underline{\mu}, N^{-1}\,Q)$$

$$\underline{\mu} = \begin{pmatrix} e^{-\gamma^2/2} \\ 0 \end{pmatrix}$$

$$Q = \begin{bmatrix} \left(\frac{1}{2} + \frac{1}{2} e^{-2\gamma^2} - e^{-\gamma^2}\right) & 0 \\ 0 & \left(\frac{1}{2} - \frac{1}{2} e^{-2\gamma^2}\right) \end{bmatrix} \qquad (27)$$

The above results are plotted in Fig. 4.

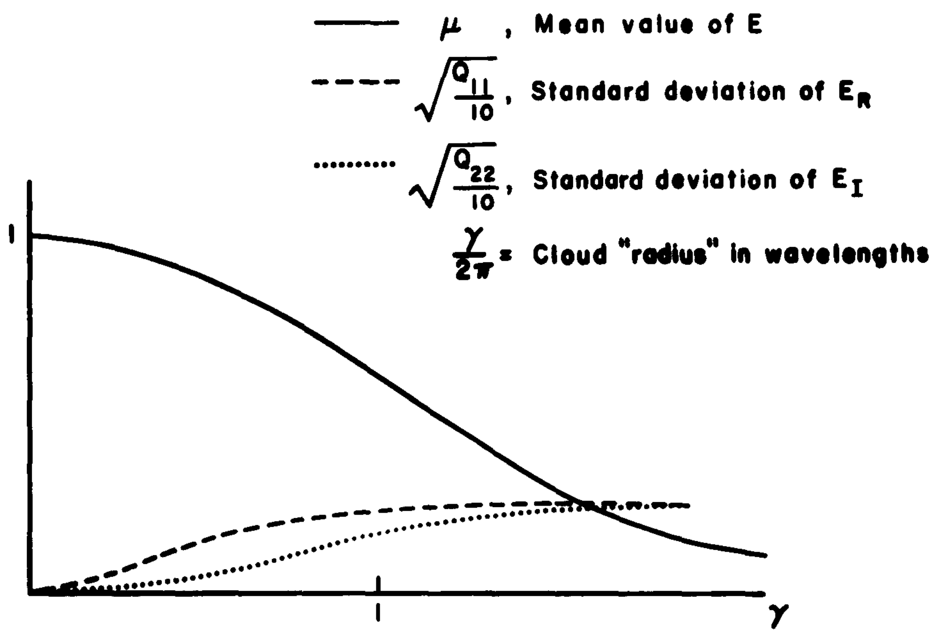


Fig. 4 — Gaussian array (N=10)

Remark 1: P(E) is independent of direction of propagation $\underline{k}$.

Remark 2: A gas of N particles diffusing from an initial point concentration produces a $\gamma$ which is proportional to the square root of the elapsed time. Now the echo statistics can be read from the graph.

**Remark 3:** For $\gamma > 1$, the envelope r of the electric field E is essentially Rayleigh distributed.

$$p(r) \doteq \frac{r}{(1/2\ N)} \exp\left[-\frac{r^2 + e^{-\gamma^2/2}}{(1/N)}\right] I_0\left(\frac{r\ e^{-\gamma^2/2}}{(1/2\ N)}\right)$$

This in turn, for $\gamma \gg 1$, becomes $P(r) \doteq 2Nr\ e^{-Nr^2}$.

UNIFORM DISTRIBUTION OVER A DISC

The statistics of the uniform distribution over the area of a circle are of interest, for example, if one has coherent transmitters scattered randomly in the ocean, desert, or on the surface of another planet, or if one has the analogous problem of analyzing echoes from clusters of points on the sea. Consider a source, the position of which is a random variable having a uniform distribution over a disc of radius $R_0$.

$$F'(\underline{r}) = \begin{cases} \dfrac{1}{\pi R_0^2}\ ; & z = 0,\ x^2 + y^2 \le R_0^2 \\ \\ 0 & ; \text{ otherwise} \end{cases} \tag{28}$$

Then[7]

$$\psi(\underline{k}) = \frac{2\ J_1(\pi\gamma)}{\pi\gamma} \tag{29}$$

where $J_1(z)$ is the Bessel function of order one; $\gamma = \dfrac{2\ R_0 \sin\theta}{\lambda}$ = "effective" aperture diameter (wavelengths); and $\theta$ is the angle between $\underline{k}$ and the z axis. Equation (29), together with Eq. (18), yields the probability density of the field as a function of $\gamma$.

OTHER DISTRIBUTIONS

As has been pointed out, the only difficulty in finding P(E) is in computing the Fourier transform $\psi(\underline{k})$ of $F(\underline{r})$. This has been done in many interesting cases, and no further examples will be enumerated.

## V. ENVELOPE DISTRIBUTION

The probability distribution of the electric field vector is useful when completely coherent reception (detection) is realized, i.e., the phase of the carrier is known at the receiver. In general, however, the phase of the return is of secondary importance, and we shall be most interested in the probability distribution of the envelope, or magnitude of the field. Unfortunately, averaging out the phase proves to be a difficult unsolved problem,[8] where satisfactory expressions exist only for special cases. In this section we attempt to determine the envelope density $p(r)$.

Assume for simplicity that $F(\underline{r})$ is symmetric in $\underline{r}$; that is, $dF(\underline{r}) = dF(-\underline{r})$. (The same analysis will follow, after diagonalization of $Q$, for the general case.) Then $\psi(\underline{k})$ is real for all $\underline{k}$ and

$$Q = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}\psi(2\underline{k}) - \psi^2(\underline{k}) & 0 \\ 0 & \frac{1}{2} - \frac{1}{2}\psi(2\underline{k}) \end{bmatrix} \tag{30}$$

which we rewrite

$$Q = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{31}$$

Then

$$Q^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix} \tag{32}$$

$$\underline{\mu} = \begin{pmatrix} \psi(\underline{k}) \\ 0 \end{pmatrix}$$

We have for the probability density of the normalized electric field vector

$$p(\underline{E}) = N(\underline{\mu}, \frac{1}{N} Q)$$

$$= \frac{N}{2\pi(\lambda_1\lambda_2)^{1/2}} \exp - \frac{N}{2}\left[\frac{\left(E_R - \psi(\underline{k})\right)^2}{\lambda_1} + \frac{E_I^{\,2}}{\lambda_2}\right] \qquad (33)$$

Let
$$E_R = r \cos \theta$$
$$E_I = r \sin \lambda \qquad (34)$$

Then after reduction

$$p(r,\theta) = \frac{1}{2\pi} C\, e^{-A \cos 2\theta + B \cos \theta} \qquad (35)$$

where

$$C = \frac{Nr}{(\lambda_1\lambda_2)^{1/2}} \exp\left[- \frac{N(\lambda_1+\lambda_2)r^2}{4\,\lambda_1\,\lambda_2} - \frac{N\,\psi^2\,(\underline{k})}{2\,\lambda_1}\right]$$

$$A = \frac{N(\lambda_2 - \lambda_1)r^2}{4\,\lambda_1\,\lambda_2}$$

$$B = \frac{N\,r\,\psi(\underline{k})}{\lambda_1}$$

$$\lambda_1 = \frac{1}{2} + \frac{1}{2}\,\psi(2\underline{k}) - \psi^2(\underline{k})$$

$$\lambda_2 = \frac{1}{2} - \frac{1}{2}\,\psi(2\underline{k})$$

and

$$\psi(\underline{k}) = \iiint_{-\infty}^{\infty} \cos \underline{k}\cdot\underline{r} \, dF(\underline{r})$$

In order to proceed in the determination of p(r), we list the following pertinent Bessel function relationships:

Bessel's integrals for the Bessel coefficients (Ref. 9, p. 20)

$$\begin{cases} J_n(z) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} e^{in\Phi - iz\sin\Phi} \, d\Phi \qquad , \quad n \geq 0 \\ I_n(z) = \dfrac{(-1)^n}{2\pi} \displaystyle\int_0^{2\pi} e^{in\Phi + z\cos\Phi} \, d\Phi \qquad , \quad n \geq 0 \end{cases} \tag{36}$$

The series definitions of the coefficients (Ref. 9, pp. 15, 77):

$$\begin{cases} J_n(z) = \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2} z)^{n+2m}}{m!\,(n+m)!} \qquad , \quad n \geq 0 \\ J_{-n}(z) = (-1)^n J_n(z) \qquad , \quad n \geq 0 \\ I_n(z) = \displaystyle\sum_{m=0}^{\infty} \frac{(\frac{1}{2} z)^{n+2m}}{m!\,(n+m)!} \qquad , \quad n \geq 0 \\ I_{-n}(z) = I_n(z) \qquad , \quad n \geq 0 \\ I_n(z) = (-i)^n J_n(iz) \end{cases} \tag{37}$$

The Fourier series expansions:

$$\left\{\begin{aligned} e^{iz\sin\Phi} &= \sum_{n=-\infty}^{\infty} e^{in\Phi} J_n(z) \\ e^{z\cos\Phi} &= \sum_{n=-\infty}^{\infty} e^{in\Phi} I_n(z) \end{aligned}\right. \tag{38}$$

Then the marginal density of the envelope is given

$$p(r) = \int_0^{2\pi} p(r,\theta)\, d\theta$$

$$= \frac{1}{2\pi} C \int_0^{2\pi} e^{-A\cos 2\theta + B\cos\theta}\, d\theta$$

By Eq. (38)

$$p(r) = \frac{1}{2\pi} C \int_0^{2\pi} e^{B\cos\theta} \sum_{n=-\infty}^{\infty} e^{2in\theta} I_n(-A)\, d\theta$$

Rearranging

$$p(r) = C \sum_{n=-\infty}^{\infty} I_n(-A) \frac{1}{2\pi} \int_0^{2\pi} e^{2in\theta + B\cos\theta}\, d\theta$$

By Eq. (36)

$$p(r) = C \sum_{n=-\infty}^{\infty} I_n(-A)\, I_{2n}(B)\, (-1)^n$$

Putting $z = -A$ in Eq. (37)

$$p(r) = C \sum_{n=-\infty}^{\infty} I_n(A)\, I_{2n}(B)$$

Finally, by Eq. (37)

$$p(r) = C\, I_0(A)\, I_0(B) + 2\, C \sum_{n=1}^{\infty} I_n(A)\, I_{2n}(B) \tag{39}$$

The expression given here for the density of the envelope r is the principal result of this section. Although p(r) is expressed as an infinite sum, much information can be obtained from closer examination.

Consider, for example, the case where $\lambda_1 = \lambda_2 = \lambda$, i.e., the real and imaginary components of the electric field E, are independent random variables with identical variances $\lambda$ and with means $\psi(\underline{k})$ and 0, respectively. (Observe that this case occurs for the field from a random array only when $\psi(2\underline{k}) = \psi^2(\underline{k})$.) In such a case the parameter A in Eq. (39) is equal to zero. Since

$$I_n(0) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

all terms but the first drop out of Eq. (39) leaving

$$\begin{aligned} p(r) &= C\, I_0(B) \\ &= \frac{Nr}{\lambda} \exp - \frac{N}{2\lambda}\left[r^2 + \psi^2(\underline{k})\right] I_0\left(\frac{Nr\, \psi(\underline{k})}{\lambda}\right) \text{ for } r \geq 0 \end{aligned} \tag{40}$$

This is just the Rayleigh distribution with parameters $\psi(\underline{k})$ and $\lambda/N$, which is discussed, for example, by Rice (Ref. 10, p. 106).

Consider next the case where the mean value of E is zero, i.e., $\psi(\underline{k}) = 0$. Here we have E obeying an elliptical bivariate normal law with mean zero, corresponding to an expected node in the antenna pattern.

Now the parameter B in Eq. (39) is zero; all terms of the sum but the first are zero. Thus

$$p(r) = C\ I_0(A)$$

$$= \frac{Nr}{(\lambda_1\lambda_2)^{1/2}} \exp\left[-\frac{N(\lambda_1+\lambda_2)r^2}{4\ \lambda_1\lambda_2}\right] I_0\left(\frac{N(\lambda_2-\lambda_1)r^2}{4\ \lambda_1\lambda_2}\right) \text{ for } r \geq 0 \tag{41}$$

Finally, for comparison purposes, we append an approximation that is good when $\frac{\lambda_1}{\lambda_2}$ is on the order of one. We propose to reduce the elliptical gaussian bivariate distribution of $\underline{E}$ to an "equivalent" circular distribution. This is then integrated directly to give the Rayleigh distribution. There is no restriction to diagonal Q.

A locus of constant probability of $\underline{E}$ (see Eq. (13)) is the ellipse $(E - \mu)'\ Q^{-1}\ (E - \mu) = 1$. The area of this ellipse is $\pi\ \lambda_1\lambda_2$, where $\lambda_1$ and $\lambda_2$ are the first and second eigenvalues of Q; that is, $\lambda_1$ and $\lambda_2$ are the lengths of the major and minor axes of the ellipse.

A circle of the same area would have radius $r = \sqrt{\lambda_1\lambda_2}$ (the harmonic mean of the lengths of the axes of the ellipse). Finally, recall that $\lambda_1\lambda_2 = \|Q\| = \|Q^{-1}\|^{-1}$. Now simply replace the elliptical

$$Q^{-1} = P' \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix} P \tag{42}$$

by the circular

$$\tilde{Q}^{-1} = P' \begin{bmatrix} \frac{1}{(\lambda_1\lambda_2)^{1/2}} & 0 \\ 0 & \frac{1}{(\lambda_1\lambda_2)^{1/2}} \end{bmatrix} P = \frac{1}{(\lambda_1\lambda_2)^{1/2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{43}$$

Hence Eq. (13) becomes

$$p(\underline{E}) = \frac{N}{2\pi\|Q\|^{1/2}} \exp - \left[\frac{N(E-\mu)^t(E-\mu)}{2\;\|Q\|^{1/2}}\right] \tag{44}$$

When the phase is integrated out, the probability density of the envelope r is

$$p(r) = \frac{Nr}{\|Q\|^{1/2}} \exp - \left[\frac{N\left(r^2 + |\underline{\mu}|^2\right)}{2\;\|Q\|^{1/2}}\right] I_0\left(\frac{Nr\;|\underline{\mu}|}{\|Q\|^{1/2}}\right) \tag{45}$$

## VI. AN EXACT CASE

In the preceding sections we have been forced to consider a random walk with unit steps having an arbitrary distribution of angle at each step. This differs from the classical case, treated by Kluyver[9] and Rayleigh,[11] where all angles are equiprobable. However, we do get the isoperiodic case when our source distribution $F(\underline{r})$ satisfies certain conditions (roughly, a uniform distribution over a volume whose dimension along the direction of propagation is an integral number of wavelengths). To illustrate this, consider a random linear array where N sources are distributed independently, uniformly over a length L (see Fig. 2). Then whenever $\underline{k} \cdot \underline{L}$ is an integer, i.e., $\frac{2\pi L \cos \alpha}{\lambda} = n$, it is apparent that the distribution of the angle of the unit electric field vector from each source is uniform over $2\pi$. Hence Kluyver's[9] results apply (see also Ref. 10, p. 242 and Ref. 11), and the probability density of the envelope $r = |E|$ is exactly

$$p(r) = r \int_0^\infty x \, J_0(rx) \, J_0^N(x) \, dx \tag{46}$$

This has been shown to approach a Rayleigh distribution in the limit for large N.

## VII. CORRELATION IN RADIATION AT DIFFERENT ANGLES AND FREQUENCIES

The knowledge of the radiation field from a random array at one direction and frequency will allow an improved estimate of the radiation field at another direction or another frequency. It is intuitively clear that specification of the radiation field in a given direction puts constraints on the positions of the sources. This information induces a conditional probability distribution on the random field at new angles and frequencies.

Applications of the generalizations in this section are numerous. An example of interest is the correlation between radar returns from a cloud of fixed random scatterers when the carrier frequency is changed. An "invisible" cloud at one frequency will, with a certain probability, become visible at another. Let us also consider a certain scintillating target model composed of N randomly distributed scatterers. The observation angle will change as this target moves past, and we might wish to use the past information in the optimal way in order to predict future fading and provide for a good tracking capability. In the former example, we consider a direction of propagation vector $\underline{k}$ changing in magnitude only; in the latter, $\underline{k}$ changes only in angle. We shall treat both cases at once.

Let $\underline{k}_1, \underline{k}_2, \ldots, \underline{k}_m$ be m direction of propagation vectors for which we should like to specify the radiation field vectors $\underline{E}(\underline{k}_1), \underline{E}(\underline{k}_2), \ldots,$ $\underline{E}(\underline{k}_m)$. Recall that $|\underline{k}_i| = \frac{2\pi}{\lambda_i}$. Then for large N, so that the individual $\underline{E}(\underline{k}_i)$ are each approximately normally distributed as expressed in Eq. (18), it is obvious that

$$p\left(\underline{E}(\underline{k}_1), \underline{E}(\underline{k}_2), \ldots, \underline{E}(\underline{k}_m)\right) = N\left(\underline{\mu}, \frac{1}{N} Q\right) \tag{47}$$

where now

$$\underline{\mu} = \left(\underline{\mu}(\underline{k}_1),\ \underline{\mu}(\underline{k}_2),\ \ldots\ ,\ \underline{\mu}(\underline{k}_m)\right)^t$$

and Q is the composite matrix

$$Q = \left[\begin{array}{c:c:c} Q(\underline{k}_1,\underline{k}_1) & Q(\underline{k}_1,\underline{k}_2) & \\ \hdashline Q(\underline{k}_2,\underline{k}_1) & Q(\underline{k}_2,\underline{k}_2) & \\ \hdashline Q(\underline{k}_m,\underline{k}_1) & & Q(\underline{k}_m,\underline{k}_m) \end{array}\right] \tag{48}$$

and where each $Q(\underline{k}_i,\underline{k}_j)$ is the 2 x 2 correlation matrix

$$Q(\underline{k}_i,\underline{k}_j) = E\left\{\left(\underline{E}(\underline{k}_i) - \underline{\mu}_i\right)\left(\underline{E}(\underline{k}_j) - \underline{\mu}_j\right)^t\right\} \tag{49}$$

and is shown explicitly in Eq. (51). We evaluate the upper left-hand element of a typical component matrix

$$\begin{aligned} Q_{11}(\underline{k}_i,\underline{k}_j) &= E\left\{\left(E_R(\underline{k}_i) - \bar{E}_R(\underline{k}_i)\right)\left(E_R(\underline{k}_j) - \bar{E}_R(\underline{k}_j)\right)\right\} \\ &= E\left\{E_R(\underline{k}_i)\ E_R(\underline{k}_j)\right\} - \psi_R(\underline{k}_i)\ \psi_R(\underline{k}_j) \\ &= \iiint_{-\infty}^{\infty} \cos\ (\underline{k}_i\cdot\underline{r})\ \cos\ (\underline{k}_j\cdot\underline{r})\ dF(\underline{r}) \\ &\qquad - \psi_R(\underline{k}_i)\ \psi_R(\underline{k}_j) \qquad (50) \\ &= \iiint_{-\infty}^{\infty}\left(\frac{1}{2}\cos\left((\underline{k}_i+\underline{k}_j)\cdot\underline{r}\right) + \frac{1}{2}\cos\left((\underline{k}_i-\underline{k}_j)\cdot\underline{r}\right)\right) dF(\underline{r}) \\ &\qquad - \psi_R(\underline{k}_i)\ \psi_R(\underline{k}_j) \end{aligned}$$

$$= \frac{1}{2}\,\psi_R(\underline{k}_i+\underline{k}_j) + \frac{1}{2}\,\psi_R(\underline{k}_i-\underline{k}_j) - \psi_R(\underline{k}_i)\,\psi_R(\underline{k}_j)$$

Proceeding similarly we obtain

$$\underline{\mu}(\underline{k}_i) = \left(\psi_R(\underline{k}_i),\ \psi_I(\underline{k}_i)\right)^t \tag{51}$$

$$Q(\underline{k}_i,\underline{k}_j) = \begin{bmatrix} \frac{1}{2}\psi_R(\underline{k}_i+\underline{k}_j)+\frac{1}{2}\psi_R(\underline{k}_i-\underline{k}_j)-\psi_R(\underline{k}_i)\psi_R(\underline{k}_j), & \frac{1}{2}\psi_I(\underline{k}_i+\underline{k}_j)-\frac{1}{2}\psi_I(\underline{k}_i-\underline{k}_j)-\psi_R(\underline{k}_i)\psi_I(\underline{k}_j) \\ \\ \frac{1}{2}\psi_I(\underline{k}_i+\underline{k}_j)+\frac{1}{2}\psi_I(\underline{k}_i-\underline{k}_j)-\psi_I(\underline{k}_i)\psi_R(\underline{k}_j), & \frac{1}{2}\psi_R(\underline{k}_i-\underline{k}_j)-\frac{1}{2}\psi_R(\underline{k}_i+\underline{k}_j)-\psi_I(\underline{k}_i)\psi_I(\underline{k}_j) \end{bmatrix}$$

Thus, the general joint distribution of the $\underline{E}(\underline{k}_i)$ has been demonstrated. From Eqs. (47), (48), and (51) the conditional distributions may straightforwardly be obtained (see, for example, Ref. 6, p. 315).

## VIII. DISCUSSION

The problem we have studied in this paper is essentially the following. What is the far field from an array of N (isotropic) sources (of equal amplitude and operating in phase), the positions of which are independent, identically distributed vector-valued random variables? The phrases in parentheses are noncrucial assumptions, made primarily to simplify the exposition and are easily eliminated for greater generality. However, the underlined assumptions are essential in that they are necessary for the simplicity of the results.

Isotropy and equal amplitude of source radiation are noncrucial assumptions if the probability distributions on these parameters are independent of position. As was mentioned in Section II, distributions of the phases of the sources as a function of position may be admitted by a mere reinterpretation of the notation. However, without the far field assumption, $p(\underline{E})$ would not have the nice explicit dependence on $\psi(\underline{k})$, although it would still be gaussian.

The limiting field distribution is given in Eq. (18); the envelope distribution in Eq. (39); the envelope distribution at expected nodes of the antenna pattern in Eq. (41); the joint distribution of the field in several directions in Eqs. (47), (48), (51); the field from a gaussian array in Fig. 4; and the field from a linear array in Fig. 3.

In summary it can be said that the limiting form of the probability density of the resultant electric field vector arising from an array of N sources, the positions of which are independent, identically distributed random variables, is bivariate normal. If the Fourier transform of the distribution function $F(\underline{r})$ can be performed in closed form, then the probability density of the resultant electric field vector is immediately obvious (Eq. (18)) for <u>every</u> frequency and direction of propagation.

It is suggested that examination of antenna statistics in the near field is of interest and that the methods of Chandrasekhar (Ref. 5, Chap. IV) and Kolmogorov (Ref. 12, p. 171) might be of use.

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