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ON A NONLINEAR THEORY  
OF  
ELASTIC SHELLS

BY  
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**ON A NONLINEAR THEORY OF ELASTIC SHELLS**

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# ON A NONLINEAR THEORY OF ELASTIC SHELLS

By

W. L. Wainwright

## Summary

This paper is concerned with a nonlinear theory of elastic shells with small deformations whose material response is nonlinear. The developments are carried out under the Love-Kirchhoff hypothesis. General constitutive equations are derived in which the geometrical properties (due to deformation) and the material characteristics are separable. Through this separability, it is shown how to extract constitutive equations of predetermined types. Particular examples are followed by a discussion of the membrane theory.

## 1. INTRODUCTION

In a recent paper, Zerna [1960] dealt with a nonlinear theory of elastic shells which has as its basis the proposal of constitutive equations which are at once geometrically linear and physically nonlinear. The work of Zerna [1960], however, is somewhat limited in scope since, in particular, it contains only a third order correction to the linear theory with no indication of how a second order or higher order corrections are to be obtained. In addition, Zerna's constitutive equations upon complete linearization reduce to a version of those commonly referred to as Love's first approximation which, as brought out in more recent investigations of Naghdi [1962], do not satisfy all requisite invariance requirements.

In the present paper, through the artifice of a series expansion for the strain energy function, we derive in Section 3 general nonlinear constitutive equations for isotropic elastic shells whose mechanical behavior is (in the sense of Zerna) geometrically linear and physically nonlinear. The expansion of the strain energy function just mentioned is effected with the aid of certain results deduced in the Appendix (Section 6), where an expression for a potential is derived relating two  $3 \times 3$  symmetric matrices. The notation employed throughout the paper is essentially that of the paper by Naghdi [1962] mentioned above.

Specifically, the content of the paper is as follows: Certain preliminary results which are needed subsequently are given in Section 2. As mentioned above, Section 3 contains the derivation of the general constitutive equations. Here, in addition to invoking the Love-Kirchhoff hypothesis, it is assumed that the displacements and their gradients are small but compatible with this the constitutive equations are fully nonlinear. Also included in Section 3 is an investigation of the invariance requirements of the general constitutive equations. In Section 4, we first discuss a systematic procedure for determining approximate constitutive equations of various order in a manner which results in separation (in product form) of the purely geometrical effects (due to deformation) and the material response of the shell. Comparison is made with the results of Zerna [1960] and the special cases of the first and second order theories are discussed; the first order (or linear) theory yields the Flügge-Lur'e-Byrne equations. In Section 5, we specialize the nonlinear constitutive equations to those appropriate for the membrane theory of shells. We further show that for static problems of the membrane theory, even when the constitutive equations are (physically) nonlinear, the determination of displacements may be reduced to the solution of a system of linear differential equation.

In deriving the general constitutive equations, the use of power series form of the strain energy function offers certain advantages which should be elaborated upon. Firstly, in order to obtain the (two-dimensional) shell constitutive equations from the appropriate strain energy function in 3-space, the power series representation permits an explicit integration across the thickness of the shell. Secondly, through this procedure, there results a separation between the geometrical properties (due to deformation)

and those representing the material response so that desired geometrical and physical approximations may be introduced independently of each other. Also, it should be noted that the power series representation of the strain energy function (discussed in general in Section 6) is independent of the present application and may be used in other nonlinear problems.

2. Preliminary results. For later developments, it is convenient to recall certain results concerning the geometry of the shell which here will be referred to its deformed state. We will assume at the outset that the coordinate system is normal relative to the middle surface of the shell and hence use freely such simplifications as evolve (see, Synge and Schild [1949], Sec. 2.6 and Naghdi [1962], Sec. 3).

Let  $h$  denote the thickness of the shell and  $x^3$  be the coordinate along the normal to the middle surface. Further, with reference to Euclidean 3-space, let  $\underline{g}_i = (\underline{g}_\alpha, \underline{g}_3)$  be the base vectors of the shell space (defined by its middle surface and  $-\frac{h}{2} \leq x^3 \leq \frac{h}{2}$ ) and  $\underline{a}_\alpha$  those of its middle surface. Then

$$\underline{g}_\alpha = \mu^\beta_\alpha \underline{a}_\beta, \quad (2.1)$$

where

$$\mu^\alpha_\beta = \delta^\alpha_\beta - x^3 b^\alpha_\beta, \quad (2.2)$$

$b^\alpha_\beta$  is the second fundamental tensor of the surface and  $\delta^\alpha_\beta$  is the Kronecker delta.

In all that follows, we assume

$$|x^3| < R_{\min} \quad (2.3)^*$$

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\* For details of this restriction on the space of normal coordinates see Naghdi [1962].

with  $R_{\min}$  denoting the least radius of curvature to the middle surface. It then follows immediately that the characteristic roots of  $x^3 b_\beta^\alpha$  are less than 1 and hence (see Mirsky [1955] pg. 332)  $\mu_\beta^\alpha$  is given by the convergent series

$$\mu_\beta^\alpha = \sum_{P=0}^{\infty} (x^3)^P b_\beta^\alpha{}^P, \quad (2.4)$$

where the notation  $b_\beta^\alpha{}^P$  stands for the P factors

$$b_\beta^\alpha{}^P \equiv b_\mu^\alpha b_\nu^\mu \dots b_\lambda^\rho b_\beta^\lambda. \quad (2.5)$$

Here, we also note that the determinant  $\mu$  of  $\mu_\beta^\alpha$  is given by

$$\mu = 1 - 2Hx^3 + K(x^3)^2, \quad (2.6)$$

$$2H = b_\alpha^\alpha \quad K = |b_\beta^\alpha|,$$

and that if  $T_j^i$  is an arbitrary space tensor with subtensor  $T_\beta^\alpha$ , then the corresponding surface tensor  $\bar{T}_\beta^\alpha$  is defined through (see Naghdi [1962], Sec. 3.3),

$$\bar{T}_\beta^\alpha = \mu_\lambda^\alpha T_\rho^\lambda \mu_\beta^\rho \quad (2.7)$$

which characterizes the transformation relation between space and surface tensors.

We recall that the constitutive equations for an elastic body in the sense of Green, i.e.,

$$\sigma^{ij} = \frac{\partial \bar{\Sigma}}{\partial e_{ij}} \quad (2.8)$$

for an isotropic material may be shown to have the form (see e.g.,

Truesdell and Toupin [1960], p. 730)

$$\sigma^{ij} = \phi_0 g^{ij} + \phi_1 e^{ij} + \phi_2 e^i_k e^{kj}, \quad (2.9)$$

where  $\sigma^{ij}$  and  $e_{ij}$  are, respectively, the stress and the strain tensors,  $\Sigma$  is the strain energy and in (2.9)  $\phi_A$ , ( $A = 0,1,2$ ), are functions of three independent invariants of  $e_{ij}$ . For the purposes of this paper, the most convenient set of invariants is the set

$$K_1 = e^i_i, \quad K_2 = \frac{1}{2} e^i_j e^j_i, \quad K_3 = \frac{1}{3} e^i_j e^j_k e^k_i. \quad (2.10)$$

As stated previously, it is the purpose of the present paper to derive general constitutive equations wherein such nonlinearities as exist are purely physical and are cast into the coefficients  $\phi_A$  of (2.9). Accordingly, we confine attention to linearized strain-displacement relations

$$2 e_{ij} = u_{i|j} + u_{j|i} \quad (2.11)$$

and in keeping with this plan set  $\phi_2 = 0$  in (2.9). It then follows from (A.6) of the Appendix that  $\Sigma$  is independent of  $K_3$  so that the relevant invariants are

$$\begin{aligned} K_1 &= u^\alpha|_\alpha + u^3|_3, \\ K_2 &= \frac{1}{4} [ u^\alpha|_\beta u^\beta|_\alpha + u^\alpha|_\beta u_\alpha|^\beta \\ &\quad + 4 e^\alpha_3 e^3_\alpha + 2(u^3|_3)^2 ], \end{aligned} \quad (2.12)$$

where  $u_i$  are the components of displacements and  $( )|_i$  denotes covariant differentiation.

The analysis that follows is developed under the Love-Kirchhoff hypothesis, although it may also be carried out so as to incorporate the effect of transverse shear. It has been shown (Naghdi [1962]) that

under the Love-Kirchhoff hypothesis the usual displacement assumptions in the form

$$\begin{aligned} \bar{u}^\alpha &= v^\alpha(x^1, x^2) + x^3 \beta^\alpha(x^1, x^2), \\ u^3 &= w(x^1, x^2) \end{aligned} \quad (2.13)$$

are in fact exact. With (2.13) and the use of (2.11), we have

$$\begin{aligned} u^\alpha|_\beta &= \mu^{-1\alpha}_\rho (\gamma^\rho_\beta + x^3 \kappa^\rho_\beta), \\ 2 e_{\alpha\beta} &= \mu^\rho_\alpha (\gamma_{\rho\beta} + x^3 \kappa_{\rho\beta}) + \mu^\rho_\beta (\gamma_{\rho\alpha} + x^3 \kappa_{\rho\alpha}), \\ e_{33} &= e_{\alpha 3} = 0, \end{aligned} \quad (2.14)$$

and

$$\begin{aligned} \gamma^\alpha_\beta &= v^\alpha|_\beta - w b^\alpha_\beta, & \kappa^\alpha_\beta &= \beta^\alpha|_\beta, \\ \beta_\alpha &= - (w|_\alpha + b^\theta_\alpha v_\theta), \end{aligned} \quad (2.15)$$

where  $( )|_\alpha$  denotes covariant derivative with respect to the first fundamental tensor of the surface  $a_{\alpha\beta}$ .

In view of (2.15), the invariants  $K_1$  and  $K_2$  take the form

$$\begin{aligned} K_1 &= \mu^{-1\alpha}_\rho (\gamma^\rho_\alpha + x^3 \kappa^\rho_\alpha), \\ K_2 &= \mu^\theta_\alpha \mu^{-1\beta}_\rho [ A^{\theta\rho}_{\alpha\beta} + x^3 B^{\theta\rho}_{\alpha\beta} + (x^3)^2 E^{\theta\rho}_{\alpha\beta} ], \end{aligned} \quad (2.16)$$

where

$$\begin{aligned}
A_{\alpha\beta}^{\theta\rho} &= \frac{1}{4} [ \gamma_{\beta}^{\theta} \gamma_{\alpha}^{\rho} + \gamma_{\alpha}^{\phi} \gamma_{\phi\beta}^{\theta\rho} ] , \\
B_{\alpha\beta}^{\theta\rho} &= \frac{1}{4} [ \gamma_{\beta}^{\theta} \kappa_{\alpha}^{\rho} + \kappa_{\beta}^{\theta} \gamma_{\alpha}^{\rho} + (\gamma_{\alpha}^{\phi} \kappa_{\phi\beta} + \kappa_{\alpha}^{\phi} \gamma_{\phi\beta}) a^{\theta\rho} ] , \\
E_{\alpha\beta}^{\theta\rho} &= \frac{1}{4} [ \kappa_{\beta}^{\theta} \kappa_{\alpha}^{\rho} + \kappa_{\alpha}^{\phi} \kappa_{\phi\beta} a^{\theta\rho} ]
\end{aligned} \tag{2.17}$$

exhibit the symmetries

$$A_{\alpha\beta}^{\theta\rho} = A_{\beta\alpha}^{\rho\theta} , \quad B_{\alpha\beta}^{\theta\rho} = B_{\beta\alpha}^{\rho\theta} , \quad E_{\alpha\beta}^{\theta\rho} = E_{\beta\alpha}^{\rho\theta} . \tag{2.18}$$

3. The stress-strain relations. We have already admitted the linearized strain-displacement relation (2.11), the displacements in the form (2.13) which are exact under the Love-Kirchhoff hypothesis, and moreover have required  $\phi_2 = 0$ . On this basis, we now proceed to derive exact constitutive equations for elastic shells which are nonlinear in the sense stated in Section 1.

First, we recall the definition of the stress resultants  $N^{\alpha\beta}$  in the form

$$\begin{aligned}
N^{\alpha\beta} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \mu \sigma_{\alpha\lambda}^{\beta} \mu_{\lambda}^{\beta} dx^3 \\
&= \int_{-\frac{h}{2}}^{\frac{h}{2}} \mu \frac{\partial \Sigma}{\partial e_{\alpha\lambda}} \mu_{\lambda}^{\beta} dx^3 ,
\end{aligned} \tag{3.1}$$

where (2.8) has been used. Observing from (2.15) that

$$\begin{aligned}
\frac{\partial \Sigma}{\partial \gamma_{\alpha\beta}} &= \frac{\partial \Sigma}{\partial e_{\mu\nu}} \frac{\partial e_{\mu\nu}}{\partial \gamma_{\alpha\beta}} \\
&= \frac{\partial \Sigma}{\partial e_{\mu\nu}} \frac{1}{2} ( \mu_{\mu}^{\alpha} \delta_{\nu}^{\beta} + \mu_{\nu}^{\alpha} \delta_{\mu}^{\beta} ) \\
&= \frac{\partial \Sigma}{\partial e_{\beta\nu}} \mu_{\nu}^{\alpha} ,
\end{aligned} \tag{3.2}$$

then (3.1) may be expressed as

$$N^{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \mu \frac{\partial \Sigma}{\partial \gamma_{\beta\alpha}} dx^3 = \frac{\partial}{\partial \gamma_{\beta\alpha}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \mu \Sigma dx^3 \quad (3.3)$$

and if we define

$$\chi = \int_{-\frac{h}{2}}^{\frac{h}{2}} \mu \Sigma dx^3, \quad (3.4)$$

then

$$N^{\alpha\beta} = \frac{\partial \chi}{\partial \gamma_{\beta\alpha}}, \quad M^{\alpha\beta} = \frac{\partial \chi}{\partial \kappa_{\beta\alpha}}. \quad (3.5)$$

The expression (3.5)<sub>2</sub> involving the stress couples  $M^{\alpha\beta}$  follows from the same procedure used to obtain (3.5)<sub>1</sub>.

We now proceed to obtain a suitable representation for the scalar potential  $\chi$  in terms of the invariants  $K_1$  and  $K_2$ . Because of the restriction  $\phi_2 = 0$ , we have from (A.9) that  $C_{\Delta\Gamma\Lambda} \equiv 0$ . It then follows from (A.15) that  $\Sigma$  has the form

$$\Sigma = \frac{1}{\Gamma + 1} F_{\Delta\Gamma} K_1^\Delta K_2^{\Gamma+1} + \frac{1}{\Delta + 1} L_\Delta K_1^{\Delta+1}. \quad (3.6)$$

Consider now

$$\begin{aligned} K_1 &= \mu^{-1} \alpha_\beta (\gamma^\beta_\alpha + x^3 \kappa^\beta_\alpha) \\ &= \gamma^\beta_\alpha \sum_{P=0}^{\infty} (x^3)^P b^\beta_\alpha^P + \kappa^\beta_\alpha \sum_{P=0}^{\infty} (x^3)^{P+1} b^\beta_\alpha^P, \end{aligned} \quad (3.7)$$

where (2.4) has been used. In the last series we replace  $P+1$  by  $P$  and write

$$K_1 = \gamma^\beta_\alpha + \gamma^\beta_\alpha \sum_{P=1}^{\infty} (x^3)^P b^\beta_\alpha^P + \kappa^\beta_\alpha \sum_{P=1}^{\infty} (x^3)^P b^\beta_\alpha^{P-1} \quad (3.8)$$

so that if we define

$$M = \begin{cases} \gamma_{\beta}^{\beta}, & \text{for } P = 0, \\ \gamma_{\alpha}^{\beta} b_{\beta}^{\alpha} + \kappa_{\alpha}^{\beta} b_{\beta}^{\alpha-1}, & \text{for } P \geq 1, \end{cases} \quad (3.9)$$

then (3.8) becomes

$$K_1 = \sum_{P=0}^{\infty} (x^3)^P M^P, \quad (3.10)$$

The right hand side of (3.10) is a converging series since the series for  $\mu^{-1} \alpha_{\beta}$  converges. A parallel procedure gives for  $K_2$  the expression

$$K_2 = \sum_{P=0}^{\infty} (x^3)^P N^P, \quad (3.11)$$

where

$$N = \begin{cases} A_{\theta\rho}^{\theta\rho}, & \text{for } P = 0, \\ 2A_{\theta\beta}^{\theta\alpha} b_{\alpha}^{\beta} + B_{\theta\rho}^{\theta\rho}, & \text{for } P = 1, \\ C_{\theta\rho}^{\alpha\beta} A_{\alpha\beta}^{\theta\rho} + C_{\theta\rho}^{\beta\alpha} B_{\alpha\beta}^{\theta\rho} + C_{\theta\rho}^{\alpha\beta} E_{\alpha\beta}^{\theta\rho}, & \text{for } P \geq 2. \end{cases} \quad (3.12)$$

In (3.12),  $\underline{A}$ ,  $\underline{B}$  and  $\underline{E}$  are given by (2.17) and

$$C_{\theta\rho}^{\alpha\beta} = \sum_{Q=0}^P Q_{\theta}^{\alpha} b_{\beta}^{P-Q} = \sum_{Q=0}^P Q_{\rho}^{\beta} b_{\theta}^{P-Q}, \quad (3.13)$$

where it should be noted that C is symmetric, i.e.,

$$C_{\theta\rho}^{\alpha\beta} = C_{\rho\theta}^{\beta\alpha}. \quad (3.14)$$

By a procedure similar to that used in the foregoing, it can be shown that the powers of the invariants  $K_1$  and  $K_2$  may be expressed as

$$(K_1)^\Delta = \sum_{P=0}^{\infty} (x^3)^P \begin{matrix} P \\ C \\ \Delta \end{matrix}, \quad (K_2)^\Delta = \sum_{P=0}^{\infty} (x^3)^P \begin{matrix} P \\ D \\ \Delta \end{matrix}, \quad (3.15)$$

where

$$\begin{matrix} P \\ C \\ \Delta \end{matrix} = \sum_{Q=0}^P \sum_{R=0}^Q \dots \sum_{S=0}^T \begin{matrix} S & T-S \\ M & M \end{matrix} \dots \begin{matrix} Q-R & P-Q \\ M & M \end{matrix}, \quad (3.16)$$

$$\begin{matrix} P \\ D \\ \Delta \end{matrix} = \sum_{Q=0}^P \sum_{R=0}^Q \dots \sum_{S=0}^T \begin{matrix} S & T-S \\ N & N \end{matrix} \dots \begin{matrix} Q-R & P-Q \\ N & N \end{matrix}.$$

The formalism of (3.16) is intended to indicate that there are  $\Delta$  factors and  $\Delta - 1$  summations. It also follows from (3.16) that

$$\begin{matrix} P \\ C \\ \Delta \end{matrix} = \sum_{Q=0}^P \begin{matrix} Q \\ C \\ \Delta-1 \end{matrix} \begin{matrix} P-Q \\ M \end{matrix} \quad (3.17)$$

For the product  $(K_1)^\Delta (K_2)^\Gamma$ , we write

$$(K_1)^\Delta (K_2)^\Gamma = \sum_{P=0}^{\infty} (x^3)^P \begin{matrix} P \\ \Lambda \\ \Delta \quad \Gamma \end{matrix}, \quad (3.18)$$

$$\begin{matrix} P \\ \Lambda \\ \Delta \quad \Gamma \end{matrix} = \sum_{Q=0}^P \begin{matrix} Q \\ C \\ \Delta \end{matrix} \begin{matrix} P-Q \\ D \\ \Gamma \end{matrix} = \sum_{Q=0}^P \begin{matrix} Q \\ D \\ \Gamma \end{matrix} \begin{matrix} P-Q \\ C \\ \Delta \end{matrix},$$

and in order that the representation (3.18) be applicable to the special cases  $\Delta = 0$  and  $\Gamma = 0$ , we append to (3.16) the definitions

$$\begin{aligned}
\begin{matrix} 0 \\ C \\ 0 \end{matrix} &= \begin{matrix} 0 \\ D \\ 0 \end{matrix} = 1, & \begin{matrix} P \\ C \\ 0 \end{matrix} &= \begin{matrix} P \\ D \\ 0 \end{matrix} = 0, & \text{for } P \geq 1, \\
\begin{matrix} P \\ C \\ \Delta \end{matrix} &= \begin{matrix} P \\ D \\ \Delta \end{matrix} = 0, & \text{for } P < 0
\end{aligned} \tag{3.19}$$

and note in passing that  $\Delta \Lambda_{\Gamma}^P$  is independent of  $x^3$ .

With the aid of (3.18) and (2.6), (3.4) may now be written as

$$\begin{aligned}
x &= \int_{-\frac{h}{2}}^{\frac{h}{2}} [1 - 2Hx^3 + K(x^3)^2] \left[ \frac{1}{\Gamma+1} F_{\Delta\Gamma} \sum_{P=0}^{\infty} (x^3)^P \Delta \Lambda_{\Gamma+1}^P \right. \\
&\quad \left. + \frac{1}{\Delta+1} L_{\Delta} \sum_{P=0}^{\infty} (x^3)^P \Delta+1 \Lambda_0^P \right] dx^3
\end{aligned} \tag{3.20}$$

Introducing the notation

$$\eta(Q) = \frac{1}{Q} \left(\frac{h}{2}\right)^Q (1 - (-1)^Q) \tag{3.21}$$

and observing that

$$\begin{aligned}
\eta(Q) &= 0, & \text{for } Q \text{ even,} \\
\eta(Q) &= \frac{h^Q}{Q} \left(\frac{1}{2}\right)^{Q-1}, & \text{for } Q \text{ odd,}
\end{aligned} \tag{3.22}$$

then the integration of (3.20) yields

$$\begin{aligned}
x &= \sum_{P=0}^{\infty} \left\{ [ \eta(P+1) - 2H\eta(P+2) + K\eta(P+3) ] \right. \\
&\quad \left. \left( \frac{F_{\Delta\Gamma}}{\Gamma+1} \Delta \Lambda_{\Gamma+1}^P - \frac{L_{\Delta}}{\Delta+1} \Delta+1 \Lambda_0^P \right) \right\}.
\end{aligned} \tag{3.23}$$

In order to express the stress and couple resultants  $N^{\alpha\beta}$  and  $M^{\alpha\beta}$  in terms of the invariants  $\overset{0}{C}$  and  $\overset{0}{D}$ , we need to obtain certain additional results. To this end, from the definitions (3.9) we obtain

$$\begin{aligned} \frac{\partial \overset{0}{M}}{\partial \gamma_{\mu\nu}} &= a^{\mu\nu} \quad , \quad \frac{\partial \overset{0}{M}}{\partial \kappa_{\mu\nu}} = 0 \quad , \\ \frac{\partial \overset{P}{M}}{\partial \gamma_{\mu\nu}} &= b^{\mu\nu} \quad , \quad \frac{\partial \overset{P}{M}}{\partial \kappa_{\mu\nu}} = b^{\mu\nu} \quad , \quad (P \geq 1). \end{aligned} \quad (3.24)$$

It follows from (3.12) and (2.17) that

$$\overset{0}{N} = \frac{1}{4} [ \gamma_{\beta}^{\theta} (\gamma^{\beta}_{\theta} + \gamma_{\theta}^{\beta}) ] \quad (3.25)$$

and hence,

$$\frac{\partial \overset{0}{N}}{\partial \gamma_{\mu\nu}} = \frac{1}{2} ( \gamma^{\mu\nu} + \gamma^{\nu\mu} ) , \quad \frac{\partial \overset{0}{N}}{\partial \kappa_{\mu\nu}} = 0 \quad . \quad (3.26)$$

It is convenient to introduce the quantity

$$\lambda^{\alpha}_{\beta} = \gamma^{\alpha}_{\nu} b^{\nu}_{\beta} + \kappa^{\alpha}_{\beta} \quad (3.27)$$

in terms of which

$$\overset{1}{N} = \frac{1}{2} ( \gamma^{\rho}_{\phi} + \gamma_{\phi}^{\rho} ) \lambda^{\phi}_{\rho} \quad (3.28)$$

so that

$$\begin{aligned} \frac{\partial \overset{1}{N}}{\partial \gamma_{\mu\nu}} &= \frac{1}{2} [ b^{\nu}_{\rho} ( \gamma^{\rho\mu} + \gamma^{\mu\rho} ) + ( \lambda^{\nu\mu} + \lambda^{\mu\nu} ) ] , \\ \frac{\partial \overset{1}{N}}{\partial \kappa_{\mu\nu}} &= \frac{1}{2} ( \gamma^{\mu\nu} + \gamma^{\nu\mu} ) \quad . \end{aligned} \quad (3.29)$$

Similarly, for  $P \geq 2$ , we have

$$\begin{aligned}
 \frac{P}{N} &= \frac{1}{4} \left[ 2 \lambda_{\nu}^{\phi} b_{\alpha}^{P-1} \nu_{\alpha} (\gamma_{\phi}^{\alpha} + \gamma_{\phi}^{\alpha}) + \right. \\
 &+ (P-1) \lambda_{\alpha}^{\phi} \lambda_{\phi}^{\nu} b_{\nu}^{P-2} \alpha_{\nu} + \\
 &\left. + \sum_{Q=0}^{P-2} \lambda_{\nu}^{\theta} b_{\rho}^{Q\nu} \lambda_{\lambda}^{\rho} b_{\theta}^{P-Q-2} \lambda_{\theta} \right]
 \end{aligned} \tag{3.30}$$

so that for  $P \geq 2$ ,

$$\begin{aligned}
 \frac{\partial N}{\partial \gamma_{\mu\nu}} &= \frac{1}{2} \left[ \frac{P}{b_{\alpha}^{\nu}} (\gamma^{\alpha\mu} + \gamma^{\mu\alpha}) + P \lambda^{\mu\lambda} b_{\nu}^{P-1} \lambda_{\lambda} \right. \\
 &+ a^{\mu\theta} \sum_{Q=0}^{P-1} b_{\rho}^{Q\nu} \lambda_{\lambda}^{\rho} b_{\theta}^{P-Q-1} \lambda_{\theta} \left. \right],
 \end{aligned} \tag{3.31}$$

$$\begin{aligned}
 \frac{\partial N}{\partial \kappa_{\mu\nu}} &= \frac{1}{2} \left[ \frac{P-1}{b_{\alpha}^{\nu}} (\gamma^{\alpha\mu} + \gamma^{\mu\alpha}) + (P-1) \lambda^{\mu\lambda} b_{\nu}^{P-2} \lambda_{\lambda} \right. \\
 &+ a^{\mu\theta} \sum_{Q=0}^{P-2} b_{\rho}^{Q\nu} \lambda_{\lambda}^{\rho} b_{\theta}^{P-Q-2} \lambda_{\theta} \left. \right].
 \end{aligned}$$

We now turn to  $\frac{P}{\Delta \Lambda \Gamma}$  in (3.18) and proceed to establish the following identities:

$$\begin{aligned}
 \frac{\partial}{\partial \Delta} \frac{P}{\Lambda \Gamma} / \frac{\partial}{\partial M} &= \Delta \frac{P-Z}{\Delta-1 \Lambda \Gamma}, \\
 \frac{\partial}{\partial \Delta} \frac{P}{\Lambda \Gamma} / \frac{\partial}{\partial N} &= \Gamma \frac{P-Z}{\Delta \Lambda \Gamma-1}.
 \end{aligned} \tag{3.32}$$

To prove the first of (3.32), we first note that

$$\frac{P}{C_1} = M. \tag{3.33}$$

Hence,

$$\partial \underset{1}{C} / \partial \underset{M}{Z} = \delta \underset{Z}{P} = \underset{0}{C}^{\underset{P}{-Z}} \quad (3.34)$$

Next, from (3.17)

$$\underset{2}{C}^{\underset{P}{P}} = \sum_{Q=0}^{\underset{P}{P}} \underset{1}{C}^{\underset{Q}{Q}} \underset{M}^{\underset{P-Q}{P-Q}} \quad (3.35)$$

so that

$$\begin{aligned} \partial \underset{2}{C} / \partial \underset{M}{Z} &= \sum_{Q=0}^{\underset{P}{P}} (\delta \underset{Z}{Q}^{\underset{P-Q}{P-Q}} + \underset{1}{C}^{\underset{Q}{Q}} \delta \underset{Z}^{\underset{P-Q}{P-Q}}) \\ &= \underset{2}{M}^{\underset{P-Z}{P-Z}} = \underset{2}{C}^{\underset{1}{P-Z}} \end{aligned} \quad (3.36)$$

and by induction we arrive at

$$\partial \underset{\Delta}{C} / \partial \underset{M}{Z} = \Delta \underset{\Delta-1}{C}^{\underset{P-Z}{P-Z}} \quad (3.37)$$

Now, since by (3.18)

$$\underset{\Delta}{\Lambda}^{\underset{P}{P}} \underset{\Gamma}{\Gamma} = \sum_{Q=0}^{\underset{P}{P}} \underset{\Delta}{C}^{\underset{Q}{Q}} \underset{\Gamma}{D}^{\underset{P-Q}{P-Q}} \quad (3.38)$$

and since  $\underset{\Gamma}{D}$  as given by (3.16) is dependent only on the  $\underset{N}{N}$ 's, we have in view of (3.37)

$$\partial \underset{\Delta}{\Lambda}^{\underset{P}{P}} / \partial \underset{M}{Z} = \sum_{Q=0}^{\underset{P}{P}} \Delta \underset{\Delta-1}{C}^{\underset{Q-Z}{Q-Z}} \underset{\Gamma}{D}^{\underset{P-Q}{P-Q}} \quad (3.39)$$

If we replace  $Q$  by  $Q + Z$ , (3.39) becomes

$$\partial_{\Delta}^{\Lambda} \Gamma / \partial M^Z = \Delta \sum_{Q=-Z}^{P-Z} \begin{matrix} Q & P-Z-Q \\ C & D \\ \Delta-1 & \Gamma \end{matrix} \quad (3.40)$$

and then, through the definitions (3.19), the first of (3.32) follows. The second of (3.32) may be established in a similar manner where in particular it is convenient to use (3.18)<sub>3</sub>.

Explicit forms for the constitutive equations can now be obtained from (3.5) and (3.23) and read

$$N^{\alpha\beta} = \sum_{P=0}^{\infty} \sum_{Z=0}^P \left\{ [\eta(P+1) - 2H\eta(P+2) + K\eta(P+3)] [L_{\Delta}^{\Lambda} \Delta^{\Lambda} \partial M^Z / \partial \gamma_{\beta\alpha} \right. \\ \left. + \frac{\Delta}{\Gamma+1} F_{\Delta\Gamma} \Delta^{-1} \Lambda^{\Gamma+1} \frac{\partial M^Z}{\partial \gamma_{\beta\alpha}} + F_{\Delta\Gamma} \Delta^{\Lambda} \Gamma \frac{\partial N^Z}{\partial \gamma_{\beta\alpha}} \right] \} , \quad (3.41)$$

$$M^{\alpha\beta} = \sum_{P=0}^{\infty} \sum_{Z=0}^P \left\{ [\eta(P+1) - 2H\eta(P+2) + K\eta(P+3)] [L_{\Delta}^{\Lambda} \Delta^{\Lambda} \frac{\partial M^Z}{\partial \kappa_{\beta\alpha}} \right. \\ \left. + \frac{\Delta}{\Gamma+1} F_{\Delta\Gamma} \Delta^{-1} \Lambda^{\Gamma+1} \frac{\partial M^Z}{\partial \kappa_{\beta\alpha}} + F_{\Delta\Gamma} \Delta^{\Lambda} \Gamma \frac{\partial N^Z}{\partial \kappa_{\beta\alpha}} \right] \} . \quad (3.42)$$

The constitutive relations (3.41) and (3.42) may be put in more explicit forms by further substitution from results of the type (3.24), (3.26), etc., but we delay this until the next section, where special cases are considered.

Before closing this section, we examine the invariance properties of the above constitutive equations.<sup>†</sup> It is evident that the equations (3.41) and (3.42) are tensorially invariant. Furthermore, it is easily verified that they are also consistent with the virtual work theorem for shells. Next, it is necessary to show that the sixth equation of equilibrium,

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<sup>†</sup> For an account of invariance requirements in shell theory, see Naghdi [1962].

namely

$$\epsilon_{\alpha\beta} ( N^{\alpha\beta} - b^{\alpha}_{\nu} M^{\nu\beta} ) = 0 , \quad (3.43)$$

or equivalently

$$\epsilon_{\alpha\beta} \left( \frac{\partial X}{\partial \gamma_{\beta\alpha}} - b^{\alpha}_{\nu} \frac{\partial X}{\partial \kappa_{\beta\nu}} \right) = 0 , \quad (3.44)$$

is identically satisfied. To this end, observing that  $X$  may be regarded as a function of  $\overset{Z}{M}$  and  $\overset{Z}{N}$ , by (3.44) we have

$$\begin{aligned} & \sum_{Z=0}^{\infty} \epsilon_{\alpha\beta} \left\{ \left( \frac{\partial X}{\partial \overset{Z}{M}} \right) \frac{\partial \overset{Z}{M}}{\partial \gamma_{\beta\alpha}} + \left( \frac{\partial X}{\partial \overset{Z}{N}} \right) \frac{\partial \overset{Z}{N}}{\partial \gamma_{\beta\alpha}} \right. \\ & \quad \left. - b^{\alpha}_{\nu} \left[ \left( \frac{\partial X}{\partial \overset{Z}{M}} \right) \frac{\partial \overset{Z}{M}}{\partial \kappa_{\beta\nu}} + \left( \frac{\partial X}{\partial \overset{Z}{N}} \right) \frac{\partial \overset{Z}{N}}{\partial \kappa_{\beta\nu}} \right] \right\} \\ & = \sum_{Z=0}^{\infty} \epsilon_{\alpha\beta} \left\{ \frac{\partial X}{\partial \overset{Z}{M}} \left( \frac{\partial \overset{Z}{M}}{\partial \gamma_{\beta\alpha}} - b^{\alpha}_{\nu} \frac{\partial \overset{Z}{M}}{\partial \kappa_{\beta\nu}} \right) \right. \\ & \quad \left. + \frac{\partial X}{\partial \overset{Z}{N}} \left( \frac{\partial \overset{Z}{N}}{\partial \gamma_{\beta\alpha}} - b^{\alpha}_{\nu} \frac{\partial \overset{Z}{N}}{\partial \kappa_{\beta\nu}} \right) \right\} . \end{aligned} \quad (3.45)$$

Consider now the coefficient of  $\frac{\partial X}{\partial \overset{Z}{M}}$  in (3.45) in conjunction with equations (3.24). For  $Z \geq 1$ , it is identically zero while for  $Z = 0$ , it is symmetric in  $\alpha$  and  $\beta$  and hence vanishes. Next, consider the coefficient of  $\frac{\partial X}{\partial \overset{Z}{N}}$ . A little manipulation of (3.31) will show that

$$\frac{\partial \overset{Z}{N}}{\partial \gamma_{\beta\alpha}} - b^{\alpha}_{\nu} \frac{\partial \overset{Z}{N}}{\partial \kappa_{\beta\nu}} = \frac{1}{2} \left( \overset{Z-1}{b}_{\theta}^{\alpha} \lambda^{\beta\theta} + \overset{Z-1}{b}_{\theta}^{\beta} \lambda^{\alpha\theta} \right) , \quad (3.46)$$

an expression symmetric in  $\alpha$  and  $\beta$ . Thus, for  $Z \geq 2$ , in view of the presence of  $\epsilon_{\alpha\beta}$  in (3.45), this coefficient also vanishes. The vanishing of this coefficient for  $Z = 0, 1$ , follows with the aid of equations (3.26)

and (3.29). The final invariance requirement which should be examined is invariance under rigid rotation. It may be shown that the value of each of the invariants  $M$  and  $N$  is unchanged by infinitesimal rigid rotations and thus  $\chi$  retains its value under such a transformation. The calculations are somewhat lengthy and hence are not included here.

4. Approximate constitutive equations. From a practical point of view, it is of interest to consider constitutive equations which are approximations to those given by (3.41) and (3.42). The procedure to be used in obtaining the approximate equations will be one of simply terminating the series after a finite number of terms and we proceed to do this in a systematic manner.

First, we recall that there are two types of summations involved in (3.41) and (3.42), summation over the letter  $P$  and summation (implied) over the indices  $\Delta$  and  $\Gamma$ . These two summations are effected independently and it is this fact which leads to the independent consideration of geometric properties and material response. In order to systematize our discussion of this matter, we will use the notation  $\frac{1}{R}$  to represent terms of the type  $b_{\beta}^{\alpha}$  and  $\gamma$  to represent terms of the type  $\gamma_{\beta}^{\alpha}$ . Terms of the type  $\kappa_{\beta}^{\alpha}$  and  $\lambda_{\beta}^{\alpha}$  will be regarded as  $\frac{\gamma}{R}$ . If we return to the definitions of the various elements which are the constituents of (3.41) and (3.42), we can easily see to what power  $\gamma$  and  $\frac{1}{R}$  occur in them. These results are summarized in Table I below:

TABLE I

Element	Power to which $\gamma$ occurs	Power to which $\frac{1}{R}$ occurs
$P$		
$M$	1	$P$
$P$		
$N$	2	$P$
$P$		
$C$	$\Delta$	$P$
$\Delta$		
$P$		
$D$	$2\Gamma$	$P$
$\Gamma$		
$P$		
$\Delta \wedge \Gamma$	$\Delta + 2\Gamma$	$P$

We observe from this table that the powers to which  $\gamma$  and  $\frac{1}{R}$  occur in the various elements are independent in the sense that the former is determined by  $\Delta$  and  $\Gamma$  while the latter is determined by  $P$ .

The quantities  $L_{\Delta}$  and  $F_{\Delta\Gamma}$  in (3.41) and (3.42) are material coefficients, and the power to which  $\gamma$  occurs in terms involving these coefficients is determined solely by the value of the index of those coefficients. This fact is seen by applying the information of Table I to (3.41) and (3.42) and the result is summarized in Table II which, together with Table I, reveals the manner in which the  $\gamma$  and  $\frac{1}{R}$  terms occur in the constitutive equations.

TABLE II

Coefficient of	Power to which $\gamma$ occurs
$L_{\Delta}$	$\Delta$
$F_{\Delta\Gamma}$	$\Delta + 2\Gamma + 1$

With the information gleaned from the above two tables, the procedure for determining various approximate constitutive equations is now clear. First, we select the highest power to which we wish  $\frac{h}{R}$  to occur in the expansion and we then sum over  $P$  in such a manner as to obtain all terms up to and including that power. We emphasize, however, that because of the presence of  $\eta (P + 3)$  in (3.41) and (3.42), we cannot choose the highest power of  $\frac{h}{R}$  to be less than 2. After the choice on  $\frac{h}{R}$  has been made, we are free to select the power to which  $\gamma$  shall appear in the constitutive equations.\* The term "order" will be used in this connection, i.e., equations linear in  $\gamma$  will be called first order, those containing both linear terms and terms of second degree in  $\gamma$  will be called second order, etc. It is clear that once the choices on  $\frac{h}{R}$  and  $\gamma$  have been made, the determination of the specific equations is a routine matter.

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\* The sequence of choices used here in selecting the powers of  $\frac{h}{R}$  and  $\gamma$  may be interchanged. The one adopted here appears to be convenient.

In the examples which follow, we shall always choose the highest power to which  $\frac{h}{R}$  occurs to be 2. Consequently, we will first expand the relations (3.41) and (3.42) imposing this restriction and then, by making use of relations of the type (3.46), obtain the desired expressions. Thus expansion of (3.41) in the manner just described gives

$$\begin{aligned}
N^{\alpha\beta} = & \eta(1) \left\{ L_{\Delta}^0 \frac{\partial M^0}{\partial \gamma_{\beta\alpha}} + F_{\Delta\Gamma} \left[ \frac{\Delta}{\Gamma+1} \frac{\partial M^0}{\partial \gamma_{\beta\alpha}} + \frac{\partial N^0}{\partial \gamma_{\beta\alpha}} \right] \right\} \\
& + \eta(3) \left\{ L_{\Delta}^2 \frac{\partial M^0}{\partial \gamma_{\beta\alpha}} + \Delta^1 \frac{\partial M^1}{\partial \gamma_{\beta\alpha}} + \Delta^0 \frac{\partial M^2}{\partial \gamma_{\beta\alpha}} \right. \\
& - 2H \left( \Delta^1 \frac{\partial M^0}{\partial \gamma_{\beta\alpha}} + \Delta^0 \frac{\partial M^1}{\partial \gamma_{\beta\alpha}} \right) + K \Delta^0 \frac{\partial M^0}{\partial \gamma_{\beta\alpha}} \left. \right] \\
& + \frac{\Delta}{\Gamma+1} F_{\Delta\Gamma} \left[ \Delta^2 \frac{\partial M^0}{\partial \gamma_{\beta\alpha}} + \Delta^1 \frac{\partial M^1}{\partial \gamma_{\beta\alpha}} + \Delta^0 \frac{\partial M^2}{\partial \gamma_{\beta\alpha}} \right. \\
& - 2H \left( \Delta^1 \frac{\partial M^0}{\partial \gamma_{\beta\alpha}} + \Delta^0 \frac{\partial M^1}{\partial \gamma_{\beta\alpha}} \right) + K \Delta^0 \frac{\partial M^0}{\partial \gamma_{\beta\alpha}} \left. \right] \\
& + F_{\Delta\Gamma} \left[ \Delta^2 \frac{\partial N^0}{\partial \gamma_{\beta\alpha}} + \Delta^1 \frac{\partial N^1}{\partial \gamma_{\beta\alpha}} + \Delta^0 \frac{\partial N^2}{\partial \gamma_{\beta\alpha}} \right. \\
& - 2H \left( \Delta^1 \frac{\partial N^0}{\partial \gamma_{\beta\alpha}} + \Delta^0 \frac{\partial N^1}{\partial \gamma_{\beta\alpha}} \right) + K \Delta^0 \frac{\partial N^0}{\partial \gamma_{\beta\alpha}} \left. \right] \left. \right\}. \tag{4.1}
\end{aligned}$$

The Cayley-Hamilton theorem when applied to  $b^{\alpha}_{\beta}$  with the use of (3.24) may be expressed as

$$\frac{\partial M^2}{\partial \gamma_{\beta\alpha}} - 2H \frac{\partial M^1}{\partial \gamma_{\beta\alpha}} + K \frac{\partial M^0}{\partial \gamma_{\beta\alpha}} = 0 \tag{4.2}$$

and if we write  $b^{\alpha}_{\beta}$  for the adjoint matrix of  $b^{\alpha}_{\beta}$ , we have

$$\begin{aligned}
{}^*b^{\alpha\beta} &= 2H a^{\alpha\beta} - b^{\alpha\beta} \\
&= 2H \frac{\partial M}{\partial \gamma_{\beta\alpha}} - \frac{\partial M}{\partial \gamma_{\beta\alpha}} .
\end{aligned} \tag{4.3}$$

Further, combination of (4.3), (3.26), (3.29) and (3.31) yields

$$\begin{aligned}
\frac{\partial N}{\partial \gamma_{\beta\alpha}} &= 2H \frac{\partial N}{\partial \gamma_{\beta\alpha}} + K \frac{\partial N}{\partial \gamma_{\beta\alpha}} \\
&= \frac{1}{2} [ b^{\alpha}_{\nu} \lambda^{\beta\nu} + b^{\beta}_{\nu} \lambda^{\alpha\nu} - {}^*b^{\alpha}_{\nu} (\lambda^{\nu\beta} + \lambda^{\beta\nu}) ]
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
\frac{\partial N}{\partial \gamma_{\beta\alpha}} &= 2H \frac{\partial N}{\partial \gamma_{\beta\alpha}} \\
&= \frac{1}{2} [ \lambda^{\alpha\beta} + \lambda^{\beta\alpha} - {}^*b^{\alpha}_{\nu} (\gamma^{\beta\nu} + \gamma^{\nu\beta}) ] \\
&= \frac{1}{2} [ (b^{\alpha}_{\nu} \gamma^{\beta\nu} + b^{\beta}_{\nu} \gamma^{\alpha\nu}) - {}^*b^{\alpha}_{\nu} (\gamma^{\beta\nu} + \gamma^{\nu\beta}) + (\kappa^{\alpha\beta} + \kappa^{\beta\alpha}) ] .
\end{aligned} \tag{4.5}$$

Now, with the use of (4.2) to (4.5) and with some simplification and combination, (4.1) reduces to

$$\begin{aligned}
N^{\alpha\beta} &= h \left\{ L_{\Delta} \Delta^{\Lambda_0} a^{\alpha\beta} + F_{\Delta\Gamma} \left[ \frac{\Delta}{\Gamma+1} \Delta^{-\Gamma} \Gamma+1 a^{\alpha\beta} + \Delta^{\Lambda_{\Gamma}} \frac{1}{2} (\gamma^{\alpha\beta} + \gamma^{\beta\alpha}) \right] \right\} \\
&+ \frac{h^3}{12} \left\{ L_{\Delta} \left[ \Delta^{\Lambda_0} a^{\alpha\beta} - \Delta^{\Lambda_0} b^{\alpha\beta} \right] + \frac{\Delta}{\Gamma+1} F_{\Delta\Gamma} \left[ \Delta^{-\Gamma} \Gamma+1 a^{\alpha\beta} - \Delta^{-\Gamma} \Gamma+1 b^{\alpha\beta} \right] \right\} \\
&+ \frac{1}{2} F_{\Delta\Gamma} \left[ \Delta^{\Lambda_{\Gamma}} (\gamma^{\alpha\beta} + \gamma^{\beta\alpha}) + \Delta^{\Lambda_{\Gamma}} \left( (\lambda^{\alpha\beta} + \lambda^{\beta\alpha}) - {}^*b^{\alpha}_{\nu} (\gamma^{\nu\beta} + \gamma^{\beta\nu}) \right) \right. \\
&\left. + \Delta^{\Lambda_{\Gamma}} \left( (b^{\alpha}_{\nu} \lambda^{\beta\nu} + b^{\beta}_{\nu} \lambda^{\alpha\nu}) - {}^*b^{\alpha}_{\nu} (\lambda^{\nu\beta} + \lambda^{\beta\nu}) \right) \right] \left. \right\} ,
\end{aligned} \tag{4.6}$$

and, similarly, the corresponding expression for  $M^{\alpha\beta}$  is

$$\begin{aligned}
M^{\alpha\beta} = & \frac{h^3}{12} \left\{ L_{\Delta} \left[ \Delta^{\Lambda_0} a^{\alpha\beta} - \Delta^{\Lambda_0} b^{\alpha\beta} \right] \right. \\
& + \frac{\Delta}{\Gamma+1} F_{\Delta\Gamma} \left[ \Delta^{-1\Lambda_{\Gamma+1}} a^{\alpha\beta} - \Delta^{-1\Lambda_{\Gamma+1}} b^{\alpha\beta} \right] \\
& \left. + \frac{1}{2} F_{\Delta\Gamma} \left[ \Delta^{\Lambda_{\Gamma}} (\gamma^{\alpha\beta} + \gamma^{\beta\alpha}) + \Delta^{\Lambda_{\Gamma}} \left( (\lambda^{\alpha\beta} + \lambda^{\beta\alpha}) - b^{\alpha}_{\nu} (\gamma^{\beta\nu} + \gamma^{\nu\beta}) \right) \right] \right\}. \tag{4.7}
\end{aligned}$$

Equations (4.6) and (4.7) are of second degree in  $\frac{h}{R}$  but of arbitrary order in  $\gamma$ . Such subsequent examples as are to be given will follow directly from these equations after a choice has been made on the order of  $\gamma$ . As a first example, let us obtain the first order or linear constitutive equations. For this case, Table II instructs us to retain  $L_1$  and  $F_{00}$  and to set all other  $L_{\Delta}$ ,  $F_{\Delta\Gamma}$  equal to zero. (In order that the state of zero stress and zero strain coincide, we also set  $L_0 = 0$ .) In fact, if we set

$$L_1 = \frac{\nu E}{1 - \nu^2}, \quad F_{00} = \frac{E}{1 + \nu}, \tag{4.8}$$

we then obtain the Flügge-Lur'e-Byrne constitutive equations. For future reference, we record representative examples of these equations, written in lines of curvature coordinates and in physical components. Thus,

$$\begin{aligned}
N_{(11)} &= h \left\{ (L_1 + F_{00}) \gamma_{(11)} + L_1 \gamma_{(22)} + \frac{h^2}{12} (L_1 + F_{00}) \left( \frac{1}{R_2} - \frac{1}{R_1} \right) (\kappa_{(11)} - \frac{1}{R_1} \gamma_{(11)}) \right\}, \\
N_{(12)} &= \frac{h}{2} F_{00} \left\{ (\gamma_{(12)} + \gamma_{(21)}) + \frac{h^2}{12} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) (\kappa_{(21)} - \frac{1}{R_1} \gamma_{(21)}) \right\} \\
M_{(11)} &= \frac{h^3}{12} \left\{ (L_1 + F_{00}) \kappa_{(11)} + L_1 \kappa_{(22)} + (L_1 + F_{00}) \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \gamma_{(11)} \right\} \\
M_{(12)} &= \frac{h^3}{24} F_{00} \left\{ (\kappa_{(12)} + \kappa_{(21)}) + \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \gamma_{(21)} \right\}
\end{aligned} \tag{4.9}$$

where the constants  $L_1$  and  $F_{00}$  are given by (4.8).

We now return to equations (4.6) and (4.7) and proceed to obtain the

constitutive equations for the second order theory. For this purpose, it is convenient to introduce the notation

$$k^N{}^{\alpha\beta} , \quad k^M{}^{\alpha\beta} , \quad (4.10)$$

where the subscript  $k$  in each case designates that resultant which arises from (4.6) and (4.7) from exactly those terms which are of degree  $k$  in  $\gamma$ . Thus,  ${}_1N^{\alpha\beta}$  and  ${}_1M^{\alpha\beta}$  would refer to the right hand sides of the Flügge-Lur'e-Byrne equations (4.9). If the theory sought is  $Q^{\text{th}}$  order, then we have

$$N^{\alpha\beta} = \sum_{k=1}^Q k^N{}^{\alpha\beta} , \quad M^{\alpha\beta} = \sum_{k=1}^Q k^M{}^{\alpha\beta} , \quad (4.11)$$

while for a completely general theory,

$$N^{\alpha\beta} = \sum_{k=1}^{\infty} k^N{}^{\alpha\beta} , \quad M^{\alpha\beta} = \sum_{k=1}^{\infty} k^M{}^{\alpha\beta} . \quad (4.12)$$

With this understanding, we need only determine  ${}_2N^{\alpha\beta}$  and  ${}_2M^{\alpha\beta}$  to complete the description of the second order theory since  ${}_1N^{\alpha\beta}$  and  ${}_1M^{\alpha\beta}$  are given by (4.9). The representative results are as follows:

$$\begin{aligned} {}_2N_{(11)} = & \ln \left\{ L_2(\gamma_{(11)} + \gamma_{(22)})^2 + F_{10} \left[ \frac{1}{2}(\gamma_{(11)}^2 + \gamma_{(22)}^2) + \frac{1}{4}(\gamma_{(12)} + \gamma_{(21)})^2 + \gamma_{(11)}(\gamma_{(11)} + \gamma_{(22)}) \right] \right\} \\ & + \frac{h^3}{12} \left\{ L_2 [2(\gamma_{(11)} + \gamma_{(22)}) \lambda_{(11)} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) + (\lambda_{(11)} + \lambda_{(22)})^2] \right. \\ & + F_{10} \left[ \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \left( 4 \lambda_{(11)} \gamma_{(11)} + 2 \lambda_{(21)} (\gamma_{(12)} + \gamma_{(21)}) \right) \right. \\ & \quad \left. \left. + 2(\lambda_{(11)}^2 + \lambda_{(22)}^2) + (\lambda_{(12)} + \lambda_{(21)})^2 \right. \right. \\ & \left. \left. + \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \lambda_{(11)} (2\gamma_{(11)} + \gamma_{(22)}) + (\lambda_{(11)} + \lambda_{(22)}) \lambda_{(11)} \right] \right\} , \quad (4.13) \end{aligned}$$

$$\begin{aligned}
2N_{(12)} &= \frac{h}{2} F_{10} \left\{ (\gamma_{(11)} + \gamma_{(22)}) (\gamma_{(12)} + \gamma_{(21)}) \right. \\
&\quad \left. + \frac{h^2}{12} \left[ (\lambda_{(11)} + \lambda_{(22)}) (\lambda_{(12)} + \lambda_{(21)}) + (\gamma_{(11)} + \gamma_{(22)}) \lambda_{(21)} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \right] \right\}, \tag{4.14}
\end{aligned}$$

and

$$\begin{aligned}
2M_{(11)} &= \frac{h^3}{12} \left\{ L_2 (\gamma_{(11)} + \gamma_{(22)}) \left[ 2(\lambda_{(11)} + \lambda_{(22)}) + \frac{1}{R_2} (\gamma_{(11)} + \gamma_{(22)}) \right] \right. \\
&\quad + F_{10} \left[ (\lambda_{(11)} \gamma_{(11)} + \lambda_{(22)} \gamma_{(22)}) + \frac{1}{2} (\lambda_{(12)} + \lambda_{(21)}) (\gamma_{(12)} + \gamma_{(21)}) \right. \\
&\quad \left. \left. + \frac{1}{R_2} \left( \frac{1}{2} (\gamma_{(11)}^2 + \gamma_{(22)}^2) + \frac{1}{4} (\gamma_{(12)} + \gamma_{(21)})^2 \right) \right. \right. \\
&\quad \left. \left. + (\lambda_{(11)} + \lambda_{(22)}) \gamma_{(11)} + (\gamma_{(11)} + \gamma_{(22)}) \left( \kappa_{(11)} + \gamma_{(11)} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \right) \right] \right\}, \tag{4.15}
\end{aligned}$$

$$\begin{aligned}
2M_{(12)} &= \frac{h^3}{24} F_{10} \left\{ (\lambda_{(11)} + \lambda_{(22)}) (\gamma_{(12)} + \gamma_{(21)}) \right. \\
&\quad \left. + (\gamma_{(11)} + \gamma_{(22)}) \left[ \kappa_{(12)} + \kappa_{(21)} + \gamma_{(21)} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \right] \right\}. \tag{4.16}
\end{aligned}$$

The foregoing expressions simplify somewhat for special configurations. For example, in the case of a flat plate, the representative equations reduce to

$$\begin{aligned}
2N_{(11)} &= h \left\{ L_2 (\gamma_{(11)} + \gamma_{(22)})^2 + F_{10} \left[ \frac{1}{2} (\gamma_{(11)}^2 + \gamma_{(22)}^2) \right. \right. \\
&\quad \left. \left. + \frac{1}{4} (\gamma_{(12)} + \gamma_{(21)})^2 + \gamma_{(11)} (\gamma_{(11)} + \gamma_{(22)}) \right] \right\} \\
&\quad + \frac{h^3}{12} \left\{ L_2 (\kappa_{(11)} + \kappa_{(22)})^2 + F_{10} \left[ 2(\kappa_{(11)}^2 + \kappa_{(22)}^2) \right. \right. \\
&\quad \left. \left. + (\kappa_{(12)} + \kappa_{(21)})^2 + \kappa_{(11)} (\kappa_{(11)} + \kappa_{(22)}) \right] \right\}, \tag{4.17}
\end{aligned}$$

$$2M_{(12)} = \frac{h}{2} F_{10} \left\{ (\gamma_{(11)} + \gamma_{(22)}) (\gamma_{(12)} + \gamma_{(21)}) + \frac{h^2}{12} (\kappa_{(11)} + \kappa_{(22)}) (\kappa_{(12)} + \kappa_{(21)}) \right\}, \quad (4.18)$$

$$2M_{(11)} = \frac{h^3}{12} \left\{ 2L_2(\gamma_{(11)} + \gamma_{(22)})(\kappa_{(11)} + \kappa_{(22)}) + F_{10} \left[ 3 \kappa_{(11)} \gamma_{(11)} + \kappa_{(22)} \gamma_{(22)} + (\kappa_{(12)} + \kappa_{(21)}) (\gamma_{(12)} + \gamma_{(21)}) + \gamma_{(11)} \kappa_{(22)} + \gamma_{(22)} \kappa_{(11)} \right] \right\}, \quad (4.19)$$

$$2M_{(12)} = \frac{h^3}{24} F_{10} \left\{ (\kappa_{(11)} + \kappa_{(22)}) (\gamma_{(12)} + \gamma_{(21)}) + (\gamma_{(11)} + \gamma_{(22)}) (\kappa_{(12)} + \kappa_{(21)}) \right\}, \quad (4.20)$$

Our purpose in recording (4.17) to (4.20) is to observe the coupling, i.e., the dependence of the stress resultants on the strains  $\kappa_{\alpha\beta}$  as well as the dependence of the stress couples on the strains  $\gamma_{\alpha\beta}$ . In contrast, it may be recalled that when (4.9) is specialized to the case of a flat plate, no such coupling is present.

It is of interest to compare the constitutive equations (3.41) and (3.42) upon specialization with those obtained by Zerna [1960] whose constitutive equations contain only first and third degree terms.\*\* If in the expressions for  $N^{\alpha\beta}$  and  $M^{\alpha\beta}$ , we neglect, as Zerna does, terms involving  $(\frac{h}{R})^2$  with respect to the remaining ones as well as terms of the type  $\frac{\gamma}{R}$ , then (4.6) and (4.7) become

$$N^{\alpha\beta} = h \left\{ \left[ L_{\Delta} \Delta^{\Lambda 0} + \frac{\Delta}{\Gamma + 1} F_{\Delta\Gamma} \Delta^{-\Lambda \Gamma + 1} \right] a^{\alpha\beta} + F_{\Delta\Gamma} \Delta^{\Lambda \Gamma} \gamma^{\alpha\beta} \right\}, \quad (4.21)$$

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\*\* Although Zerna [1960] does not explicitly invoke the Love-Kirchhoff hypothesis, he neglects the effect of  $\gamma_{\alpha 3}$  and sets  $\sigma^{33} = 0$  but retains the effect of  $\gamma_{33}$ .

and

$$M^{\alpha\beta} = \frac{h^3}{12} \left\{ \left[ L_{\Delta}^1 \Delta^{\Lambda_0} + \frac{\Delta}{\Gamma+1} F_{\Delta\Gamma} \Delta^{-1\Lambda_{\Gamma}+1} \right] a^{\alpha\beta} + F_{\Delta\Gamma}^1 \Delta^{\Lambda_{\Gamma}} \tilde{\gamma}^{\alpha\beta} + F_{\Delta\Gamma}^0 \Delta^{\Lambda_{\Gamma}} \tilde{\kappa}^{\alpha\beta} \right\}, \quad (4.22)$$

where we have written

$$\tilde{\gamma}^{\alpha\beta} = \frac{1}{2} (\gamma^{\alpha\beta} + \gamma^{\beta\alpha}), \quad \tilde{\kappa}^{\alpha\beta} = \frac{1}{2} (\kappa^{\alpha\beta} + \kappa^{\beta\alpha}). \quad (4.23)$$

Now if, as in Zerna's case, we wish to retain only the first and third degree terms, it follows from Table II that the only nonvanishing coefficients are  $L_1$ ,  $L_3$ ,  $F_{00}$ ,  $F_{20}$ , and  $F_{01}$ . Here expansion of (4.21) gives

$$N^{\alpha\beta} = h \left\{ \left[ L_1 (\tilde{\gamma}_{\theta}^{\theta}) a^{\alpha\beta} + F_{00} \tilde{\gamma}^{\alpha\beta} \right] + \left[ L_3 (\tilde{\gamma}_{\theta}^{\theta})^3 + F_{20} (\tilde{\gamma}_{\theta}^{\theta}) (\tilde{\gamma}_{\rho}^{\lambda} \tilde{\gamma}_{\lambda}^{\rho}) \right] a^{\alpha\beta} + \left[ F_{20} (\tilde{\gamma}_{\theta}^{\theta})^2 + \frac{1}{2} F_{01} (\tilde{\gamma}_{\rho}^{\lambda} \tilde{\gamma}_{\lambda}^{\rho}) \right] \tilde{\gamma}^{\alpha\beta} \right\} \quad (4.24)$$

which is of the same form as the corresponding expression given by Zerna. It should be noted, however, that while the coefficients of  $(\tilde{\gamma}_{\theta}^{\theta})(\tilde{\gamma}_{\lambda}^{\rho} \tilde{\gamma}_{\rho}^{\lambda}) a^{\alpha\beta}$  and  $(\tilde{\gamma}_{\theta}^{\theta})^2 \tilde{\gamma}^{\alpha\beta}$  are identical in (4.24), they are not the same in the corresponding expression given by Zerna [1960]. A similar conclusion can be arrived at upon the comparison of (4.22), when only first and third degree terms are retained, with Zerna's expression for the stress couples after a misprint in Zerna's equation (45) has been corrected.

5. Membrane theory. In this section, we specialize the previous results to those appropriate for the membrane theory of shells. We recall that for equilibrium problems of the membrane theory, where the shear stress resultants and the stress couples are absent, the state of stress is statically determinate and the stress resultants  $N^{\alpha\beta}$  are obtained from the equations

$$N^{\alpha\beta} \parallel_{\alpha} + p^{\beta} = 0 ,$$

$$b_{\alpha\beta} N^{\alpha\beta} + p = 0 , \quad (5.1)$$

$$\epsilon_{\alpha\beta} N^{\alpha\beta} = 0 .$$

For the subsequent determination of the displacements, we turn to the constitutive equations (4.6) and, consistent with the membrane theory, neglect the terms with the factor  $(\frac{h}{R})^2$ . Thus, for the membrane theory, instead of (4.6), we have

$$N^{\alpha\beta} = h \left\{ \left( L_{\Delta}^{\Delta} \Delta^{\Delta} \Lambda_0^{\Delta} + \frac{\Delta}{\Gamma+1} F_{\Delta\Gamma} \Delta^{-\Gamma} \Lambda_{\Gamma+1}^{\Delta} \right) a^{\alpha\beta} + F_{\Delta\Gamma} \Delta^{\Delta} \Lambda_{\Gamma}^{\Delta} \frac{1}{2} (\gamma^{\alpha\beta} + \gamma^{\beta\alpha}) \right\} . \quad (5.2)$$

Because of the symmetry of the right-hand side of (5.2), it is seen that (5.1)<sub>3</sub> is identically satisfied. Also, in view of (3.18)<sub>2</sub>, all  $\Lambda$ 's appearing in (5.2) may be written in terms of  $M$  and  $N$  alone so that

$$N^{\alpha\beta} = h \left\{ \left[ L_{\Delta}^{\Delta} (M)^{\Delta} + \frac{\Delta}{\Gamma+1} F_{\Delta\Gamma} (M)^{\Delta-1} (N)^{\Gamma+1} \right] a^{\alpha\beta} + F_{\Delta\Gamma} (M)^{\Delta} (N)^{\Gamma} \frac{1}{2} (\gamma^{\alpha\beta} + \gamma^{\beta\alpha}) \right\} . \quad (5.3)$$

Since  $M$  and  $N$  are functions of  $\gamma^{\alpha\beta}$ , it is clear that the combination of (5.3) and (2.11) constitutes a system of nonlinear differential equations in the displacements. Since  $N^{\alpha\beta}$  as solutions of (5.1) can be regarded as known, it is convenient to express  $M$  and  $N$  in terms of  $N^{\alpha\beta}$ , or more specifically, in terms of the invariants of  $N^{\alpha\beta}$ . Let

$$P_1 = N^{\alpha}_{\alpha} , \quad P_2 = \frac{1}{2} N^{\alpha}_{\beta} N^{\beta}_{\alpha} , \quad (5.4)$$

and introduce further the notation

$$\psi = F_{\Delta\Gamma} (M)^{\Delta} (N)^{\Gamma} . \quad (5.5)$$

Contraction on (5.3) gives

$$P_1 = h \left\{ \left[ L_{\Delta} (M)^{\Delta} + \frac{\Delta}{\Gamma+1} F_{\Delta\Gamma} (M)^{\Delta-1} (N)^{\Gamma+1} \right]^2 + \psi M \right\} \quad (5.6)$$

or

$$\begin{aligned} & \left[ L_{\Delta} (M)^{\Delta} + \frac{\Delta}{\Gamma+1} F_{\Delta\Gamma} (M)^{\Delta-1} (N)^{\Gamma+1} \right] \\ & = \frac{1}{2} \left( \frac{P_1}{h} - \psi M \right) \end{aligned} \quad (5.7)$$

in terms of which  $N^{\alpha\beta}$  becomes

$$N^{\alpha\beta} = h \left\{ \frac{1}{2} \left( \frac{P_1}{h} - \psi M \right) a^{\alpha\beta} + \psi \frac{1}{2} (\gamma^{\alpha\beta} + \gamma^{\beta\alpha}) \right\}. \quad (5.8)$$

By (5.4) and (5.8), we arrive at the additional relation

$$4 P_2 - P_1^2 = h^2 \psi^2 \left[ 4 N^{\alpha\beta} - (M)^2 \right]. \quad (5.9)$$

Equations (5.6) and (5.9) give  $M$  and  $N$  in terms of  $P_1$  and  $P_2$ . In the linear case, the solution for  $M$  is straightforward, and since in this case  $\psi = F_{00}$ ,  $M$  is determined by

$$h (2L_1 + F_{00}) M = P_1 = {}_1 N^{\alpha}_{\alpha}. \quad (5.10)$$

It then follows from (5.3), after the values of  $F_{00}$  and  $L_1$  are supplied from (4.8), that

$$\frac{1}{2} (\gamma^{\alpha\beta} + \gamma^{\beta\alpha}) = \frac{1}{2h} \left[ (1 + \nu) {}_1 N^{\alpha\beta} - \nu {}_1 N^{\gamma}_{\gamma} a^{\alpha\beta} \right] \quad (5.11)$$

which is the usual result of the linear theory.

In the second order theory,

$$\psi = F_{00} + F_{10} M \quad (5.12)$$

so that by (5.6) and (5.9),  $P_1$  and  $P_2$  reduce to

$$P_1 = h \left\{ 2 [L_1 M + L_2 (M)^2 + F_{10} N] + (F_{00} + F_{10} M) M \right\} \quad (5.13)$$

$$P_2 = h^2 (F_{00} + F_{10} M)^2 [4N - (M)^2] . \quad (5.14)$$

Although  $N$  may be easily eliminated between these equations, giving  $M$  in terms of  $P_1$  and  $P_2$ , the resulting equation is of the 4<sup>th</sup> degree in  $M$ . However, once  $M$  is known in terms of  $P_1$  and  $P_2$ , (5.3) yields a system of linear differential equations for the displacements. In fact, even in the general case, so long as (5.6) and (5.9) can be solved for  $M$  and  $N$  in terms of  $P_1$  and  $P_2$ , the differential equations for the displacements are linear.

## 6. Appendix: Power series forms for the elastic potential.

Equations (2.9) include the most general isotropic stress-strain relations of elasticity, but since the  $\phi_A$  being arbitrary functions of the invariants of  $e_{ij}$  depend also on  $x^3$ , it is desirable to express  $\Sigma$  in (2.8) in a form which will readily lend itself to integration across the thickness of the shell. The developments here, however, are entirely general in that they are applicable whenever two symmetric second order tensors are related through a potential.

Let the symmetric tensors under discussion be  $Q_j^i$  and  $A_j^i$ , where the first is an analytic isotropic function of the second. We further assume that a potential  $\Gamma(A_j^i)$  exists\* such that

$$Q_j^i = \frac{\partial \Gamma}{\partial A_j^i} . \quad (A.1)$$

Let

$$\begin{aligned} K_1 &= A_j^i , & K_2 &= \frac{1}{2} A_j^i A_i^j , \\ K_3 &= \frac{1}{3} A_k^i A_j^k A_i^j , \end{aligned} \quad (A.2)$$

---

\* The necessary and sufficient conditions for the existence of a potential  $\Gamma$  may be found in Truesdell's [1952] paper. Truesdell's conditions, however, are given in terms of a set of invariants which are different from those utilized in this paper.

be selected as the three independent invariants, then

$$\Gamma = \Gamma (K_1, K_2, K_3). \quad (\text{A.3})$$

Our choice of the invariants (A.2) is motivated by the fact that their derivatives have the form

$$\frac{\partial K_1}{\partial A_j^i} = \delta_j^i, \quad \frac{\partial K_2}{\partial A_j^i} = A_j^i, \quad \frac{\partial K_3}{\partial A_j^i} = A_k^i A_j^k, \quad (\text{A.4})$$

and hence (A.1) may be written as

$$Q_j^i = C_1 \delta_j^i + C_2 A_j^i + C_3 A_k^i A_j^k, \quad (\text{A.5})$$

where the coefficients

$$C_A = \frac{\partial \Gamma}{\partial K_A}, \quad (A = 1, 2, 3) \quad (\text{A.6})$$

are power series in the invariants  $K_A$ . Since the existence of a potential  $\Gamma$  has been assumed, it follows that

$$\frac{\partial^2 \Gamma}{\partial K_B \partial K_A} = \frac{\partial^2 \Gamma}{\partial K_A \partial K_B} \quad (\text{A.7})$$

or equivalently

$$\frac{\partial C_A}{\partial K_B} = \frac{\partial C_B}{\partial K_A} \quad (\text{A.8})$$

which are in fact the necessary and sufficient conditions for (A.1) in terms of  $K_A$ .

Since the  $C_A$  may be written as power series in terms of  $K_B$ , we write  $C_3$  in the form

$$C_3 = C_{\Delta \Gamma \Lambda} K_1^\Delta K_2^\Gamma K_3^\Lambda, \quad (\text{A.9})$$

where  $\Delta, \Gamma, \Lambda = 1, 2, 3, \dots$  and summation is intended even in those terms (which will arise later) in which an index appears more than twice. With  $C_3$  given by (A.9), we proceed to determine  $C_1$  and  $C_2$  with the use of (A.8). Thus

$$\frac{\partial C_2}{\partial K_3} = \frac{\partial C_3}{\partial K_2} = \Gamma C_{\Delta\Gamma\Lambda} K_1^\Delta K_2^{\Gamma-1} K_3^\Lambda \quad (\text{A.10})$$

so that

$$C_2 = \frac{\Gamma}{\Lambda+1} C_{\Delta\Gamma\Lambda} K_1^\Delta K_2^{\Gamma-1} K_3^{\Lambda+1} + F_{\Delta\Gamma} K_1^\Delta K_2^\Gamma, \quad (\text{A.11})$$

where the second term on the right-hand side represents the arbitrary function of integration. Similarly, with the two remaining conditions of (A.8) and with (A.9) and (A.11), we obtain

$$\begin{aligned} C_1 = & \frac{\Delta}{\Lambda+1} C_{\Delta\Gamma\Lambda} K_1^{\Delta-1} K_2^\Gamma K_3^{\Lambda+1} \\ & + \frac{\Delta}{\Gamma+1} F_{\Delta\Gamma} K_1^{\Delta-1} K_2^{\Gamma+1} + L_\Delta K_1^\Delta. \end{aligned} \quad (\text{A.12})$$

Having determined  $C_\Lambda$ , we compute  $\Gamma$  from (A.6) and obtain

$$\begin{aligned} \Gamma = & \frac{1}{\Lambda+1} C_{\Delta\Gamma\Lambda} K_1^\Delta K_2^\Gamma K_3^{\Lambda+1} \\ & + \frac{1}{\Gamma+1} F_{\Delta\Gamma} K_1^\Delta K_2^{\Gamma+1} + \frac{1}{\Delta+1} L_\Delta K_1^{\Delta+1}, \end{aligned} \quad (\text{A.13})$$

where the arbitrary constant of integration has been set equal to zero without loss in generality. The choice of the set of invariants (A.2) and the starting point (A.9) was motivated by the desire to depress the coefficient  $C_3$ . However, as should be apparent, the procedure is the same for any choice of independent invariants such as

$$\begin{aligned} \text{I} &= A^i_1, & \text{II} &= \frac{1}{2} \delta^{im}_{jn} A^j_1 A^n_m, \\ \text{III} &= \frac{1}{3!} \delta^{imp}_{jnk} A^j_1 A^n_m A^k_p = |A^i_j|. \end{aligned} \quad (\text{A.14})$$

If, instead of (A.2), (A.14) is taken as independent, then different (although equivalent) integrability conditions will result (Truesdell [1952, p. 133]). In this case, the potential has the same form as (A.13) except with different coefficients  $\bar{C}_{\Delta\Gamma\Lambda}$ ,  $\bar{F}_{\Delta\Gamma}$ , and  $\bar{L}_{\Delta}$  while the coefficients  $C_{\Lambda}$  of (A.5) are [compare with (A.11) and (A.12)]

$$\begin{aligned}
 c_1 &= \bar{C}_{\Delta\Gamma\Lambda} I^{\Delta} II^{\Gamma+1} III^{\Lambda} + \frac{\Delta}{\Lambda+1} \bar{C}_{\Delta\Gamma\Lambda} I^{\Delta-1} II^{\Gamma} III^{\Lambda+1} \\
 &+ \frac{\Gamma}{\Lambda+1} \bar{C}_{\Delta\Gamma\Lambda} I^{\Delta+1} II^{\Gamma-1} III^{\Lambda+1} + \bar{F}_{\Delta\Gamma} I^{\Delta+1} II^{\Gamma} \\
 &\quad + \frac{\Delta}{\Gamma+1} \bar{F}_{\Delta\Gamma} I^{\Delta-1} II^{\Gamma+1} + \bar{L}_{\Delta} I^{\Delta}, \\
 c_2 &= -\bar{C}_{\Delta\Gamma\Lambda} I^{\Delta+1} II^{\Gamma} III^{\Lambda} - \frac{\Gamma}{\Lambda+1} \bar{C}_{\Delta\Gamma\Lambda} I^{\Delta} II^{\Gamma-1} III^{\Lambda+1} \\
 &\quad - \bar{F}_{\Delta\Gamma} I^{\Delta} II^{\Gamma}, \\
 c_3 &= \bar{C}_{\Delta\Gamma\Lambda} I^{\Delta} II^{\Gamma} III^{\Lambda}.
 \end{aligned} \tag{A.15}$$

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