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THE BOUNDARY LAYER FLOW OF A NON-NEWTONIAN FLUID NEAR A SPINNING CONE

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ABSTRACT

The steady motion of an incompressible inelastic non-Newtonian Reiner-Rivlin fluid near a spinning cone has been studied and a similarity solution has been presented. It has been shown that the flow patterns can be obtained from Srivastava-Jain's work. But the pressure distribution is not the same and we have given the numerical values of the pressure in two tables and also shown their variation in two figures.

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Introduction. The boundary layer flows of non-Newtonian fluids have drawn increasing attention in the past few years due to their importance in many technological fields. Several authors (Srivastava (1958), Jain (1961), Jones (1961) and others) have solved the steady boundary layer flows near a stagnation point, near a rotating disc or over a plane. Some unsteady boundary layer flows have also been solved (Srivastava (1960), author (1961)). But to my knowledge the three dimensional boundary layer over a spinning cone of a non-Newtonian fluid has not been studied. The problem of three dimensional boundary layer near a spinning cone for an ordinary viscous liquid has been studied by Wu(1959). He showed that with a suitable choice of coordinate axes and independent variable the flow functions can be made to satisfy the same set of equations as were obtained previously by Cochran-Karman, but the pressure does not satisfy the same equations. In considering the same problem in this paper for a non-Newtonian fluid (incompressible, inelastic and of the Reiner-Rivlin type) we have come to the same conclusion that the flow functions satisfy the same set of equations which were obtained by Srivastava and Jain but the pressure does not, when we make a suitable choice of axes.

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We have computed the pressure within the boundary layer and shown the variation in the figures I and II.

Equations. The constitutive equations for an inelastic non-Newtonian Reiner-Rivlin fluid are

$$\tau_{ij} = -p\delta_{ij} + \tau'_{ij} \,, \qquad i,j = 1,\ 2,\ 3 \qquad (1)$$
$$\tau'_{ij} = 2\mu\, e_{ij} + 4\mu_c\, e_{ik}\, e_{kj} \,,$$

where τ_{ij} and e_{ij} are the components of stress and rate of strain respectively, p is the pressure, μ and μ_c are the coefficients of viscosity and cross viscosity respectively. We know that

$$e_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \,, \qquad (2)$$

where v_i's are the components of velocity.

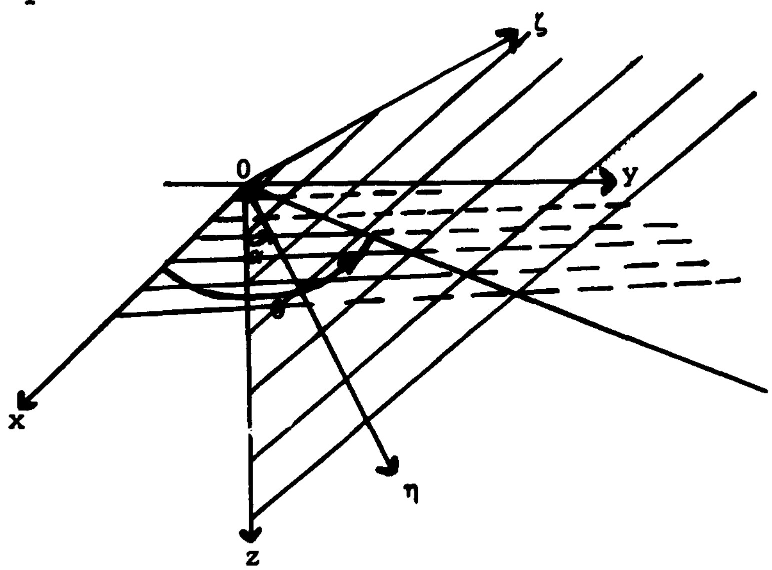


Figure 1

We choose cartesian coordinates (x, y, z) with origin 0 at the vertex of the cone and z-axis along the axis of the cone. Let α be the semi-opening angle of the cone. Then if we take coordinates (η, ζ, θ) such that

$$x = (\eta \sin\alpha + \zeta \cos\alpha) \cos\theta, \quad y = (\eta \sin\alpha + \zeta \cos\alpha) \sin\theta, \quad z = \eta \cos\alpha - \zeta \sin\alpha, \quad (3)$$

then the physical components of rate of strain are

$$e_{\eta\eta} = \frac{\partial v_\eta}{\partial \eta}, \quad e_{\zeta\zeta} = \frac{\partial v_\zeta}{\partial \zeta}, \quad e_{\theta\theta} = \frac{v_\eta \sin\alpha + v_\zeta \cos\alpha}{\eta \sin\alpha + \zeta \cos\alpha},$$

$$e_{\eta\zeta} = \frac{1}{2}\left(\frac{\partial v_\eta}{\partial \zeta} + \frac{\partial v_\zeta}{\partial \eta}\right), \quad e_{\eta\theta} = \frac{1}{2}\left(\frac{\partial v_\theta}{\partial \eta} - \frac{v_\theta \sin\alpha}{\eta \sin\alpha + \zeta \cos\alpha}\right), \quad (4)$$

$$e_{\zeta\theta} = \frac{1}{2}\left(\frac{\partial v_\theta}{\partial \zeta} - \frac{v_\theta \cos\alpha}{\eta \sin\alpha + \zeta \cos\alpha}\right),$$

where we have taken the symmetry into consideration.

Thus the equation of continuity is

$$\frac{\partial v_\eta}{\partial \eta} + \frac{\partial v_\zeta}{\partial \zeta} + \frac{v_\eta \sin\alpha + v_\zeta \cos\alpha}{\eta \sin\alpha + \zeta \cos\alpha} = 0 . \quad (5)$$

The equation of momentum will give after the transformation (3) ,

$$\rho\left(v_\eta \frac{\partial v_\eta}{\partial \eta} + v_\zeta \frac{\partial v_\eta}{\partial \zeta} - \frac{v_\theta^2 \sin\alpha}{\eta \sin\alpha + \zeta \cos\alpha}\right) = -\frac{\partial p}{\partial \eta} + \frac{\partial \tau'_{\eta\eta}}{\partial \eta} + \frac{\partial \tau'_{\eta\zeta}}{\partial \zeta} + $$

$$+ \frac{\sin\alpha(\tau'_{\eta\eta} - \tau'_{\theta\theta}) + \cos\alpha\, \tau'_{\eta\zeta}}{\eta \sin\alpha + \zeta \cos\alpha}, \quad (6)$$

$$\rho\left(v_\eta \frac{\partial v_\zeta}{\partial \eta} + v_\zeta \frac{\partial v_\zeta}{\partial \zeta} - \frac{v_\theta^2 \cos\alpha}{\eta\sin\alpha + \zeta\cos\alpha}\right) = -\frac{\partial p}{\partial \zeta} + \frac{\partial \tau'_{\zeta\zeta}}{\partial \zeta} + \frac{\partial \tau'_{\eta\zeta}}{\partial \eta} + $$

$$+ \frac{\cos\alpha(\tau'_{\zeta\zeta} - \tau'_{\theta\theta}) + \sin\alpha\,\tau'_{\eta\zeta}}{\eta\sin\alpha + \zeta\cos\alpha}, \quad (7)$$

$$\rho\left(v_\eta \frac{\partial v_\theta}{\partial \eta} + v_\zeta \frac{\partial v_\theta}{\partial \zeta} + \frac{v_\eta \sin\alpha + v_\zeta \cos\alpha}{\eta\sin\alpha + \zeta\cos\alpha} v_\theta\right) = \frac{\partial \tau'_{\eta\theta}}{\partial \eta} + \frac{\partial \tau'_{\zeta\theta}}{\partial \zeta} + \frac{2(\sin\alpha\,\tau'_{\eta\theta} + \cos\alpha\,\tau'_{\zeta\theta})}{\eta\sin\alpha + \zeta\cos\alpha}, \quad (8)$$

taking the symmetry into consideration. τ'_{ij}'s are given by the second equation of (1) and equations (4) .

2. Boundary Layer Approximation. In order to solve equations (5) - (8) we assume that

$$v_\eta \sim 0(1), \qquad v_\zeta \sim 0(\delta/\eta_0), \qquad v_\theta \sim 0(1), \quad (9)$$

which are the ordinary boundary layer assumptions (of interest) compatible with the equation of continuity (5) . These assumptions are generally valid at a large distance from the vertex of the cone. Assuming

$$\frac{\delta}{\eta_0} \approx 0\left(\frac{1}{\eta_0}\sqrt{\frac{\nu}{\omega\sin\alpha}}\right), \quad \nu_c \approx 0(\nu), \quad (10)$$

$$\nu = \mu/\rho, \qquad \nu_c = \mu_c/\rho,$$

where ω is the angular velocity of rotation, we have from equations (5) - (8) , neglecting all terms smaller than $0(1)$,

$$\frac{\partial v_\eta}{\partial \eta} + \frac{\partial v_\zeta}{\partial \zeta} + \frac{v_\eta}{\eta} = 0 \ , \tag{11}$$

$$v_\eta \frac{\partial v_\eta}{\partial \eta} + v_\zeta \frac{\partial v_\eta}{\partial \zeta} - \frac{v_\theta^2}{\eta} = -\frac{1}{\rho}\frac{\partial p}{\partial \eta} + \nu \frac{\partial^2 v_\eta}{\partial \zeta^2} + \nu_c\Big[\frac{\partial}{\partial \eta}\Big(\frac{\partial v_\eta}{\partial \zeta}\Big)^2 +$$

$$+ \frac{\partial}{\partial \zeta}\Big\{\frac{\partial v_\theta}{\partial \eta}\cdot\frac{\partial v_\theta}{\partial \zeta} - \frac{v_\theta}{\eta}\frac{\partial v_\theta}{\partial \zeta} - 2\frac{v_\eta}{\eta}\frac{\partial v_\eta}{\partial \zeta}\Big\} + \frac{1}{\eta}\Big\{\Big(\frac{\partial v_\eta}{\partial \zeta}\Big)^2 - \Big(\frac{\partial v_\theta}{\partial \zeta}\Big)^2\Big\}\Big] \ , \tag{12}$$

$$v_\eta \frac{\partial v_\theta}{\partial \eta} + v_\zeta \frac{\partial v_\theta}{\partial \zeta} + \frac{v_\eta v_\theta}{\eta} = \nu \frac{\partial^2 v_\theta}{\partial \zeta^2} + \nu_c\Big[\frac{\partial}{\partial \eta}\Big(\frac{\partial v_\theta}{\partial \zeta}\cdot\frac{\partial v_\eta}{\partial \zeta}\Big) +$$

$$+ \frac{\partial}{\partial \zeta}\Big\{\frac{\partial v_\theta}{\partial \eta}\cdot\frac{\partial v_\eta}{\partial \zeta} - \frac{v_\theta}{\eta}\cdot\frac{\partial v_\eta}{\partial \zeta} - 2\frac{\partial v_\theta}{\partial \zeta}\cdot\frac{\partial v_\eta}{\partial \eta}\Big\} + \frac{2}{\eta}\frac{\partial v_\theta}{\partial \zeta}\cdot\frac{\partial v_\eta}{\partial \zeta}\Big] \ , \tag{13}$$

$$-\frac{v_\theta^2}{\eta}\cot\alpha = -\frac{1}{\rho}\frac{\partial p}{\partial \zeta} + \nu_c\Big[\frac{\partial}{\partial \zeta}\Big\{\Big(\frac{\partial v_\eta}{\partial \zeta}\Big)^2 + \Big(\frac{\partial v_\theta}{\partial \zeta}\Big)^2\Big\} + \cot\alpha\frac{1}{\eta}\Big(\frac{\partial v_\eta}{\partial \zeta}\Big)^2\Big] \ . \tag{14}$$

In deducing (12) – (14) we have used the reduced equation of continuity (11) .

So, now we are to solve the system of equations (11) – (14) subject to the boundary conditions

$$\begin{gathered} v_\eta(\eta,0) = 0, \quad v_\zeta(\eta,0) = 0, \quad v_\theta(\eta,0) = \eta\omega\sin\alpha \ , \\ \lim_{\zeta\to\infty} v_\eta(\eta,\zeta) = 0, \quad \lim_{\zeta\to\infty} v_\theta(\eta,\zeta) = 0 \ . \end{gathered} \tag{15}$$

In the next section we present a similar solution for the system of equations (11) – (14) subject to the conditions (15) .

3. <u>Similarity Solution.</u> We assume as solution of equations (11) – (14)

$$v_\eta = f(\eta)\,F(\zeta), \quad v_\zeta = g(\eta)\,G(\zeta), \quad v_\theta = \phi(\eta)\,\Phi(\zeta) \ ,$$

substitute these in equation (11) and get as a result

$$\frac{F(\zeta)}{G'(\zeta)} = -\frac{\eta\, g(\eta)}{f(\eta) + \eta f'(\eta)} = -\frac{\lambda}{2} \ , \tag{16}$$

where λ is a constant.

To satisfy the third condition (15) we set $\phi(\eta) = \eta$. Thus we have, from equations (12) – (14) ,

$$\begin{aligned} f(\eta)\,f'(\eta)\,F^2(\zeta) + g(\eta)\,f(\eta)\,F'(\zeta)\,G(\zeta) - \eta\Phi^2 &= -\frac{1}{\rho}\frac{\partial p}{\partial \eta} + \nu f(\eta)\,F''(\zeta) + \\ &+ \nu_c\Big[2f(\eta)\,f'(\eta)\,(F'(\zeta))^2 - 2\frac{\partial}{\partial\zeta}\Big\{\frac{(f(\eta))^2}{\eta}F(\zeta)\,F'(\zeta)\Big\} + \\ &+ \frac{1}{\eta}\{(f(\eta))^2\,(F'(\zeta))^2 - \eta^2(\Phi'(\zeta))^2\}\Big] \ , \end{aligned} \tag{17}$$

$$-\eta(\Phi(\zeta))^2 \cot\alpha = -\frac{1}{\rho}\frac{\partial p}{\partial\zeta} + \nu_c\Big[\frac{\partial}{\partial\zeta}\{(f(\eta))^2\,(F'(\zeta))^2 + \eta^2(\Phi'(\zeta))^2\} + \frac{\cot\alpha}{\eta}(f(\eta))^2(F'(\zeta))^2\Big], \tag{18}$$

$$\begin{aligned} f(\eta)\,F(\zeta)\,\Phi(\zeta) + g(\eta)\,G(\zeta)\,\Phi'(\zeta) + f(\eta)\,F(\zeta)\,\Phi(\zeta) &= \nu\eta\Phi''(\zeta) + \\ + \nu_c\Big[\frac{\partial}{\partial\eta}(\eta f(\eta)\,F'(\zeta)\,\Phi'(\zeta)) + \frac{\partial}{\partial\zeta}\{-2\eta f'(\eta)\,\Phi'(\zeta)\,F(\zeta)\} &+ 2\Phi'(\zeta)\,f(\eta)\,F'(\zeta)\Big] \ . \end{aligned} \tag{19}$$

It appears from the set of equations (17) – (19) that the similarity solution is possible if

$$f(\eta) = \eta, \quad p(\eta,\zeta) = \eta^2 p_1(\zeta) + \eta p_2(\zeta)\cot\alpha, \quad p_1 \sim 0(1), \quad p_2 \sim 0(\frac{\delta}{\eta_0}) \quad . \tag{20}$$

Then from (16) we obtain $g(\eta) = \lambda$. Substituting these in (17) – (19) we get, equating coefficients of η and η^2 ,

$$F^2(\zeta) + \lambda G(\zeta)\, F'(\zeta) - \Phi^2(\zeta) = \nu F''(\zeta) - \nu_c[(F'(\zeta))^2 + 2F(\zeta)\, F''(\zeta) + 3(\Phi'(\zeta))^2] \quad , \tag{21}$$

$$\frac{1}{\rho}\, p_1(\zeta) = \nu_c[(F'(\zeta))^2 + (\Phi'(\zeta))^2] \tag{22}$$

$$\frac{1}{\rho}\, p_2'(\zeta) = \nu_c(F'(\zeta))^2 + \Phi^2(\zeta) \quad , \tag{23}$$

$$2F(\zeta)\, \Phi(\zeta) + \lambda G(\zeta)\, \Phi'(\zeta) = \nu\Phi''(\zeta) + 2\nu_c[F'(\zeta)\, \Phi'(\zeta) - F(\zeta)\, \Phi''(\zeta)] \quad . \tag{24}$$

Equations (21), (24) and (16) give us the functions $F(\zeta)$, $G(\zeta)$ and $\Phi(\zeta)$, while equations (22) and (23) give us the pressure distribution within the boundary layer. Changing to dimensionless variables,

$$\begin{aligned}
\lambda G(\zeta) &= H(\zeta) \quad , \\
H(\zeta) &= \sqrt{\nu\omega \sin\alpha}\; H_1(\zeta_1) \quad , \\
F(\zeta) &= \omega \sin\alpha\, F_1(\zeta_1) \quad , \\
\Phi(\zeta) &= \omega \sin\alpha\; \Phi_1(\zeta_1) \quad , \\
p_2(\zeta) &= \rho\omega\sqrt{\nu\omega\sin\alpha}\; \sin\alpha\, P_2(\zeta_1) \quad . \\
p_1(\zeta) &= \rho\omega^2 \sin^2\alpha\, P_1(\zeta_1), \quad \zeta_1 = \zeta\sqrt{\frac{\omega\sin\alpha}{\nu}}
\end{aligned} \tag{25}$$

we obtain,

$$F_1^2(\zeta_1) + H_1(\zeta_1)\, F_1'(\zeta_1) - \Phi_1^2(\zeta_1) = F_1''(\zeta_1) - K[(F_1'(\zeta_1))^2 + 2F_1(\zeta_1)\, F_1''(\zeta_1) + 3(\Phi_1'(\zeta_1))^2] \ , \quad (26)$$

$$P_1(\zeta_1) = K[(F_1'(\zeta_1))^2 + (\Phi_1'(\zeta_1))^2] \ , \quad (27)$$

$$P_2'(\zeta_1) = (\Phi_1(\zeta_1))^2 + K(F_1'(\zeta_1))^2 \ , \quad (28)$$

$$2F_1(\zeta_1)\, \Phi_1(\zeta_1) + H_1(\zeta_1)\, \Phi_1'(\zeta_1) = \Phi_1''(\zeta_1) + 2K[F_1'(\zeta_1)\, \Phi_1'(\zeta_1) - F_1(\zeta_1)\, \Phi_1''(\zeta_1)], \quad (29)$$

$$2F_1(\zeta_1) + H_1'(\zeta_1) = 0 \ , \quad (30)$$

$$K = \nu_c\, \omega \sin \alpha / \nu \ .$$

Equations (26), (29) are the same as obtained by Srivastava* (1958) for the rotational motion of a plane lamina in a non-Newtonian fluid. The boundary conditions of our present problem are:

$$F_1(0) = 0, \quad \Phi_1(0) = 1, \quad H_1(0) = 0 \ , \quad (31)$$

$$\lim_{\zeta_1 \to \infty} F_1(\zeta_1) = 0, \quad \lim_{\zeta_1 \to \infty} \Phi_1(\zeta_1) = 0 \ ,$$

and these also correspond to the boundary conditions of his problem. The expression for P_1 in (27) is also same as obtained by him. Only the expression for P_2 in (28) differs from that of his. This shows that the nature of the flow will be the same as in the case of rotation of a disk. The values

* Equations (12), (13) .

of F_1, F_1', Φ_1 and $-H_1$ have been given in tables II–III by Jain (1961) in his paper solving Srivastava's problem in a different way. We borrow these results from his paper and since the pressure distribution is different in our case, so we compute the values of

$$p_1/\rho\omega^2 \sin^2\alpha = P_1 = K\,[(F_1'(\zeta_1))^2 + (\Phi_1'(\zeta_1))^2] \;,$$

and also of

$$\frac{p_2 - p_{20}}{\rho\omega^{3/2}\sin\alpha\sqrt{\nu\sin\alpha}} = P_2 - P_{20} = \int_0^{\zeta_1} \{\Phi_1^2(\xi) + K(F_1'(\xi))^2\}\,d\xi$$

for $K = 0.05$ and 0.1, and give these in the following tables I and II and show their variations in figures 2 and 3 .

Table I

$p_1/\rho\omega^2\sin^2\alpha$

ζ_1 \ K	0.05	0.1
0.0	0.0323	0.0646
0.5	0.0160	0.0320
1.0	0.0081	0.0161
1.5	0.0039	0.0076
2.0	0.0018	0.0035
2.5	0.0008	0.0015

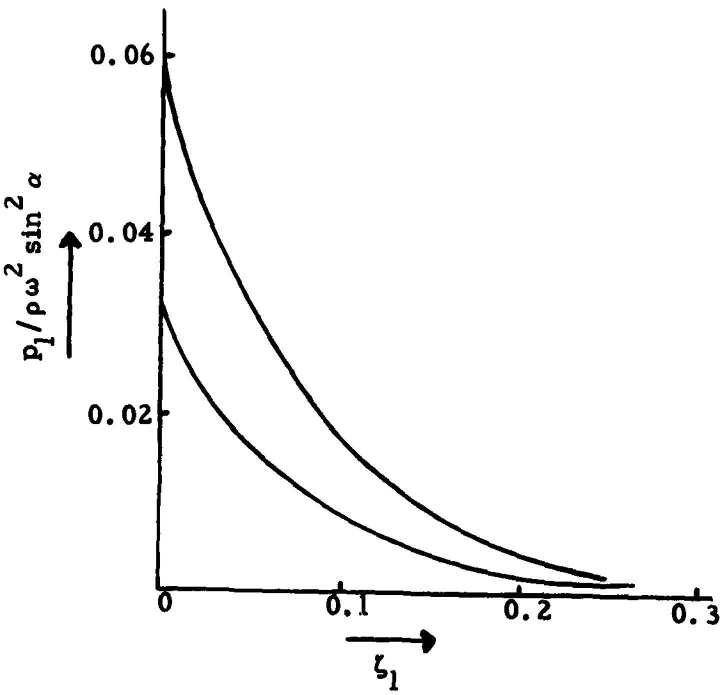
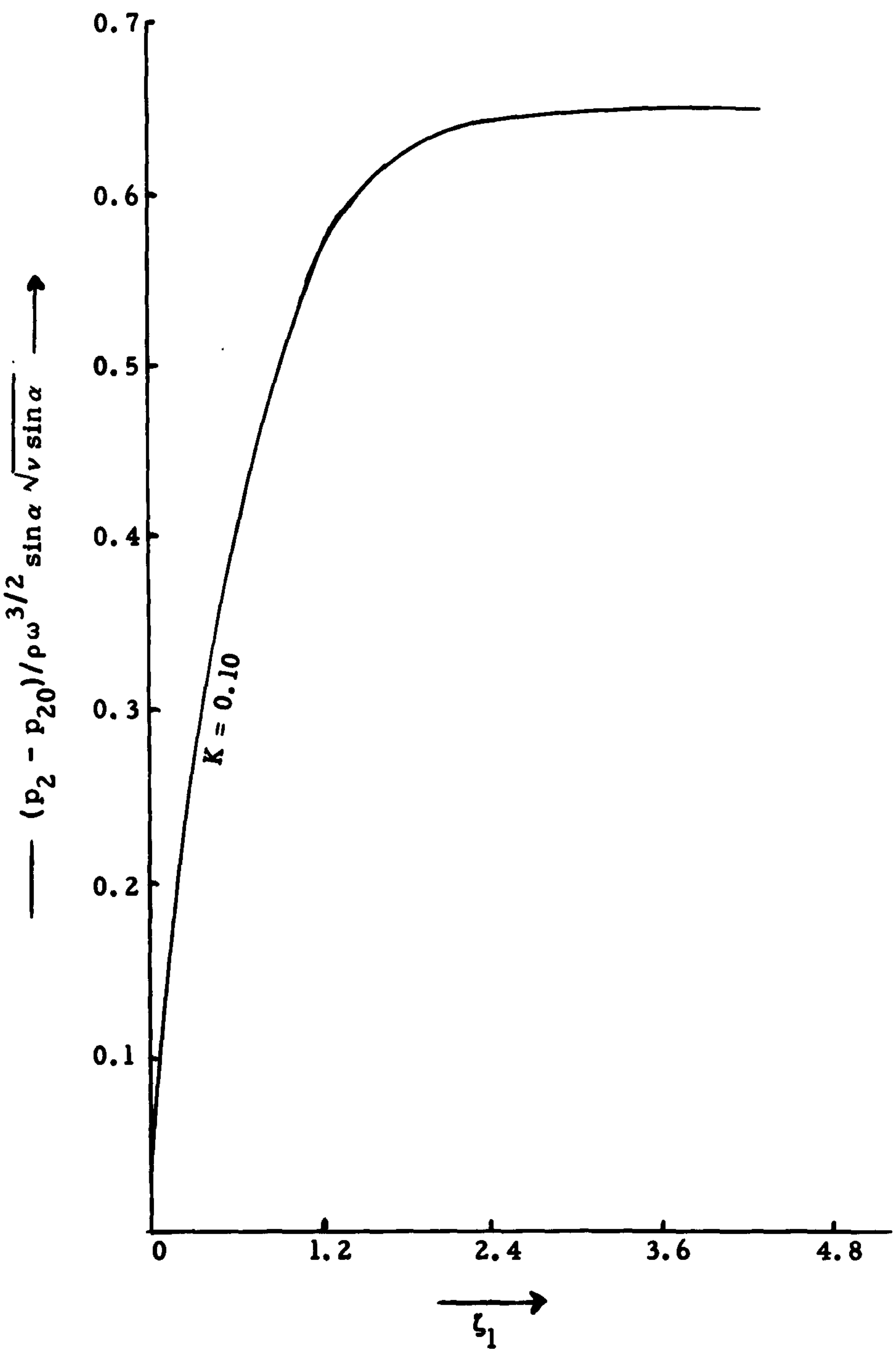


Figure 2

Table II

ζ_1 \ K	0.05	0.1
0.60	0.4098	0.4111
1.20	0.5687	0.5691
1.80	0.6238	0.6240
2.40	0.6418	0.6421
3.00	0.6476	0.6479
3.60	0.6493	0.6497
4.20	0.6499	0.6502

Figure 3


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1.2
2.4
3.6
4.8
$(p_2 - p_{20})/\rho\omega^{3/2}\sin\alpha\sqrt{\nu\sin\alpha}$
$K = 0.10$
ζ_1

Conclusion. It has been pointed out before that the flow functions F_1, Φ_1 and H_1 satisfy the same set of equations as was obtained by Jain when he considered the flow due to a rotating plane lamina in a Reiner-Rivlin fluid. So, the flow pattern will be same as in his case. This means that the values of F_1, Φ_1 and $-H_1$ decrease as K increases. Besides, the boundary layer thickness also decreases with increasing K .

Regarding the pressure distribution it may be seen from table I that P_1 increases as K increases, but it dies away vary rapidly as we go away from the wall of the cone. Also, from table II we find that $(P_2 - P_{20})$ (P_{20} being the value of P_2 at the wall of the cone) tends to its asymptotic value earlier as K increases (which corroborates with the fact that the boundary layer thickness decreases). Table I shows that P_1 dies away within (almost) half the thickness of the boundary layer. Thus the effect of the cross viscisity is most prominent only near the wall and this too becomes more magnified as we go farther from the vertex of the cone.

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