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Contract No.: DA-11-022-ORD-2059

CONJUGATE FUNCTIONS IN ORLICZ SPACES

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MRC Technical Summary Report # 367
December 1962

Madison, Wisconsin

ABSTRACT

The purpose of this report is to show that the mapping of a function on the unit circle into its conjugate is a bounded operation in an Orlicz space if and only if the Orlicz space is reflexive.

CONJUGATE FUNCTIONS IN ORLICZ SPACES

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1. The purpose of this paper is to prove the following results:

<u>Theorem 1.</u> Let $\widetilde{f}(x) = -\frac{1}{\pi}\int_0^\pi \frac{f(x+t)-f(x-t)}{2\tan\frac{1}{2}t}\,dt = \lim_{\epsilon\to+0}\left\{-\frac{1}{\pi}\int_\epsilon^\pi\right\}$.

The mapping $f \to \widetilde{f}$ is a bounded mapping of an Orlicz space into itself if and only if the space is reflexive.

Beginning with the classical result by M. Riesz for the L_p spaces [6; vol. I, p. 253] several authors have proved this theorem in one direction or the other for various special classes of Orlicz spaces. We mention in particular the papers by J. Lamperti [2] and S. Lozinski [4] and the results given in A. Zygmund's book [6; vol. II, pp. 116-118]. In our proof we use inequalities and techniques due to S. Lozinski [3, 4] to show that boundedness of the mapping implies that the space is reflexive. We use the theorem of Marcinkiewicz on the interpolation of operations [6; vol. II, p. 116] to prove that reflexivity implies the boundedness of $f \to \widetilde{f}$. Our results are more general than Lozinski's results since we use the definitions of an Orlicz space given by A. C. Zaanen [5] which includes, for example, the space L_1.

Sponsored by the Mathematics Research Center, U.S. Army, Madison, Wisconsin under Contract No.: DA-11-022-ORD-2059.

Section 2 contains preliminary material about Orlicz spaces. In Section 3 we prove that boundedness implies reflexivity and in Section 4 we prove the converse.

2. Let $v = \varphi(u)$ be a non-decreasing real valued function defined for $u \geq 0$. Assume that $\varphi(0) = 0$, that φ is left continuous and that that φ does not vanish identically. Let $u = \psi(v)$ be the left continuous inverse of φ. If $\lim_{u \to \infty} \varphi(u) = \ell$ is finite then $\psi(v) = \infty$ for $v > \ell$; otherwise $\psi(v)$ is finite for all $v \geq 0$. The complementary Young's functions Φ and Ψ are defined by

$$\Phi(u) = \int_0^u \varphi(t)\,dt \quad , \qquad \Psi(v) = \int_0^v \psi(s)\,ds .$$

Φ is an absolutely continuous convex function for $0 \leq u < \infty$ and Ψ is absolutely continuous and convex in the internal where it is finite. If $\lim_{u\to\infty} \varphi(u) = \infty$ this internal is $0 \leq v < \infty$. If $\lim_{u \to \infty} \varphi(u) = \ell$ is finite we say that Ψ jumps to infinity at $v = \ell$.

Φ is said to satisfy the Δ_2 - condition if there is a constant $k > 0$ and a $u_0 \geq 0$ such that $\Phi(2u) \leq k\,\Phi(u)$ for $u \geq u_0$. This is equivalent to satisfying the inequality $\Phi(\ell u) \leq k\ell\,\Phi(u)$ for all sufficiently large u, where ℓ is any number greater than one (for a proof and further details see [1; p. 23]).

The Orlicz space $L_\Phi = L_\Phi(0, 2\pi)$ consists, by definition, of all measurable complex functions f defined on the unit circle for which $\|f\|_\Phi = \sup \int_0^{2\pi} |f(t)g(t)|dt < \infty$, where the supremum is taken over all functions g with $\int_0^{2\pi} \Psi|g(t)|dt \leq 1$. The space L_Ψ is defined by interchanging Φ and Ψ. The Orlicz space $L_{M\Phi}$ is defined to be the set of all measurable complex functions f for which $\|f\|_{M\Phi} = \sup \int_0^{2\pi} |f(t)g(t)|dt < \infty$, where the supremum is taken over all g with $\|g\|_\Psi \leq 1$. $L_{M\Psi}$ is similarly defined. The spaces L_Φ, L_Ψ, $L_{M\Phi}$ and $L_{M\Psi}$ are all Banach spaces with their respective norms when functions equal almost everywhere are identified. The spaces L_Φ and $L_{M\Phi}$ consist of the same functions and $\|f\|_{M\Phi} \leq \|f\|_\Phi \leq 2\|f\|_{M\Phi}$. The same is true replacing Φ by Ψ. The space L_Φ is reflexive with dual space $L_{M\Psi}$ if and only if both Φ and Ψ satisfy the Δ_2-condition.

Two Young's functions Φ_1 and Φ_2 are said to be equivalent $(\Phi_1 \sim \Phi_2)$ if and only if there exist positive constants k_1, k_2, and u_0 such that $\Phi_1(k_1 u) \leq \Phi_2(u) \leq \Phi_1(k_2 u)$ for $u \geq u_0$. It is clear that $\sim$ is an equivalence relation and that the Δ_2-condition is an equivalence class property. If $\Phi_1 \sim \Phi_2$ then L_{Φ_1} and L_{Φ_2} consist of the same functions and the norms $\| \|_{\Phi_1}$ and $\| \|_{\Phi_2}$ are equivalent. Conversely, if L_{Φ_1} and L_{Φ_2} have the same elements then $\Phi_1 \sim \Phi_2$ [1; p. 112].

3. In this section we will show that if $f \to \tilde{f}$ is bounded then L_Φ is reflexive. Let $S_n(f)$ denote the n^{th} partial sum of the Fourier series of f and write $D_n(t) = \sin(n+\frac{1}{2})t/2\sin\frac{1}{2}t$. If $\|\tilde{f}\|_\Phi \le C\|f\|_\Phi$ for all $f \in L_\Phi$ then it follows [6; vol. I, p. 266] that $\|S_n(f)\|_\Phi \le A\|f\|_\Phi$ for all $f \in L_\Phi$ and all n, where A is a positive constant independent of n and f. Thus, the following result is ostensibly more general than the corresponding part of Theorem 1.

<u>Theorem 2.</u> If $\|S_n(f)\|_\Phi \le A\|f\|_\Phi$ for all $f \in L_\Phi$ and all n then L_Φ is reflexive.

The proof of Theorem 2 uses the following two lemmas given by S. Lozinski in [3]. Lozinski proved these lemmas under more restrictive conditions on φ than we have assumed. Nevertheless, Lozinski's proofs remain valid for the functions as we have defined them.

<u>Lemma 1.</u> $\frac{\varphi(u)}{250}\log\frac{n}{u\varphi(u)} \le \|D_n\|_\Phi$ for $u\varphi(u) \ge 1$.

<u>Lemma 2.</u> If $\|S_n(f)\|_\Phi \le A\|f\|_\Phi$ for all $f \in L_\Phi$ and all n then $\|D_n\|_\Phi \le 2\pi A\,\frac{n+\Phi(u)}{u}$ for $0 < u < \infty$.

Proof of Theorem 2. Our proof is a variation of the one given by Lozinski in [4]. From Lemmas 1 and 2 we have

$$(1) \qquad \varphi(v) \log \frac{n}{v\varphi(v)} \leq k \frac{n + \Phi(u)}{u}$$

for $v\varphi(v) \geq 1$ and $0 < u < \infty$. $k = \frac{2\pi A}{250}$. Our immediate aim is to show that for all sufficiently large $\lambda > 1$,

$$(2) \qquad \log \left(\frac{\lambda}{2}\right) \leq 2k \frac{\varphi(v)}{\varphi\left(\frac{v}{\lambda}\right)}$$

for $v \geq v_0$, where v_0 depends upon λ.

For any $\lambda > 1$, $\Phi(u) = \int_0^u \varphi(t)\,dt > \int_{u/\lambda}^u \varphi(t)\,dt$

and hence $\Phi(u) > \left(u - \frac{u}{\lambda}\right) \varphi\left(\frac{u}{\lambda}\right) = (\lambda - 1) \frac{u}{\lambda} \varphi\left(\frac{u}{\lambda}\right)$.

Thus

$$(3) \qquad \log \frac{(\lambda - 1)n}{\Phi(v)} < \log \frac{n}{\frac{v}{\lambda}\varphi\left(\frac{v}{\lambda}\right)} .$$

By combining (3) and (1) we see that

$$(4) \qquad \varphi(\frac{v}{\lambda}) \log \frac{(\lambda - 1)n}{\Phi(v)} \leq k \, \frac{n + \Phi(v)}{v}$$

whenever $\frac{v}{\lambda}\,\varphi(\frac{v}{\lambda}) \geq 1$. Let $n = [\Phi(v)]$ = greatest integer in $\Phi(v)$. Then (4) becomes

$$(5) \qquad \varphi(\frac{v}{\lambda}) \log \left\{ (\lambda - 1) \frac{[\Phi(v)]}{\Phi(v)} \right\} \leq k \frac{[\Phi(v)] + \Phi(v)}{v} \leq 2k \frac{\Phi(v)}{v}.$$

For every sufficiently large λ there exist a $v_0 \geq 0$ such that for $v \geq v_0$

$$(6) \qquad 1 < \frac{\lambda}{2} \leq (\lambda - 1) \frac{[\Phi(v)]}{\Phi(v)} \quad \text{and}$$

$$(7) \qquad \frac{v}{\lambda}\varphi(\frac{v}{\lambda}) \geq 1 .$$

Using (5), (6) and the fact that $\Phi(v) \leq v\varphi(v)$ we get inequality (2) for $v \geq v_0$. Since λ can be arbitrarily large (2) implies that $\lim_{u \to \infty} \varphi(u) = \infty$ and hence that Ψ does not jump to infinity. We next show that Ψ satisfies the Δ_2-condition.

Let λ be large but fixed and write $\ell = \frac{1}{2k} \log (\frac{\lambda}{2})$. Then (2) states that

$$(8) \qquad \ell\varphi(\frac{t}{\lambda}) \leq \varphi(t)$$

for $t \geq v_0$.

This implies, on taking inverses, that there is a number s_0 such that for $s \geq s_0$

$$(9) \qquad \psi(s) \leq \lambda \psi(\tfrac{s}{\ell}).$$

Thus $\int_{s_0}^{v} \psi(s)\,ds \leq \lambda \int_{s_0}^{v} \psi(\tfrac{s}{\ell})\,ds = \lambda\ell \int_{\frac{s_0}{\ell}}^{\frac{v}{\ell}} \psi(s)\,ds$ or

$$(10) \qquad \Psi(v) - \Psi(s_0) \leq \lambda\ell[\Psi(\tfrac{v}{\ell}) - \Psi(\tfrac{s_0}{\ell})].$$

This shows that for sufficiently large v

$$(11) \qquad \Psi(\ell v) \leq 2\lambda\ell\,\Psi(v)$$

and hence proves that Ψ satisfies the Δ_2-condition.

If $\|s_n(f)\|_\Phi \leq A\|f\|_\Phi$ for all $f \in L_\Phi$ then it follows that $\|s_n(g)\|_{M\Psi} \leq A\|g\|_{M\Psi}$ for all $g \in L_{M\Psi}$ or, equivalently, that $\|s_n(g)\|_\Psi \leq 2A\|g\|_\Psi$ for all $g \in L_\Psi$. Since we have shown that Ψ does not jump to ∞ we can interchange the rôle of Φ and Ψ in the above argument to show that Φ satisfies the Δ_2-condition. This proves that L_Φ is reflexive and completes the proof of Theorem 2.

4. In this section we prove a general result about reflexive Orlicz spaces which combined with the classical results of M. Riesz [6; vol. I, p. 256 and p. 266] yields the unproved half of Theorem 1 as well as the converse of Theorem 2.

Theorem 3. Suppose that T is a bounded linear operator on L_p into L_p for $1 < p < \infty$. Then if L_Φ is reflexive T is defined and bounded on L_Φ into L_Φ.

Proof. The proof consists of showing that Φ can be replaced by an equivalent function Φ_1 $(\Phi \sim \Phi_1)$ such that Φ_1 satisfies the conditions of the Marcinkiewicz theorem on the interpolation of operations i.e. such that

$$(12) \qquad \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}}\, dt = O\left\{\frac{\Phi(u)}{u^\beta}\right\} \quad \text{and}$$

$$(13) \qquad \int_1^u \frac{\Phi_1(t)}{t^{\alpha+1}}\, dt = O\left\{\frac{\Phi(u)}{u^\alpha}\right\}$$

for $u \to \infty$, where $1 < \alpha < \beta < \infty$.

The assumption that L_Φ is reflexive implies that $\lim_{u\to\infty} \varphi(u) = \infty$ and hence that $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$. By [1; p. 16] Φ is equal for sufficiently large values of u to a function M of the form $M(u) = \int_0^u p(t)\,dt$ where p is a non-decreasing right continuous function with $\lim_{u\to 0} p(u) = 0$ and $\lim_{u\to\infty} p(u) = \infty$. Clearly $\Phi \sim M$.

By [1; p. 46] the function M_1 defined by $M_1(u) = \int_0^u \frac{M(t)}{t}\,dt$ is equivalent to M and hence to Φ. The derivative of M_1 is continuous and strictly increasing.

Since L_Φ is reflexive both Φ and Ψ satisfy the Δ_2-condition. Thus both M_1 and its conjugate Young's function N_1 satisfy the Δ_2-condition [1; p. 23]. According to [1; pp. 26-27] this implies the existence of numbers a, b, and $u_0 \geq 0$ with $1 < a < b < \infty$ such that

$$1 < a < \frac{uM_1'(u)}{M_1(u)} < b$$

for all $u \geq u_0$. If we define Φ_1 by

$$\Phi_1(u) = \begin{cases} \dfrac{M_1(u_0)}{u_0^a}\, u^a & \text{for } u \leq u_0 \\ \\ M_1(u) & \text{for } u \geq u_0 \end{cases}$$

we obtain a function $\Phi_1 \sim \Phi$ such that

$$(14) \qquad 1 < a \leq \frac{u\varphi_1(u)}{\Phi_1(u)} \leq b$$

for all $u \geq 0$.

We next show that Φ_1 satisfies (12) and (13) for suitably chosen α and β. In particular choose α and β such that $1 < \alpha < a \leq b < \beta < \infty$. This is clearly possible. In what follows all of the integrals will exist as finite numbers because of (14).

Integration by parts shows that

$$(15) \qquad \int_u^\infty \frac{\varphi_1(t)}{t^\beta}\,dt = \beta \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}}\,dt - \frac{\Phi_1(u)}{u^\beta} \qquad \text{and}$$

$$(16) \qquad \int_0^u \frac{\varphi_1(t)}{t^\alpha}\,dt = \alpha \int_0^u \frac{\Phi_1(t)}{t^{\alpha+1}}\,dt + \frac{\Phi_1(u)}{u^\alpha} \quad .$$

From (14) we obtain

$$(17) \qquad \int_u^\infty \frac{\varphi_1(t)}{t^\beta}\,dt \leq b \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}}\,dt \qquad \text{and}$$

$$(18) \qquad \int_0^u \frac{\varphi_1(t)}{t^\alpha}\,dt \geq a\int_0^u \frac{\Phi_1(t)}{t^{\alpha+1}}\,dt.$$

Combining (15) with (17) and (16) with (18) shows that

$$(19) \qquad \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}}\,dt \leq \frac{1}{\beta - b}\left\{\frac{\Phi_1(u)}{u^\beta}\right\} \qquad \text{and}$$

$$(20) \qquad \int_0^u \frac{\Phi_1(t)}{t^{\alpha+1}}\,dt \leq \frac{1}{a - \alpha}\left\{\frac{\Phi_1(u)}{u^\alpha}\right\} \qquad .$$

This shows that Φ_1 satisfies (12) and (13). Thus by the Marcinkiewicz theorem and Theorem 10.14 of [6; vol I, p.174] there exists a constant K_1 such that $\|Tf\|_{\Phi_1} \leq K_1\|f\|_{\Phi_1}$ for all $f \in L_{\Phi_1}$. Since $\Phi \sim \Phi_1$ there is a constant K such that $\|Tf\|_\Phi \leq K\|f\|_\Phi$ for all $f \in L_\Phi$. This completes the proof of Theorem 3.

Statements of the standard corollaries of Theorem 1 can be found in [2].

REFERENCES

[1] M.A. Krasnosel'skii and Ya. B. Rutickii, Convex Functions and Orlicz Spaces, Groningen, 1961.

[2] J. Lamperti, A note on conjugate functions, Proc. Amer. Math. Soc. vol. 10 (1959) 71-76.

[3] S. Lozinski, On convergence and summability of Fourier series and interpolation processes, Mat. Sbornik N.S. 14 (1944) 175-262.

[4] ______________, On convergence in mean of Fourier series, Doklady, vol.51 (1946) 7-10.

[5] A. C. Zaanen, Linear Analysis, New York-Amsterdam-Groningen, 1953.

[6] A. Zygmund, Trigonometric Series, Vols. I, II, Cambridge, 1959 .