UNCLASSIFIED

AD 295 644

Reproduced by the

ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA

UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

63-2-3

NWL Report No.1835

BOUNDARY LAYER ALONG A FLAT SURFACE NORMAL TO A VORTEX FLOW

by

Ernst W. Schwiderski and Hans Lugt
Computation and Analysis Laboratory

DEPARTMENT OF THE NAVY
BUREAU OF NAVAL WEAPONS

U. S. NAVAL WEAPONS LABORATORY
DAHLGREN, VIRGINIA

Date: DECEMBER 20 1962

U. S. Naval Weapons Laboratory
Dahlgren, Virginia

Boundary Layer Along a Flat Surface Normal to a Vortex Flow

by

Ernst W. Schwiderski and Hans Lugt
Computation and Analysis Laboratory

NWL REPORT NO. 1835

Task Assignment
NO. R360FR103/2101/R01101001

December 1962

TABLE OF CONTENTS

ABSTRACT

The axisymmetric vortex motion of a viscous fluid over a flat surface is investigated. Paradoxical results obtained by Moore and Goldshtik are resolved. Guided by a useful extension of Prandtl's boundary layer theory an appropriate similarity transformation is found which reduces the Navier-Stokes equations to three ordinary differential equations. The solution of the remaining boundary value problem reveals significant properties of vortex flows and explains several phenomena observed in hurricanes. A complete numerical result is displayed.

FOREWORD

This work was sponsored by the Naval Weapons Laboratory Foundational Research Program. It was performed in the Computation and Analysis Laboratory as part of the Meteorological and Oceanographical project (R360FR103/2101/R01101001).

The authors wish to extend their thanks to Dr. C. J. Cohen for many critical and stimulating discussions.

The date of completion was 10 December 1962.

Approved for Release:

/s/ R. H. LYDDANE
Technical Director

1. Introduction

In recent years several investigations have been devoted to the mathematical description of the steady vortex motion of a viscous fluid which is governed by the Navier-Stokes equations. Taylor [9] , Cooke [2], Goldshtik [3], and Long [4] investigated the vortex motion normal to a flat surface, which is asymptotic to a coaxial potential vortex at large distances from the surface and the core of the vortex. Taylor and Cooke excluded the core of the vortex and presented certain solutions of the corresponding classical boundary layer equations by applying the method of Kármán-Pohlhausen. Moore [5] showed that the axisymmetric vortex flow considered is nonasymptotic to certain so-called "similarity solutions" of the boundary layer equations used by Taylor and Cooke. Nevertheless, Long ignored the presence of the surface and constructed a certain similarity solution of the classical boundary layer equations, which presumably describe the vortex flow approximately in the core. Goldshtik reexamined the problem and attempted to integrate the exact Navier-Stokes equations by a similarity solution of Moore's type. This interesting investigation led to the paradoxical results presented in [3] .

Since the paradoxical results of Moore and Goldshtik appear to be physically questionable, the same vortex flow problem will be reinvestigated in the present paper. It will be shown that the discrepancies discovered by Moore and Goldshtik are due to inconsistent and incomplete boundary data used to determine a unique solution of the differential equations considered. Moore's nonexistence result is also a consequence of Prandtl's boundary layer theory which destroys the elliptic character of the Navier-Stokes equations without examining the existence of a solution to the remaining parabolic boundary value problem.

On the basis of a specified physical problem, complete and consistent boundary data will be found in order to define a useful solution of the Navier-Stokes equations. Guided by an extended boundary layer theory, which preserves the full elliptic Navier-Stokes equations, a set of ordinary differential equations will be derived by an appropriate similarity transformation. This transformation is based upon the concept of the limiting line of a boundary layer the existence of which is suggested by the extended boundary layer theory.

The investigations are supported by a complete numerical solution, which has been established by means of the Runge-Kutta method. All results appear to be physically plausible and permit the explanation of several phenomena observed in hurricanes outside their cores.

2. Definition of a Solution of the Navier-Stokes Equations

In the cylindrical coordinate system (r, φ, z) let (u, v, w) denote the corresponding velocity vector of an axisymmetric vortex flow over the solid surface $z = 0$. If ν, ρ, and p are the constant kinematic viscosity, the constant density, and the variable pressure of the fluid, the flow is governed by the Navier-Stokes equations

$$uu_r + wu_z - \frac{v^2}{r} = -\frac{1}{\rho} p_r + \nu\left[u_{rr} + \left(\frac{u}{r}\right)_r + u_{zz}\right] \tag{1}$$

$$uv_r + wv_z + \frac{uv}{r} = \qquad + \nu\left[v_{rr} + \left(\frac{v}{r}\right)_r + v_{zz}\right] \tag{2}$$

$$uw_r + ww_z \qquad = -\frac{1}{\rho} p_z + \nu\left[w_{rr} + \frac{1}{r}w_r + w_{zz}\right] \tag{3}$$

$$(ru)_r + (rw)_z = 0 \quad . \tag{4}$$

Since the nonlinear elliptic partial differential equations (1) through (4) are singular at the axis $r = 0$, it is not possible to specify a solution in the region $(r \geq 0,\ z \geq 0)$ by an arbitrary set of boundary data and singularity conditions. In order to find a consistent and complete set of boundary values, it is helpful to examine the characteristic properties of the following physical flow model (see Figure 1).

The vortex flow over the surface at $z = 0$ may be produced by a very long rod of small diameter d which is rotating around its axis at $r = 0$ with the angular velocity ω. If all friction forces at the surface at $z = 0$ were eliminated, then the rod would produce the potential vortex flow

$$u \equiv 0, \qquad v \equiv \frac{\Gamma}{r}, \qquad w \equiv 0, \qquad \frac{p}{\rho} \doteq -\frac{\Gamma^2}{r^2} \tag{5}$$

with the vortex strength

$$\Gamma = \omega\left(\frac{d}{2}\right)^2 , \tag{6}$$

where an additive pressure constant is neglected. In order to investigate exclusively the influence of the friction forces at the surface $z = 0$ on the potential vortex (5), it is necessary to prevent any additional exterior disturbances of the flow. This condition may be fulfilled by providing the rod with a flexible surface which glides freely with the flow in the axial direction. For the rigorous mathematical model the rod must extend from $z = 0$ to $z = \infty$. The diameter of the rod must shrink to zero, while the angular velocity increases to infinity so that the vortex strength (6) remains a constant. Since the fluid is to be at rest at large distances from the rod, adequate sources and sinks of equal strength must be located at large distances from the surface. A circular sink around the axis $r = 0$ is necessary at $z = \infty$ to absorb the fluid which is ejected by the vortex motion from the boundary layer at the surface. This sink must change at some $r = r_0$ to an annular source of equal strength, which supplies the boundary layer with the necessary fluid in return.

With this description of the physical flow model, it is now easy to determine consistent singularities and complete boundary data which define a useful solution of the Navier-Stokes equations. At the surface and at large distances from the rod the solution is subject to the "regular boundary conditions":

$$\left.\begin{matrix} r > 0 \\ z = 0 \end{matrix}\right\} : u = 0, \qquad v = 0, \qquad w = 0 \tag{7}$$

$$\left.\begin{matrix} r = \infty \\ z < \infty \end{matrix}\right\} : u = 0, \qquad v = 0, \qquad w = 0 \ . \tag{8}$$

At the rotating rod and at large distances from the surface the solution is determined by the following "singular boundary conditions":

$$\left.\begin{matrix} r \to 0 \\ z > 0 \end{matrix}\right\} : u \to 0 \ , \qquad \frac{rv}{\Gamma} \to 1 \ , \qquad \frac{w}{\Gamma A \log \frac{r_0}{r}} \to 1 \tag{9}$$

$$\left.\begin{matrix} r < \infty \\ z \to \infty \end{matrix}\right\} : u \to 0 \ , \qquad \frac{rv}{\Gamma} \to 1 \ , \qquad \frac{w}{\Gamma A \log \frac{r_0}{r}} \to 1 \ , \tag{10}$$

where Γ and A are constant parameters at one's disposal. The constant r_0 must be determined simultaneously with the solution. No conditions are imposed at the two singular points ($r = 0$, $z = 0$) and ($r = \infty$, $z = \infty$).

The consistency of the foregoing singularities (9) and (10) may be demonstrated on the basis of the following principles:

(I) The singularities required for the primary tangential flow v are admissible, because the Navier-Stokes equations (1) through (4) yield the potential solution (5) which has the same singularities.

(II) The sink and source distribution admitted for the secondary axial flow w is compatible with the singularities of the primary tangential flow, as the Navier-Stokes equations yield the potential solution

$$u = 0 \ , \qquad v = \frac{\Gamma}{r} \ , \qquad w = \Gamma A \log \frac{r_0}{r} \tag{11}$$

with the same singularities. Thus the exchange of fluid between the boundary layer at $z = 0$ and the sink and source at $z = \infty$ can proceed freely without friction forces acting. Since the undisturbed flow (5) is not regular at $r = \infty$, the amount of fluid revolved by the vortex flow depends upon the radius $r = \infty$ of the surface at $z = 0$. This indicates the freedom of the parameter A which, vice versa, determines the radius of the surface, i. e., it defines the exact meaning of the limit $r = \infty$ (see [8]).

In this connection it is significant to compare the boundary data posed by M. A. Goldshtik for the same physical problem (see [3] and also [2, 5, 9]). Goldshtik subjected the flow to the same regular boundary values (7) and (8), to which he added the singular boundary conditions

$$\left.\begin{matrix} r \to 0 \\ z \to \infty \end{matrix}\right\} : \frac{rv}{\Gamma} \to 1 , \qquad -\frac{2r^2}{\Gamma^2}\frac{p}{\rho} \to 1 \tag{12}$$

$$\left.\begin{matrix} r \to 0 \\ z > 0 \end{matrix}\right\} : u \to 0 , \qquad w \to w_0(z)\,(\text{bounded!}) \tag{13}$$

Since these conditions were incomplete, he required later a special "similarity property" for the solution, by which he tacitly added the missing data

$$\left.\begin{matrix} r > 0 \\ z \to \infty \end{matrix}\right\} : w \to 0 , \ (u \to 0) , \tag{14}$$

$$\left.\begin{matrix} r \to 0 \\ z > 0 \end{matrix}\right\} : \frac{rv}{\Gamma} \to 1 . \tag{15}$$

Without going into any process of integration, it is easy to show the inconsistency of these conditions which led Goldshtik to his "paradoxical results". His special similarity assumption requires the vortex flow to produce a sort of source in the axial flow w at the point (r = 0, z = 0). This source is characterized by an unbounded axial velocity w which must remain unbounded along the entire axis r = 0, where the tangential and radial flows are at their asymptotic frictionless state. In fact, Goldshtik's existence proof for small Reynolds numbers is based on the second condition (13), which is not fulfilled by his solution (see [3, page 927]):

$$w \approx \frac{c}{\sqrt{r^2 + z^2}} \approx \frac{c}{z} \quad . \tag{16}$$

It may be mentioned that the authors' attempt to solve Goldshtik's equations (1.3) through (1.8) under the boundary conditions (1.7) and (1.8) (see [3]) failed for small and large Reynolds numbers because of nonconvergence of the Runge-Kutta method. The numerical results demonstrated very clearly that the friction forces at the surface are not sufficient to produce a source of the strength (16) in the axial flow. Further exterior forces are needed, to reduce the axial velocity from w = ∞ to w = 0 along the axis r = 0. This observation led to the introduction of the logarithmic singularity (11) which is free of effective friction forces.

3. A Reduction of the Navier-Stokes Equations

For an approximate integration of the Navier-Stokes equations (1) through (4) under the boundary conditions (7) through (10), it is useful to simplify the singular boundary conditions (9) and (10) by introducing dimensionless quantities by the following conventions:

$$u = \frac{\Gamma}{r} U, \qquad v = \frac{\Gamma}{r} V, \qquad w = \Gamma A \log \frac{r_o}{r} W, \tag{17}$$

$$\frac{p}{\rho} = -\frac{\Gamma^2}{r^2} P, \qquad R = \frac{\Gamma}{\nu}, \tag{18}$$

where R may be called the Reynolds number of the flow. After carrying out this transformation one arrives at the new Navier-Stokes equations:

$$UU_r + Ar \log \frac{r_o}{r} WU_z - \frac{1}{r}(U^2 + V^2) = P_r - \frac{2}{r} P + \frac{1}{R}[r(U_{rr} + U_{zz}) - U_r] \tag{19}$$

$$UV_r + Ar \log \frac{r_o}{r} WV_z = \frac{1}{R}[r(V_{rr} + V_{zz}) - V_r] \tag{20}$$

$$r \log \frac{r_o}{r}[UW_r + Ar \log \frac{r_o}{r} WW_z] - UW = \frac{1}{A} P_r + \frac{1}{R}[r^2 \log \frac{r_o}{r}(W_{rr} + W_{zz} + \frac{1}{r}W_r) - 2rW_r] \tag{21}$$

$$U_r + Ar \log \frac{r_o}{r} W_z = 0 \tag{22}$$

The transformed boundary conditions are:

$$\left.\begin{matrix} r > 0 \\ z = 0 \end{matrix}\right\} : U = 0, \qquad V = 0, \qquad W = 0, \tag{23}$$

$$\left.\begin{matrix} r = \infty \\ z < \infty \end{matrix}\right\} : U = 0, \qquad V = 0, \qquad W = 0, \tag{24}$$

$$\left.\begin{matrix} r = 0 \\ z > 0 \end{matrix}\right\} : U = 0, \qquad V = 1, \qquad W = 1, \tag{25}$$

$$\left.\begin{matrix} r < \infty \\ z = \infty \end{matrix}\right\} : U = 0, \qquad V = 1, \qquad W = 1 \quad . \tag{26}$$

The solution of this equivalent elliptic boundary value problem may be guided by the essential hypothesis of Prandtl's boundary layer theory in the following form.

<u>Weak Boundary Layer Assumption:</u> Friction forces, which are caused by the nonslip condition at a solid surface in a primary real flow without effective friction forces, are essentially acting only within a "boundary layer" that is bounded by a certain "limiting line." The asymptotic flow beyond the boundary layer is the primary flow on which may be superposed a secondary flow without effective friction forces.

It is significant to note that the "extended boundary layer theory" based on the weak assumption preserves the character of the

Navier-Stokes equations. Thus the validity of existence and uniqueness theorems, which are known for elliptic boundary value problems, remains untouched by this theory. This most important requirement of admissible simplifications of mathematical problems is, however, ignored by the complete "classical boundary layer theory." Indeed, the well-known boundary layer assumptions, which neglect certain first and second order partial derivatives, lead to parabolic differential equations, for which the existence and uniqueness theorems of elliptic boundary value problems do not hold in the same generality. This is reason enough to abandon such risky assumptions. In this connection it may be interesting to compare other simplifications of this sort which have been suggested in the past and which have been criticized in a similar manner (see, for instance, [1]).

The additional assumptions of the classical boundary layer theory may also be abandoned as they do not simplify the constructive solution of the remaining mathematical problem in general. This statement will be verified for the present problem by the following derivations and for further examples by other investigations (see [8]). It may be worthwhile to mention that the extended boundary layer theory introduced above leads to exactly the same asymptotic solution of the classical flat plate problem as the common boundary layer theory without any mathematical complications. This reveals a clear superiority of the extended boundary layer theory over the classical theory which may be well founded in special cases.

In order to find an approximate solution of the boundary value problem considered, it is helpful to utilize the limiting line $z = \delta(r)$ of the boundary layer the existence of which is suggested by the extended boundary layer theory. For an accuracy parameter ϵ, which is at one's disposal, the limiting line of the boundary layer along the surface at $z = 0$ may be characterized by the following flow data:

$$z = \delta(r) \; : \; V = 1 - \epsilon \tag{27}$$

$$z \geq \delta(r) \; : \; U \approx 0 \; , \qquad V \approx 1 \; , \qquad W \approx 1 \; . \tag{28}$$

The existence of a line $z = \delta(r)$, which satisfies all conditions (27) and (28) is plausible from the boundary data (23) through (26). Since V is a solution of an elliptic differential equation, $\delta(r)$ must be an analytic function of r which connects the two singular boundary points $(r = 0, \; z = 0)$ and $(r = \infty, \; z = \infty)$ for any accuracy parameter $0 < \epsilon < 1$ (see Figure 1).

According to the conditions (23) through (28) it is useful to transform the Navier-Stokes equations (19) through (22) by the "similarity relation"

$$r = r, \qquad \zeta = \frac{z}{\delta(r)} \; , \tag{29}$$

which maps the boundary layer $0 \leq z \leq \delta(r)$ onto the parallel strip $0 \leq \zeta \leq 1$. The boundary lines $(r > 0, \; z = 0)$ and $(r < \infty, \; z = \infty)$ are mapped onto the corresponding lines ($r > 0$, $\zeta = 0$) and ($r < \infty$, $\zeta = \infty$). While the lines ($r = \infty$, $z < \infty$) and ($r = 0$, $z > 0$) reduce to the points ($r = \infty$, $\zeta = 0$) and ($r = 0$, $\zeta = \infty$), the singular points ($r = 0$, $z = 0$) and ($r = \infty$, $z = \infty$) stretch into the lines ($r = 0$, $\zeta < \infty$) and ($r = \infty$, $\zeta > 0$). Thus, in the (r, ζ) plane the solution considered is not governed by any boundary data at $r = 0$ and $r = \infty$.

After evaluating the transformed conditions (23) through (28) one is tempted to seek a solution of the corresponding Navier-Stokes equations which is independent of r. Indeed, if a boundary layer exists, which satisfies the conditions (27) and (28), then all partial derivatives with respect to r vanish at $\zeta = 0$ and are almost zero at $\zeta = 1$ at least for some values of r, for instance, around $r = r_0$. The same remains true within the boundary layer $0 \leq \zeta \leq 1$, because the lines ζ = const. may be considered as first order approximations of the lines of constant velocity (U, V, W) around $r = r_0$. Thus, the exact solution of the problem considered must yield a first order approximation which is independent of r around $r = r_0$ within the strip $0 \leq \zeta \leq 1$.

These results reveal the limits of the classical boundary layer theory. After differentiating the essential boundary layer assumption (27) along the line $z = \delta(r)$ one arrives at

$$\frac{dV}{dr} = \frac{\partial V}{\partial r} + \frac{\partial V}{\partial z}\frac{d\delta}{dr} \equiv 0 \tag{30}$$

Thus, $\frac{\partial V}{\partial r}$ and $\frac{\partial V}{\partial z}$ display the same asymptotic behavior, if $|\delta'(r)|$ has a positive lower bound. Unless r can be confined to a proper vicinity of a point $r = r_0$ at which $\delta'(r_0) = 0$, $\delta'(r)$ may even grow beyond any upper bounds, if the accuracy of condition (27) is increased. After differentiation of equation (30) along the line $z = \delta(r)$ one sees in the same manner that, in the Navier-Stokes equations (19) through (22), partial derivatives with respect to r cannot be neglected up to second order, unless the derivatives $\delta'(r)$ and $\delta''(r)$ vanish at some point $r = r_0$. Hence, the classical boundary layer theory is consistent for an "almost parallel" boundary layer which has a limiting line that assumes the slope and the curvature of the corresponding surface in the region of interest. This condition is fulfilled in the classical flat plate problem, but it is violated in the present problem as the following derivations will show (see also [8]).

After carrying out the similarity transformation (29) one arrives at the reduced Navier-Stokes equations:

$$(1 + \delta'^2\zeta^2)\ddot{U} + [\zeta(2\delta'^2 - \delta\delta'' + \frac{\delta\delta'}{r}) + \frac{R\delta}{r}(\delta'\zeta U - Ar\log\frac{r_0}{r}W)]\dot{U}$$
$$+ R\frac{\delta^2}{r^2}[U^2 + V^2 - 2P - \frac{r\delta'}{\delta}\zeta\dot{P}] = 0 \qquad (31)$$

$$(1 + \delta'^2\zeta^2)\ddot{V} + [\zeta(2\delta'^2 - \delta\delta'' + \frac{\delta\delta'}{r}) + \frac{R\delta}{r}(\delta'\zeta U - Ar\log\frac{r_0}{r}W)]\dot{V} = 0 \qquad (32)$$

$$UW + \frac{1}{A\delta}\dot{P} = \frac{r^2}{R\delta^2}\log\frac{r}{r_0}[(1 + \delta'^2\zeta^2)\ddot{W} + \zeta(2\delta'^2 - \delta\delta'' - \frac{\delta\delta'}{r})]\dot{W}$$
$$+ \frac{r}{\delta}\log\frac{r}{r_0}[\delta'\zeta U - Ar\log\frac{r_0}{r}W]\dot{W} - \frac{2r\delta'}{R\delta}\dot{W} \qquad (33)$$

$$\delta'\zeta\dot{U} - Ar\log\frac{r_0}{r}\dot{W} = 0 \qquad (34)$$

In these equations the partial derivatives of U, V, W, and P with respect to r are omitted because only ζ-dependent solutions are of interest. The partial derivatives with respect to ζ are indicated by a dot, while the primes denote the derivatives of the limiting line of the boundary layer.

The continuity equation (34) displays very clearly that the search for an approximate solution of the boundary value problem considered can be successful in the present form, only if $\delta'(r)$ yields the expansion

$$\delta' = Br_0\left(\frac{r}{r_0}\right) \log \frac{r_0}{r} + \ldots \qquad (B \neq 0) \tag{35}$$

for instance around $r = r_0$, where terms of second and higher order may be neglected. The limiting line of the boundary layer has then the expansion

$$\delta = B\left(\frac{r_0}{2}\right)^2 \left(\frac{r}{r_0}\right)^2 \left[2 \log \frac{r_0}{r} + 1\right] + \ldots \qquad (B \neq 0) \tag{36}$$

around $r = r_0$.

The general features of the limiting line of the boundary layer determined by equation (36) are displayed in Figure 1. It is remarkable to note that the boundary layer thickness reaches a relative maximum at the point $r = r_0$ where the axial velocity w changes its direction. Hence the boundary layer thickness must assume also a relative minimum before it increases rapidly beyond any bounds. This surprising phenomenon is physically plausible, as the positive and negative secondary axial flows deform the boundary layer which is produced by the friction forces along the surface at $z = 0$.

After substituting the approximation (36) in the equations (31) through (34) and neglecting terms of higher order one obtains the following system of ordinary differential equations:

$$\ddot{U} + \left(\frac{Br_o}{2}\right)^2 \zeta\dot{U} + \frac{R}{4}\left(\frac{Br_o}{2}\right)^2 \left(U^2 + V^2 - 2P\right) = 0 \tag{37}$$

$$\ddot{V} + \left(\frac{Br_o}{2}\right)^2 \zeta\dot{V} = 0 \tag{38}$$

$$UW + \frac{B}{A}\left(\frac{2}{Br_o}\right)^2 \dot{P} = 0 \tag{39}$$

$$B\zeta\dot{U} - A\dot{W} = 0 \quad . \tag{40}$$

These equations may be simplified by the introduction of the stream function $G(\zeta)$ and the characteristic number σ which are defined by the relations

$$U = -\dot{G} \ , \qquad W = \frac{B}{A}(G - \zeta\dot{G}) \ , \qquad \sigma = \frac{Br_o}{2} \quad . \tag{41}$$

This substitution leads to the following solution of the problem considered:

The functions $G(\zeta)$, $V(\zeta)$, and $P(\zeta)$ represent an approximate solution of the Navier-Stokes equations (1) through (4) under the boundary conditions (12) through (15), which is valid within the boundary layer $0 \le \zeta \le 1$ in the vicinity of $r = r_0$, provided:

(A) $G(\zeta)$, $V(\zeta)$, and $P(\zeta)$ fulfill the differential equations

$$\ddot{G} + \sigma^2 \zeta \dot{G} = \frac{R\sigma^2}{4} (\dot{G}^2 + V^2 - 2P) \tag{42}$$

$$\ddot{V} + \sigma^2 \zeta \dot{V} = 0 \tag{43}$$

$$\dot{G}(G - \zeta \dot{G}) = \frac{1}{\sigma^2} \dot{P} \tag{44}$$

under the boundary conditions

$$\zeta = 0: \qquad G = 0 , \qquad \dot{G} = 0 , \qquad V = 0 \tag{45}$$

$$\zeta = \infty: \qquad G = G_\infty, \qquad V = 1 , \qquad P = \frac{1}{2} , \tag{46}$$

(B) the characteristic number σ satisfies the accuracy condition

$$\zeta = 1: \qquad V = 1 - \epsilon , \tag{47}$$

(C) $G(\zeta)$ and $P(\zeta)$ satisfy the conditions

$$\zeta = 1: \quad \begin{cases} G \approx G_\infty , & \dot{G} \approx 0 , \quad \ddot{G} \approx 0 , \quad \dddot{G} \approx 0 \\ P \approx \frac{1}{2} , & \dot{P} \approx 0 \end{cases} \tag{48}$$

with sufficient accuracy,

(D) the characteristic constants B and r_0 of the limiting line of the boundary layer

$$\delta(r) \approx \frac{B}{4} r^2 (2 \log \frac{r_0}{r} + 1) \tag{49}$$

are determined by the relations

$$B = \frac{A}{G_\infty} \,, \qquad r_0 = \frac{2\sigma}{B} \,. \tag{50}$$

This solution is based upon the "joining properties" (B) and (C), which constitute the mathematical justification of the entire boundary layer theory. The exact integral (11) of the Navier-Stokes equations (1) through (4) is utilized as an approximate solution outside the boundary layer $0 \le \zeta \le 1$ of the problem under consideration. An approximate solution of the Navier-Stokes equations is constructed within the boundary layer which yields the correct boundary data at $\zeta = 0$. Both solutions are asymptotically equal for large values of ζ. They are joined along the limiting line of the boundary layer $\zeta = 1$ under a small violation of the analyticity of the exact solution. This violation is controlled by the properties (B) and (C), which is extended to all derivatives of the dependent variables that occur in the Navier-Stokes equations.

The existence of an integral, which fulfills the differential equations (42), (43), and (44) and the boundary conditions (45) and (46) indicates the simplified system of differential equations

$$\dddot{G} + \sigma^2 \zeta \ddot{G} = \frac{R\sigma^2}{4} (V^2 - 1) \tag{51}$$

$$\ddot{V} + \sigma^2 \zeta \dot{V} = 0 \,, \qquad P = \frac{1}{2} \tag{52}$$

In fact, the system (51) and (52) is integrable by quadratures and yields a solution under the unaltered boundary conditions (45) and (46). Although this system represents no approximate substitute for the system (42), (43), and (44), it retains all its characteristic properties at both ends of the interval $0 \le \zeta \le \infty$. Thus, the solution of the simplified system (51) and (52) may be used as an initial integral for an appropriate iteration procedure (see section 4) which converges to the solution of the complete system.

The existence of the integral defined by the conditions (A) through (D) justifies the general boundary layer assumption which guided the foregoing derivations. Furthermore, it demonstrates the consequences of ignoring the restrictive limits of Prandtl's boundary layer theory. If all assumptions of the classical boundary layer theory were applied then the differential equations (42), (43), and (44) would reduce to the equations

$$\dddot{G} = \frac{R\sigma^2}{4}\left(\dot{G}^2 + V^2 - 2P\right) \tag{53}$$

$$\ddot{V} = 0 \tag{54}$$

$$\sigma^2 \dot{G}(G - \zeta\dot{G}) = \dot{P} \quad , \tag{55}$$

which have no solution under the boundary conditions (45) and (46). It is obviously false to neglect the term $\sigma^2 \zeta\ddot{G}$ against all other terms of the differential equations (42) and (43) no matter how large the

Reynolds number R may be assumed. It is exactly this term which characterizes the boundary layer directly at the surface and at large distances from the surface. If the term $\sigma^2 \zeta \ddot{G}$ is deleted, then the differential equations assume a totally different character within and outside the boundary layer and no longer permit a proper solution. As was pointed out above, this is not surprising as the classical boundary layer theory destroys the elliptic character of the Navier-Stokes equations without examining the existence of a proper solution to the remaining parabolic equations.

4. Characteristic Properties of Vortex Flows

In section 3 the vortex flow problem defined in section 2 has been reduced to an ordinary boundary value problem the integral of which represents an approximate solution of the exact problem. Without complete integration of the remaining boundary value problem the foregoing derivations and the equations obtained display significant phenomena of the vortex flows considered.

The derivations in the sections 2 and 3 led to two important properties of vortex flows, which may be summarized as follows:

Phenomenon 1: The boundary layer along a flat surface normal to an axisymmetric vortex flow may be considered as a logarithmic source and sink pair for the secondary axial flow outside the boundary layer. The radial extension r_0 of the circular axial source is a characteristic flow parameter in addition to the vortex Reynolds number $R = \Gamma/\nu$.

Phenomenon 2: The secondary axial flow deforms the monotonic structure of the friction boundary layer along the flat surface as sketched in figure 1.

The equation (43), which is integrable by the error function leads to the following properties of the vortex flow considered.

Phenomenon 3: Within the accuracy of a first order approximation the dimensionless velocity (U, V, W) is a vector function of the dimensionless variable $\zeta = z/\delta(r)$, such that:

(a) the tangential velocity $V(\zeta)$ is independent of both characteristic flow parameters R and r_0 ,

(b) the radial and axial velocities $u(\zeta)$ and $w(\zeta)$ depend solely upon the Reynolds number R of the vortex flow.

<u>Phenomenon 4</u>: Within a first order approximation the characteristic number σ ($\sigma \approx 2.5$ for a relative accuracy $\epsilon_r = 1\%$) is independent of both characteristic flow parameters R and r_0. Hence the boundary layer thickness

$$z = \delta(r) \approx \sigma \frac{r_0}{2}\left(\frac{r}{r_0}\right)^2\left(1 + 2 \log \frac{r_0}{r}\right) \tag{56}$$

is independent of the Reynolds number R of the vortex flow.

The statement (a) of the phenomenon (3) is an obvious consequence of the integral

$$V(\zeta) = \operatorname{erf}\left(\frac{\sigma\zeta}{\sqrt{2}}\right) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{\sigma\zeta}{\sqrt{2}}} e^{-t^2} dt \ , \tag{57}$$

which solves equation (43) under the corresponding boundary conditions (45) and (46). The equations (47) and (57) lead directly to a constant number σ as was stated in phenomenon 4. From this result one deduces easily the statement (b) of phenomenon 3 by examining the equation (42) together with the corresponding boundary conditions (45) and (46).

In this connection a significant analogy between the present problem and the classical flat plate problem may be pointed out. In both cases the decay of the corresponding boundary layers along flat surfaces is of the same strength, i. e., the disturbances of the primary velocities decay essentially with the same logarithmic order two. This analogy seems to indicate a very general property of boundary layers along solid surfaces (see also [8]). This is physically plausible as the occurrence of a friction boundary layer along a solid surface represents a transport phenomenon.

For an investigation of further phenomena of vortex flows, it is helpful to examine complete solutions of the remaining boundary value problems defined in the previous chapter. Such explicit solutions may be obtained through numerical integrations after introducing the following new variables

$$\eta = \sigma\zeta\ , \qquad g = \sigma G\ , \qquad \text{and } h = 2P\ . \tag{58}$$

This substitution leads to the transformed system of differential equations

$$\dddot{g} + \eta\ddot{g} = \frac{R}{4}(\dot{g}^2 + V^2 - h) \tag{59}$$

$$\ddot{V} + \eta\dot{V} = 0 \tag{60}$$

$$\dot{g}(g - \eta\dot{g}) = \frac{1}{2}\dot{h} \tag{61}$$

which must be integrated under the boundary conditions

$$\eta = 0: \quad g = 0\ , \qquad \dot{g} = 0\ , \qquad V = 0 \tag{62}$$

$$\eta = \infty: \quad g = g_\infty, \qquad V = 1\ , \qquad h = 1\ . \tag{63}$$

A numerical example, which is determined by the Reynolds number R = 10, is displayed in figure 2. The solution has been obtained by the Runge-Kutta method which started the integration with an assumed set of initial data that had to be improved successively until the accuracy conditions (B) and (C) (see section 3) were sufficiently met.

A more efficient iteration procedure, which improves successively the crude solution of the system (51) and (52) by integrating the linear equations

$$\dddot{g}_n + \eta\ddot{g}_n = \frac{R}{4}\left(\dot{g}_{n-1}^2 + V^2 - h_{n-1}\right) \tag{64}$$

$$\ddot{V} + \eta\dot{V} = 0 \tag{65}$$

$$\dot{g}_n(g_n - \eta\dot{g}_n) = \frac{1}{2}\dot{h}_n\ , \tag{66}$$

will be described in another paper in preparation. Numerical results will be presented and discussed for various Reynolds numbers R.

All numerical calculations reveal a very strong dependence of the secondary flow (U, W) upon the Reynolds number R. Hence, it is useful to introduce a "boundary layer of the secondary flow" the thickness of which will depend upon the Reynolds number R. This boundary layer may be determined by the parameter value ζ_s such that at

$$\zeta = \zeta_s : \qquad W(\zeta) = 1 - \epsilon \quad . \tag{67}$$

The limiting line $z = \delta_s(r)$ of the boundary layer of the secondary flow is then defined by

$$z = \delta_s(r) = \zeta_s \delta(r) \ ,$$

where $z = \delta(r)$ is the limiting line of the boundary layer of the entire flow defined by the similar condition (47) (see figure 1 and 2).

After simple numerical calculations one arrives at the following two significant properties of vortex flows.

Phenomenon 5: The boundary layer thickness of the secondary flow decreases as the Reynolds number R increases.

Phenomenon 6: At the solid surface the radial shear stress increases much faster with the Reynolds number R than the tangential shear stress.

The phenomena 1 through 6, which appear to be new in the theory of real flows, are physically plausible. This has been pointed out for the phenomena 1 and 2 in the previous sections. The phenomena 3 through 6 are physically also feasible, as an increasing tangential velocity tends to decrease and to increase the boundary layer thickness at the same time. Indeed, when the particles of the fluid are forced to remain closer to their undisturbed circular motion they are simultaneously forced to remain longer under the influence of the friction forces at the surface. Thus, an increasing tangential velocity can neither increase nor decrease the thickness of the friction layer at the surface. However, these arguments do not apply to the secondary flow. Hence the boundary layer of the secondary flow must behave in the usual manner, i.e., its thickness must decrease as the Reynolds number increases.

The properties of vortex flows over a flat surface found above may be compared with phenomena observed outside the cores of hurricanes. Such a comparison is generally feasible , as the driving core of a hurricane may roughly be replaced by a rotating rod that produces a similar vortex flow. Of course exterior disturbances of the vortex flow other than those caused by the surface of the earth must be neglected. The observations have fully confirmed the source and sink character of the boundary layer at the surface for the secondary axial flow. The logarithmic increase of the axial velocity toward the vortex axis explains the rapidly increasing rainfall toward the core of a hurricane (see [10, p. 130]). In this connection it appears feasible to identify the radius of the rainfall area roughly with the characteristic flow parameter r_0, which measures the radial extent of the logarithmic source around the axis. The radial shear stress at the surface, which is large compared with the tangential shear stress, explains the very strong radial ocean waves produced by hurricanes (see [6, p. 298]). Finally, it may be mentioned that the temperature observed in hurricanes (see [6, p. 319]) exhibits a similar nonmonotonic behavior as the axial velocity found above. This should be expected, because the energy equation, which governs the temperature of the fluid, has the same second order terms as the third Navier-Stokes equation for the axial velocity

REFERENCES

[1] Bers, L., Survey in applied mathematics III, Mathematical aspects of subsonic and transonic gas dynamics, John Wiley & Sons, Inc., New York, 1958, p. 20.

[2] Cooke, J. C., On Pohlhausen's method with application to a swirl problem of Taylor, Journ. Aer. Sci. 19, 1952, 486 - 490.

[3] Goldshtik, M. A., A paradoxical solution of the Navier-Stokes equations, Prikladnaia Matematika i Mekhanika, 24, 1960, No. 4, 610 - 621. (Engl. transl. 913- 929).

[4] Long, R. R., A vortex in an infinite viscous fluid, Journ. Fluid Mech., 11, 1961, 611 - 624.

[5] Moore, F. K., Three-dimensional boundary layer theory, Advances in Applied Mechanics, IV, 1956, 159 - 228.

[6] Riehl, H., Tropical Meteorology, McGraw-Hill Book Co., Inc., 1954

[7] Schlichting, H., Boundary Layer Theory, Fourth Edition, 1960

[8] Schwiderski, E. W. and Lugt, H., Rotating Flows of Kármán-Bödewadt and Stagnation Flows, U. S. Naval Weapons Laboratory Report No. 1836, 1962.

[9] Taylor, G. I., The boundary layer in the converging nozzle of a swirl atomizer, Quart. Journ. Mech. Appl. Math. 3, 1950, 129 - 139.

[10] Handbuch der Physik, Bd. XLVIII, Geophysik II, Springer Verlag 1957.

APPENDIX A

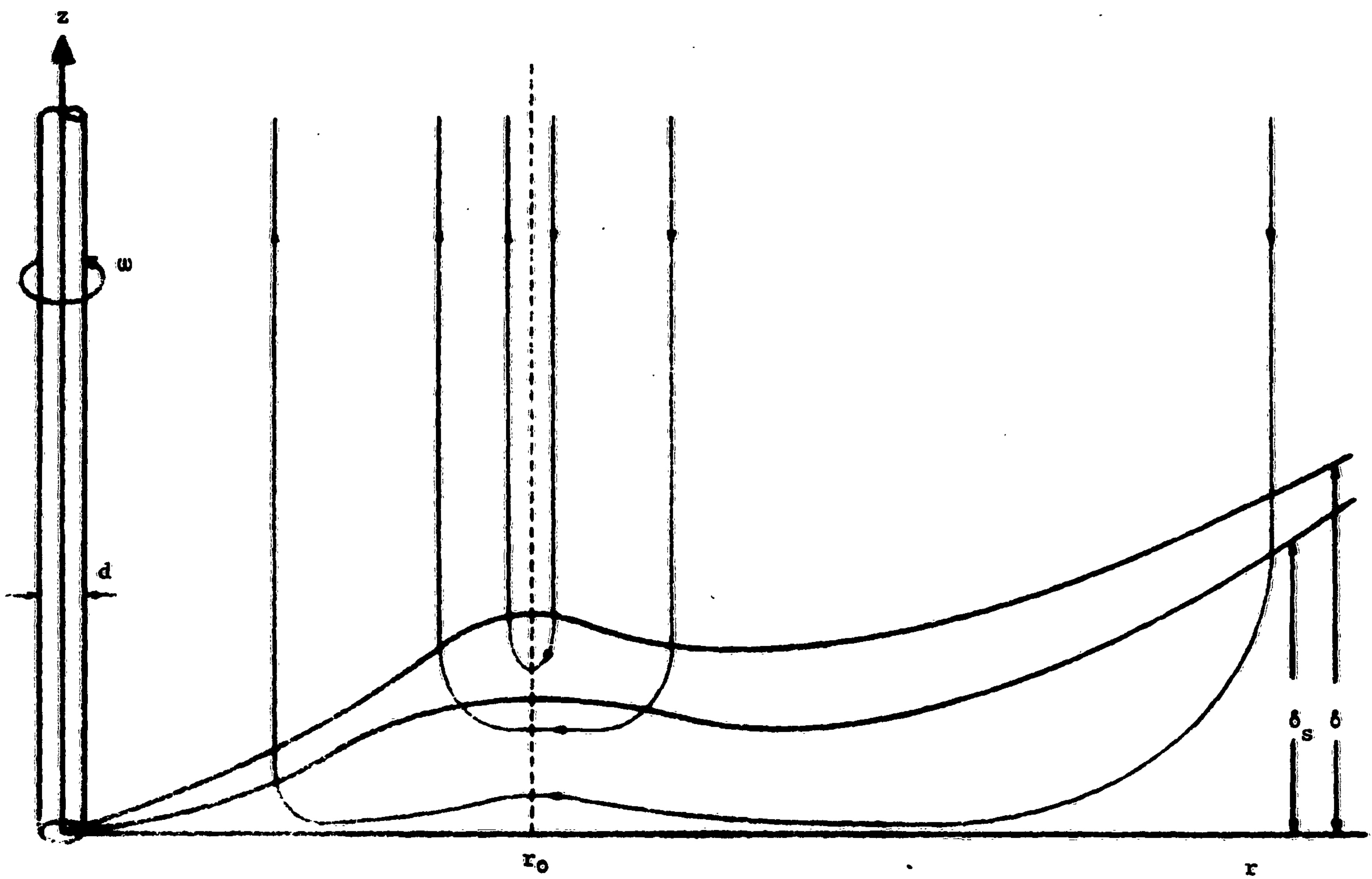

FIGURE 1 : Scheme of the secondary flow produced by a vortex motion normal to a flat surface.

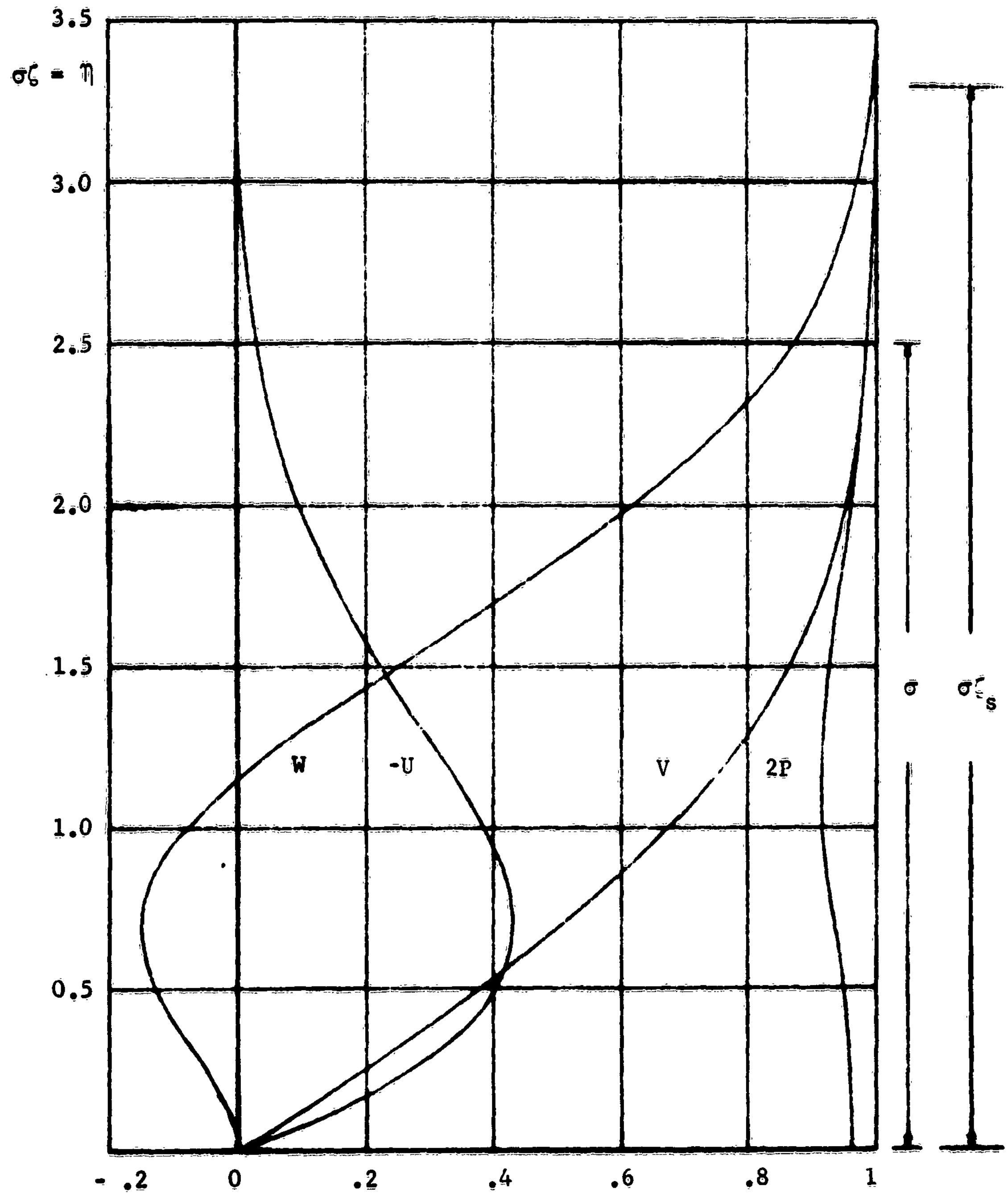

FIGURE 2 : The dimensionless velocity (U, V, W) and the pressure 2P vs. the dimensionless variable η for the Reynolds number 10.

APPENDIX B

DISTRIBUTION

Bureau of Naval Weapons	
DLI-3	1
R-14	1
RREN	1
RRRE	1
RT	1
RM	1
R-12	1
Special Projects Office Department of the Navy Washington 25, D. C.	
SP-20	1
SP-43	2
Commander Armed Services Technical Information Agency Arlington Hall Station Arlington 12, Virginia Attn: TIPDR	10
Commanding General Aberdeen Proving Ground Aberdeen, Maryland Attn: Technical Information Section Development and Proof Services	2
Commander, Operational and Development Force U. S. Atlantic Fleet, U. S. Naval Base Norfolk 11, Virginia	1
Chief of Naval Research Department of the Navy Washington 25, D. C.	
Attn: Code 438	2
Attn: Mathematical Sciences Division	1
Attn: Dr. F. J. Weyl	1
Attn: Mathematics Branch	1
Attn: Fluid Dynamics Branch	1

DISTRIBUTION (Continued)

Director Naval Research Laboratory Washington 25, D. C.	3
Commander Naval Ordnance Laboratory White Oak, Maryland	
Attn: Dr. R. Roberts	1
Attn: Dr. R. E. Wilson	2
Attn: Dr. A. VanTuyl	1
Attn: Technical Library	1
Chief, Bureau of Ships Department of the Navy Washington 25, D. C.	2
Director David Taylor Model Basin Washington 7, D. C.	
Attn: Dr. Daniel Shanks	1
Attn: Dr. John W. Wrench, Jr.	1
Attn: Dr. F. D. Murnaghan	1
Attn: Dr. H. Polachek	1
Attn: Dr. Elizabeth Cuthill	2
Attn: Library	2
Commander U. S. Naval Ordnance Test Station China Lake, California	
Attn: Dr. D. E. Zilmer	1
Attn: Library	2
Superintendent U. S. Naval Postgraduate School Monterey, California	
Attn: Library, Technical Reports Section	1
Director of the Institute of Naval Studies 185 Alewife Brook Parkway Cambridge 38, Massachusetts	1

DISTRIBUTION (Continued)

Commanding General White Sands Proving Ground La Cruces, New Mexico Attn: Flight Determination Laboratory	1
Commander, 3206th Test Group Building 100 Eglin Air Force Base, Florida Attn: Mr. H L. Adams	1
Commander Wright Air Development Center Wright-Patterson Air Force Base Dayton, Ohio Attn: WCRRN-4 Attn: Dr. K. G. Guderley	 1 1
U. S. Atomic Energy Commission Washington, D. C. Attn: Technical Library	1
Los Alamos Scientific Laboratory Los Alamos, New Mexico	2
Superintendent U. S. Naval Academy Annapolis, Maryland Attn: Dept. of Mathematics Attn: Library	 1 1
U. S. Naval Observatory Washington 25, D. C. Attn: Dr. G. M. Clemence	1
U. S. Weather Bureau Washington 25, D. C. Attn: Dr. J. Smagorinsky	1
Commander Naval Ordnance Test Station Pasadena Annex 3202 Foothill Boulevard Pasadena, California	1

DISTRIBUTION (Continued)

Bureau of Naval Weapons Representative P. O. Box 594 Sunnyvale, California Attn: BUWEPSREP-314	1
Army Rocket and Guided Missile Agency U. S. Army Ordnance Missile Command Redstone Arsenal, Alabama Attn: Capt. Robert H. C. Au	1
National Aeronautics and Space Administration 1520 H Street, N. W. Washington 25, D. C.	6
Commander Ballistic Missile Division ARDC P. O. Box 262 Inglewood, California Attn: Col. Ebelke (WDTVR)	2
Commanding General Army Ballistic Missile Agency Redstone Arsenal Huntsville, Alabama	
Attn: Mr. H. G. Paul (ORDAB - DS)	1
Attn: Dr. W. Lucas (ORDAB - DS)	1
Attn: Mr. Dale L. Burrows (ORDAB - DSDA)	1
Attn: Technical Library	1
National Science Foundation 1520 H Street, N. W. Washington, D. C.	
Attn: Engineering Sciences Division	1
Attn: Mathematical Sciences Division	1
Director National Bureau of Standards Washington 25, D. C.	
Attn: Fluid Mechanics Division	1
Attn: Dr. G. B. Schubauer	1
Attn: Dr. G. H. Keulegan	1
Attn: Mr. J H. Wegstein, Computation Laboratory	1
Attn: Dr. Phillip Davis	1
Attn: Dr. E. W. Cannon, Applied Mathematics Division	1
Attn: Dr. S. N. Alexander, Data Processing Division	1
Attn: Dr. R. J. Arms	1
Attn: Technical Library	2

DISTRIBUTION (Continued)

Office of Technical Services Department of Commerce Washington 25, D. C.	1
Guggenheim Aeronautical Laboratory California Institute of Technology Pasadena 4, California	
Attn: Prof. Lester Lees	1
Attn: Prof. J. D. Cole	1
California Institute of Technology Pasadena 4, California	
Attn: Prof. John Todd	1
University of California Berkeley 4, California	
Attn: Department of Mathematics	1
Institute of Mathematical Sciences New York University 25 Waverly Place New York 3, New York	
Attn: Prof. K. O. Friedrichs	1
Attn: Prof. J. J. Stoker	1
Attn: Dr. Max Goldstein	1
Attn: Prof. B. Haurwitz	1
Attn: AEC Computing Facility	1
Harvard University Cambridge 38, Massachusetts	
Attn: Prof. G. Birkhoff	1
Attn: Division of Applied Sciences	1
Attn: Computation Laboratory	1
Attn: Prof. G. F. Carrier	1
Attn: Prof. H. M. Stommel	1
University of Maryland College Park, Maryland	
Attn: Department of Mathematics	1
Attn: Institute for Fluid Dynamics and Applied Mathematics	1

DISTRIBUTION (Continued)

Massachusetts Institute of Technology Cambridge, Mass.	
Attn: Prof. C. C. Lin	1
Attn: Prof. A. H. Shapiro	1
Attn: Prof. J. G. Charney	1
The Johns Hopkins University Baltimore 18, Maryland	
Attn: Prof. R. R. Long	1
Applied Physics Laboratory Johns Hopkins University Silver Spring, Maryland	
Attn: Librarian	2
University of California San Diego, California	
Attn: Prof. W. H. Munk	1
University of Chicago Chicago, Illinois	
Attn: Prof. H. Riehl	1
AVCO Manufacturing Corporation Research and Advanced Development Division 201 Lowell Street Wilmington, Massachusetts	
Attn: J. P. Wamser Via: INSMAT, Boston, Massachusetts	2
General Electric Co. Missile and Space Vehicle Department 3198 Chestnut Street Philadelphia 4, Pennsylvania	
Attn: Mr. R. J. Kirby Via: INSMAT, Philadelphia, Upper Darby, Pa.	2
Lewis Flight Propulsion Laboratory National Aeronautics and Space Administration Cleveland, Ohio	
Attn: F. K. Moore	1
Attn: S. H. Maslen	1
Attn: W. E. Moeckel	1

DISTRIBUTION (Continued)

Lockheed Aircraft Corporation Palo Alto, California Attn: Dr. W. C. Griffith	1
Prof. M. Van Dyke Department of Aeronautical Engineering Stanford University Stanford,California	1
Dr. Bernd Zondek Computer Usage Company, Inc. 18 East 41st Street New York 17, N. Y.	1
Prof. J. Siekmann University of Florida Gainesville, Florida	1
Dr. L. M. Mack Jet Propulsion Laboratory California Institute of Technology Pasadena, California	1
Prof. W. D. Hayes Department of Aeronautical Engineering Princeton University Princeton, N. J.	1
Prof. R. F. Probstein Division of Engineering Brown University Providence 12, R. I.	1
Dr. C. C. Bramble 145 Monticello Annapolis, Maryland	1
Dr. M. J. Lighthill Royal Aircraft Establishment Farnborough, Hampshire, England Via: BUWEPS (DSC-3)	1
Prof. G. K. Batchelor University of Cambridge Cambridge, England Via: BUWEPS (DSC-3)	1

DISTRIBUTION (Continued)

Prof. C. R. Illingworth University of Manchester Manchester, England Via: BUWEPS (DSC-3)	1
Prof. Dr. H. Goertler University of Freiburg Freiburg, Germany Via: BUWEPS (DSC-3)	1
Prof. Dr. H. Schlichting Technische Hochschule Braunschweig, Germany Via: BUWEPS (DSC-3)	1
Prof. Dr. W. Tollmien Max Planck Institute Goettingen, Germany Via: BUWEPS (DSC-3)	1
Prof. Dr. A. Naumann Technische Hochschule Aachen Aachen, Germany Via: BUWEPS (DSC-3)	1
Prof Dr. J. Ackeret Eidgenoessische Technische Hochschule Zurich, Switzerland Via: BUWEPS (DSC-3)	1
Local:	
D	1
K	1
K-1	1
KB	1
KC	1
KBX	1
KBX-1	1
KBP	1
KCM	2
KBX-3	25
KCX-1	50
ACL	5
File	1

LIBRARY CATALOGING INPUT
PRNC-NWL-5 /15 (7-62)

BIBLIOGRAPHIC INFORMATION

DESCRIPTOR	CODE	DESCRIPTOR	CODE
SOURCE NWL report	NPGA	SECURITY CLASSIFICATION AND CODE COUNT UNCLASSIFIED	U015
REPORT NUMBER 1835	1835	CIRCULATION LIMITATION	
REPORT DATE September 1962	0962	CIRCULATION LIMITATION OR BIBLIOGRAPHIC	
		BIBLIOGRAPHIC *(Suppl., Vol., etc.)*	

SUBJECT ANALYSIS OF REPORT

DESCRIPTOR	CODE	DESCRIPTOR	CODE	DESCRIPTOR	CODE
Boundary layer	BOUL	Core	CORE		
Flat	FLAT				
Surfaces	SURA				
Vortex	VORT				
Flow	FLOW				
Motion	MOTI				
Prandtl	PRAN				
Differential	DIFE				
Equations	EQUA				
Navier-Stokes	NAVE				
Fluids	FLUI				
Transformation	TRAO				
Axial	AXIA				
Symmetrical	SYMM				

Naval Weapons Laboratory, Dahlgren, Virginia.
(NWL Report No. 1835)
BOUNDARY LAYER ALONG A FLAT SURFACE NORMAL TO A VORTEX FLOW, by E. W. Schwiderski and Hans Lugt. December 20, 1962. 21 p.
UNCLASSIFIED

The limiting line of a boundary layer leads to an extension of Prandtl's boundary layer theory. The solution is defined as the integral of three ordinary differential equations which result from a similarity transformation of the Navier-Stokes equations. Without numerical integration the derivations display significant features of the flow. A complete numerical example is discussed.

1. Boundary layer - Theory
2. Meteorology
3. Vortex reactor
4. Partial differential equations
I. Schwiderski, E. W.
II. Lugt, H.

Task: R360FR103/2101/R01101001

UNCLASSIFIED

Naval Weapons Laboratory, Dahlgren, Virginia.
(NWL Report No. 1835)
BOUNDARY LAYER ALONG A FLAT SURFACE NORMAL TO A VORTEX FLOW, by E. W. Schwiderski and Hans Lugt. December 20, 1962. 21 p.
UNCLASSIFIED

The limiting line of a boundary layer leads to an extension of Prandtl's boundary layer theory. The solution is defined as the integral of three ordinary differential equations which result from a similarity transformation of the Navier-Stokes equations. Without numerical integration the derivations display significant features of the flow. A complete numerical example is discussed.

1. Boundary layer - Theory
2. Meteorology
3. Vortex reactor
4. Partial differential equations
I. Schwiderski, E. W.
II. Lugt, H.

Task: R360FR103/2101/R01101001

UNCLASSIFIED

Naval Weapons Laboratory, Dahlgren, Virginia.
(NWL Report No. 1835)
BOUNDARY LAYER ALONG A FLAT SURFACE NORMAL TO A VORTEX FLOW, by E. W. Schwiderski and Hans Lugt. December 20, 1962. 21 p.
UNCLASSIFIED

The limiting line of a boundary layer leads to an extension of Prandtl's boundary layer theory. The solution is defined as the integral of three ordinary differential equations which result from a similarity transformation of the Navier-Stokes equations. Without numerical integration the derivations display significant features of the flow. A complete numerical example is discussed.

1. Boundary layer - Theory
2. Meteorology
3. Vortex reactor
4. Partial differential equations
I. Schwiderski, E. W.
II. Lugt, H.

Task: R360FR103/2101/R01101001

UNCLASSIFIED

Naval Weapons Laboratory, Dahlgren, Virginia.
(NWL Report No. 1835)
BOUNDARY LAYER ALONG A FLAT SURFACE NORMAL TO A VORTEX FLOW, by E. W. Schwiderski and Hans Lugt. December 20, 1962. 21 p.
UNCLASSIFIED

The limiting line of a boundary layer leads to an extention of Prandtl's boundary layer theory. The solution is defined as the integral of three ordinary differential equations which result from a similarity transformation of the Navier-Stokes equations. Without numerical integration the derivations display significant features of the flow. A complete numerical example is discussed.

1. Boundary layer - Theory
2. Meteorology
3. Vortex reactor
4. Partial differential equations
I. Schwiderski, E. W.
II. Lugt, H.

Task: R360FR103/2101/R01101001

UNCLASSIFIED

BOUNDARY LAYER ALONG A FLAT SURFACE NORMAL TO A VORTEX FLOW

NWL REPORT NO. 1835