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RESEARCH IN THE AREA OF MATHEMATICAL ANALYSIS

S. Agmon, A. Dvoretzky, A. Robinson
and group

The Hebrew University,
Jerusalem, Israel

TECHNICAL (FINAL) REPORT

CONTRACT No. AF 61(052)-187

October 1962

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RESEARCH IN THE AREA OF

MATHEMATICAL ANALYSIS

Part I

1 November 1958 - 30 September 1959

Abstract

The research in the area of mathematical analysis (Contract AF 61(052)-187) was conducted at the Department of Mathematics of the Hebrew University. Participating in the Project were: Professor Shmuel Agmon, Professor Aryeh Dvoretzky, Professor Abraham Robinson, Professor Elisha Netanyahu, Dr. Amnon Jakimowski and Dr. Michael Mashler. The research covered various topics in analysis. As a result of these investigations the following eight Technical (scientific) Notes were issued:

Analytic functions of the classes L^2 and l^2 and their kernel functions, by M. Mashler, Tech. Note No. 1.

On the iteration product of summability methods; new classes of transformations and their properties (Parts I & II), Tech. Notes No. 2 & 8.

On divergence of random power series, by A. Dvoretzky and P. Erdös, Tech. Note No. 3.

Local differential algebra, by A. Robinson, Tech. Note No. 4.

On some compositions of Hadamard type in classes of analytic functions, by E. Netanyahu and C. Loewner, Tech. Note No. 5.

On the concept of a differentially closed field, by A. Robinson, Tech. Note No. 6.

The L_p approach to the Dirichlet problem (Part I), by S. Agmon, Tech. Note No. 7.

Professor S. Agmon in his studies of partial differential equations introduced a new L_p approach to general boundary value problems for general elliptic equations. In particular the Dirichlet problem was treated by this method. The method is based on strong regularity theorems (in L_p) for very weak solutions of the Dirichlet problem. The new method is described in detail in Tech. Note No. 7. The method also yields interior regularity theorems for weak solutions of overdetermined elliptic systems and a priori L_p estimates for such solutions.

Professor A. Dvoretzky carried out investigations in two main directions. a) Investigations on rearrangement of series in general vector spaces and, in particular, on rearrangement of power series. A paper on the last topic, written jointly with P. Erdős, was issued as Tech. Note No. 3. b) Investigations on relations between various densities of sequences.

The investigations of Prof. A. Robinson on differential algebra were divided into two parts. One part of the work is concerned with the adaptation of the established methods of differential algebra to initial value problems for systems of algebraic differential equations. The results here were issued as Technical Note No. 4. The second part of the work is concerned with the introduction of the concept of a differentially closed field. The results of this research were published as Technical Note No. 6.

Professor E. Netanyahu investigated a number of problems on univalent functions and in particular a number of conjectures concerning a Hadamard type composition of univalent functions. In a joint paper with C. Loewner, issued as Technical Note No. 5, a conjecture of Mandelbrojt and Schiffer was disproved. (Independently and at the same time the same conjecture was disproved by Epstein and Schoenberg).

Dr. A. Jakimowski continued with his investigations on different summability methods. His investigations deal with the following main problem: to find pairs of linear transformations T_1, T_2, such that T_1 summability of a sequence implies $T_1 \cdot T_2$ summability of the same sequence. A very general theorem yielding such pairs of transformations was obtained. The detailed results of these investigations were published as Tech. Note No.2, and Tech. Note No. 8.

Dr. M. Maschler continued his investigations on the Bergman kernel for a plan domain. Some interesting results on the kernel function for 'essentially simply connected domains' were established. The details of this investigations were published in Technical Note No. 1.

Main Report

The investigations of S. Agmon

Subject: Elliptic partial differential equations and related boundary value problems.

The work done during the year dealt mainly with a new L_p approach to the Dirichlet problem and to related regularity problems for higher order elliptic equations. Although this approach is not as simple as the well known Hilbert space approach as developed by Vishik, Gårding, Browder, Nirenberg, and others, it has the advantage of a greater generality. For instance, the method enables to treat the Dirichlet problem for elliptic equations which are not strongly elliptic and without any uniqueness assumptions. The method is not restricted to the Dirichlet problem and could be used to solve a very general class of boundary value problems.

The L_p approach to the Dirichlet problem is based on a L_p regularity theory for very weak solutions of the Dirichlet problem. The theory uses some of the ideas used in a well known paper of Nirenberg with the following essential modifications: instead of using Gårding's inequality one uses the explicit solution of the Dirichlet problem for elliptic operators with constant coefficients in a half-space, and L_p estimates for such solutions derived in a joint paper by Agmon - Donglis - Nirenberg to be published shortly. The details of the regularity theory were published in the report: <u>The L_p approach to the Dirichlet problem (Part I)</u>, Tech. Note No. 7, August 1959. We shall content ourselves here by describing a typical regularity result.

Let G be sufficiently smooth bounded domain in E_n, and let A be an elliptic operator of order 2m in $\bar{G}$ with sufficiently smooth coefficients (the operator also satisfies a 'roots condition' for n=2). Let u be a function belonging to $L_q(G)$ for some $q>1$ which is a <u>weak</u> solution of the Dirichlet problem:

$$(*) \qquad Au = f \quad \text{in} \quad G ,$$

$$D^{\alpha} u = o \quad \text{on the boundary}, \quad |\alpha| \leqq m-1,$$

in the sense that

$$\int_G u \, \overline{A^* v} dx = \int_G f \bar{v} dx,$$

for all functions $v \in C^{2m}(\overline{G})$ with a zero Dirichlet data (A^* being the adjoint operator).

We have the following

THEOREM: If $f \in L_p(G)$, then all distribution derivatives of u of order $\leq 2m$ are functions belonging to $L_p(G)$.

Using the regularity theory it is shown that the null space N(A) of solutions of the homogeneous Dirichlet problem corresponding to (*) is finite dimensional: that the dimension of N(A) equals that of $N(A^*)$ and that the following theorem holds (also for non-strongly elliptic operators):

THEOREM: The Dirichlet problem (*) admits a solution u if and only if f is orthogonal to the null space $N(A^*)$.

The investigations of A. Dvoretzky

Subject: Rearrangement of series; comparison of densities of sequences.

The main work completed is a paper written jointly with P. Erdös etitled: On divergence of random power series, Technical (Scientific) Note No. 3, issued February 1959.

Let $\gamma_n(t)$, $(n = 0,1,2,\dots)$ be the Rademacher functions, i.e. $\gamma_n(t) = (-1)^j$ for $j/2^n \leq t < (j+1)/2^n$

$(j = 0,1,\ldots,2^n - 1)$. Given any sequence of complex numbers $(a_n) = (a_0, a_1, \ldots, a_n, \ldots)$ we consider the family of power series

(1) $\quad P(z;t) = \sum_{n=0}^{\infty} \varphi_n(t) a_n z^n, \qquad (0 \leq t < 1)$, i.e. the

family of power series obtained from a given one by applying arbitrary changes of sign to the coefficients.

It is well known that if

$$(2) \qquad \sum_{n=0}^{\infty} |a_n|^2 = \infty$$

then almost all series (1) diverge almost everywhere on the unit circle $|z| = 1$, while if (2) does not hold then almost all series (1) converge almost everywhere on $|z| = 1$. Here almost all refers to the set of t (in the usual Lebesgue sense) while almost everywhere refers to the set of z on $|z| = 1$ (again in the usual sense).

It has been remarked recently [A. Dvoretzky, Proc. Nat. Acad. Sci. U.S.A. 42 (1956) 199-203] that a strengthening of (2) implies the divergence of almost all series (1) everywhere, on $|z| = 1$. In the present work this phenomenon of everywhere divergence is studied further. The main theorem proved asserts that if $|a_n| \geq c_n$ where c_n $(n = 0,1,2,\ldots)$ is a monotone sequence of positive numbers satisfying

$$(3) \qquad \limsup_{n=\infty} \left(\sum_{j=0}^{n} c_j^2\right) / (-\log c_n) > 0$$

then almost all series (1), diverge everywhere on $|z| = 1$.

An important special case for which (3) holds is $c_n = c/\sqrt{n}$. In the above assertions the words <u>have unbounded partial sums</u> may be substituted instead of the word <u>diverge.</u>

It is not known whether condition (3) is best possible, however, it can be shown that (3) cannot be replaced by (2). Indeed, there exsists a monotone sequence satisfying (2) such that almost all series of the family (1) have on every arc of $|z|=1$ a set of points of convergence whose power is that of the continuum.

Other lines of investigations which were pursued concern rearrangement of series in topological vector spaces and comparison of different densities of sequences of numbers. The problems involved were described in the Quarterly Technical Status Reports of Contract AF 61 (052)-187. These investigations, however, are not yet complete.

<u>The investigations of A. Robinson</u>

Subject: <u>Initial value problems and differential algebra.</u>

The work done during the year fall naturally into two parts. The results were published in the following two reports: 1) <u>Local differential algebra,</u> Tech. Note No. 4, May 1959. 2) <u>On the concept of a differentially closed field,</u> Tech. Note No. 6, July 1959.

1. One part of the work is concerned with the adaptation of the established methods of differential algebra to <u>initial value problems</u> for systems of algebraic differential equations. The value of a function (i.e. an element of a differential ring) at a given point is represented by a homomorphism of the ring into an ordinary ring or field. In place of the ideals of differential polyomials, which determine differential varieties in ordinary differential algebra, we introduce double ideals representing conditions on the functions (elements of the differential ring) on one hand and on their initial values on the other. A theory of these double ideals and of their varieties can be developed, which is to some extent parallel to the theories of ordinary polynomial ideals and of differential polynominal ideals. Among other things, one obtains a consistency conditions for a system of algebraic differential equations with given initial values.

Systems which satisfy certain natural additional conditions have also been considered. In particualr if the fact that an element of the differential field as well as all of its derivatives are mapped on 0 implies that the element is itself equal to 0 then the system is said to be regular. In that case the standard initial value problem for a single algebraic differential equation of arbitrary degree and order can be shown to possess a solution.

Further developments can be expected in the following directions: (i) a corresponding theory for partial differential equations, (ii) a theory for two-point boundary value problems, (iii) the inclusion of singularities.

2. The second part of the research is concerned with the introduction of the concept of a differential closed field. A number of conditions, either of a purely algebraic or of a metamathematical character, are laid down, which are satisfied by the algebraically closed fields in the field theory on one hand, and by the real-closed fields in the theory of ordered fields on the other. It is shown by means of Seidenberger's elimination theory for differential algebra that for characteristic 0, there exists just one (arithmetical) class of differential fields which satisfies these conditions, and these fields are said to be differentially closed. Every differential field **M** (of characteristic 0) can be embedded in a differentially closed field M', and if a finite set of equations and inequalities possesses a solution in some extension of **M** then it possesses a solution also in M'. However, it does not seem to be possible to define (uniquely) the differential closure of any given differential field. The universal extensions introduced by Kolchin are special case of the differentially closed field described here. In addition to its intristic interest, the concept enables to prove a theorem on the

specialization of parameters in differential algebra, and to establish a constructive version of Ritt's Nullstellensatz. The later result is not new having been given previously by Seidenberg.

It is possible that the methods used in this connection may throw some light on Ritt's decomposition problem. There also remains the question of formulating a corresponding concept for fields of positive characteristic.

The investigations of E. Netanyahu

Subject: Univalent functions.

Let

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

and

$$g(z) = \sum_{n=1}^{\infty} b_n z^n$$

be regular schlicht functions in the unit circle.

We form the composition

$$h(z) = \sum_{n=1}^{\infty} \frac{a_n b_n}{n} z^n \quad .$$

The following conjecture of Mandelbrojt and Schiffer was mentioned in [1] .

Conjecture I: h(z) is schlicht in the unit circle.

A weaker conjecture by the author was

Conjecture II: $h'(z) \neq 0$ in $|z| < 1$.

The interest in these conjectures is that if even conjecture II was correct Bieberbach's conjecture on the coefficients of schlicht functions would have followed.

However, these conjectures were disproved by Prof. C. Loewner and the author (A different disproof was given by B. Epstein and I.J. Schoenberg), and published as a report entitled: On some compositions of Hadamard type in classes of analytic functions, Tech. Note No. 5, June 1959.

Still the author believes that II is correct for schlicht functions with real coefficients, though he could prove it only for very special cases.

Another conjecture belonging to the same domain is the following:

Conjecture III: Let $f(z)$ and $g(z)$ be convex functions in $|z| < 1$ then $\sum_{n=1}^{\infty} a_n b_n z^n$ is also a convex function there.

This was conjectured by Polya and Schoenberg [1] , and independently by the author (see report to U.S.A.F. from the year 1958).

Some progress has been made. Indeed, the conjecture can be proved for convex functions in the "vicinity" of $\frac{z}{1-z}$ $(\frac{z}{1-z})$

will serve as a unit in the semi-group if the conjecture is correct).

Let $\quad f(z) \equiv \frac{z}{1-z}$.

Loewner and Netanyahu determine the mappings which are convex and infinitesimally close to the mapping $\frac{z}{1-z}$.

A mapping by g(z) is convex if

$$\mathrm{Re}\left\{1 + z\frac{g''(z)}{g'(z)}\right\} \geqslant 0$$

$$\left| z\frac{g''(z)}{g'(z)} \Big/ 2 + z\frac{g''(z)}{g'(z)}\right| \leqslant 1$$

for $|z| < 1$.

Denoting by $\varphi(z)$ a variation of f(z), then since

$$\left|\frac{z\frac{f''(z)}{f'(z)}}{2 + z\frac{f''(z)}{f'(z)}}\right| \equiv z \quad \text{for } f(z) \equiv \frac{z}{1-z} \text{ we get}$$

that $\varphi(z)$ must satisfy.

$$\mathrm{Re}\left\{\frac{2}{\left(2 + z\frac{f''(z)}{f'(z)}\right)^2}\left[\frac{\varphi''}{f'} - \frac{f''}{f'^2}\varphi'\right]\right\} \leqslant 0$$

for $|z| \leqslant 1$.

From this the author concludes that if $f+\varepsilon\varphi(z)$ and $f+\varepsilon\psi(z)$, $\varepsilon > 0$ and sufficiently small, are two convex functions then their composition is convex as well.

This result was proved in a different way by Polya and Schoenberg [1].

Similarly it follows that the composition of the convex functions $\frac{z}{1-\eta^2} + \varepsilon\, \Upsilon(z)$ and $\frac{z}{1-\delta^2} + \varepsilon\, \Psi(z)$, where $|\delta| = |\eta| = 1$ and $\varepsilon > 0$ sufficiently small, is again a convex function in $|z| < 1$.

References

[1] G. Polys and I.J. Schoenberg, Remarks on De La Vallee Poussin means and convex conformal maps of the circle, Pac. Journal Vol. 8, 1958, pp. 295-334.

The investigations of A. Jakimowski

Subject: Tauberian theorems and summability.

The work done during the last year is divided in a natural way into two parts. The full results were published in the following report: On the iteration product of summability methods; new classes of transformations and their properties, Parts I & II, Tech. Notes No. 2 & 8 (respectively).

The first part deals with the problem of finding pairs of linear transformations T_1, T_2 such that (always, or under suitable conditions) T_1 summability of a sequence implies the $T_1 \cdot T_2$

summability of the same sequence. All the results were unified in a heuristic theorem, which will be stated below, in the sense That if to each particular pair of transformations T_1, T_2 satisfying the suppositions of the heuristic theorem we add some additional suppositions we obtain a rigorous theorem.

A HEURISTIC THEOREM: T_1 summability implies $T_1 T_2$ summability if the following three conditions are satisfied:

1) $T_2(N\ ;\ S_m) = \int_a^b T_y(N\ ;\ S_m)\,d\alpha(y).$

2) $(T_1 \circ T_y)(N\ ;\ S_m) = T_1(\xi_{x,y}\ ;\ S_m)$ where $\lim_{x\uparrow\infty} \xi_{x,y} = +\infty$ for each y, $a \leq y \leq b$.

3) The transformation defined $t(x) = \int_a^b T_1(\xi_{x,y}\ ;\ S_m)\,d\alpha(y)$ is regular for $x \uparrow \infty$.

Given T_1 and $\xi_{x,y}$ or T_2 and $\xi_{x,y}$ we can, in many cases, obtain the other missing transformation. Examples of pairs of transformations obtained in this way are given in Tech. Note No. 8.

Other methods were devised for finding pairs of linear transformations T_1, T_2 such that T_1 summability would imply $T_1 \cdot T_2$ summability. These methods uses known as Abelian or Tauberian theorems. The following is an example.

It is known that if $\alpha > 0$ then

$$\lim_{x\uparrow 1}(1-x)^{\alpha+1}\sum_{n=0}^{\infty}\binom{n+\alpha}{n}s_n x^n = 0 \quad \left(A^{(\alpha)} \text{ summability}\right)$$

implies

$$\lim_{x\uparrow 1}(1-x)\sum_{n=0}^{\infty} s_n x^n = s \quad \text{(Abel's summability).}$$

Now, it is easily established that

$$(1-x)\sum_{n=0}^{\infty} s_n x^n = (1-x)^{\alpha+1}(1-x)^{-\alpha}\sum_{n=0}^{\infty} s_n x^n$$

$$= (1-x)^{\alpha+1}\sum_{n=0}^{\infty}\binom{n+\alpha}{n}\sigma_n^{(\alpha)} x^n,$$

where $\sigma_n^{(\alpha)}$ is the (C,α) transform of $\{s_n\}$. Thus the above Abelian theorem might be stated in the form:

THEOREM: <u>If $\{s_n\}$ is summable $A^{(\alpha)}$ $(\alpha > 0)$ to s then the (C,α) transform of $\{s_n\}$ is also summable $A^{(\alpha)}$ to s.</u>

In the second part of the research six classes of transformations were defined and investigated. Two classes are generalized sequence-to-sequence Hausdorff transformations. Two classes of transformations might be looked upon as the sequence-to-function analogues of the Hausdorff and quasi-Hausdorff transformations. The last two classes of transformations might be looked upon as sequence to function analogues of the generalized Hausdorff and quasi-Hausdorff transformations. It was shown that the problem of regularity of all these transformations is closely connected with the Hausdorff moment problem.

The investigations of M. Maschler

Subject: The Bergman kernel.

M. Maschler was engaged in finding properties of the Bergman kernel function $K_D(Z,\xi)$ of a planar domain D. The results obtained were issued as a report entitled: Analytic functions of the classes L^2 and l^2 and their kernel frunctions, Tech. Note No. 1 (Dec. 1958).

For the sake of convenience call a domain 'essentially simply-connected' if its boundary contains at most one component which is not completely point-like in the sense of H. Grotzsch (i.e., under no univalent conformal mapping can the domain be mapped onto a domain whose boundary contains two or more proper continua). Maschler introduces the domain function

$$(1)\quad J_D(Z,\xi) = \frac{1}{[K_D(Z,\xi)]^3} \begin{vmatrix} K_D(Z,\xi) & \dfrac{K_D(Z,\xi)}{\partial Z} \\ \dfrac{\partial K_D(Z,\xi)}{\partial \xi} & \dfrac{\partial^2 K_D(Z,\ )}{\partial Z \partial \xi} \end{vmatrix} ,$$

which, if $\zeta = Z$, reduces to the Riemannian curvature based on the Bergman conformally invariant metric:

$$(2) \qquad ds^2 = K_D(Z,\bar{Z}) \quad dZ^2 .$$

$J_D(Z,\zeta)$ is a conformally invariant domain-function which becomes constant if D is a simplyconnected domain. Moreover, if D is <u>not</u> essentially simply-connected, then theequation

$$(3) \qquad J_D(Z,\bar{t}) = \text{constant, } t \text{ fixed, } \quad t \in D ,$$

is never satisfied.

Thus, if (3) is satisfied, then D is essentially simply-connected. It is conjectured that in this case D is even a simply-connected domain punctured by a null set, but the efforts to prove it in the case of an infinitely-connected domain remained fruitless.

The above results are based on the fact, interesting in itself, that if

$$(4) \qquad \oint_C K_D(Z,\bar{t})dZ \doteq 0$$

for a fixed point t in D, and for each closed path C in D, then D is essentially simply-connected. This is a consequence of a theorem which states that for a non essentially simply-connected domain there does not exist a point t which is critical for all the harmonic measures of the boundary components. This theorem was proved by M. Maschler in the case of finitely-connected domain and by S. Agmon in the general case.

Various results follow from the above theorem which were described in Tech. Note No. 1. Other investigations concerning the properties of the kernel function of multipy-connected domains are being conducted. An attempt is being made to map the domains onto canonical domains which are circles "with identified points", the kernels of which have special properties.

RESEARCH IN THE AREA OF MATHEMATICAL ANALYSIS

Part II

1 October 1959 - 30 September 1960

Abstract

The group of mathematicians working on Contract AF 61(052) - 187 during its second year consisted of the following: Prof. S.Agmon, Prof. A.Dvoretzky, Prof. A.Robinson, Dr. A.Jakimowski and Mr. S.Halfin. In this period the following additional Technical (Scientific) Notes were issued:

<u>The angular distribution of eigenvalues of non self-adjoint elliptic boundary value problems of higher order,</u> By S.Agmon, Technical Note No. 8.

<u>Local differential algebra - the analytical case,</u> by A.Robinson and S.Halfin, Technical Note No. 9.

<u>The product of summability methods: Part 3.</u> by A.Jakimowski, Technical Note No. 10.

<u>Some results on convex bodies and Banach spaces,</u> by A.Dvoretzky, Technical Note No. 11.

Also additional investigations of Robinson and Halfin are near completion and might be issued shortly as a report.

Professor S. Agmon investigated the problem of distribution in the complex plane of eigenvalues of general non self-adjoint elliptic boundary value problems. Very general results were obtained on the angular distribution of eigenvalues. The results apply in particular to higher order "Oblique derivative" boundary value problems. Also, the results furnish a general sufficient condition in order that the spectrum of a normal elliptic boundary value problem be discrete. A Preliminary report entitled: "The angular distribution of eigenvalues of non self-adjoint elliptic boundary value problems of higher order" was issued as Technical Note No. 8.

Prof. A. Dvoretzky investigated the consequences of the following:

Theorem: For every positive integer k and every $\varepsilon > 0$ there exists and integer $N = N(k,\varepsilon)$ such that if C is any symetric body in E^N there exists a k-dimensional subspace E^k such that $E^k \cap C$ contains a sphere and is contained in a concentric sphere whose radius is $1+\varepsilon$ times the radius of the inner sphere.

The details of proof of the theorem and various applications were given in Technical Note No. 11 of this Contract.

The investigations of Prof. A. Robinson (jointly with Mr. S. Halfin) during the second year of the Contract dealt with the following two topics.

(i) Local differential algebra - the analytic case.

(ii) Local partial differential algebra.

The first topic deals with the question when solutions given by the general theory are analytic. Sufficient conditions for this to hold were given. The results were published in Technical Note No. 9.

The second mentioned topic deals with the question of developing a theory of <u>partial</u> <u>differential</u> algebra. The beginning of such a theory was actually developed. It is expected the results be completed and a report be published.

Dr. Jakimowski continued with his investigations on properties of various sequence - to - function transformations which are analogous to sequence transformations. Among the results obtained were small - o tauberians. Technical Note No. 10, which was issued gives the details of these investigations.

Main Report

The investigations of S. Agmon

Subject: **Elliptic partial differential equations and related boundary value problems.**

The work done during the second year of the project dealt mainly with the angular distribution in the complex plane of eigenvalues of non self-adjoint elliptic boundary value problems of higher order. Among the results obtained are the following:

(i) Characterization of a general class of elliptic boundary value problems whose spectrum consists of a discrete set of eigenvalues.

(ii) Characterization of a subclass of non self-adjoint elliptic boundary value problems whose spectrum clusters along a sigle ray in the complex plane. (In particular, 'oblique derivative' boundary value problems belong to this class).

All these results are connected with growth properties of the resolvent operator along certain rays in the complex plane, and as a matter of fact follow from a single theorem dealing with 'directions of regular growth' of the resolvent. Details of the above and other results were given in the report: <u>The angular distribution of eigenvalues of non self-adjoint elliptic boundary value problems of higher order</u>, Technical Note No. 8, June 1960. Here we shall mention only two typical results.

Consider an eigenvalue problem:

$$(*) \qquad Au - \lambda u = 0 \quad \text{in } G\,,$$
$$B_j u = 0 \quad \text{on } \partial G\,, \quad j = 1,\dots,m\,,$$

where A is an elliptic operator of order 2m in a bounded domain G, where $\{B_j\}$ is a normal system of boundary (differential) operators of distinct orders $\leq$ 2m-1. G and the differential operators are sufficiently smooth. Denote by A', B'_j the principal parts of the operators. We have the following

<u>Theorem:</u> In order that the spectrum of the eigenvalue problem $(*)$ will be discrete it is sufficient that for some real number θ

the following conditions would hold:

(i) $$(-1)^m \frac{A'(x,\xi)}{A'(x,\xi)} \neq e^{i\theta}$$

for all real vectors $\xi \neq 0$ and all $x \in \bar{G}$.

(ii) At any point x of ∂G let ν be the normal vector and let $\xi \neq 0$ be any real vector parallel to the boundary. Denote by $t_k^+(\xi;\lambda)$ the m roots with positive imaginary parts of the polynomial in t:

$$(-1)^m A'(x;\xi + t\nu) - \lambda ,$$

where λ is any number on the ray $\arg \lambda = \theta$. Then one should have that the polynomials (in t) $B_j'(x;\xi + t\nu)$, $j=1,\dots,m$, be linearly independent modulo the polynomial

$$\prod_{k=1}^{m} (t - t_k^+ (\xi;\lambda)) .$$

Another result is the following

<u>Theorem:</u> Suppose that $(-1)^m A'(x,\xi) > 0$ for all real vectors $\xi \neq 0$, $x \in \bar{G}$, and let $B_j'(x,D) = \partial^{k+j-1}/\partial\mu^{k+j-1}$, $j=1,\dots,m$, where $\partial/\partial\mu$ denotes differentiation along a non-tangential (variable) direction μ at the boundary and where $0 \leq k \leq m$. Then, the eigenvalues of the corresponding problem (*) form a discrete set clustering along the positive axis.

The investigations of A. Dvoretzky

Subject: Rearrangement of series in Banach spaces and related questions.

During the second year of the Contract the investigations centered around a theorem on convex bodies and its applications to various problems in Banach spaces. The full details of the results were published in Technical Note No. 11: Some results on convex bodies and Banach spaces, by A. Dvoretzky.

In the following we shall give a summary of the main results.

Progress in the theory of linear spaces depends on the discovery of new geometrical properties of Euclidean space. This is due to the fact that many problems in Banach spaces may be reduced, in more or less standard ways, to the finite dimentional case. The present study is devoted to establishing some new properties of convex bodies in Euclidean space and applying them to Banach spaces.

We say that a convex set C in a linear metric space is spherical to within ε if there exist, in the flat space generated by C, two concentric balls B_1 and B_2 of radii r and $(1+\varepsilon)r$ such that $B_1 \subset C \subset B_2$. The g.l.b. of the ε having the above property may be called the asphericity of C and will be denoted by $\alpha(C)$.

If C is a compact convex set in Euclidean E^N having the origin as an inner point we denote by $\beta(C)$ the l.u.b. of the ratio of the lengths of two parts into which the origin divides chords passing through it. $\beta(C) = 1$ if and only if C is symmetric (about the origin).

The following are some of the main results on convex bodies.

I. Let $\varepsilon > 0$, $d \geq 1$ and a positive integer k be given. Then there exists an integer $N = N(k; \varepsilon, d)$ such that if C is any convex body in E^N with $\beta(C) \leq d$ there is a k-dimensional subspace E^k with $\alpha(C \cap E^k) < \varepsilon$.

It is possible to give estimates of $N(k; \varepsilon, d)$, we give the result only for the symmetric case:

II. $N(k; \varepsilon, 1) < \exp(c\varepsilon^{-2} k^2 \log^2 k)$,

c being an absolute constant (one may take $c = 2^{15}$).

Actually not only are there E^k for which $E^k \cap C$ is nearly spherical, but there are many such. To this end we must use the rotation invariant measure of E^k in E^n which we normalize to be a probability measure and denote by $\mu_{n,k}$. As a sample result we mention:

III. Let $\varepsilon > 0$, $\delta > 0$, a positive integer k and a real function $g(n)$ defined for every $n \geq 1$ and satisfying $g(n) = o(\log^{1/2} n)$ be given. Then exists an integer N^1 depending only on k, ε, δ and g such that

$$\mu_{n,k}\left\{E^k; \alpha(E^k \cap C) < \varepsilon\right\} > 1 - \delta$$

for all $n > N^1$ and all C symmetric with respect to the origin and satisfying $B_n \subset C \subset g(n)Q_n$ where B_n is the unit ball in E_n and Q_n a cube circumscribing B_n.

The proof of these results depends heavily on integral geometry in spherical spaces.

IV. Let A be a symmetric subset of $\tilde{B}_n$, the boundary of B_n, then we have for all integers $1 < k < n$

$$\nu_{n,1}(A_t) \geq \nu_{n,k}(A) \quad (1 - \exp(-c(k)\ t\ n^{\frac{1}{2}}) \quad)^k$$

where $c(k)$ is a positive number depending only on k, $\nu_{n,k}(A) = \mu_{n,k}\left\{E^k;\ E^k \cap A = \emptyset\right\}$ and A_t is the set of points on $\tilde{B}_n$ whose (geodesic) distance from A is $\leq t$.

Another result needed is the following:

V. Let $U_{n,\varepsilon}$ be the set of directions for which the distance from the origin to Q_n^* (the boundary of Q_n) is between the limits

$n^{\frac{1}{2}}$ $(2 \log n - (1 \pm \varepsilon) \log \log n)^{\frac{1}{2}}$ then $\mu_{n,1}(U_{n,\varepsilon}) \to 1$ as $n \to \infty$ for every $\varepsilon > 0$.

A result of C.A. Rogers and the author [Proc. Nat. Acad. Sci. U.S.A. 36 (1950) 192-197] is also needed.

As immediate applications to Banach spaces we mention the following:

VI. Let B be an infinite dimensional Banach space and assume that for some fixed $k \geq 2$ all k-dimensional subspaces are isometric, then B is a Hilbert space.

This answers a question raised by Banach, while the following proves a conjecture of Grothendieck:

VII. Infinite dimensional Hilbert spaces are characterized by having the least metric type among all infinite dimensional Banach spaces.

Finally we mention a result on series:

VIII. Let c_n, d_n $(n = 1,2, \ldots)$ be positive numbers satisfying $\sum c_n^2 < \infty$, $\sum d_n^2 = \infty$. Then there exist in every finite dimensional Banach space points x_n $(n = 1,2, \ldots)$ of norm 1 such that $\sum \pm c_n x_n$ is convergent (thus commutatively convergent) and $\sum \pm d_n x_n$ is commutatively divergent for every choice of signs.

The investigations of A. Robinson

Subject: Initial value problems and differential algebra.

During the second year A. Robinson cooperated with S. Halfin in dealing chiefly with two topics (i) the analytic case in Local Differential Algebra (ii) initial value problems in Partial differential algebra.

(i) Local differential algebra - the analytic case.

In a report issued under the present contract (ref. 1) Robinson developed an algebraic theory for the solution of initial value problems for systems of algebraic differential equations. The question arose to what extent the solutions provided by the general theory are analytic, supposing that the coefficients of the equations are analytic. While in Ritt's theory, where no initial value problems are considered, the formal solutions can always be realised by solutions which are analytic in a certain neighbourhood, it is easy to find examples which show that the opposite may be true for particular initial value conditions. Thus, in certain cases, the solution provided by the formal theory may correspond to an asymptotic expansion. It is, therefore, of interest to find classes of problems for which the formal solutions are analytic. This was done in an investigation whose results are published in the Technical Note No. 9. "Local differential algebra - the analytic case" which was issued under the present contract. The principle result which was obtained in this connection is as follows: (For the terminology see ref. 1 and 2).

Let K be a chain of differential polynomials

$$p_{r+1}\{y_1,\ldots,y_n\},\ldots,p_n\{y_1,\ldots,y_n\}$$

where the variables $y_1,\ldots,y_n$ are numbered so that p_j introducs y_j, $j = r+1,\ldots,n$. Let s_j be the separant of P_j with respect to y_j, $j = r+1,\ldots,n$, and let

$$s_0(z_{10}, \ldots z_{20}, z_{21}, \ldots) = H^*(s_{r+1}\cdot \ldots \cdot s_n) .$$

Furthermore, let K_0 be a set of initial value conditions

$$q_j(z_{10},\ldots z_{20}, z_{21},\ldots) = 0$$

Then a sufficient condition for the existence of an anlytic solution to the system $p_k = 0$, $q_j = 0$ is that

$$s_0 \notin \sqrt{(H^*([K]) \cup K_0)}$$

It can be shown that the above test is satisfied by wide class of differential equations with given initial conditions. In spite of its special nature it can be applied also when the set of equations K is not originally given in the form of a chain.

(ii) Local partial differential algebra. The aim of this part of the joint research was to develop an extension to partial differential algebra of the concepts and results of ref. 1.

A localized partial differential algebra is defined as a homomorphism H of a partial differential algebra R with n derivations D_i, $n \geq 2$, into a (partial) differential algebra R_o with n-1 derivations Δ_i such that the following condition is satisfied.

There exists a matrix of elements of R, (a_{ik}), $i=1,\ldots,n-1$, $k=1,\ldots,n$ such that $(H(a_{ik}))$ is of rank $(n-1)$ and such that

$$H\left(\sum a_{ik} D_k a\right) = \Delta_i H(a) \quad , \quad i=1,\ldots,n-1 \quad , \quad a \in R.$$

However, the coefficients of the matrixm a_{ik}, have to satsfy a certain set of differential conuitions which hold authomatically if the a_{ik} are constants and also in the concrete case, when the derivations Δ_i represent inner derivations on an (n-1) dimensional surface in n-dimensional space.

The only serious difficulty in adapting the theory of ref. 1 to the case under consideration arises in the course of the definition of a polynomial extension. Having overcome this difficulty, one may define bi-ideals as in ref. 1 and develop a theory of varieties for these bi-ideals

In addition to the concrete results which were obtained during the year arose a number of further interesting questions. While some of these seem to require only a straight forward development, others still require the introduction of new fundamental concepts. The former class includes a theory of regular local partial differential algebras (compare ref. 1 for the case of an ordinary differential algebra), the latter - a theory of characteristics in partial differential algebra.

References

1. A. Robinson — Local Differential Algebra, Technical (Scientific) Note No. 4. Contract No. AF 61(052)-187 - 1959.

2. J.F. Ritt — Differential Algebra, American Mathematical Society Colloquium Publications, Vol. 33, 1950.

The investigations of A. Jakimowski

Subject: Tauberian theorems and summability.

The work during the last year dealt mainly, with properties of the $[J,\alpha;\ f(x)]$, $[QH,\alpha;\mu_n]$ and the $[K,\alpha;\ c_p]$ transformations. Of these classes of transformations the first two were defined in a previous report: "The product of summability methods; new classes of transformations and their properties", Part 2, Technical (Scientific) Note No. 4. The third class of transformations - the $\left[K,\alpha;\ c_p\right]$ transformations were defined in the third part of the last mentioned report. A report entitled: "The product of summability methods; new classes of transformations and their properties", Part 3, Technical Note No. 10 was issued.

1. It was shown that the class of $\left[K,\alpha;\ c_p\right]$ transformations is the sequence - to - function analogues to the class of sequence - to - sequence $\left[QH,\alpha\ ;\mu_n\right]$ transformations. The definition of the $\left[K,\alpha;\ c_p\right]$ transformations is:

Definition: Let α be a fixed number. For a given sequence $\left\{c_p\right\}$, $t(x)$, the $\left[K,\alpha;\ c_p\right]$ transform of $\left\{s_n\right\}$ is defined by

$$t(x) = x^{\alpha+1} \sum_{n=0}^{\infty} \binom{n+\alpha}{n} s_n \sum_{m=0}^{n} (-1)^m \binom{n}{m} c_m x^m \quad , \quad 0 < x < x_0 < \infty .$$

We say that $\left\{s_n\right\}$ is summable $\left[K,\alpha;\ c_p\right]$ to s if

$$\lim_{x \downarrow 0} t(x) = s$$

If we choose $c_p = a^{p+\alpha+1}$ $(a > 0)$ then the $\left[K,\alpha;\ c_p\right]$ transformation is the $A^{(\alpha)}$ transformation. Sufficient conditions for the regularity of the $\left[K,\alpha;\ c_p\right]$ transformations were given.

Theorem: Let $\alpha\ (>-1)$ be a fixed number. If

$$c_p = a^{p+\alpha+1} \int_0^1 t^{p+\alpha+1}\, d\beta(t) \quad , \quad p \geq 0 \ ,$$

where $a > 0$ is a fixed number and $\beta(t)$ is of bounded variation in $\langle 0,1\rangle$ then the $\left[K,\alpha;\ c_p\right]$

transformation is convergence preserving.

Under the above suppositions the $\left[K,\alpha;\ c_p\right]$ transformation is regular if, and only if $\beta(1) - \beta(0) = 1$.

About the interation product of the $\left[K,\alpha;\ c_p\right]$ and the $\left[QH,\alpha;\mu_n\right]$ transformations the following result was proved:

<u>Theorem:</u> Suppose $\alpha(>-1)$ is a fixed number. Let the $\left[K,\alpha;\ c_p\right]$ transformation be convergence preserving. If $\{s_n\}$ is bounded and summable $\left[K,\alpha;\ c_p\right]$ to s, then any regular $\left[QH,\alpha;\mu_n\right]$ transformation $\{t_n\}$ of $\{s_n\}$,

$$t_n \equiv \sum_{m=n}^{\infty} \binom{m+\alpha}{m-n} (\Delta^{m-n}\mu_m)s_m \quad , \quad n \geqslant 0 \quad ,$$

is summable $\left[K,\alpha;\ c_p\right]$ to s.

2. Another result proved was a generalization of a theorem of Widder about the inversion of the Laplace transform. By means of this result it is possible to obtain necessary and sufficient conditions for the regularity of the $\left[J,\alpha;\ f(x)\right]$ transformations. For instance we have the following

<u>Theorem:</u> Let $\alpha\ (\geqslant 0)$ be a fixed number. If $f(x)$ is defined and has derivatives of all orders for $x > x_0 > 0$ and

$$\sum_{k=0}^{\infty} \frac{x^{k+\alpha}}{\Gamma(k+\alpha+1)} \left|f^{(k)}(x)\right| \leqslant M < +\infty$$

uniformly for $x > x_0$ then $f(x)$ has the representation

$$f(x) = \int_0^\infty e^{-xt}\, t^\alpha \, d\beta(t) \quad , \quad \text{for} \quad x > 0 \quad ,$$

where $\beta(t)$ is a function of bounded variation in $\langle 0, \infty \rangle$. The converse is also true.

An open question is whether it is possible to suppose in the last theorem α = a real number.

By means of the last theorem it is quite easy to obtain the following result:

<u>Theorem:</u> The $\left[J, \quad ; \quad f(x)\right]$ transformation, for a fixed $\alpha \geq 0$, defined by

$$t(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{m+\alpha}}{\Gamma(m+\alpha+1)} f^{(m)}(x) s_m \quad , \; x > x_0 \geq 0 \quad , \quad x \longrightarrow \infty \quad ,$$

is regular if, and only if,

$$f(x) = \int_0^\infty e^{-xt}\, t^\alpha \, d\beta(t) \; , \qquad x > 0,$$

where $\beta(t)$ is a function of bounded variation in $\langle 0, \infty \rangle$ and $\beta(\infty-) - \beta(0+) = 1$.

RESEARCH IN THE AREA OF MATHEMATICAL ANALYSIS

Part III

1 October 1959 - 30 September 1961

Abstract

The research in the area of mathematical analysis under contract AF 61(052)-187, was conducted at the Department of Mathematics of the Hebrew University. Participating in the contract were:
Professor Aryeh Dvoretzky, Dr. Shaul Foguel, Dr. Amnon Jakimowski and Dr. Henry Kesten. Professors S. Agmon and A Robinson had been on a leave for one year. Also Mr. J. Lindenstrauss was employed as a Junior Assistant to Professor Dvoretzky.

The following Technical (Scientific) notes were issued:

Local partial differential algebra, by Halfin and A. Robinson, Tech. Note No. 12.

Limits at ∞ of semi groups of Contraction, by S.R. Foguel, Tech. Note No. 13.

A subspace of l^1 which is not isomorphic to l^1, by Joram Lindenstrauss Tech. Note No. 14.

Tauberian constants for Hausforff transformations, by Amnon Jakimowski, Tech. Note No. 20.

<u>Occupation times for Markov and semi Markov chains,</u> by Harry Kesten, Tech. Note No. 16.

<u>Markov processes with stationery measures,</u> by S.R. Foguel, Tech. Note No. 17.

<u>Existence of invariant measures for Markov Processes,</u> by S.R. Foguel, Tech. Note No. 18.

<u>Some probabilistic theorems on diophantine approximations,</u> by Harry Kesten, Tech. Note No. 19.

<u>Tauberian Constants for the $[J, f(x)]$ transformations,</u> by Amnon Jakimowsky, Tech. Note No. 20.

<u>Positive contractions,</u> by S.R. Foguel, Tech. Note No. 21.

Professor A. Dvoretzky continued his research reported in the Final Technical Report of October 1960. Also under his direction Mr. J. Lindenstrauss investigated problems about the extension of compact operators. Results are given in Technical Note No. 14.

Dr. S.R. Foguel studied applications of Hilbert space methods to the theory of Markov processes. Some results on Markov chain were extended. Also the problem of existence of invariant measure for Markob processes was studied. His results appear in Tech. Notes No. 13, 17, 18, 21.

Dr. A. Jakimowsky investigated the problem of Tauberian constants for the Hausdorff transformation. Also he studied the inversion formula for the Laplace transform. His results appear in Technical Notes No. 15, 20.

Dr. H. Kesten studied the following problems: Positive intervals of stable processes; occupation times for Markov and semi Markov chains; probabilistic theorems on diophantine approximations. His results appear in Technical Notes No. 16,19.

Main Report

The investigations of A. Dvoretzky

Professor A. Dvoretzky continued his studies of linear spaces. We mention the following result on series in Frécht spaces. Let F be an infinitely dimensional Fréchet space and C_n $(n = 1,2,\ldots)$ a sequence of positive numbers tending to 0, then there exist x_n in F, with $\|x_n\| = C_n$ (for all sufficiently large n) and such that $\sum x_n$ in commutatively convergent. As an immediate corrollary we see that in every such F there exist commutatively convergent series which are not absolutely convergent.
This last fact was recently proved by S. Rolezicz (Colloq. Math. 8 (1961)).

Mr. J. Lindenstrauss investigated the extension of compact operators. He gave new characterizations of the Banach spaces X for which every compact operator T from an arbitrary Banach space Y to X can be extended to a compact operator $\tilde{T}$ from every Banach space Z containing Y to X, with an arbitrary small increase characterized by a certain geometrical property concerning the intersection of spheres. In this connection a conjecture of Grothendieck was disproved and a problem raised by Nachbin solved.

The investigation of S.R. Foguel

The research done in the period 10.60 - 9.61 was mainly concerned with generalizations of results in [1] on Markov Processes. In [1] we studied the behaviour of $\mu(x_n \in A \cap x_o \in B)$ for $n \to \infty$ where x_n is a Markov Process and μ is a finite stationary measure.

The first problem studied was to find what happens if the assumption that μ is finite is dropped. In addition we studied the expressions

$\mu(x_n \in A$ for some $n \cap x_o \in B)$ $\mu(x_n \in A$ infinitely often $\cap x_o \in B)$

It was found that under certain condition which is similar to Doeblin's condition See Doof stochastic Processes) then many of the results on Markov chains remain true.

It is noted that if μ is finite then the results in [1] follow from a theorem on contractions in Hilbert space. The theorem proved is :

Given a contraction operator P on a Hilbert space H there exists a subspace K with :

1. K and $K^{\perp}$ are invariant under P.
2. On K P is a unitary operator.
3. On $K^{\perp}$ P^n tends weakly to zero.

Also if $H = L_2(\Omega, \Sigma, \mu)$ where $\mu(\Omega) < \infty$ and $Pf \geqslant o$ whenever

$f \geq o$ and $P1 = 1$ then K is generated by a σ subalgebra of Σ and P acts on it like an invertible measure preserving transformation. This last result is a generalization of theorem on positive matrices.

Another problem studied was the existence of invariant measure for a given Markov transition function. We let $\underline{X}$ be a locally compact topological space. Let $P(x,A)$ be a transition function where $x \in \underline{X}$ and A a Borel set of $\underline{X}$. Let $P(x,A)$ be continuous when A is open. There exists a closed set $\underline{X}_1$ with:

1. If μ is an invariant measure then μ vanishes outside $\underline{X}_1$.
2. For some invariant measure, μ , $\underline{X}_1$ is the kernel of μ .
3. If $x \in \underline{X}_1$ then $P(x, \underline{X}_1) = 1$.
4. If C is a compact set disjoint to $\underline{X}_1$ and μ any

positive measure then

$$\liminf \int_{\underline{X}} P^n(x,C)\mu(dx) = 0$$

Bibliography

1 S.R. Foguel, Weak and strong convergence for Markov processes. Pacific J. of Math. Vol. 10 (1960) p. 1221-1234.

The investigations of A. Jakimowski

In the first half of the last year the problem of Tauberian constants for the Hausdorff transformations and for the class of $[J, f(x)]$ transformations was studied. In the second half of the last year Dr. Jakimowski was investigating (together with Z. Ditzian) what is known as the real inversion formula for the Laplace-transform.

1. The problem of Tauberian constants is the following: Abelian and Tauberian theorems give information about one of $\lim_{n\to\infty} s_n$, $\lim_{n\to\infty} t_n$ ($\{t_n\}$ is a linear transform of $\{s_n\}$) when the other exists. Now, it is possible to find estimate of $\{s_n - t_n\}$ when neither $\lim s_n$ nor $\lim t_n$ is assumed to exist, but $\{a_n\}$ ($s_n = a_o + \dots + a_n$) is supposed to satisfy the Tauberian condition $\limsup_{n\to\infty} |na_n| < \infty$.

I obtained the following estimate when $\{t_n\}$ is a Hausdorff transform.

<u>Theorem:</u> Suppose $\beta(t)$ is a non-decreasing function in $o \leq t \leq 1$, $\beta(o) = \beta(o+) = o, \beta(1) = 1$ and $\int_o \frac{\beta(t)}{t} dt$ exists as an L-integral.

Denote $\mu_n = \int_o^1 t^n \, d\beta(t)$ $(n \geq 0)$. For any sequence $\{s_n\}$ define

$$t_n \equiv \sum_{m=o}^{n} \binom{n}{m} (\Delta^{n-m} \mu_m) \, s_m, \quad n \geq 0 .$$

If $\limsup_{n\to\infty} |na_n| < \infty$ then for each $q > o$ we have

(1) $$\lim_{n\to\infty, \frac{m}{n}\to q} \sup |s_m - t_n| \leq A_q \cdot \limsup_{n\to\infty} |na_n|$$

where

(2) $$A_q = \begin{cases} \int_o^q \frac{\beta(t)}{t} dt + \int_q^1 \frac{1-\beta(t)}{t} dt & \text{if } o < q \leq 1 \\ \int_o^1 \frac{\beta(t)}{t} dt + \log q & \text{if } q \geq 1 \end{cases}$$

Moreover the constant A_q is the best in the following sense. There, is a sequence $\{s_n\}$ such that $o < \limsup_{n \to \infty} |na_n| < \infty$ and the members of inequality (1) are equal.

A_q is called the Tauberian constant for the transformation defined by $\{t_n\}$.

If $\beta(t) = 1 - (1-t)^{\gamma}$ $(\gamma > o)$ then $\{t_n\}$ is the Cesáro transform of order γ. Substituting the value of $\beta(t)$ in (2) we obtain the value of A_q. For integral values of γ this constant was obtained by Garten.

Similarly we can obtain the Tauberian constant for the Holder transformation.

If $\beta(t) = o$ for $o \leqslant t < \alpha < 1$ and $\beta(t) = 1$ for $\alpha \leqslant t \leqslant 1$ then $\{t_n\}$ is the Euler transform of order α, $\{e_n^{(\alpha)}\}$, of $\{s_n\}$. In this case we obtain that if $q = \alpha$ then $A_q = o$. This means that

<u>Theorem</u>: If $\limsup_{n \to \infty} |na_n| \infty$ then for $o < \alpha < 1$

$$\lim_{n \to \infty,\ \frac{m}{n} \to \alpha} |s_n - e_n^{(\alpha)}| = 0$$

In particular the sets of limit points of $\{s_n\}$ and $\{e_n^{(\alpha)}\}$ are the same.

If instead of $\{t_n\}$ we take the $[J, f(x)]$ transform $t(x)$ then we have the following result.

<u>Theorem:</u> Suppose $\beta(t)$ is a non-decreasing function in $o \leqslant t \leqslant 1$ satisfying $\beta(o+) = o, \beta(1) = \beta(1-o) = 1$. Suppose $f(x) \equiv \int_0^1 t^x \, d\beta(t$ for $x > o$ and the integrals

$$\int_{0+} \frac{\beta(t)}{t \log \frac{1}{t}}\, dt \qquad \int^{\infty-} \frac{1-(t+1)f(t)}{t(t+1)}\, dt$$

exist. If $t(x)$ is the $[J, f(x)]$ transform of $\{s_n\}$, that is

$$t(x) \equiv \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} f^{(n)}(x)\, s_n, \qquad x > 0,$$

then for any sequence $\{s_n\}$ satisfying: $\limsup_{n\to\infty} |na_n| < \infty$ and for any fixed $q > 0$ we have

$$(3) \qquad \lim_{n\to\infty,}\ \sup_{x\to\infty,\ \frac{n}{x}\to q} \left| s_n - \sum_{m=0}^{\infty} (-1)^m \frac{x^m}{m!} f^{(m)}(x) s_m \right| \leq A_q \limsup_{n\to\infty} |na_n|$$

where

$$A_q = (\text{Euler's constant}) + \log q - \int_0^{\infty-} \frac{1-(t+1)f(t)}{t(t+1)}\, dt + 2 \int_{0+}^{e^{-q}} \frac{\beta(t)}{t \log \frac{1}{t}}\, d$$

$$= \log q + \int_0^{\infty-} \frac{f(t)-e^{-t}}{t}\, dt + 2 \int_{0+}^{e^{-q}} \frac{\beta(t)}{t \log \frac{1}{t}}\, dt$$

Moreover, the constant A_q is the best in the sense of the previous theorem.

By choosing the last theorem $\beta(t) = t$ we obtain a theorem of R.P. Agnero for the Abel transformqtion. By choosing

$$\beta(t) = \frac{1}{\Gamma(\alpha+1)} \int_0^t \left(\log \frac{1}{t}\right)^{\alpha} dt \quad (\alpha > -1)$$

we obtain the value of A_q for the $A^{(\alpha)}$ transformations. If we chose $\beta(t) = 0$ for $0 \leqslant t < e^{-1}$, $\beta(t) = 1$ for $e^{-1} \leqslant t \leqslant 1$ we obtain the following result for the Borel transformation.

Theorem: Suppose $q > o$. If $\{s_n\}$ satisfies $\limsup_{n\to\infty} |na_n| < \infty$ then

$$\lim_{n\to\infty,}\ \sup_{x\to\infty,\ \frac{n}{x}\to q} \left|s_n - e^{-x}\sum_{m=o}^{\infty} \frac{s_n}{n!} x^n\right| \leqslant |\log q| \cdot \limsup_{n\to\infty} |na_n| .$$

Moreover the constant $|\log q|$ is the best in the sense of the previous theorems.

(If $q = 1$ then $A_q = 0$).

2. As a consequence of the investigation of the real inversion formula for the Laplace-transform some improved forms of the Post-Widder formula were obtained. As an example we quote the following result:

Theorem; Suppose $f(x) \equiv \int_o^{\infty-} e^{-xt}\varphi(t)\,dt$ exists for $x > x_o$ for some $\varphi(x) \in L$ in $o \leq x \leq R$, for each $R > o$. If, for some $t > o$, $\varphi(t-o)$ and $\varphi(t+o)$ exist then for each $\lambda > o$ and any vanishing sequence $\{b_k\}$ $(b_k \to o)$ we have

$$(4)\qquad \lim_{k\to\infty} (-1)^k \frac{1}{k!} \left\{\frac{k(\lambda)}{t}\right\}^{k+1} f^{(k)}\left(\frac{k(\lambda)}{t}\right) = A_\lambda \cdot \varphi(t+o) + (1-A_\lambda)\varphi(t-o)$$

when $k(\lambda) = k + \lambda\sqrt{k} + b_k\sqrt{k}$ and $A_\lambda = \frac{1}{\sqrt{2\pi}} \int_\lambda^\infty e^{-\frac{1}{2}u^2}\,du.$

The last result for $\lambda = o$, $b_k \equiv o$, is due to Widder. The same result (4) remains valid if t is a Lebesque-point of $\varphi(t)$. In this case we have

$$\lim_{k\to\infty} (-1)^k \frac{1}{k!} \left\{\frac{k(\lambda)}{t}\right\}^{k+1} f^{(k)}\left(\frac{k(\lambda)}{t}\right) = \varphi(t) .$$

In the same way it is possible to improve most of the real inversion formula for the Laplace-transform and also to obtain some new inversion formula.

The investigation of H. Kesten

Three topics were considered. Abstracts of the results follow below.

Positivity intervals of stable processes. Let $X(t)$ be separable stable process with independent increments with $X(0) = 0$ i.e.

$$Ee^{iu(X(t)-X(s))} = e^{-(t-s)\,|u|^{\alpha}} \qquad s<t$$

for some $0<\alpha\leqslant 2$. The sample paths may be assumed right continuous with probability one. The random set

$$A \ \{t: \ 0\leqslant t\leqslant 1, X(t)> 0\}$$

(A is the set where X is positive) can be represented as a countable union of disjoint maximal intervals. The purpose was to find the limiting distribution of

$N(\varepsilon)$ = the number of intervals whose length exceeds ε in the above representation of A

as $\varepsilon \downarrow 0$. It was found that for $0<\alpha\leqslant 1$

$$\frac{2\pi\alpha N(\epsilon)}{\operatorname{tg}\frac{\pi\alpha}{2}\log\frac{1}{\epsilon}} \longrightarrow 1 \quad \text{in probability} \quad (\epsilon\downarrow 0).$$

for $\alpha = 1$

$$\lim_{\varepsilon \downarrow 0} P\left\{\frac{2\pi^2 N(\varepsilon)}{(\log\frac{1}{\varepsilon})^2} \leq x\right\} = \int_0^x \sum_{n=0}^{\infty} (-1)^k \, \pi(k+\tfrac{1}{2}) e^{\frac{-\pi^2(2k+1)^2}{8} t} \, dt \, .$$

for $\alpha \geq 1$

$$\lim_{\varepsilon \downarrow 0} P\left\{\frac{\pi\alpha \sin\frac{\pi}{\alpha}\, N(\varepsilon)}{\varepsilon^{1/\alpha-1}\Gamma(1-\frac{1}{\alpha})} \leq x\right\} = \frac{1}{\pi(1-\frac{1}{\alpha})} \int_0^x \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sin\pi(1-\tfrac{1}{\alpha})k\,\Gamma(1+k(1-\tfrac{1}{\alpha}))^{k-1}$$

<u>Occupation times for Markov and semi Markov chains.</u> Let $Y_0, Y_1, \ldots$ be a recurrent Markov chain with state space the integers. Define a semi Markov process $Y(t)$ as follows. The process remains for a time T_o in Y_o. Then it jumps to Y_1 where it remains a time T_1 after which it jumps to Y_2 etc. The times $T_o, T_1, \ldots$ are random variables. Given $Y_o, Y_1, \ldots$ the T_i are assumed independent and the distribution of T_i depends only on the value of Y_i. We then have

$$Y(t) = Y_k \quad \text{if} \quad \sum_{i=o}^{k-1} T_i < t \leq \sum_{i=o}^{k} T_i \, .$$

Let $V(j)$ be a function on the integers which vanishes outside a finite set J. The process $Y(t)$ is considered up till time λ and the limiting distribution (as $\lambda \to \infty$) of $\int_0^\lambda V(Y(t))dt$ is studied. If $M(\lambda)$ denotes the number of indices k satisfying

$$(*) \qquad Y_k \in J \quad \text{and} \quad \sum_{i=o}^{k-1} T_i < \lambda$$

i.e. if $M(\lambda) + 1$ is the number of visits to J till time λ, then also the limiting distribution of

$$\sum_{i=0}^{M(\lambda)} V(Y_{r_i})$$

is studied. Here $r_0, r_1, \ldots, r_{M(\lambda)}$ denote the successive values k satisfying (*). The results for these limiting distributions cannot be written down in a few lines but can be found in a report with the same title as this abstract. We mention only that the choice

$$V(j) = \begin{cases} 1 \text{ if } & j \in J \\ 0 \text{ if } & j \notin J \end{cases}$$

leads to

$$\int_0^\lambda V(Y(t))dt = \text{occupation time of } J \text{ up till time } \lambda.$$

If $T_0 = T_1 = \ldots = 1$ with probability 1 the $Y(t)$ process is an ordinary Markov chain with discrete time. These choices lead to generalizations of results of Takacs and Darling & Kac.

Some probabilistic theorems on diaphantine approximations. Let $x_1, x_2, \ldots$ be independent random variables with a uniform distribution on $[0,1]$. Denote by $\langle a \rangle$ the (positive) distance of a to the integer nearest to a. The limiting distribution (as $m \to \infty$) of $m. \min\limits_{1 \leq k \leq m} \langle kx_1 \rangle$ was found. Its complicated form can be found in a report with the same title as this abstract.

Define futhermore

$N(m,\gamma,p)$ = the number of $k, 1 \leq k \leq m$, for which simultaneously

$$\langle kx_1\rangle \leqslant \gamma, \langle kx_2\rangle \leqslant \gamma, \ \ldots \ , \langle kx_p\rangle \leqslant \gamma \ .$$

It is proved that as $m \longrightarrow \infty$ and $p \longrightarrow \infty$ in such a way that $m(2\gamma)^p \longrightarrow \lambda$ (γ fixed, $0 < \gamma < \frac{1}{2}$) then the distribution of $N(m,\gamma,p)$ tends to a Poisson distribution with mean λ, i.e.

$$\lim \ P\left\{M(m,\gamma,p) = k\right\} = e^{-\lambda}\frac{\lambda^k}{k!} \quad k = 0, 1, \ldots \ .$$

RESEARCH IN THE AREA OF MATHEMATICAL ANALYSIS

Part IV

1 October 1961 - 30 September 1962

Abstract

The research in the area of mathematical analysis under Contract AF 61(052)-187 was conducted at the Department of Mathematics of the Hebrew University. Participating in the contract were:
Professor S. Agnon, Professor A. Robinson, Dr. S.R. Foguel, Dr. B. Grünbaum, Dr. A. Jakimowski, Dr. Y. Katznelson and J. Lindenstrauss, (Assistant). The following Technical (Scientific) Notes were issued:

<u>On the extension property for compact operators,</u> by J. Lindenstrauss, Tech. Note No. 22.

<u>Common secants for families of polyhedra,</u> by B. Grünbaum, Tech. Note No. 23.

<u>On a paper of A. Feldzamen,</u> by S.R. Foguel, Tech. Note No. 24.

<u>Characterization of C $(\mathfrak{M})$,</u> by Y. Katznelson, Tech. Note. No. 25.

<u>Tauberian constans for the Abel and Cesaro transformations,</u> by A. Jakimowski, Tech. Note. No. 26.

<u>On the unitary part of a contraction,</u> by S.R. Foguel, Tech. Note No. 27.

<u>Extension of compact operators I,</u>
by J. Lindenstrauss, Tech. Note No. 28.

<u>Mixed problems for higher order hyperbolic equations,</u>
by S. Agmon, Tech. Note No. 29.

<u>Complex function theory over non-Archimedean fields,</u>
by A. Robinson, Tech. Note No. 30.

<u>Extension of compact operators II,</u>
by J. Lindenstrauss, Tech. Note No. 31.

<u>Extension of compact operators III,</u>
by J. Lindenstrauss, Tech. Note No. 32.

<u>Convexity theorem and lower bounds for solutions of differential equations in Banach space,</u>
by S. Agmon and L. Nirenberg, Tech. Note No. 33.

<u>Contractions and their quadratic forms,</u>
by S.R. Foguel, Tech. Note No. 34.

Professor S. Agmon studied higher order hyperbolic equations and differential equations in Banach spaces. His results were published in Technical Notes Nos. 29 and 33.

Professor A. Robinson continued his development of the methods of Non Standard Analysis. His results were published in Technical Note No. 30.

Dr. S.R. Foguel continued his research on contraction operators and limits of their powers. His results were published in Technical Notes No. 24, 27 and 34.

Dr. B. Grünbaum studied different aspects of convexity. His results were published in Technical Note No. 23.

Dr. A. Jakimowski studied inversion formula for integral transforms and problems about Tauberian constants. His results were published in Technical Note No. 26.

Dr. Y. Katznelson studied symbolic calculus in various Banach algebras. His results were published in Technical Note No. 25.

Mr. J. Lindenstrauss studied extension of compact operators. His results were published in Technical Notes No. 22, 28, 31 and 32.

Main Report

The investigations of S. Agmon

Subject: Higher order hyperbolic equations and differential equations in Banach spaces.

During the period October 1, 1961 - September 30, 1962 the research was divided in two subjects. (i) Higher order hyperbolic equations. (ii) Differential equations in Banach space.

(i) Higher order hyperbolic equations.

The problem discussed under this title was the mixed initial boundary value problem for higher order hyperbolic equations with constant coefficients where the underlying domain is a quarter space or, more generally, a semi-infinite strip. In this connection the class of well posed problems was chracterized and, moreover, the typical L_2 estimates were derived. The results obtained were described in Technical Note No. 29 entitled: "Mixed problems for higher order hyperbolic equations". We shall mention one result here. Let $x = (x_1, \ldots, x_n)$ be the generic point in n - space and let (x,t) denote a point in $n+1$ - space. Put:

$$D_t = \frac{1}{i}\frac{\partial}{\partial t}, \quad D_k = \frac{1}{i}\frac{\partial}{\partial x_k} \quad \text{and} \quad D_x = (D_1, \ldots, D_n).$$

We shall denote by D^j any derivative of order j. Let now $A(D_x, D_t)$ be a (strongly) hyperbolic linear differential operator with constant coefficients of order m and with no lower order terms. One considers the following mixed initial boundary value problem for semi-inifinte strip:

$$(1) \qquad A(D_x, D_t)\, u(x,t) = f(x,t) \quad \text{for} \quad 0 \leqslant t \leqslant T, \quad x_n \geqslant 0,$$

$$D_t^j u \Big|_{t=0} = 0 \quad \text{for} \quad x_n \geqslant 0, \quad j = 0, \ldots, m-1,$$

and

$$B_j(D_x, D_t)\, u \Big|_{x_n=0} = 0 \quad \text{for} \quad 0 \leqslant t \leqslant T, \quad j = 1, \ldots, l.$$

Here $\{B_j\}$ is a system of l differential operators with constant coefficients of orders $m_j < m$. The number l of boundary operators equals the number of ingoing characteristics into the first quadrant of the hyperbolic operator in the two variables t and x_n obtained from A by retaining only differentiations with respect to these variables. Now, in order that the problem (1) be well posed it is necessary that A and the system $\{B_j\}$ satisfy a certain algebraic "complementing condition". If this as well as another condition on A hold (both conditions are described in the Technical Note) then the following basic result is valid.

Theorem: Consider the class of functions $u(x,t)$ belonging to C^m in the semi-infinite strip: $0 \leq t \leq T$, $x_n \geq 0$, satisfying the initial conditions:

$$D_t^j u = 0 \quad \text{for} \quad t = 0, \quad x_n \geq 0, \quad j = 0,\ldots,m-1,$$

and the boundary conditions:

$$B_j u = 0 \quad \text{for} \quad x_n = 0, \quad 0 \leq t \leq T, \quad j = 1,\ldots,l.$$

Then for arbitrary number 0 and any function u in the class one has:

$$\int_0^T e^{-\gamma t} \|D^k u(\cdot,t)\|^2_{L_2(E_n^+)} \, dt \leq \frac{C}{\gamma^{2(m-k)}} \int_0^T e^{-\ t} \|Au(\cdot,t)\|_{L_2(E_n^+)} dt$$

for $0 \leq k \leq m-1$. Here E_n^+ denotes the half-space $X_n > 0$ in n-space while C is a constant not depending on γ, T or u.

The proof of this this theorem is rather involved and is achieved by successive reductions to a one dimensional analogue. The theorem in turn is basic for the existence theory for solutions of (1).

(ii) Ordinary differential equations in Banach space.

The investigations on this topic were carried out in collaboration with L. Nirenberg extending some recent results of both authors [*]. The new results were described in Technical Note No. 33 entitled: "Convexity theorems and lower bounds for solutions of differential equations in Banach space". The general problem here is to give conditions under which one has uniqueness for solutions of

$$(2) \qquad Lu = \frac{1}{i}\frac{du}{dt} - Au = 0 ,$$

as well as for perturbed equation written in an inequality form:

$$(3) \qquad \|Lu\| \leqslant \Phi(t)\ \|u\| \quad .$$

Here $u(t)$ is a function of the real valiable t taking its values in a Banach space while A is some closed (unbounded) linear operator. A strong form of a uniqueness result is to derive lower bounds for solutions of (3). In certain cases these can be derived via a convexity argument. One such convexity result is the following:

Theorem: Suppose that iA is a non-real multiple of an infinitesimal generator of a one-parameter group of bounded operators $T(t)$ and let Φ in (3) be constant. Then there exist positive numbers d and C (depending only on A and Φ) such that for every solution of (3) in an interval $[a,b]$ of length $\leqslant d$ the following convexity result holds:

$$\|u(t)\| \leq C\ \|u(a)\|^{\frac{b-t}{b-a}}\ \|u(b)\|^{\frac{t-a}{b-a}} , \qquad a < t < b \quad .$$

From the last theorem it follows easily that under the same conditions any solution on $t \geqslant 0$ satisfies a lower bound of the form:

$$\|u(t)\| \geqslant C e^{-\mu t^2} ,$$

where C and μ are certain positive constants. Better results are obtained under additional assumptions on the group of operators and additional assumptions on the group of operators and on the function Φ , and the various results admit applications to solutions of partial differential equations in cylindrical domains.

References

* S. Agmon and L. Nirenberg, Properties of solutions of ordinary differential equations in Banach space. N.Y. Univ. report, 1961.

The investigations of A. Robinson

Subject: Development of the Methods of Non Standard Analysis.

Throughout the year, the investigator's effort was directed towards the development of the methods of non-standard analysis. This is a new subject (ref. 1) which is based on the introduction of a proper extension of the field of real numbers which shares the properties of the latter within the framework of a specified formal language. It permits a new approach to classical problems in various branches of Analysis. The Theory of Functions of a Complex Variable was shown as the first topic for detailed investigation, and new results were obtained in the analytic theory of polynomials and concerning the behavior of an analytic function in the neighbourhood of an essential irregularity. The work was described in a comrehensive Technical (Scientific) Note issued under the project. After its completion, the investigator began to consider problems in Functional Analysis, in particular in the theory of summability.

The investigations of S.R. Foguel

Subject: Contraction operators and limits of their power.

Dr. Foguel continued his research on contraction operators in Hilbert Space. It was proved in [1] that if P is such a contraction then there exists a subspace K invariant under P and P^x such that:

1. On K P is a unitary operator
2. If $x \perp K$ weak $\lim\limits_{n \to \infty} P^n x = 0$

It was found that if E is the self adjoint projection on K then E is the spectral measure of the circumference of the unit circle, provided H is finite dimensional. If H is infinite dimensional examples were constructed to show that this is not the case even if P is a scalar type operator in the sense of Dunford.

It is possible to find the behavior of $P^n x$ as $n \to \infty$ from the numerical entities $(P^n x, x)$ thus:

1. If $(P^n x, x) \longrightarrow 0$ then $P^n x \xrightarrow{\omega} 0$.
2. If the sequence $(P^n x, x)$ has finitely many limits then $x = x_0 + x_1$ where $P^n x_0 \xrightarrow{\omega} 0$, for some k $P^k x_1 = P^{*k} x_1 = x_1$.

Finally a different subject was studied with relation to spectral theory:

Results of A. Feldzamen in [2] were reproved by more elementary methods for separable Hilbert spaces.

Bibliography:

[1] Foguel, S.R., Limits at ∞ of Semi Group of Contractions. Technical Note No. 13, Contract No. AF 61(052)-187.

[2] Feldzamen, A.N. Semi Similarity Invariants for Spectral Operators on Hilbert Space. Trans. Amer. Math. Soc. Vol. 100 (1961).

The investigations of B. Grünbaum

Subject: Different aspects of convexity.

The investigations of B. Grünbaum were mostly concerned with different aspects of convexity. Not all the results obtained have been written up in the final form of a Technical Report; their description will take up most of the following summary.

1. Extending earlier results of L.A. Santalo on parallelepipeds, the existence of Helly-type theorems on intersecting hyperplanes was established for certain families of convex polyhedra in E^n ("Common secants for families of polyhedra", Tech. Note No. 23).

The same paper contains also a converse result to the effect that (in the plane) Helly-type theorems on common transversals may be used to characterize convex polygons. The notion of a polyhedron 'related' to another polyhedron, introduced in the proofs of the above results, seems to be of an independent interest.

2. Another series of results deals with relatives of Helly's theorem in the following setting. Let a family of sets be called k-fixable, k a positive integer, if there exists a set of cardinality k intersecting every member of the family. Helly's theorem deals with 1-fixable families of convex sets in E^n. Considering families P^n of hypercubes with edges in the directions of the coordinate axes in Euclidean n-space, let $f(r,k,n)$ denote the least integer m with the property: If every r-membered subfamily of a family P^n is k-fixable, then the whole family is m-fixable. Also, let $r(k,n)$ denote the least r such that $f(r,k,n) = k$. The following are the main results obtained:

$$\text{(i)}\quad f(k+1,k,n) \leq \binom{k+n-1}{n} \quad ;$$

$$\text{(ii)}\quad r(2,n) = \begin{cases} 3n & \text{for odd } n \\ 3n-1 & \text{for even } n \,. \end{cases}$$

At present, this direction of research is pursued jointly by Grünbaum and L. Danzer (München), and it is hoped that a technical report shall be prepared within the next few months.

3. In connection with other investigations concerning measures of asymmetry for convex sets the following problem was considered. Let a plane convex body K be partitioned by 3 straight lines into seven parts in such a way that the area of each of the outer regions adjacent to an edge of the central triangle be at most equal to the area of the region adjacent to the opposite vertex of the triangle. Let $f(k)$ denote the maximal value (for all possible lines) of the ratio of areas of the central triangle and K. Generalizing results of M. Sholander and H.G. Eggleston it is established that $f(K) \leq 1/49$, with equality only if K is a triangle, and that $f(K)$ is a measure of asymmetry (i.e. $f(K) = 0$ if and only if k has a center of symmetry). The details, and some extensions, are given in a technical report entitled "A measure of asymmetry for plane convex sets", which is now in print.

4. In collaboration with T.S. Motzkin (University of California, Los Angeles) the existence of certain polyhedra in E^3 was investigated. Let P_n [resp. Q_n, resp. R_n] denote a trivalent convex polyhedron consisting of 12 pentagons [resp. 6 quadrangles, resp. 4 triangles] and n hexagons. The problem was to determine for which values of n do those polyhedra exist. By example it is established that P_n and Q_n exist for every value of n except $n = 1$, while the problem of the R_n's is settled by the following theorem: R_n exists if and only if n is a non-negative even integer different from 2. A technical report on this subject is in preparation.

5. Inclusion-representations for partially ordered sets were considered and the following result established: Every partially ordered set of cardinality $\leq \aleph$ $[\aleph_0]$ is inclusion-representable by [compact] convex sets in the plane. This extends a result of B. Dushnik and E.W. Miller, Amer. J. Math. 63(1941), 600-610, which deals with sets of order-dimension 2). Considering the related problem of intersection-representations of graphs a new characterization of graphs intersection-representable by segments in a line (see C.G. Lekkerkerker and J.C. Boland, Fund. Math. 51(1962), 45 - 64) is obtained: A graph is representable if and only if (i) every 4-circuit in it contains a diagonal, and (ii) the graph and its complement may be partially ordered. Other results in this area concern the existence of graphs intersection-representable by plane continue but not by convex sets in the plane, the intersection-representability of every denumerable graph by convex sets in E^3, etc. A technical report on these topics is in preparation.

The investigations of A. Jakimowski

Subject: Inversion formula for integral transforms and problems about Tauberian constants.

During the period October 1961 - September, 1962 the following problems were investigated.

The first is in the theory of real inversion formula for integral transforms (the Laplace transform and others) together with Z. Diztian. Then results are being prepared for publication.

The second problem is about Tauberian constants for the Abel transformation. The following is a result which was obtained (Techn. Note No. 26). Denote for a sequence $\{s_n\}$ ($s_n = a_0+\dots+a_n$) by $\{C_n^{(\alpha)}\}$ ($\alpha>-1$) its Cesáro transform of order α ; that is,

$$C_n^{(\alpha)} = \frac{1}{\binom{n+\alpha}{n}} \sum_{m=0}^{n} \binom{n-m+\alpha-1}{n-m} s_m, \qquad n \geq 0 \quad .$$

By $\{a_n^{(\alpha)}\}$ we denote the $(C\,\alpha)$ transform of sequence $\{n\, a_n\}$ $(n \geq 0)$.

<u>Theorem:</u> Let the three real numbers α, β, q satisfy $q>0$ and $-1<\alpha\leq\beta\leq\alpha+1$. Then for any squence $\{s_n\}$ satisfying $\left|a_n^{(\beta)}\right|\leq K<+\infty$ for $n\geq 0$ we have

$$(1) \qquad \overline{\lim_{n\to\infty,\ x\to 1,\ n(1-x)\to q}} \left|C_n^{(\alpha)} - \sum_{m=0}^{\infty} a_m x^m\right| \leq C_q^{(\alpha,\beta)} \overline{\lim_{n\to\infty}} |a_n^{(\beta)}| ,$$

where

$$C_q^{(\alpha,\beta)} = \gamma(\text{Euler's constant}) + \log q - \int_0^1 \frac{1-(1-u)^{\alpha}}{u}\, du +$$

$$+ \frac{2}{\Gamma(\beta+1)} \int_q^{\infty} v^{\beta} e^{-v} \log \frac{v}{q}\, dv \ .$$

Moreover the constant $C_q^{(\alpha,\beta)}$ is the best in the following sense. There is a real sequence $\{s_n\}$ such that $0<\overline{\lim_{n\to\infty}} |a_n^{(\beta)}|<+\infty$ and the members of inequality (1) are equal.

The last theorem includes as special cases some known results. The method of proof enables to obtain similar results for other transformations beside the Abel transformation. This technical note will appear in the Proceedings of the American Mathematical Society.

The third problem concerns the relation between the analytic continuation and the summability of an associated series of Legendre Polynomials. The following is the first result obtained (the numbers are complex numbers).

<u>Theorem:</u> Let D be a domain having the following properties: (i) D includes the unit open disk $|z|<1$. (ii) The point $z = 1$ belongs to the boundary of D. (iii) D has the property that together with $z \in D$ also $\lambda z \in D$ for $0 \leq \lambda \leq 1$. (iv) D has the property that together with $z \in D$, $|z|>1$, for each a, $|a|>1$, the closed interval $\langle z, \frac{1}{az} \rangle$ is included in D (such domains D are for example convex domains satisfying (i) and (ii); also, the union and intersection of a collection of convex domains satisfying (i) and (ii) are domains satisfying (i) - (iv)). Denote by A an infinite matrix $\|a_{nm}\|$ $(n,m = 0,1,2,\ldots$ Suppose A has the following properties. (v) $\lim_{n \to \infty} \sum_{m=\infty}^{\infty} a_{nm} = 1$. (vi) For any closed and bounded subset H of D we have uniformly in $z \in H$,

$$\lim_{n \to \infty} \sum_{m=0}^{\infty} a_{nm} z^{m+1} = 0 .$$

If t is a complex number not in the closed interval $\langle -1, 1 \rangle$ then we denote by D_t^* the domain

$$D_t^* \equiv \bigcap_{0 \leq u \leq +\infty} \left\{ (t + \sqrt{t^2-1} \cos h \quad u) D \right\}$$

(where the sign of $\sqrt{t^2 - 1}$ is chosen in such a way that $|t + \sqrt{t^2 - 1}| > 1$). If D_t is the domain

$$D_t \equiv \left\{ \tfrac{1}{2}\left(w + \frac{1}{w}\right) : w \in D_t^* \right\}$$

then the series

$$(2) \qquad \sum_{m=0}^{\infty} (2m + 1)\ P_m(z)\ Q_m(t)$$

(where $P_m(z)$, $Q_m(t)$ denote, respectively, the Legendre Polynomial of order n and the Legendre function of the second kind of order n) is summable A to $\frac{1}{t-z}$ for all z D_t. For each closed and bounded subset H of D_t the series (2) is uniformly summable $-A$, in $z \in H$, to $\frac{1}{t-z}$.

This result includes as special cases theorems by Heine, K. Prachar, V.F. Counting and J.P. King.

In order to state our second result we need the following remark.

Suppose $f(z)$ is holomorphic in some simply-connected domain G which includes the closed interval $\langle -1, 1 \rangle$. We call a hyperbolic-ray any one of the four arcs which begin at the real axis of any hyperbola with foci ± 1 and the two rays λi, $-\lambda i$, $0 \leq \lambda \leq +\infty$. We continue $f(z)$ analytically from G along a hyperbolic ray. If we encounter a singular point on this hyperbolic ray we cut the complex plane along this hyperbolic-ray from the singular point to infinity. The set which remains will be called the Mittag-Leffler hyperbolic star domain of $f(z)$ (in short $M \equiv M(f(z))$). It is easy to see that $M(f(z))$ is a simply-connected domain and that the analytic continuation of $f(z)$ into $M(f(z))$ yield a holomorphic function in $M(f(z))$. We shall denote this analytic continuation into $M(f(z))$ by $f(z)$.

We associate to the function $f(z)$ the Neumann series

$$(3)\quad f(z) \sim \sum_{n=0}^{\infty} a_n P_n(z), \qquad a_n = \frac{2n+1}{2\pi i} \int_{\gamma} f(t)\, Q_n(t)\, dt$$

where γ is a closed and rectifiable Jordan curve included in G.

The following is our second result.

<u>Theorem:</u> Let $f(z)$ be a holomorphic function in some simply connected domain which includes the closed interval $\langle -1, 1 \rangle$. Suppose D and A satisfy the suppositions of previous theorem. Denote $\Delta \equiv \bigcap_{t \in M^c} D_t$. If H is a closed and bounded set included in Δ then the Neumann series (3) is uniformly A - summable in $z \in H$, to $f(z)$; that is

$$f(z) = \lim_{n \to \infty} \sum_{m=0}^{\infty} a_{nm} \sum_{k=0}^{m} a_k P_k(z), \text{ uniformly in } z \in H .$$

The last theorem includes as special cases several known result.

The investigations of Y. Katznelson

Subject: Symbolic calculus in various Banach algebras.

The work done by Y. Katznelson during this year can be divided into three parts:

a) Symbolic calculus in homogenous Banach algebras and application to Banach algebras with one real generator, in particular to quotient algebras of $L^1(R)$. It was proved that only analytic functions operate in a large class of homogenous algebras and a corresponding class of quotient algebras of $L^1(R)$.

b) Characterization of $C(X)$. (The algebra of all continuous functions on compact Hausdorff X) as the only algebra where approximate idempotents are bounded for any restriction algebra. This is an extension of a previous paper where the same conclusion was obtained under the additional assumptions that the algebra is regular and self adjoint.

c) Symbolic calculus in non-self-adjoint Banach algebras. The following results were obtained in joint work with Karel de Leeuw.

i) Only analytic functions operate in (non-self-adjoint) sup-normed algebras or algebras of differentiable functioning.

ii) For arbitrary Banach algebras (non-self-adjoint) a function that operates continuously cannot vanish on an open set without vanishing identically and if it is real analytic in an open set it must be complex analytic throughout its domain of definition (assumed connected).

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