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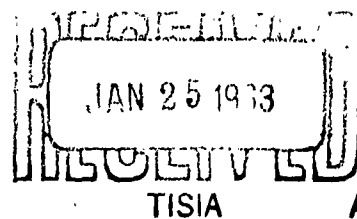
TRAINING CENTER FOR EXPERIMENTAL AERODYNAMICS

TECHNICAL NOTE   10

A ONE DIMENSIONAL MATHEMATICAL MODEL

FOR THE BOLTZMANN EQUATION

by

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October 1962

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SUMMARY

A one-dimensional model for the Boltzmann equation of kinetic theory of gases is proposed, in order to study the validity of the usual approximation method and the behavior of the solutions near the free molecule limit. The interaction term is very similar to the Fokker Planck term. The conservation of mass, momentum and energy results from the particular form assumed for the interaction term and the only equilibrium solutions are shown to be of Maxwell-type. Hydrodynamic equations similar to the Navier-Stokes- Fourier equations are derived in the usual way under the assumption of small gradients. Shock waves are shown to exist in supersonic flows.

The distribution function is obtained in the case of a spatially uniform condition. and the existence of an infinite number of relaxation modes and corresponding relaxation times is shown.

The linearized versions of the model equation are established with particular emphasis on the simplest problems of heat conduction and wave propagation and the usefulness of Fourier transformation with respect to the velocity component is indicated.

Finally, the behavior of the solutions near. the free molecule limit is briefly investigated for steady state conditions. Existence of a singular behavior for slow molecules, analogous to boundary layer phenomenon, is indicated and the equation is shown to reduce to a singular parabolic equation studied by Gevrey, in the region of interest.

The leading terms in the deviation from free molecule data are found to be of the order of the cube root of an interaction parameter, playing a role analogous to the Knudsen number.

1. Usefulness and Limitations of a Mathematical Model

The mathematical difficulties arising in the treatment of the Boltzmann equation for the velocity distribution function f in a monatomic gas at moderate or low densities are known to be formidable, a fact which has precluded, so far, the obtaining of any exact particular solution of interest for conditions far removed from equilibrium.

The Boltzmann equation may be written in the standard form (ref. 1)

$$\frac{\partial f}{\partial t} + \sum_i \xi_i \frac{\partial f}{\partial x_i} = \iiint\iiint [f^{x}f^{'x} - ff'] \cdot |\vec{\xi} - \vec{\xi'}| \cdot b\,db\,d\varepsilon\; d\xi_1' d\xi_2' d\xi_3' \qquad 1.1$$

In this equation, the distribution function f is a (unknown) function of the variables t (time), x_1, x_2, x_3 (space coordinates), ξ_1, ξ_2, ξ_3 (molecular velocity components). In the integral of the second member, the so-called collision integral, there appear the variables b and ε which define the geometry of a particular collision (b being the distance of closest approach in the absence of interaction and ε an angle defining the position, in space, of the plane of relative motion of the molecules with respect to the center of mass of the system). The dashes indicate the velocity components of the partner molecule in the encounter. The stars indicate the values of velocity components after the encounter, so that ξ_1^{x}, ξ_2^{x}, ξ_3^{x}, $\xi_1^{'x}$, $\xi_2^{'x}$ and $\xi_3^{'x}$ are functions of the initial velocity components and of b and ε. Stars and dashes attached to the symbol f indicate which velocity components should be taken as arguments. The integral is extended to all possible encounters and all possible partners.

The velocity distribution function f is, of course, defined in the usual manner as indicating the number density of molecules in configuration space (i.e. the space with coordinates x_1, x_2, x_3, ξ_1, ξ_2, ξ_3),

so that the number dn of molecules in an infinitesimal volume dx_1, dx_2, dx_3 of space, centered at the point (x_1, x_2, x_3) which have molecular velocity components in the ranges (ξ_1, $\xi_1 + d\xi_1$), (ξ_2, $\xi_2 + d\xi_2$) and (ξ_3, $\xi_3 + d\xi_3$) is given by

$$dn = f(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) \, dx_1 \, dx_2 \, dx_3 \, d\xi_1 \, d\xi_2 \, d\xi_3$$

It is seen that all the possible mathematical complications are represented in eq. 1; this equation being simultaneously non linear, partial differential, integral and functional. It is nòt surprising, therefore, that most of the available information concerning its solutions refer almost exclusively to two extreme limiting cases: the near equilibrium case and the free molecule regime (which is somewhat trivial in that the second member of 1, which contains all the mathematical difficulties, is neglected).

In the near equilibrium case, f differs but little from the well known Maxwellian velocity distribution, given by (ref. 1)

$$f_0 = n \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{m(\xi - u)^2}{2kT}} \qquad 1.2$$

Here, n represents the number density of molecules (in the ordinary sense, i.e. in physical space), m the mass of a molecule, T the absolute tempeature, u the macroscopic, or bulk, velocity and k the Boltzmann constant.

Different linearization or expansion techniques are applicable in cases where f is closely approximated by this Maxwellian distribution, the best known being, probably, the Chapman Enskog expansion (ref. 2). This technique assumes small gradients for the macroscopical quantities (bulk velocity, density, pressure or temperature) and lead to the classical Navier-Stokes-Fourier equations as a first approximation, describing the so-called continuum flow regime.

At the other end of the range, the free molecule regime, the collision term becomes negligible, and equation (1.1) reduces to a near trivial first order differential equation. The interaction between molecules and solid boundaries then represents the central problem but no major mathematical difficulties are encountered.

All other regimes, between the two extremes quoted, require a full treatment of the Boltzmann equation. Usually, a distinction is made between several such regimes: slip flow, transitional regime and near free molecule flow (ref. 3).

One must also be aware of the fact that equation (1.1) only accurately describes monatomic gases, and should only be considered as a physically simplified model for the complicated behavior of the technically important diatomic gases. In many attempts at solving eq. (1.1), one goes even farther than this and assumes a particular interaction potential for the molecules in order to simplify the collision term.

These particular potentials are sometimes very far from realistic. The most spectacular simplification, as already indicated by Maxwell, occurs when a repulsion proportional to the inverse of the fifth power of intermolecular distance is chosen (ref. 2). This Maxwell molecular model is not realistic either.

However, it appears that no physical model will lead to a collision term simple enough to allow an exact solution to be obtained in the general case or in non equilibrium cases of practical interest.

Many important mathematical questions appear in connection with equation 1, such as its behavior in the vicinity of the free molecule regime, the validity of expansion procedures and of several important

approximative methods. Such methods are usually introduced without attempting rigorous mathematical justifications (see for instance, Mott Smith's treatment of the shock wave in ref. 4 and Lee's discontinuous distribution function technique in ref. 5).

It seems, therefore, that a mathematical model without immediate physical significance, but which would present most of the main features of eq. 1.1, would be valuable if exact solutions could be obtained for significant cases.

First of all, one should know what are the most significant features of equation 1.1, from the mathematical standpoint. This is not easy because so little is known about the properties of the solutions.

The mathematical model studied in this report presents the following basic properties in common with eq. 1.1:

a) It has the same structure, involving space and time derivatives in the first member and a "collision term" in the second member.

b) There are collisional invariants as in equation 1 (the number of molecules, the momentum and the kinetic energy), so that a set of hydrodynamical equations may be derived in the usual way.

c) The only equilibrium solutions are also of Maxwellian type, as given by eq. 1.2.

d) Though no property analogous to Boltzmann's H theorem (ref. 1) could be obtained for the model equation, it appears from its analogy to the Fokker Planck equation (ref. 6) that the collision provides a dissipative mechanism.

e) The boundary conditions that have to be imposed in order to define a solution are the ones that would be physically encountered for an actual gas kinetic system.

f) It can be shown (see paragraph 4) that the model equation exhibits an infinite number of relaxation times, just as the Boltzmann equation (a condition that is not satisfied, for instance, by the simple Krook model of ref. 7).

These appear to be the widest possible similarity to the Boltzmann equation one can hope to obtain using a reasonably simple mathematical model (as distinct from a physical model).

One of the difficulties one might fear because of the absence of direct physical meaning, is that solutions for f corresponding to reasonable boundary or initial conditions would turn out to be meaningless because f would take negative values in some regions of the field* or exhibit singularities. The same difficulty, of course, could arise in applying approximate methods which are not mathematically justified to equation 1.1.

However, the analogy with a Fokker Planck equation again enables one to hope that negative f values will not occur in solutions corresponding to physically acceptable boundary and initial conditions (i.e. conditions defining entirely the solution and such that the initial and boundary values of f are everywhere positive).

It is possible, of course, that the positive definiteness

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* It is clear that such difficulty could never occur in a small disturbance treatment where $f = f_o + \delta f$ and f_o is always positive and $|\delta f| \ll f_o$

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of f could be proved for the model and for all valid boundary and initial conditions but such a proof has not been attempted.

Attempts will be made to obtain exact solutions for boundary and initial conditions of physical interest such as steady heat conduction, wave propagation, steady shock waves, etc. The behaviour close to the free molecule regime and the validity of the usual approximation methods will then be studied.

2. The One Dimensional Model

An important simplification of eq. 1.1 would arise if a fully one dimensional situation could be considered. This would lead to the physical model of a system of molecules restricted to move on a straight line, say, the x_1 axis. The number of independent variables would then be reduced from seven to three in the general case and from six to two in stationary situations.

However, such a model is useless if a physical collision process, with the usual conservation of momentum and energy, is assumed. Simple exchange of velocity would then occur for each collision ($\xi_1^* = \xi_1^!$ and $\xi_1^{!*} = \xi_1$) so that the collision term would be identically zero (a one dimensional gas is therefore always in the free molecule regime!).

The relaxation of one conservation condition could, perhaps, save the one dimensional model. However, with the energy condition dropped, one cannot expect a Maxwell type equilibrium distribution to exist, while on the other hand, to keep the conservation of energy and drop the conservation of momentum does not seem to lead to clear physical interpretations.

It thus appears that no simple one dimensional physical model can be imagined.

The mathematical model proposed here is suggested by the study of the Boltzmann collision term for weak collisions. This term, which turns out to be quite comparable to the Fokker Planck interaction term used in the treatment of ionized gases (ref. 6), is mentioned in ref. 1 without details or reference to other papers.

This term is therefore calculated in appendix I.

Initially, it was hoped to use the weak interaction term and some further simplifications (involving quite eccentric interaction laws, as shown in appendix I) in order to obtain a simple physical model. The resulting equation is still very complicated, as shown below

$$\frac{\partial f}{\partial t} + \sum_i \xi_i \frac{\partial f}{\partial x_i} = \lambda n \left\{ [(\xi - u)^2 + \overline{c^2}] \Delta_{(\xi)} f - \sum_{i,j} [(\xi_i - u_i)(\xi_j - u_j) + \overline{c_i c_j}] \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} + 6f \right\} \quad (2.1)$$

where λ is a parameter depending on the law of interaction, $m\overline{(c_i c_j)}n$ the stress tensor and $\Delta_{(\xi)} f = \sum \frac{\partial^2 f}{\partial \xi_i^2}$ is the Laplace operator in the velocity space. $\vec{c}$ denotes the random velocity of the molecules:

$$\vec{c} = \vec{\xi} - \vec{u}$$

This equation is again a non linear integro-differential equation for f, but the integral aspect is simplified by the fact that only a finite number of "moments" appear which involve integrals over f, namely the ten moments

$$n = \iiint f \, d\xi_1 d\xi_2 d\xi_3$$
$$n u_i = \iiint f \xi_i \, d\xi_1 d\xi_2 d\xi_3$$
$$n \overline{c_i c_j} = \iiint f c_i c_j \, d\xi_1 d\xi_2 d\xi_3$$

We have then tried to obtain a one dimensional mathematical model of similar structure namely

$$\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = A \frac{\partial^2 f}{\partial \xi^2} + B \frac{\partial f}{\partial \xi} + Cf \qquad (2.2)$$

where A, B, C must be functions of ξ and simple moments such as n, u and

$$n \overline{c^2} - \frac{p}{m} = \int_{-\infty}^{+\infty} f \cdot c^2 d\xi$$

The only significant difference with (2.1) appears to be the acceptance of a first derivative with respect to the velocity. It is not clear whether this has a particular physical or mathematical significance.

We now require conditions (b) and (c) of paragraph 1 to be satisfied, i.e. that eq. 2.2 would satisfy the conservation of number of particles, momentum and energy and that the equilibrium solutions (i.e. solutions of the equation obtained by equating the collision term to zero) would lead to a Maxwellian distribution (1.2). As shown in appendix II, A, B and C are entirely defined by these conditions up to a common factor. Logically, this factor could be a function of n and $\overline{c^2}$ (not of u, of course, because of the principle of Galilean relativity).

Physically, one must expect that a collision term, involving pairs of molecules, should be proportional to the square of the density. The factor is thus written as λn, where λ may be a function of $\overline{c^2}$. For simplicity we will take a constant value for this parameter. The resulting equation then becomes

$$\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = \lambda n \left\{ \overline{c^2} \frac{\partial^2 f}{\partial \xi^2} + (\xi - u) \frac{\partial f}{\partial \xi} + f \right\} \qquad (2.3)$$

with

$$\lambda = const.$$

$$n = \int_{-\infty}^{+\infty} f\,d\xi \quad , nu = \int_{-\infty}^{+\infty} f\,\xi\,d\xi \quad and \quad n\overline{c^2} = \int_{-\infty}^{+\infty} f\,c^2\,d\xi \qquad (2.4)$$

This is the mathematical model we intend to study. The interaction term is very similar to the Fokker Planck term, except that the coefficients of the derivatives involve integrals of the distribution function which greatly complicate the mathematical nature of the equation by making it non linear and integral.

3. Basic properties of the model equation 2.3

Equation 2.3 has been obtained under the conditions that the number of molecules, momentum and energy are collisional invariants. This means that the "collision term" has no global effect on total density, momentum and energy. Integration over the velocity ξ, of equation 2.3, when multiplied successively by m, $m\xi$, and $m\frac{\xi^2}{2}$ yield the usual macroscopic equations of continuity, momentum and energy.

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \qquad \left(or \quad \frac{\partial n}{\partial t} + \frac{\partial n u}{\partial x} = 0 \right) \qquad (3.1)$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial p}{\partial x} = 0 \qquad \left(or \quad \frac{\partial n u}{\partial t} + \frac{\partial n u^2}{\partial x} + \frac{\partial n\overline{c^2}}{\partial x} = 0 \right) \qquad (3.2)$$

$$\frac{\partial \left(\rho \frac{u^2}{2} + \frac{p}{2} \right)}{\partial t} + \frac{\partial \left(\frac{1}{2}\rho u^3 + \frac{3}{2} u p + \dot{q} \right)}{\partial x} = 0 \quad \left(or \quad \frac{\partial \left(n\frac{u^2}{2} + n\frac{\overline{c^2}}{2} \right)}{\partial t} + \frac{\partial \left(\frac{n u^3}{2} + \frac{3}{2} n u\overline{c^2} + \frac{1}{2} n\overline{a} \right)}{\partial x} = 0 \right) \quad (3.3)$$

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Where the density ρ , the pressure p and the heat flux $\dot{q}$ are defined as usual in kinetic theories:

$$\rho = mn = m \int_{-\infty}^{+\infty} f \, d\xi \qquad (3.4)$$

$$p = mn\overline{c^2} = m \int_{-\infty}^{+\infty} f \, c^2 d\xi \qquad (3.5)$$

$$\dot{q} = \tfrac{1}{2} m \, n \, \overline{c^3} = m \int_{-\infty}^{+\infty} f \, \frac{c^3}{2} \, d\xi \qquad (3.6)$$

Equations (3.1), (3.2), (3.3) are entirely analogous to the system of equations describing the one dimensional motion of a hypothetical gas with a ratio of specific heats γ equal to three* and without viscosity. It is clear that viscosity cannot occur in purely one dimensional molecular motions.

Temperature may be defined, of course, through the fundamental relation of statistical mechanics:

$\frac{1}{2} kT$ = average kinetic energy of random motion for a molecule

= $\frac{1}{n} \int_{-\infty}^{+\infty} \tfrac{1}{2} mc^2 f \, d\xi$, so that the usual equation of state for a perfect gas applies:

$$p = \rho \, RT. \qquad (3.7)$$

The only deviation from ideal fluid flow results, in the absence of viscosity, from the presence of the heat flux term in the energy equation.

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* This is, of course, the value to be expected, from statistical mechanics, for a gas with molecules having only one single degree of freedom (i.e. translation along the x-axis).

The macroscopic equations (3.1), (3.2) and (3.3) are studied
in more details in appendix III, where the corresponding Navier-Stokes-
Fourier equations are derived in the usual way, as a first approximation
for small gradients. Also, the existence of a shock wave is shown, and
its fine structure analyzed according to these Navier-Stokes-Fourier type
equations.

In appendix II, the condition that the equilibrium solutions
be of Maxwellian type was used to obtain a necessary condition for the
coefficients. We must now analyze whether this condition is sufficient
to ensure that these are the only equilibrium solutions.

The equilibrium solutions must satisfy

$$\overline{c^2}\, \frac{\partial^2 f}{\partial \xi^2} + (\xi-u)\frac{\partial f}{\partial \xi} + f = \frac{\partial}{\partial \xi}\left\{\overline{c^2}\frac{\partial f}{\partial \xi} + (\xi-u)f\right\} = 0$$

and the general solution of this ordinary linear differential equation for
f is immediately obtained as

$$f = c_1\, e^{-\frac{(\xi-u)^2}{2\overline{c^2}}} + c_2\, e^{-\frac{(\xi-u)^2}{2\overline{c^2}}} \int^\xi e^{-\frac{(\xi'-u)^2}{2\overline{c^2}}}\, d\xi'$$

C_1 and C_2 being arbitrary constants.

The second term is of course a particular solution of the
non homogeneous equation

$$\frac{df}{d\xi} + \frac{\xi-u}{\overline{c^2}}\, f = c_2$$

which is readily seen to be unacceptable. Indeed, f must tend to zero
for large values of $|\xi|$ and so must $\frac{df}{d\xi}$. Therefore, f would behave
asymptotically like $c_2\overline{c^2}/\xi$ for large $|\xi|$ and the momentum and total
energy would diverge.

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We may thus conclude that all meaningful equilibrium solutions of eq. (2.3), yielding finite density, momentum and energy must be of Maxwellian type.

4. Mathematical Nature of the Model Equation

The model equation (2.3) unfortunately exhibits some basic mathematical complications in common with the full Boltzmann equation as well as with some other models extensively used (such as the Krook equation of ref. 7), namely in being non linear and integral.

However, one can imagine a procedure for solution which formally bypasses these difficulties, because they both arise from the appearance of the parameters n, u, $\overline{c^i}$ in the coefficients of the equation.

If one observes, then, that equations (3.1) and (3.2), resulting from (2.3), provide two equations for these parameters, one can visualize the following formal procedure for a solution.

Assume a given set of initial and boundary conditions, sufficient to define the solution f. The character of this set is of course defined by the mathematical nature of equation (2.3), considered as a partial differential equation for f, whatever the nature of the functions n, u, $\overline{c^i}$, n being essentially positive. The linear partial differential equation in f can then be solved for arbitrary functions $n, u, \overline{c^i}$ and the resulting function f will be a functional of $n, u, \overline{c^i}$

$$f = \mathcal{F}(t, x, \xi; \; n, u, \overline{c^i})$$

Writing

$$n = \int_{-\infty}^{\infty} \mathcal{F}(t, x, \xi; \; n, u, \overline{c^i}) \, d\xi$$

we get a functionsl equation which is expected to completely define $n, u, \bar{c^2}$ when coupled with the two differential equations (3.1) and (3.2).

One could, of course, also devise an iterative scheme, whereby n, for instance, is chosen arbitrarily, or better, realistically, $u, \bar{c^2}$ calculated from (3.1) and (3.2) and the resulting set $n, u, \bar{c^2}$ substituted in (2.3) and the resulting linear partial differential equation solved for f taking into account the boundary conditions. $n, u, \bar{c^2}$ are then recalculated from this function and the process repeated indefinitely. In order to justify these procedures, one must of course analyze the nature of the functional relationship between f and $n, u, \bar{c^2}$ in the first case, or the convergence in the second case. This has not yet been attempted.

However, if one assumes at least one of the precedures to be valid, the study of the boundary and initial conditions pertinent to the problem may be made for equation (2.3) considered as a linear partial differential equation for f, assuming $n, u, \bar{c^v}$ to be known functions of x, t (with the restriction that n and $\bar{c^v}$ should always be positive). If these functions were constants, equation (2.3) would reduce to the standard Fokker-Planck equation, and it is therefore likely that the initial and boundary conditions suggested by the physical origin of the equation will apply. These are, of course, the same as would be considered natural for equation (2.3).

The question of boundary conditions is analyzed in more detail , for the steady state case in the next paragraph.

5. Mathematical Nature of the Model Equation for Steady State

For steady states (f independent of t), equation (2.3) reduces to

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$$\xi \frac{\partial f}{\partial x} = \lambda n \left\{ \overline{c^2} \frac{\partial^2 f}{\partial \xi^2} + (\xi - u) \frac{\partial f}{\partial \xi} + f \right\} \tag{5.1}$$

Furthermore, (3.1) and (3.2) simplify to

$$\frac{\partial (nu)}{\partial x} = 0 \qquad or \qquad nu = A \tag{5.2}$$

$$\frac{\partial [n(u^2 + \overline{c^2})]}{\partial x} = 0 \qquad or \qquad n(u^2 + \overline{c^2}) = B \tag{5.3}$$

Hence mA represents the mass flow and mB the total x-impulse (momentum flow plus pressure).

Finally, equation (2.3) becomes, in the steady state

$$\xi \frac{\partial f}{\partial x} = \lambda \left\{ \left(B - \frac{A^2}{n} \right) \frac{\partial^2 f}{\partial \xi^2} + (n\xi - A) \frac{\partial f}{\partial \xi} + n f \right\} \tag{5.4}$$

with the auxiliary condition

$$n = \int_{-\infty}^{+\infty} f \, d\xi \tag{5.5}$$

The constants A and B must be determined after solution by solving the equations (for some particular abscissa x_o):

$$A = \int_{-\infty}^{+\infty} f(x_o, \xi, A, B) \xi \, d\xi \qquad , B = \int \int f(x_o, \xi, A, B) \, d\xi$$

This is a set of two equations for the two unknowns A and B. If, in view of the formal procedures described in paragraph 4, one may, for a moment, consider n as a given function of x, the equation (5.4) is seen to be a partial differential equation of parabolic type. However, it

differs from the simplest and classical equation of this type, the heat
equation, by the very important fact that the sign of the coefficient of
$\frac{\partial f}{\partial x}$ changes in the field (for we have of course always to consider the
full range $(-\infty, +\infty)$ of values of ξ)*. Gevrey (ref. 8) who appears to
be the first (and, to our knowledge, the only one) to have dealt with such
parabolic equations involving a coefficient with variable sign, calls them
singular parabolic equations. He also shows, in his fundamental paper,
how the usual boundary problem pertinent to the classical heat equation must
be modified in the case of singular parabolic equation.

Before quoting his results, let us first stress that the
space variable x in equation (5.4) plays the role of the time variable in
the heat equation (because it appears in a first derivative only), while
the molecular velocity ξ plays the role of the space variable in the heat
equation (because it appears in a second derivative).

The simplest boundary value problem for the heat equation
(which corresponds to the simple infinite "wall" problem) consists in
giving the values of the unknown function at some initial time and also
at two boundaries (both sides of the "wall") at all subsequent times. In
the case of eq. (5.4) this would correspond to giving $f(0, \xi)$, and $f(x, \xi)$
for two values of ξ . Here, however, the normal boundaries for ξ being
$\pm \infty$, the boundary conditions become a condition about the asymptotic
behaviour of f for large absolute values of ξ , as already mentionned.

An important remark will immediately show the difficulties
connected with the singular parabolic equation: the heat equation cannot,
in general, be solved backwards in time.

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* In order to keep the usual meaning of the parameters, $\lambda n \overline{c^2}$ should of
  course always be chosen positive.

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Indeed, if values for the unknown functions are prescribed at some final time, as well as values at two space boundaries at earlier times, then the solution will be found to break down immediately, unless very special final values are chosen. This can be shown easily by using a Fourier series expansion of the solution in terms of the space variable. The coefficients of the expansion are found to be exponentials in the time variable, increasing for decreasing time and with an exponent increasing very rapidly with the order of the harmonics (the solution will therefore exist in a finite region of decreasing times, only if the number of initial harmonic components is finite or if the amplitude of the harmonics decreases sufficiently rapidly when their order is increased). This phenomenon is, of course, connected with the extreme "smoothing" that heat conduction produces on temperature distributions.

This remark shows then, that a classical boundary value problem, with one initial and two boundary conditions, cannot be solved, in general, for a singular parabolic equation, because the phenomenon just mentioned would certainly occur in the region where the coefficient of the time derivative* becomes negative, which is of course equivalent to a change of sign for the time variable or a permutation of past and future.

Gevrey shows in ref. 8 that the standard boundary value problem for singular parabolic equations must, as expected, be modified as follows:-

In the region of positive coefficient, initial values (for some time t_1, say) must be given, as well as a boundary condition for a

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* When referring to this coefficient, we always assume of course that the coefficient of the second derivative is equal to one or, at least, positive.

value of the space variable for $t > t_1$. In the region of negative coefficient, final values (for some $t_2 > t_1$, say) must be given, as well as a boundary condition for a value of the space variable for $t < t_2$.

The main problem arising in the solution is then seen to be a matching problem along the "singular" line, where the coefficient vanishes. Indeed, assuming arbitrary values for the unknown function on the singular line, one can obtain the solution in the region of positive coefficient just as in the classical heat conduction problem (because the initial values are given, as well as two boundary values) and the solution in the region of negative coefficient can be obtained similarly. However, we have to satisfy a matching condition along the singular line, because the normal derivative for both solutions just described must be equal at all points of this line. This condition can be written as an integral equation involving the unknown values on the singular line which are then obtained by solving the integral equation.

In the particular case of the model equation (5.4), the singular line corresponds to $\xi = 0$ and is of course the x-axis in a x, ξ diagram. The normal boundary conditions are, as already stressed, replaced by conditions defining the asymptotic behaviour of the distribution function. This, again, may be considered as a condition for $\xi = +\infty$ and another condition at $\xi \to -\infty$. The boundary conditions for the Gevrey problem must then further include the values of f at a certain abscissa x_1, for the positive values of the velocity ξ, as well as the values of f at another abscissa $x_2 > x_1$ for the negative values of ξ.

This is of course completely in agreement with what would be suggested by purely physical considerations. Consider a finite "linear volume" of one dimensional gas, on the segment $x_1 x_2$ say, and the boundary conditions at the two solid "walls" at $x = x_1$ and $x = x_2$. Interaction

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between the molecules and the solid walls may be very complicated, but will
always lead to an expression of the velocity distribution of the outgoing
molecules in terms of the distribution of the incoming molecules*. There
is, however, no mechanism described in the kinetic equation or in the
boundary conditions which enables the walls to control the velocity of
the incoming molecule, for this would involve action at a distance. Accord-
ingly, the conditions introduced by wall effects only affect outgoing
molecules, so that for the wall on the left, at $x = x_1$, we will obtain
information about the distribution function of the molecules with $\xi > 0$
and at the wall on the right, at $x = x_2 > x_1$, information about the
distribution function of the molecules with $\xi < 0$. This is similar to
a Gevrey problem. Of course, the situation is much more complicated here,
in that we must, in general, consider relationships between the values of
f for positive and negative ξ at the walls, and only in the case where
complete accommodation is assumed (i.e. outgoing molecules have no memory
of their condition at arrival) do we get the simple Gevrey type boundary
conditions.

We may, however, conclude that the model is very satisfac-
tory from the point of view of the nature of the pertinent boundary value
problems for the steady state, which are in complete agreement with the
physical nature. This again could be expected from the analogy with the
Fokker-Planck equation.

6. Unsteady Spatially Uniform Solutions for Equation (2.3).

It is interesting and comparatively easy to examine the
solutions of (2.3) depending only on t and ξ but not on x, i.e., the

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* For instance, through an integral relationship involving a stochastic
  kernel, as described in ref. 1.

solutions uniform in space. This will enable us to show that the model equation exhibits an infinite number of relaxation times and modes. The problem, in itself, has, of course, little physical significance, for the corresponding initial conditions (i.e., a non Maxwellian initial distribution of f, uniform in space) cannot easily be created and does not occur in practical cases.

The analogous problem has been treated, for the Boltzmann equation, and for the Maxwellian molecule model by Maxwell (ref. 9) and also involves an infinite number of relaxation modes.

In the spatially uniform case, the macroscopic equations (3.1), (3.2) and (3.3) reduce to

$$\frac{dn}{dt} = 0 \quad , \quad \frac{du}{dt} = 0 \quad \text{and} \quad \frac{d\,\overline{c^{\iota}}}{dt} = 0$$

so that n, u and $\overline{c^{\iota}}$ are constant and equal to their initial values, and equation (2.3) thus becomes linear.

We first introduce the non dimensional variables

$$\tau = \lambda n t \qquad\qquad \zeta = \frac{\xi - u}{\sqrt{\overline{c^{\iota}}}} \tag{6.1}$$

(reduced time and random molecular velocity) and obtain the following simplified version of (2.3):

$$\frac{\partial f}{\partial \tau} = f_{\zeta\zeta} + \zeta f_{\zeta} + f \tag{6.2}$$

while the auxiliary conditions, defining u and $\overline{c^{\iota}}$ as integrals over f, may be written

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$$\int_{-\infty}^{+\infty} (1-\zeta^2)\, f\, d\zeta = 0 \tag{6.3}$$

$$\int_{-\infty}^{+\infty} \zeta\, f\, d\zeta = 0 \tag{6.4}$$

(The equation defining n is irrelevant because the solution f will always involve an arbitrary constant factor which can be adjusted to obtain the correct density).

There are, of course, boundary conditions for $\xi = \pm\infty$ in that f must tend to zero for large $|\xi|$ and that the momentum and energy integrals must converge.

The resulting mathematical problem turns out to be entirely analogous to the problem of computing the wave functions for the quantum mechanical linear oscillator (ref. 10).

The problem is treated in a different way in appendix IV, using a Fourier transformation with respect to the velocity variable ξ, because this technique proves to be useful for the study of the linearized model equation (see also paragraph 7, eq. 7.15).

The general solution of the problem is given by

$$f = e^{-\frac{\xi^2}{2}} \left\{ a_0 + a_1 H_1(\zeta) e^{-\tau} + a_2 H_2(\zeta) e^{-2\tau} + \cdots + a_n H_n(\zeta) e^{-n\tau} + \cdots \right\}$$

where H_n is the Hermite polynomial of order n (ref. 11)

$$H_1(\zeta) = \zeta \quad , \quad H_2(\zeta) = \zeta^2 - 1 \qquad \text{etc.}$$

The auxiliary conditions must now be taken into account. Multiplying the solution obtained by ζ and ($\zeta^2 - 1$) respectively and integrating between $\xi = -\infty$ and $\xi = +\infty$ yield, because of the orthogonality properties of the Hermite polynomials (ref. 11):

$$\int_{-\infty}^{+\infty} \zeta f\, d\zeta = \int_{-\infty}^{+\infty} H_1(\zeta) f\, d\zeta = \sqrt{2\pi}\, a_1\, e^{-\tau}$$

$$\int_{-\infty}^{+\infty} (\zeta^2 - 1) f\, d\zeta = \int_{-\infty}^{+\infty} H_2(\zeta) f\, d\zeta = 2\sqrt{2\pi}\, a_2\, e^{-2\tau}$$

The auxiliary conditions therefore imply

$$a_1 = a_2 = 0$$

The other coefficients, $a_0, a_3, \ldots a_n$ etc. are well defined by the initial condition

$$f = f(0, \xi) \qquad \text{for } t = 0$$

because of the fact that the functions $e^{-\frac{\zeta^2}{2}} H_n(\zeta)$ form a complete orthogonal set.

Hence, it is clearly seen that the function f tends to a Maxwellian distribution.

$$f_0 = a_0\, e^{-\frac{(\xi - u_0)^2}{2 c_0^2}}$$

for $t \to \infty$, and this tendency is characterized by an infinite set of relaxation modes, each of which has its own relaxation time. The relaxation times are, in decreasing order

$$t_1 = \frac{1}{3\lambda n}\, , \quad t_2 = \frac{1}{4\lambda n}\, , \quad t_3 = \frac{1}{5\lambda n}\, , \cdots \qquad \text{etc.}$$

and correspond to faster and faster relaxation. The relaxation times are,

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as expected, inversely proportional to the number density.

It is also seen, from the form of the solution obtained in appendix IV, that f remains positive definite if it was initially positive definite.

7. Remarks on the Linearized Treatment of Equation (2.3)

It is useful to attempt the solution of linearized version of complicated equations before attacking the fully non linear cases. In the case of the model equation (2.3), as in the case of the Boltzmann equation the simplest example of a spatially non uniform linear problem occurs when considering a small, steady disturbance of an equilibrium, Maxwell distribution, of the type

$$f_0 = \frac{n_0}{\sqrt{2\pi\ \overline{C_0^2}}}\ e^{-\frac{(\xi-u_0)^2}{2\ \overline{C_0^2}}}$$

f_0 may be a spatially non uniform function if n_0, u_0 and $\overline{C_0^2}$ are considered to be functions of x (local Maxwellian distribution. See, for instance, appendix III).

We assume, here, that n_0, u_0 and $\overline{C_0^2}$ are constants so that we study disturbances from an absolute equilibrium (uniform initial conditions).

In this case we write, as is usual

$$f = f_0 + f'\quad,\quad n = n_0 + n'\quad,\quad u = u_0 + u'\quad,\quad \overline{C^2} = \overline{C_0^2} + \overline{C^2}'$$

where f', n' and $\overline{C^2}'$ are disturbances, which must be considered uniformly small compared to the corresponding equilibrium values f_c, n_0 and $\overline{C^2}$ while the non uniform bulk velocity component u' is limited by the condition

$$u' \ll \sqrt{\overline{c_o^2}}$$

a condition which may be expressed by saying that the Mach number based on the velocity disturbance must be very small.

The linearized version of equation (2.3) is obtained by neglecting the terms of order higher than one in the disturbances:

$$\frac{\partial f'}{\partial t} + \xi \frac{\partial f'}{\partial x} = \lambda n_o \left\{ \overline{c^2} f'_{\xi\xi} + (\xi - u_o) f'_\xi + f' \right\} + \lambda \left\{ (n\overline{c^2}) \int'_{o\xi\xi} + [n'\xi - (nu)'] f_{o\xi} + n' f_o \right\} \quad (7.1)$$

Equations (3.1) and (3.2) become

$$\frac{\partial n'}{\partial t} + \frac{\partial (nu)'}{\partial x} = 0 \quad (7.2)$$

$$\frac{\partial (nu)'}{\partial t} + \frac{\partial [(nu^2)' + (n\overline{c^2})']}{\partial x} = 0 \quad (7.3)$$

If we consider the steady state case, the time derivative drops and the equation (3.1) and (3.2) give

$$(nu)' = A' \quad (7.4)$$

$$(nu^2)' + (n\overline{c^2})' = B' \quad (7.5)$$

It is clear, however, that by a suitable choice of the reference values n_o, u_o, $\overline{c_o^2}$, one can make A' and B' vanish, which yields

$$(nu)' = n_o u' + n' u_o = 0$$

$$(nu^2)' + (n\overline{c^2})' = (nu)' u_o + n_o u_o u' + (n\overline{c^2})' = 0$$

so that, with this choice of reference values

$$u' = -\frac{u_o}{n_o} n'$$

(7.6)

$$\left(n\overline{c^2}\right)' = -n_o u_o u' = u_o^2 n'$$

(7.7)

It is useful to introduce the non dimensional variables

$$\tau = \lambda n_o t \quad , \quad \zeta = \frac{\xi}{\sqrt{\overline{c_o^2}}} \quad , \quad \eta = \lambda \frac{n_o}{\sqrt{\overline{c_o^2}}} x$$

(7.8)

as well as the parameter N', related to n' by

$$N' = \frac{n'}{\sqrt{\overline{c_o^2}}} = \int_{-\infty}^{+\infty} f' d\zeta$$

(7.9)

The reduced equation then becomes

$$\frac{\partial f'}{\partial \tau} + \zeta \frac{\partial f'}{\partial \eta} = f'_{\zeta\zeta} + \zeta f'_{\zeta} + f' + N' \left\{ (\mu_o^2 - 1)\zeta^2 - \mu_o(2\mu_o^2 - 1)\zeta + 1 - \mu_o^2 \right\} \frac{e^{-\frac{\zeta^2}{2}}}{\sqrt{2\pi}}$$

(7.10)

where μ_o is a non dimensional parameter, given by

$$\mu_o = \frac{u_o}{\sqrt{\overline{c_o^2}}}$$

(7.11)

which is, of course, closely related to the Mach number $\left(M = \frac{u_o}{a_o} = \frac{\mu_o}{\sqrt{3}}\right)$

In problems, such as heat transfer, where no bulk velocities exist, we must, of course, take $\mu_o = 0$ so that the equation is slightly simplified to

$$\frac{\partial f'}{\partial \tau} + \zeta \frac{\partial f'}{\partial \eta} = f'_{\zeta\zeta} + \zeta f'_{\zeta} + f' + N'(1 - \zeta^2) \frac{e^{-\frac{\zeta^2}{2}}}{\sqrt{2\pi}}$$

(7.12)

The Fourier integral transformation with respect to the
(reduced) velocity component ζ , as used in appendix IV, also provides
a simplification in the actual case. Let φ be the Fourier transform of
f', which exists in view of the fact that f must tend to zero for large
values of $|\zeta|$ in the case of physically valid solutions. We have

$$\varphi(\eta,\tau,\omega)= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f'(\eta,\tau,\zeta)e^{-i\omega\zeta}d\zeta \quad ; \quad f'(\eta,\tau,\zeta)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\varphi\, e^{i\omega\zeta}d\omega \quad (7.13)$$

We then transform the equation, taking into account the fundamental pro-
perties of the Fourier transformation (ref. 12)

$$i\omega\varphi= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\frac{\partial f'}{\partial\zeta}\right)e^{-i\omega\zeta}d\zeta \quad ; \quad i\frac{\partial\varphi}{\partial\omega}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(\zeta f')e^{-i\omega\zeta}d\zeta \quad (7.14)$$

and obtain

$$\frac{\partial\varphi}{\partial t} +i\frac{\partial^2\varphi}{\partial\omega\partial\eta} +\omega\frac{\partial\varphi}{\partial\omega}+\omega^2\varphi = \varphi(\eta,\tau,0)\{(1-\mu_o^2)\omega+i\mu_o(2\mu_o^2-1)\}\,\omega\, e^{-\frac{\omega^2}{2}} , \quad (7.15)$$

Using the obvious formula

$$N'= \int_{-\infty}^{+\infty} f'd\zeta = \sqrt{2\pi}\,\varphi(\eta,\tau,0) \quad (7.16)$$

and the fact that the function $e^{-\frac{\zeta^2}{2}}$ is invariant under the Fourier
transformation (ref. 12).

In the case of a steady state, it is seen that the singular
parabolic equation is reduced to a hyperbolic equation.

The simplest physically significant problems that arise in
connection with these equations are the steady state heat conduction pro-
blem and the wave problem.

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In the steady state heat conduction, one assumes $u_o = 0$ (and therefore $\mu_o = 0$), so that the following direct and transformed equations have to be used

$$\zeta \frac{\partial f'}{\partial \eta} = f'_{\zeta\zeta} + \zeta f'_{\zeta} + f' + N'(1-\zeta^2)\frac{e^{-\frac{\zeta^2}{2}}}{\sqrt{2\pi}} , \quad N' = \int_{-\infty}^{+\infty} f' d\zeta \quad (7.17)$$

$$i \frac{\partial^2 \varphi}{\partial \omega \partial \eta} + \omega \frac{\partial \varphi}{\partial \omega} + \omega^2 \varphi = \varphi(\eta, 0) \, \omega^2 e^{-\frac{\omega^2}{2}} \quad (7.18)$$

If the heat conduction is treated for a finite amount of gas between two walls, at $\eta = \eta_1$ and $\eta = \eta_2 > \eta_1$, the corresponding boundary conditions would be of the type

$$f'(\eta_1, \zeta) = g_1(\zeta) \qquad \text{for} \quad \zeta > 0$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (7.19)$$

$$f'(\eta_2, \zeta) = g_2(\zeta) \qquad \text{for} \quad \zeta < 0$$

Usually, one would assume that the molecules reflected from the wall are completely accomodated, thermally, with the wall. If the reference values $n_o, \overline{c_o^2}$ are chosen to correspond to, say, the conditions at the wall at $\eta = \eta_1$, we would then have the simplest set of possible boundary conditions

$$f'(\eta_1, \zeta) = 0 \qquad \text{for} \quad \zeta > 0$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (7.20)$$

$$f'(\eta_2, \zeta) = \frac{n_2 - n_o}{\sqrt{2\pi}\,\overline{c_o^2}} \frac{3-\zeta^2}{2} e^{-\frac{\zeta^2}{2}} \qquad \text{for} \quad \zeta < 0$$

The value for $g_2(\zeta)$ is arrived at by writing the Maxwellian distribution f_2 corresponding to the parameters $n_2 = n_o + (n_2 - n_o)$, $\overline{c_2^2} = \overline{c_o^2} + (\overline{c_2^2} - \overline{c_o^2})$ and computing the first order term of $f_2 - f_o$, taking into account the (integrated) macroscopic equation

$$n_2 \overline{c_2^2} = n_0 \overline{c_0^2}$$

The treatment of the equations for linearized heat conduction still appears to be very involved. Results obtained in this direction will be published shortly.

In the wave problem, one investigates unsteady sinusoidal solutions of the form

$$f' = g(\zeta) e^{i\nu\tau + \beta\eta} \tag{7.21}$$

where $g(\zeta)$ is an amplitude function. This is expected to lead to an eigenvalue problem, yielding, for each value of the frequency ν, to an infinite set of propagation modes. We have, of course

$$\beta = -i\frac{\nu}{a} - \alpha \tag{7.22}$$

where a is the speed of sound for a particular mode and α the corresponding attenuation.

The equation for $g(\zeta)$ becomes (assuming the medium at rest, and hence $\mu_0 = 0$)

$$g_{\zeta\zeta} + \zeta g_\zeta + (1 - i\nu - \beta\zeta) g + \left[\int_{-\infty}^{+\infty} g \, d\zeta \right] (1 - \zeta^2) \frac{e^{-\frac{\zeta^2}{2}}}{\sqrt{2\pi}} = 0 \tag{7.23}$$

and the transformed equation simplified to a first order equation

$$(\omega + i\beta) \psi_\omega + (\omega^2 + i\nu) \psi = \psi(0) \omega^2 e^{-\frac{\omega^2}{2}} \tag{7.24}$$

where ψ is the Fourier transform of g.

The reduction in order of the equation from second to first brought about by the transformation is of course very useful, for the

solution of the latter equation can be obtained at sight. It is therefore interesting to note that this reduction corresponds, here, to the rejection of (physically) meaningless solution types. We already noted, in paragraph 3, that there are two types of equilibrium solutions, one of them being rejected because it leads to infinite total momentum and energy. It is easily seen that this is just the type of function for which the Fourier transformation of the equation is not allowed, because terms such as $\zeta\, f$ lead to diverging transforms.

The Fourier transformation thus automatically eliminates the unacceptable solutions and a corresponding simplification of the transformed equation had to be expected.

The simplifications are not so obvious in the general case. The complete treatment of the wave problem will be presented in a companion report.

8. Preliminary Remarks on the Near Free Molecule Regime

One of the most interesting and difficult problems arising in the application of kinetic theory to aerodynamic problems is the study of the near free molecule regime, or in other words, the way in which solutions of the Boltzmann equation behave for very large Knudsen number.

The use of the model to clarify the situation is therefore indicated.

Actually, we cannot strictly define a Knudsen number for the model, because the collision term cannot be interpreted as representing two body collisions and the notion of mean free path therefore disappears.

However, it is clear that the free molecule regime corresponds (for given boundary conditions) to a vanishing $\overset{\wedge}{\lambda}$. We must therefore study the behaviour of solutions corresponding to given boundary conditions for λ tending to zero.

The following remarks concern the model equation for steady state i.e.

$$\xi \frac{\partial f}{\partial x} = \lambda n \left\{ \overline{c^2} \frac{\partial^2 f}{\partial \xi^2} + (\xi - u) \frac{\partial f}{\partial \xi} + f \right\} \qquad (8.1)$$

which reduces, for $\lambda = 0$ to $\qquad\qquad\qquad\qquad\qquad$ (8.2)

$$\frac{\partial f}{\partial x} = 0$$

It is clear that the type of the equation entirely changes for $\overset{\wedge}{\lambda} = 0$, so that one has a singular perturbation problem for small λ .

Let us consider the simplest case of heat conduction between two walls at $x = x_1, x_2 (x_2 > x_1)$ (u = 0). The physical boundary conditions are known to be of the type (Cf 7.19)

$$f(x_1, \xi) = g_1(\xi) \qquad for \ \xi > 0$$

$$f(x_2, \xi) = g_2(\xi) \qquad for \ \xi < 0 \qquad (8.3)$$

and the corresponding free molecule solution, (for $\lambda = 0$), is of course

$$f(x, \xi) = \begin{cases} g_1(\xi) & for \ \xi > 0 \\ g_2(\xi) & for \ \xi < 0 \end{cases} \qquad (8.4)$$

In general, we therefore expect singularities (discontinuities of f or its derivatives) at $\xi = 0$. The approximation for the free molecule solution must therefore break down in the vicinity of $\xi = 0$ in this simplest case. A small region will exist there, say from $\xi = -\epsilon$ to $\xi = +\epsilon$, where the interaction term still has to be taken into

account. The situation here is quite similar to the one leading to the occurence of boundary layers in fluid flows with small viscosity, where viscosity terms still have to be taken into account in regions of small extension close to the walls (as well as in the wakes).

One can readily evaluate the importance of the region where the interaction term is important, using an order of magnitude argument similar to the one used to compute boundary layer thickness in terms of Reynolds number in viscous flow theory.

Let L be a length scale for the problem at hand. We would take $L = x_2 - x_1$ in the heat conduction problem. We then assume $\frac{\partial f}{\partial x}$ to be of order $\frac{f}{L}$ (this in fact, leads to more rigorous definitions for L : $L = \max \left|\frac{1}{\frac{\partial \, l_{n} f}{\partial x}}\right|$, for instance). Let ε be the width of the region around the line $\xi = 0$ in which the interaction term is important. The definition of ε also has to be made more precise, by stating that $\frac{\partial f}{\partial \xi}$ is to be of order $\frac{f}{\varepsilon}$ and $\frac{\partial^2 f}{\partial \xi^2}$ of order $\frac{f}{\varepsilon^2}$ in the said region. The first member of equ. (8.1) is seen to be of order $\varepsilon \frac{f}{L}$ and the second member involves terms of the order of $\lambda n \overline{c^2} \frac{f}{\varepsilon^2}$ and $\lambda n f$ in the region between $\xi = \pm \varepsilon$. If ε is small compared to the velocity scale $\sqrt{\overline{c^2}}$ for the problem, the first term of the second member is seen to be dominant and an order of magnitude comparison then follows :

$$\varepsilon \frac{f}{L} \sim \lambda n \overline{c^2} \frac{f}{\varepsilon^2}$$

and, hence

$$\varepsilon \sim \left(\lambda n \overline{c^2} L\right)^{1/3}$$

so that ε is small of order $\frac{1}{3}$ when λ tends to zero. Furthermore,

equ. (8.1) reduces, in first approximation, to

$$\xi \frac{\partial f}{\partial x} = \lambda n \, \overline{c^i} \frac{\partial^2 f}{\partial \xi^2}$$

(8.5)

in the singular region. This equation is   analogue to the boundary layer equation in viscous flow theory. It is seen to be equivalent to the typical singular parabolic equation treated by Gevrey in ref.8, because $n \overline{c^i}$ reduces to a constant. It is, of course, useful to introduce suitable non dimensional variables

$$\Xi = \frac{\xi}{\varepsilon} = \frac{\xi}{(\lambda n \overline{c^i} L)^{1/3}} \qquad\qquad X = \frac{x}{L}$$

and (8.5) then reduces to the Gevrey canonical form

$$\Xi \frac{\partial f}{\partial X} = \frac{\partial^2 f}{\partial \Xi^2}$$

(8.6)

The boundary conditions (8.3) in the singular region, become for λ tending to zero

$$f(0, \Xi) = g_1(o) \qquad for \quad \Xi > 0$$

$$f(1, \Xi) = g_2(o) \qquad for \quad \Xi < 0$$

(8.7)

(taking $L = x_2 - x_1$, the origin of the x axis being displaced to the point $x = x_1$).

This is valid for $g_1(o) \neq g_2(o)$. If $g_1(o) = g_2(o)$ but $g_1'(o) \neq g_2'(o)$ we must write, in the singular region

$$f = g(o) + f^*$$

f^* obviously satisfies the same equation (8.6) but the pertinent boundary conditions are now

$$f^*(0, \Xi) = \left(\frac{dg_1}{d\xi}\right)_{\xi=0} \varepsilon \, \Xi \qquad for \quad \Xi > 0$$

$$f^*(1, \Xi) = \left(\frac{dg_2}{d\xi}\right)_{\xi=0} \varepsilon \, \Xi \qquad for \quad \Xi < 0$$

(8.8)

The contribution to f from the interaction term is therefore of order f in the singular region of thicknessε, and of order $\frac{\lambda n \overline{c^i} L}{|\xi|^3}$ elsewhere (i e for $|\xi| \gg \varepsilon$). If we then compute the influence of the interaction term on the deviation from free molecule values, for physically significant macroscopic quantities, such as $n, \overline{c^i}, \dot{q}$ etc., the contribution of the "boundary layer" will be of the order $\lambda^{1/3}$ and the contribution of the other regions of order λ. The first contribution is thus seen to be the main term in deviation from free molecule values. This deviation is then proportional to $\lambda^{1/3}$ for $\lambda \to 0$.

The details of the calculations, based on the solution of equ.(8.6) with the boundary layer conditions (8.7) or (8.8) will be presented in a forthcoming report.

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APPENDIX I - The Boltzmann Equation for Weak Interactions

We write, using the notation defined in paragraph 1, the momentum and energy relations for a collision, between molecules of equal mass :

$$\vec{\xi}^* + \vec{\xi}'^* = \vec{\xi} + \vec{\xi}'$$

$$\vec{\xi}^{*2} + \vec{\xi}'^{*2} \pm \vec{\xi}^2 + \vec{\xi}'^2 \tag{I.1}$$

Introducing the relative velocities $\vec{\delta}$ and $\vec{\delta}^*$ with respect to the center of mass of the molecule pair before and after collision

$$\vec{\delta} = \frac{\vec{\xi} - \vec{\xi}'}{2} \qquad \vec{\delta}^* = \frac{\vec{\xi}^* - \vec{\xi}'^*}{2} \tag{I.2}$$

we get, from the momentum equation :

$$\vec{\xi}^* - \vec{\xi} = -(\vec{\xi}'^* - \vec{\xi}') = \vec{\delta}^* - \vec{\delta}$$

and from the energy equation

$$|\vec{\delta}^*| = |\vec{\delta}| \tag{I.3}$$

If we put

$$\vec{\phi} = \vec{\delta}^* - \vec{\delta} \tag{I.4}$$

(variation of relative velocity caused by the collision), we may use the following series expansion, which will be useful to analyse weak interactions, corresponding to small values of $|\vec{\phi}|$ [1]

$$f(\vec{\xi}^*) = f(\vec{\xi}) + \phi_i \frac{\partial f}{\partial \xi_i} + \frac{1}{2}\left(\phi_i \frac{\partial}{\partial \xi_i}\right)^2 f + \frac{1}{6}\left(\phi_i \frac{\partial}{\partial \xi_i}\right)^3 f + \frac{1}{24}\left(\phi_i \frac{\partial}{\partial \xi_i}\right)^4 f + \cdots$$

and a similar expression for $f(\vec{\xi}'^*)$, replacing $\vec{\xi}$ by $\vec{\xi}'$ and $\vec{\phi}$ by $-\vec{\phi}$

[1] The Einstein summation convention is used throughout this appendix : If an index appears once in a term, one should take the sum of the terms corresponding to the values 1,2,3 of the index, unless stated otherwise : $\phi_i \frac{\partial f}{\partial \xi_i}$ therefore means $\sum_{i=1,2,3} \phi_i \frac{\partial f}{\partial \xi_i}$

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After some algebra, the following expansion is obtained
for the main factor in the Boltzmann collision term

$$f^*f'^* - ff' = \phi_i\,(f'f_i - ff_i') + \phi_i\phi_j(-f_if_j' + \tfrac{1}{2}ff_{ij}' + \tfrac{1}{2}f'f_{ij}) + \phi_i\phi_j\phi_k\left(\tfrac{1}{2}f_if_{jk}'\right.$$

$$-\tfrac{1}{2}f_i'f_{jk} + \tfrac{1}{6}f'f_{ijk} - \tfrac{1}{6}ff_{ijk}'\right) + \phi_i\phi_j\phi_k\phi_\ell\left(\tfrac{1}{4}f_{ij}f_{k\ell}' + \tfrac{1}{24}f'f_{ijk\ell} + \tfrac{1}{24}ff_{ijk\ell}'\right.$$

$$\left.-\tfrac{1}{6}f_if_{jk\ell}' - \tfrac{1}{6}f_i'f_{jk\ell}\right) + \cdots \tag{I.5}$$

Fig. 1 indicates the geometry of a collision, as observed from the
center of mass of the system formed by the two colliding molecules
(this center of mass of course moves with uniform velocity during the
whole collision process, because of the principle of equality of
action and reaction). θ is the scattering angle and we have, clearly

$$\vec{\phi} = \vec{\delta}^* - \vec{\delta} = \vec{\delta}(\cos\theta - 1) + \vec{\zeta} \qquad ; \; |\vec{\zeta}| = |\vec{\delta}|\sin\theta \tag{I.6}$$

where $\vec{\zeta}$ is a vector perpendicular to $\vec{\delta}$. If we assume weak
interactions, θ will be very small and we use the expansions

$$\vec{\phi} = -\vec{\delta}\left(\tfrac{\theta^2}{2} - \tfrac{\theta^4}{24} + \cdots\right) + \vec{\zeta} \qquad\qquad |\vec{\zeta}| = |\vec{\delta}|\left(\theta - \tfrac{\theta^3}{6} + \cdots\right) \tag{I.7}$$

Now, the collision term obviously involves an integration
with respect to $\mathcal{E}$, the angle defining the position of the plane Π
of the relative motion (relative to a reference plane). We have,
by symmetry :

$$\int_0^{2\pi} \vec{\zeta}\, d\mathcal{E} = 0$$

which leads to the important conclusion that no term of first order
in θ is going to appear in the collision integral.

In order to compute further terms of this integral, we
need to integrate products of components of $\vec{\zeta}$ with respect to $\mathcal{E}$.

It is clear that many such products will yield vanishing integrals

A mathematical trick greatly simplifies these laborious computations. Let $\vec{a}$ be an arbitrary vector, α the angle between the vectors $\vec{a}$ and $\vec{\delta}$ and β the angle between $\vec{\zeta}$ and the projection of $\vec{a}$ on a plane perpendicular to $\vec{\delta}$ (see fig 1) We may choose the arbitrary origin for the angle ε to coincide with this projection so that $\varepsilon = \beta$ and

$$\vec{a}_o\vec{\zeta} = |\vec{a}| \sin\alpha \, |\vec{\zeta}| \cos\varepsilon = |\vec{a}| \sin\alpha \, |\vec{\delta}| \sin\theta \cos\varepsilon = |\vec{a}\wedge\vec{\delta}| \sin\theta \cos\varepsilon$$

The method suggested for the simplification of the integrals consists in using the following identity, for a product of n components of $\vec{\zeta}$ (some of which may be identical)

$$\zeta_i \zeta_j \zeta_k \cdots = \left(\frac{\partial^n}{\partial a_i \partial a_j \partial a_k \cdots} \, e^{\vec{a}\vec{\zeta}}\right)_{\vec{a}=0}$$

From this, one immediately concludes that

$$\int_0^{2\pi} (\zeta_i \zeta_j \zeta_k \cdots)\, d\varepsilon = \left(\frac{\partial^n}{\partial a_i \partial a_j \partial a_k \cdots} \int_0^{2\pi} e^{\vec{a}.\vec{\zeta}}\, d\varepsilon\right)_{\vec{a}=0}$$

The integral can be transformed into a tractable form as follows [1]

$$\int_0^{2\pi} e^{\vec{a}.\vec{\zeta}}\, d\varepsilon = \int_0^{2\pi} e^{|\vec{a}\wedge\vec{\delta}|\sin\theta\cos\varepsilon}\, d\varepsilon = \int_0^{2\pi} \sum_{\nu=0}^{\infty} \frac{|\vec{a}\wedge\vec{\delta}|^\nu \sin^\nu\theta \cos^\nu\varepsilon}{\nu!}\, d\varepsilon = 2\pi \sum_{\mu=0}^{\infty} \frac{\{\vec{a}^2\vec{\delta}^2-(\vec{a}.\vec{\delta})^2\}^\mu \sin^{2\mu}\theta}{2^{2\mu}(\mu!)^2}$$

using the identity

$$(\vec{a}\wedge\vec{\delta})^2 = \vec{a}^2\vec{\delta}^2 - (\vec{a}.\vec{\delta})^2$$

---

[1] The integral can actually be expressed in terms of Bessel functions but this does not seem useful for the present purpose.

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Computation of the integrals over ε of $\zeta_i\zeta_j$ and $\zeta_i\zeta_j\zeta_k\zeta_\ell$ then becomes immediate :

$$\int_0^{2\pi}\zeta_i\zeta_j\,d\varepsilon = 2\pi\,\tfrac{1}{4}\left\{\frac{\partial^2}{\partial a_i\partial a_j}[\vec{a}^2\vec{\delta}^2-(\vec{a}.\vec{\delta})^2]\sin^2\theta\right\}_{\vec{a}=0} = \pi\left(\vec{\delta}^2\delta_{ij}-\delta_i\delta_j\right)\sin^2\theta$$

$$\int_0^{2\pi}\zeta_i\zeta_j\zeta_k\zeta_\ell\,d\varepsilon = 2\pi\,\tfrac{1}{64}\left\{\frac{\partial^4}{\partial a_i\partial a_j\partial a_k\partial a_\ell}[\vec{a}^2\vec{\delta}^2-(\vec{a}.\vec{\delta})^2]^2\sin^4\theta\right\}_{\vec{a}=0} \qquad\qquad (I.8)$$

$$= \frac{\pi}{4}\left\{(\delta_{k\ell}\delta^2-\delta_k\delta_\ell)(\delta_{ij}\delta^2-\delta_i\delta_j)+(\delta_{i\ell}\delta^2-\delta_i\delta_\ell)(\delta_{jk}\delta^2-\delta_j\delta_k)+(\delta_{j\ell}\delta^2-\delta_j\delta_\ell)(\delta_{ik}\delta^2-\delta_i\delta_k)\right\}\sin^4\theta$$

(δ_{ij} denotes the Kronecker symbol: δ_{ij} = 0 for $i\neq j$, 1 for $i=j$).

After integration with respect to ε, we therefore get the following expansion in powers of θ :

$$\int_0^{2\pi}(f^*f'^*-ff')\,d\varepsilon = 2\pi\,\theta^2\left(-\tfrac{1}{2}\delta_i\,(f'f_i-ff_i')+\tfrac{1}{2}(\delta^2\delta_{ij}-\delta_i\delta_j)(-f_if_j'+\tfrac{1}{2}ff_{ij}'+\tfrac{1}{2}f'f_{ij})\right)$$
$$+\cdots \qquad\qquad (I.9)$$

The remaining terms are of order θ^4 or higher

The first approximation for weak interaction therefore would lead to a collision integral of the form

$$\frac{\pi}{2}\iiint_{-\infty}^{+\infty}\left\{-(\xi_i-\xi_i')(f'f_i-ff_i')+\tfrac{1}{2}[(\vec{\xi}-\vec{\xi}')^2\delta_{ij}-(\xi_i-\xi_i')(\xi_j-\xi_j')](-f_if_j'+\tfrac{1}{2}ff_{ij}'+\tfrac{1}{2}f'f_{ij})\right\}\left\{\int_0^{\infty}|\vec{\xi}-\vec{\xi}'|\theta^2b\,db\right\}d\xi_1'd\xi_2'd\xi_3'$$

$$(I.10)$$

(the next term being of order 4)

The dynamics of a collision must now be analyzed in more detail in order to obtain the relationship connecting the scattering angle θ with the collision parameter b and the relative initial velocity $|\vec{\xi}-\vec{\xi}'| = 2\delta$. In the spirit of the weak interaction assumption, one should again try to obtain an expansion in powers of a parameter indicating the order of magnitude of the force of interaction

The first term in such an expansion is very simple for it can be computed assuming that the trajectories are undisturbed. The final lateral momentum in the relative motion, $m\vec{\zeta}$, induced by the central interaction force $\vec{F}$ is simply obtained as the total impulse produced by this force in the direction perpendicular to the undisturbed trajectory; i.e.

$$m|\vec{\zeta}| \cong \int_{-\infty}^{+\infty} F \cdot \frac{b}{r} dt \cong \frac{b}{|\vec{\xi}-\vec{\xi'}|} \int_{b}^{\infty} F \frac{dr}{\sqrt{r^2-b^2}}$$

Now,

$$|\vec{\zeta}| = |\vec{\delta}| \sin\theta \sim |\vec{\xi}-\vec{\xi'}| \cdot \theta/2$$

So that, in the first approximation

$$\theta \sim \frac{2|\vec{\zeta}|}{|\vec{\xi}-\vec{\xi'}|} = const / |\vec{\xi}-\vec{\xi'}|^2$$

This leads to a singular factor $\frac{1}{|\vec{\xi}-\vec{\xi'}|^3}$ in the integral (I.10) which is, however, immediately seen to converge (because of the first factor which involves terms of order $|\vec{\xi}-\vec{\xi'}|$ and $|\vec{\xi}-\vec{\xi'}|^2$)

If $\vec{F}$ is a power law interaction

$$|\vec{F}| = K r^{-s} \qquad (S>0)$$

then

$$\int_{b}^{\infty} F \frac{dr}{\sqrt{r^2-b^2}} = K b^{-s} \int_{1}^{\infty} \frac{\rho^{-s} d\rho}{\sqrt{\rho^2-1}} \qquad (\rho = r/b)$$

so that θ is proportional to b^{1-s} and $\theta^2 b$ proportional to b^{3-2s}. Therefore, the integral $\int_{0}^{\infty} \theta^2 b\, db$ would converge for $S>2$ and diverge for $S \leqslant 2$, the second case certainly showing a predominance of weak interactions. The divergence of the integral has to be remedied by considering a finite cut off distance, as is done in the theory of ionized gases.

We may, of course, consider a force law which would nowhere lead to strong interactions, for instance by considering that $\vec{F}$ is bounded for $r \to 0$.

The collision integral (I.10) is still very complicated, however, because of the coupling factor

$$|\vec{\xi}-\vec{\xi'}| \int_o^\infty \theta^2 b\, db$$

which is proportional to $|\vec{\xi}-\vec{\xi'}|^{-3}$ .Maxwell has shown that this coupling factor disappears, in the full Boltzmann collision term, when the interaction force is proportional to r^{-5} (Maxwell's molecular model).

It would, of course, be meaningless to consider weak interactions for such a power law in view of the obvious predominance of the strong interactions in the collision term.

However, one could imagine a rather artificial interaction model which would induce in the collision term (I.10), the same simplification as the Maxwell interaction induces in the Boltzmann collision integral. In order to do this, one must assume the interaction force to depend on the relative velocity of the colliding molecules; i.e. $\vec{F}$ must be proportional to the cube of this relative velocity. If we then put

$$\frac{\pi}{8} \int_o^\infty |\vec{\xi}-\vec{\xi'}| \, \theta^2 b\, db = \lambda$$

λ being now a constant, integration with respect to $\vec{\xi}$ ' becomes feasible in I.10. The derivatives of f ' may be eliminated by partial integration and the terms at the limit will vanish because of the required asymptotic behaviour of f

The following integrals are then seen to appear as coefficients of f and its various derivatives in I.10 :

$$\iiint f' \, d\xi_1' d\xi_2' d\xi_3' = n$$

$$\iiint f' \xi_i' \, d\xi_1' d\xi_2' d\xi_3' = n u_i$$

$$\iiint f' \xi_i' \xi_j' \, d\xi_1' d\xi_2' d\xi_3' = n u_i u_j + \frac{\tau_{ij}}{m} = n u_i u_j + n \overline{c_i c_j}$$

n being the number density, $\vec{u}$ the bulk (or macroscopic) velocity, τ_{ij} the stress tensor and $\vec{c}$ the random molecular velocity.

The collision integral for this model reduces to

$$\lambda n \left\{ \left[(\vec{\xi}-\vec{u})^2 + \overline{c^2} \right] \Delta_{\xi} f - \left[(\xi_i - u_i)(\xi_j - u_j) + \overline{c_i c_j} \right] f_{ij} + 6f \right\} \qquad (\text{I}.11)$$

where

$$\Delta_{\xi} = \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \xi_i}$$

is the Laplace operator in velocity space.

The collision integral therefore appears as a linear second order differential form in f , the coefficients of which are, in fact, integrals over f . This collision term is accordingly still non linear and integro differential with respect to f. However, the non linearity and integral dependance here appear through a limited number of parameters, namely the 10 moments n, $n u_i$ and τ_{ij}, which might produce considerable simplification in the treatment of this model.

APPENDIX II - Derivation of the Interaction Term Satisfying the

Condition of Paragraph 1.

The interaction term is of the form

$$A \frac{\partial^2 f}{\partial \xi^2} + B \frac{\partial f}{\partial \xi} + C f$$

where A, B, C should be functions of ξ and simple moments of the distribution, such as n, nu and p

The conservation of total mass, momentum and energy in presence of an interaction of this type requires the following integral conditions to be satisfied

$$\int [A \frac{\partial^2 f}{\partial \xi^2} + B \frac{\partial f}{\partial \xi} + C f] d\xi = 0$$

$$\int [A \frac{\partial^2 f}{\partial \xi^2} + B \frac{\partial f}{\partial \xi} + C f] \xi \, d\xi = 0$$

$$\int [A \frac{\partial^2 f}{\partial \xi^2} + B \frac{\partial f}{\partial \xi} + C f] \xi^2 \, d\xi = 0 \qquad \text{II.1}$$

The derivatives of f may be eliminated by partial integration (assuming all integrals to exist), and the following conditions for A, B, C are then obtained :

$$\int [\frac{d^2 A}{d\xi^2} - \frac{dB}{d\xi} + C] f \, d\xi = 0$$

$$\int [\frac{d^2}{d\xi^2}(\xi A) - \frac{d}{d\xi}(\xi B) + \xi C] f \, d\xi = 0$$

$$\int [\frac{d^2}{d\xi^2}(\xi^2 A) - \frac{d}{d\xi}(\xi^2 B) + \xi^2 C] f \, d\xi = 0 \qquad \text{II.2}$$

It is immediately clear that A, B, C must depend on some moments of f, for if A, B, C were completely independant of f, and II.2 was to be satisfied for all f, one would be led to a trivial solution A = B = C = 0

The assumption that A, B, C are to depend only on n, mu and p is equivalent to satisfying condition II.2 for all f satisfying the

$$\frac{d^2 A}{d\zeta^2} - \frac{dB^*}{d\zeta} + C^* \qquad = \alpha^* \zeta + \alpha'^*(\zeta^2-1) = L$$

$$\frac{d^2(A\zeta)}{d\zeta^2} - \frac{d(B^*\zeta)}{d\zeta} + \zeta C^* \quad = \beta^* \zeta + \beta'^*(\zeta^2-1) = M \qquad \text{II.6}$$

$$\frac{d^2(A\zeta^2)}{d\zeta^2} - \frac{d(B^*\zeta^2)}{d\zeta} + \zeta^2 C^* = \gamma^* \zeta + \gamma'^*(\zeta^2-1) = N$$

with

$$B^* = \sqrt{\bar{c^2}}\, B, \; C^* = \bar{c^2} C, \; \alpha^* = \alpha\, \bar{c^2}, \alpha'^* = \alpha'\bar{c^2}, \; \beta^* = (\beta-\alpha u)\sqrt{\bar{c^2}}, \beta'^* = (\beta'-\alpha' u)\sqrt{\bar{c^2}}, \; \gamma^* = \gamma - 2\beta u + \alpha u^2$$

$$\gamma'^* = \gamma' - 2\beta' u + \alpha' u^2$$

Combining the equations II.6 linearly among themselves (with respective coefficients ζ^2, -2ζ and 1, we get

$$2\, A = \zeta^2 L - 2\zeta M + N \qquad \text{II.7}$$

Multiplying the first equation by ζ and substracting the second yields

$$B^* = 2 \frac{dA}{d\zeta} + L\zeta - M \qquad \text{II.8}$$

and finally

$$C^* = L + \frac{dB^*}{d\zeta} - \frac{d^2 A}{d\zeta^2} \qquad \text{II.9}$$

We must now return to the third condition set in paragraph 1, namely, that the Maxwellian distribution

$$f = \text{Const.}\, e^{-\frac{(\xi-u)^2}{2\bar{c^2}}} = \text{Const}\; e^{-\zeta^2/2}$$

be a solution corresponding to equilibrium, or, in other words, that substitution of this function in the interaction term will yield zero. In terms of the unknown coefficients A, B*, C*, this reduces to the condition

$$A(\zeta^2-1) - B^* \zeta + C^* = 0 \qquad \text{II.10}$$

2

Now, according to (II.6) (II.7) (II.8) and (II.9), A, B*, and C* are seen to be polynomials, the degrees of which are respectively 4, 3 and 2 at most (L, M, N being quadratic in ζ). According to (II.10), however, A can only be of degree 2 at most, so that we must have (see II.7)

$$L \equiv 0 \quad \text{or} \quad \alpha^* = \alpha'^* = 0 \;,\; \beta'^* = 0$$

But the last result shows B* to be a linear function of ζ (see II.8) and we conclude then, from II.10 that A must be a constant

$$A = \tfrac{1}{2}(N - 2\zeta M) = \text{const.}$$

Introducing the explicit expressions for N and M then yields

$$\gamma^* \zeta + (\gamma'^* - 2\beta^*)\zeta^2 - \gamma'^* = 2A$$

and hence

$$\gamma^* = 0 \;,\quad \gamma'^* = -2A = 2\beta^*$$

We have then, finally

$$A = \text{const.}$$
$$B^* = A\,\zeta$$
$$C^* = A$$

These values obviously satisfy the system of equations (II.6) and (II.10). Reverting to the original variables and coefficients we get, for the interaction term

$$\frac{A}{c^2}\left\{ \bar{c}^2\, \frac{\partial^2 f}{\partial \xi^2} + (\xi - u)\frac{\partial f}{\partial \xi} + f \right\} \tag{II.11}$$

A is a function of $\bar{c}^2$ and n which is taken to be $\lambda\, n\,\bar{c}^2$ in paragraph 2.

APPENDIX III - Hydrodynamic Equations Corresponding to the Model

The following macroscopic equations are immediately obtained when the model equation (2.3) is multiplied successively by m, $m\,\xi$ and $m\dfrac{\xi^2}{2}$ and integrated over the velocity component ξ from $-\infty$ to $+\infty$ taking into account the conservation properties for the interaction term :

Continuity :

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \qquad\qquad \text{III.1}$$

Momentum :

$$\rho\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) u = -\frac{\partial p}{\partial x} \qquad\qquad \text{III.2}$$

Energy

$$\rho\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left(\frac{u^2}{2} + \frac{1}{2}\frac{p}{\rho}\right) + \frac{\partial p u}{\partial x} = -\frac{\partial \dot{q}}{\partial x} \qquad\qquad \text{III.3}$$

with the heat flux

$$\dot{q} = \int_{-\infty}^{+\infty} \frac{1}{2}\,m c^3 f\, d\xi \qquad\qquad \text{III.4}$$

If we consider a situation for which the deviation from equilibrium is negligible, the heat flux disappears (for f is then a Maxwellian distribution) and we obtain the equations of one dimensional motion for ideal gas with $\gamma = 3$, as expected from statistical mechanics.

As already stressed, the only effect of a deviation from equilibrium is the appearance of a heat flux, because no viscosity can appear in a hypothetical gas with purely one dimensional molecular motion.

In order to derive from (2.3) a complete system describing the hydrodynamics of the one dimensional gas, similar to the classical Navier - Stokes - Fourier equations, a technique similar to the various expansion procedures applied to the Boltzmann equation will be used (Cf Ref.2)

Obviously we require a formula for $\dot{q}$ in terms of the gradients of the other physical parameters (n, u, p)

To obtain an equation for $\dot{q}$, we transform equ. (2.3) using the variable c rather than ξ. This yields

$$\frac{\partial f}{\partial t} + (c+u)\frac{\partial f}{\partial x} - \frac{\partial f}{\partial c}\left[\frac{\partial u}{\partial t} + (c+u)\frac{\partial u}{\partial x}\right] = \lambda\, n\left(\overline{c^2}\frac{\partial^2 f}{\partial c^2} + c\frac{\partial f}{\partial c} + f\right)$$

(III.5)

Multiplication of both members by $\frac{1}{2}\, mc^3$ and integration with respect to C produces the required equation of transfer for $\dot{q}$. Partial integrations must be carried out on the interaction term in order to eliminate the derivatives of f with respect to C. The result is :

$$\frac{\partial \dot{q}}{\partial t} + u\frac{\partial \dot{q}}{\partial x} + \frac{\partial}{\partial x}\int_{-\infty}^{+\infty}\tfrac{1}{2}mc^4 f\,dc + 2\dot{q}\frac{\partial u}{\partial x} + \frac{3}{2}\,nm\overline{c^2}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -3\lambda n\dot{q}$$

We now look for a first approximation to this relationship under the assumption that the space and time derivatives of f are small and that f is well approximated by a local equilibrium distribution f_0 It is then immediately seen that $\dot{q}$ is a first order term in the derivatives, (so that the derivatives of $\dot{q}$ and $\dot{q}\frac{\partial u}{\partial x}$ may be neglected) and that

$$\frac{\partial}{\partial x}\int_{-\infty}^{+\infty}\tfrac{1}{2}mc^4 f\,dc \sim \frac{\partial}{\partial x}\int_{-\infty}^{+\infty}\tfrac{1}{2}mc^4 f_0\,dc$$

neglecting higher order terms.

But

$$f_0 = \frac{n}{\sqrt{2\pi \overline{c^2}}}\, e^{-\frac{c^2}{2\overline{c^2}}}$$

so that

$$\int_{-\infty}^{+\infty} c^4 f_0\,dc = \frac{n}{\sqrt{2\pi \overline{c^2}}}\left(\frac{\partial^2}{\partial \chi^2}\int_{-\infty}^{+\infty}e^{-\chi c^2}dc\right)_{\chi=\frac{1}{2\overline{c^2}}} = \frac{n}{\sqrt{2\pi \overline{c^2}}}\left(\frac{\partial^2}{\partial \chi^2}\frac{\sqrt{\pi}}{\sqrt{\chi}}\right)_{\chi=\frac{1}{2\overline{c^2}}} = 3n(\overline{c^2})^2$$

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The first approximation to the equation of transfer for $\dot{q}$ is, accordingly, given by

$$\frac{\partial}{\partial x}\left(\tfrac{3}{2}\,nm\overline{(c^2)}^2\right) + \tfrac{3}{2}\,nm\,\overline{c^2}\left(-\tfrac{1}{\rho}\,\frac{\partial p}{\partial x}\right) = -3\lambda n\dot{q}$$

so that

$$\dot{q} = -\frac{1}{2}\,\frac{m\overline{c^2}}{\lambda}\,\frac{\partial\overline{c^2}}{\partial x} = -\frac{R^2m}{2\lambda}\,T\frac{\partial T}{\partial x}$$

(III.6)

(The temperature T being defined by Equ. 3.7)

Formula (III.6) is equivalent to the Newton-Fourier law of heat conduction, the heat transfer coefficient k being given for the one dimensional model gas, by

$$k = \frac{mR^2T}{2\lambda}$$

It is seen to be proportional to the temperature and independent of pressure, just as in the case of the Maxwell molecular model with inverse fifth power interaction.

Equation (2.3) is thus seen to provide a dissipative mechanism for $\lambda > 0$, at least if the disturbance, from equilibrium is small.

Let us now consider some simple steady state problems for the model equation

In the steady state, equations III.1, 2 and 3 reduce to

$$\frac{\partial\rho u}{\partial x} = 0$$

$$\rho u\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x}$$

$$\rho u\frac{\partial}{\partial x}\left(\frac{u^2}{2} + \tfrac{1}{2}\frac{p}{\rho}\right) + \frac{\partial\rho u}{\partial x} = -\frac{\partial\dot{q}}{\partial x}$$

These may be immediately integrated as follows (see also equ. (5.2) (5.3)

$$\rho u = \mathcal{M} = m A$$
$$\rho u^2 + p = \mathcal{L} = m B$$
$$\mathcal{M}\left(\frac{u^2}{2} + \frac{3}{2}\frac{p}{\rho}\right) + \dot{q} = \mathcal{K}$$

$$(\text{III}.7)$$

($\mathcal{L}, \mathcal{M}, \mathcal{K}$ being constants)

The simple heat conduction problem, with $u = 0$, is immediately solved as follows, in the first approximation

$$p = \mathcal{L} = const$$
$$\dot{q} = -\frac{R^2 m}{2\lambda} T \frac{\partial T}{\partial x} = \mathcal{K} = const.$$

and hence

$$T = \sqrt{T_1^2 + (T_2^2 - T_1^2)\frac{x - x_1}{x - x_2}} \cdot$$

$$(\text{III}.8)$$

where T_1 and T_2 are the temperatures at the "walls", located at the abscissae x_1 and x_2

The ideal fluid approximation (corresponding to $\dot{q} = 0$) leads to the classical shock condition, but for the case $\gamma = 3$

$$\rho_1 u_1 = \rho_2 u_2 = \mathcal{M}$$
$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 = \mathcal{L}$$
$$\frac{u_1^2}{2} + \frac{3}{2}\frac{p_1}{\rho_1} = \frac{u_2^2}{2} + \frac{3}{2}\frac{p_2}{\rho_2} = \mathcal{K}/\mathcal{M}$$

$$(\text{III}.9)$$

If we introduce the Mach number M, and the speed of sound a :

$$M = u/a = u/\sqrt{3 p/\rho}$$

the following relations are easily derived from the shock conditions :

$$3 M_1^2 M_2^2 - M_1^2 - M_2^2 - 1 = 0$$
$$u_2 = \frac{1}{2}\left(u_1 + \frac{a_1^2}{u_1}\right)$$

$$(\text{III}.10)$$

The detailed structure for a weak shock may be obtained approximately by using the first approximation described by the expression (III.6) for $\overset{\bullet}{q}$ and substituting this in equation (III.7).

We have

$$RT = p/\rho = (\mathcal{L} - \mathcal{H}u)u/\mathcal{H} = (\mathcal{L}/\mathcal{H})u - u^2$$

$$\overset{\bullet}{q} = -(m/4\lambda)\frac{d(RT)^2}{dx} = \mathcal{H}u^2 - (3/2)\mathcal{L}u + \mathcal{K}$$

An ordinary differential equation for u is readily obtained by eliminating RT

$$(m/4\lambda)\frac{du}{dx} = -\frac{\mathcal{H}u^2 - (3/2)\mathcal{L}u + \mathcal{K}}{2\left(u - \frac{\mathcal{L}}{\mathcal{H}}\right)\left(2u - \frac{\mathcal{L}}{\mathcal{H}}\right)u}$$

$\frac{du}{dx}$ must vanish at upstream and downstream infinity, so that we must have

$$\mathcal{H}u^2 - (3/2)\mathcal{L}u + \mathcal{K} \equiv \mathcal{H}(u - u_1)(u - u_2)$$

(this identity can of course be derived directly from the shock conditions III.9)

From this identity we also get

$$(3/2)(\mathcal{L}/\mathcal{H}) = u_1 + u_2$$

and the equation for the shock structure then reduces to

$$(m/4\lambda\mathcal{H})\frac{du}{dx} = -\frac{(u - u_1)(u - u_2)}{4\,u[u - (2/3)(u_1 + u_2)][u - (1/3)(u_1 + u_2)]} \qquad (III.11)$$

It is important to note that we have, in view of III.10 and assuming u_1 to represent the upstream supersonic conditions :

$$u_1 > u_2 > u_1/2$$

and therefore

$$(2/3)(u_1 + u_2) > u_1 > u_2 \quad , \quad (1/3)(u_1 + u_2) < u_2 < u_1$$

so that the denominator of III.11 is always positive for $u_1 \geqslant u \geqslant u_2$

This is typical for $\gamma = 3$. For lesser values of γ, as always encountered in physical situations, this condition does not hold, in general, and one can expect a sufficiently strong wave still to exhibit purely kinematic discontinuity in its fine structure in the absence of viscosity, as shown in ref. 13. In the present case, however, the velocity profile is continuous and given, implicitly by simple integration of III.11, i.e. (using III.10)

$$-\frac{u^2}{2} + \frac{a_1^2}{g} \; \frac{3M_1^2-1}{M_1^2-1} \left\{ \ln(u_1-u) - \frac{M_1^2+1}{2M_1^2} \ln(u-u_2) \right\} = \left(\lambda \mathcal{K}/m \right) x + const. \quad \text{(III.12)}$$

The first member being a monotonic decreasing function of u, the plot of u as a function of x can easily be derived and we have $u = u_1$ for $x = +\infty$ and $u = u_2$ for $x = -\infty$, which indicates the irreversibility of the shock-process.

APPENDIX IV - Application of the Fourier Transformation to the Solution

of the Spatially Homogeneous Case :

We use the transformed equation (7.15). In the spatially homogeneous case, the derivative with respect to x disappears. Furthermore we can always take $u_0 = 0$ and hence, $\mu_0 = 0$, because u_0 is a constant in the present case and may therefore be reduced to zero by means of a Galilean transformation which does not change the spatially homogeneous character of the problem. The term $\varphi(\tau, 0) \omega^2 e^{-\omega^2/2}$, which arises because of the linearization does not occur here because the spatially homogeneous case is essentially linear.

The equation to be treated then reduces to :

$$\frac{\partial \varphi}{\partial \tau} + \omega \frac{\partial \varphi}{\partial \omega} + \omega^2 \varphi = 0 \tag{IV.1}$$

We first solve the problem using the eigenvalue method, which reduces here to the computation of the relaxation times τ_ν and the corresponding relaxation modes. We thus have to look for solutions of (IV.1) of the type

$$\psi(\omega) \, e^{\sigma \tau}$$

(IV.1) then reduces to an ordinary equation for ψ ;

$$\omega \frac{d\psi}{d\omega} + (\omega^2 + \sigma) \psi = 0 \tag{IV.2}$$

This represents an eigenvalue problem in the parameter σ, ψ representing the corresponding eigenfunction, as will now be shown.

The general solution of the first order equation IV.2, without second member is immediately found to be

$$\psi = A \, e^{-\omega^2/2} . \, \omega^{-\sigma} \tag{IV.3}$$

A being a constant.

The eigenvalue nature of the problem results from the condition that ψ will be the Fourier transform of a function which is expected to behave like $e^{-k\zeta^2}$ for large $|\zeta|$. Therefore $\psi(\omega)$ exists for all finite complex values of ω, because

$$\int_{-\infty}^{+\infty} e^{-k\zeta^2 - i\omega\zeta} f(\zeta)\, d\zeta$$

exists even if ω has an imaginary part. $\psi(\omega)$ is thus seen to be an integral analytic function and this is a strong condition on the solution IV.3. In fact, this solution, in general exhibits a singularity at $\omega = 0$ except for zero or negative integral values of the parameter σ This remark solves our eigenvalue problem, for we must have

$$\sigma = \sigma_\nu = -\nu \quad \text{for} \quad \nu = 0, 1, 2, \ldots$$

The corresponding eigenfunctions are

$$\psi_\nu = const\ \omega^\nu e^{-\omega^2/2} \qquad (\nu = 0,1,2,\ldots) \quad \text{(IV.4)}$$

Note that the auxiliary conditions

$$\int_{-\infty}^{+\infty} f\, \zeta\, d\zeta = 0 \qquad \int_{-\infty}^{+\infty} f\, (\zeta^2 - 1)\, d\zeta = 0$$

translate, after Fourier transformation, into the simple conditions

$$\frac{d\psi}{d\omega} = 0 \quad for \ \omega = 0\ ; \qquad \frac{d^2\psi}{d\omega^2} + \psi = 0 \quad for \ \omega = 0$$

These conditions are obviously satisfied for $\nu \geqslant 3$, so that the values 0,1 and 2 must be rejected in IV.4.

As shown in ref 12, the inverse Fourier transforms of these eigenfunctions are the Hermite functions indicated in paragraph 6.

We now give an alternative treatment, using Laplace transformations with respect to ζ and based on a mathematical technique of interest for more complicated cases. The final formula so obtained directly shows the positive definiteness of the distribution function.

52.

If we apply the Laplace transformation with respect to τ:

$$\phi(p, \omega) = \int_0^\infty e^{-p\tau} \varphi(\tau, \omega) \, d\tau$$

to the function φ, equation IV.1 becomes

$$p\phi - \varphi(0, \omega) + \omega \frac{\partial \phi}{\partial \omega} + \omega^2 \phi = 0 \qquad (IV.5)$$

Again, the general solution of this ordinary differential equation in ω is immediately obtained as

$$\phi = e^{-\omega^2/2} \left\{ A\omega^{-p} + \omega^{-p} \int_0^\omega \varphi(0, \widetilde{\omega}) e^{-\widetilde{\omega}^2/2} \widetilde{\omega}^{p-1} d\widetilde{\omega} \right\} \quad (IV.6)$$

The lower integration limit, which is arbitrary, has been taken as zero for convenience (as will appear) and this, of course, corresponds to a particular choice for the constant A.

Just as in the eigenvalue problem treated above, ϕ must be an integral analytic function of the complex variable ω, for all values of p. This condition will define the constant A.

$\varphi(0, \omega)$ being an integral analytic function of ω, we may write,

$$\varphi(0, \omega) e^{\omega^2/2} = \sum_{\nu=0}^\infty c_\nu \omega^\nu \qquad (IV,7)$$

The expression being valid for all ω.

iV.7, then becomes

$$\phi = e^{-\omega^2/2} \left\{ A\omega^{-p} + \sum_{\nu=0}^\infty c_\nu \frac{\omega^\nu}{p+\nu} \right\} \qquad (IV.8)$$

For arbitrary p, the only singularity appears in the A term, and therefore [1]

$$A = 0$$

---

(1) The simplicity of this condition of course results from the special choice of lower limit of integration in formula iV.6.

Going back to the original function φ , we get, using IV.7

$$\varphi = \varphi(0, \omega e^{-\tau}) e^{\frac{\omega^2(e^{-2\tau}-1)}{2}}$$

(IV.9)

The inverse Fourier transformation then yields the following integral representation for $f(\zeta, \tau)$:

$$f(\zeta, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega\zeta - (1-e^{-2\tau})\omega^2/2} \varphi(0, \omega e^{-\tau}) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega\zeta - (1-e^{-2\tau})\omega^2/2} \left\{ \int_{-\infty}^{+\infty} e^{-i\omega e^{-\tau}\tilde{\zeta}} f(0,\tilde{\zeta}) d\tilde{\zeta} \right\} d\omega$$

the latter expression resulting from the definition of $\varphi(0,\omega)$ as Fourier transform of $f(0,\zeta)$.

The above formula may be written as

$$f(\zeta, \tau) = \int_{-\infty}^{+\infty} K(\zeta, \tilde{\zeta}, \tau) f(0, \tilde{\zeta}) d\tilde{\zeta}$$

(IV.10)

with the kernel

$$K(\zeta, \tilde{\zeta}, \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega(\zeta - \tilde{\zeta}e^{-\tau}) - \frac{\omega^2}{2}(1 - e^{-2\tau})} d\omega$$

which is the Green function for the problem.

Introducing the variable

$$\Omega = \omega \sqrt{1 - e^{-2\tau}}$$

the kernel can be written

$$K(\zeta, \tilde{\zeta}, \tau) = \frac{e^{-\frac{(\zeta - \tilde{\zeta}e^{-\tau})^2}{2(1-e^{-2\tau})}}}{2\pi\sqrt{1-e^{-2\tau}}} \int_{-\infty}^{+\infty} e^{-[\Omega - i\frac{\zeta-\tilde{\zeta}e^{-\tau}}{\sqrt{1-e^{-2\tau}}}]^2/2} d\Omega = \frac{e^{-\frac{(\zeta - \tilde{\zeta}e^{-\tau})^2}{1-e^{-2\tau}}}}{\sqrt{2\pi(1-e^{-2\tau})}}$$

(IV.11)

$K(\zeta,\tilde{\zeta},\tau)$ is clearly a positive definite kernel and IV.10 thus shows that f will always be positive if it was initially positive for all values of ζ
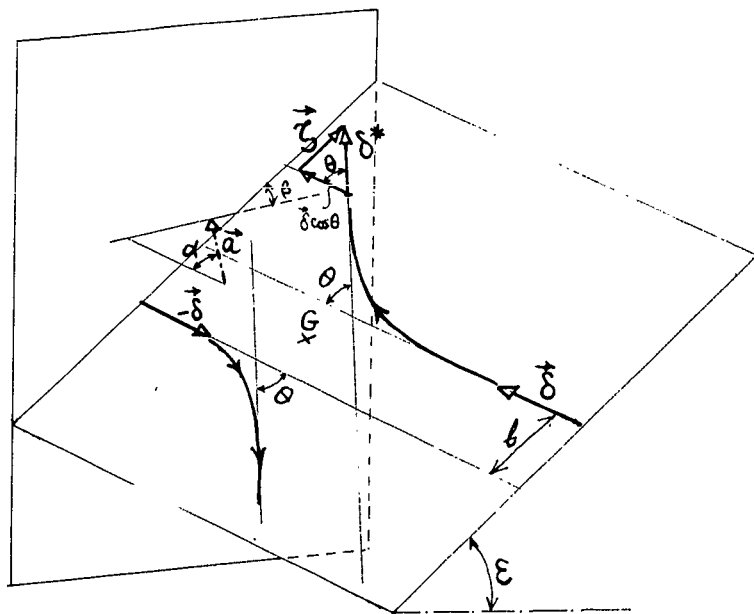
fig 1.

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The corresponding fluid dynamic equations are derived and the existence and structure of shocks is studied.

The distribution function is obtained for the unsteady, spatially uniform problem and the existence of an infinite number of relaxation times and modes is proved.

Linearized versions of the model equation are derived.

The near free molecule limit for solutions of the full model equation is briefly investigated for steady state, and a boundary layer type of behaviour is found to exist, the corresponding "boundary layer" approximation leading to a Gevrey type singular parabolic equation.

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