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A Note on Generating Chi Random Numbers

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by

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Introduction

Marsaglia [1] has given a simple method for generating exponential random numbers on a digital computer. We present a similar method for generating random numbers with the chi distribution. Such random numbers may be used to generate normal random numbers.

I. The chi distribution (of rank two) F is

\[ F(a) = 0, \ a \leq 0, \]
\[ F(a) = 1 - e^{-a^2/2}, \ 0 \leq a. \]

Let \( x \) be a random variable with the distribution \( F \). Let \( G_c(a) = \text{Prob}(x \leq a \mid x \leq c) \) where \( 0 < c \). Then for \( 0 \leq a \leq c \),

\[
G_c(a) = \frac{(1 - e^{-a^2/2})/(1 - e^{-c^2/2})}{\sum_{k=1}^{\infty} q_k (1 - a^2/c^2)^k},
\]

where

\[ q_k = (c^2/2)^k/[k!(e^{c^2/2} - 1)]. \]

Let \( H_c(a) = 1 - e^{-(a^2-c^2)/2}, \ c \leq a, \)
\[ = 0, \quad a \leq c. \]

Then

\[ F(a) = (1 - e^{-c^2/2})G_c(a) + e^{-c^2/2}H_c(a). \]

Thus a random number with the distribution \( F \) may be generated as follows. Generate a uniform random number \( u \), i.e., a random number uniformly distributed on \((0,1)\). If \( u < (1 - e^{-c^2/2}) \),
generate a random number with the distribution \( G_c \); otherwise generate one with distribution \( H_c \).

A random number \( y \) with the distribution \( H_c \) may be generated by setting \( y = \sqrt{ar + c^2} \), where \( r \) is a random number with the exponential distribution.

A random number \( x \) with the distribution \( G_c \) can be generated by setting
\[
x = c \cdot \min[\max(u_1, u_2), \ldots, \max(u_{2z-1}, u_{2z})],
\]
where the \( u_i \) are independent uniform random numbers, and \( z \) is a random integer taking on the value \( k \) with probability \( q_k \). This fact is easily verified by noting that the distribution of \( x \) is just the series (1).

For a binary computer the best choice for \( c \) is \( c = 2 \). On the IBM 7090 computer the average time to generate a chi random number \( x \) by this method is 112 cycles. (A cycle is 2.14 microseconds on this computer). This assumes that the exponential random numbers are generated by the method given in [1].

If \( x \) is generated by setting \( x = \sqrt{2r} \), the average time is 165 cycles.

II. To generate normal random numbers we make use of the following well-known fact. Let \((\alpha, \beta)\) be the rectangular coordinates of a random point uniformly distributed on the unit circle. Then if \( x \) is a chi random number \( y = \alpha x \) and \( z = \beta x \) are independent standard normal random numbers.
The following methods for generating such a pair \((\alpha, \beta)\) are well known.

**Method 1.** Test independent pairs of uniform numbers \((u, v)\) until a pair is found which satisfies \(u^2 + v^2 \leq 1\). Then set \(\alpha = \frac{u}{\sqrt{u^2 + v^2}}\) and \(\beta = \frac{v}{\sqrt{u^2 + v^2}}\).

**Method 2.** Test independent pairs \((u, v)\) until a pair is found which satisfies \(u^2 + v^2 \leq 1\). Then set \(\alpha = \frac{2uv}{u^2 + v^2}\) and \(\beta = \frac{v^2 - u^2}{u^2 + v^2}\).

To generate normal random numbers we can use the following procedure. Generate a chi random number \(x\). If \(x < c\), use method 2 to generate \((\alpha, \beta)\). If \(c \leq x\) use method 1 to generate \((\alpha, \beta)\). Note that we can decide if \(c \leq x\) before we set \(x = \sqrt{c^2 + 2r}\). Therefore this square root operation can be combined with that used to generate \((\alpha, \beta)\). In effect when \(c \leq x\), we compute a pair of normal random numbers \(y\) and \(z\) by

\[
y = xx = u\left[(2r + c^2)/(u^2 + v^2)\right]^\frac{1}{2}
\]

and

\[
z = x\beta = v\left[(2r + c^2)/(u^2 + v^2)\right]^\frac{1}{2}.
\]

This procedure takes 156 cycles to generate one normal random number on the 7090.
REFERENCES