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# Transient Response of Linear Antennas Driven from a Coaxial Line*

by

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## Abstract

The initial transient response of straight wires connected to coaxial lines is studied theoretically for the case where a pulse is applied to the coaxial line. The wave form of the return pulse is first found approximately for the case of a pulse of zero rise time. Since this does not correspond to any feasible experimental situation, the effect of a finite rise time is considered in detail. Numerical results are obtained for several special cases.

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** Alfred P. Sloan Foundation Fellow.

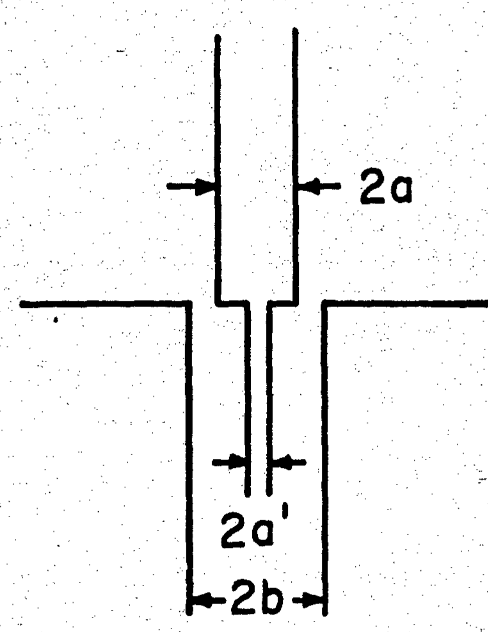
## 1. Introduction

In principle, the properties of a linear system can be completely specified by giving its response at all real frequencies. The response of such a system to a given input may then be determined by the principle of superposition. In practice, however, this procedure may not be straightforward, either because the frequency response is imperfectly known, or because it is not convenient to carry out the necessary calculation for the linear superposition. In the case of the problem of the dipole antenna, the frequency response has been the subject of investigation for several decades, but relatively little is known about the transient behavior. For the idealized problem of the infinite dipole driven by a delta-function generator, the solution can be obtained explicitly.[1] This solution unfortunately cannot be checked experimentally, because in virtually every experimental setup a transmission line is involved. King and Schmitt[2] have measured and computed theoretically the effect of the coaxial transmission line connecting the antenna to the generator. The experiment was a rather difficult one, and probably will be repeated soon. In view of the experimental inaccuracy, only an average reflection coefficient is calculated theoretically. There is an ambiguity as to how this average should be taken, but it is found that the results of taking different averages are quite close to each other, and are in good agreement with the measurements.

With the help of a high-speed oscilloscope, it seems quite possible that the accuracy of the measurement can be improved significantly. With this possibility in mind, it may be argued that an average reflection coefficient is no longer adequate in describing the result of observation. It is therefore the purpose of this paper to study in greater detail the problem of the linear antenna considered by King and Schmitt.[2]

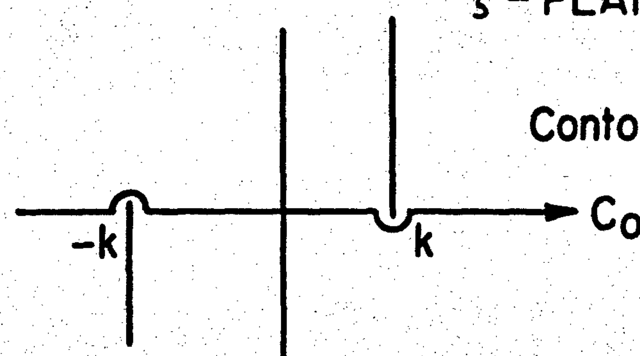
In connection with the problem of the frequency response of a linear antenna driven from a coaxial line, it has been shown[3] that, to a good approximation when the wavelength is not too small, the apparent terminal admittance can be decomposed additively into two parts, one being characteristic of the antenna and independent of the dimensions of the transmission line, and the other being a frequency-independent capacitance. In the experiment of King and Schmitt, it has been found that the effect of this capacitance is at least one order of magnitude too small to be detectable. The approximation is hence made throughout this paper that this capacitance can be neglected. Indeed, the calculation becomes immensely more complicated without this approximation. Moreover, in the treatment of King and Schmitt, only the case of the infinite antenna is considered; this approximation is also retained here. Physically, this means that only the initial behavior in time is given correctly by the present theory. In view of these assumptions, it is felt that it is not meaningful to ask for expressions of great accuracy; instead, the purpose is to find formulas that are relatively simple and are yet accurate to perhaps several per cent. The geometry of the problem is shown in Fig. 1 .

FIG. 1




Antenna driven from coaxial line over a ground screen

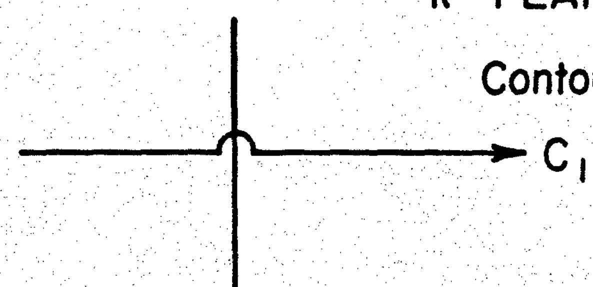
FIG. 2




$\zeta$ - PLANE

Contour of integration

FIG. 3




k - PLANE

Contour of integration

## 2. Formulation of the Problem

Let I denote the current on the inner conductor of the coaxial line, then for a pure TEM mode the incident current is given by

$$I^{inc}(z, t) = f^{inc}(t - z/c) \quad , \tag{2.1}$$

where c is the velocity of light. It is assumed that $f^{inc}$ has a Fourier representation

$$f^{inc}(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\omega\, F(\omega)\, e^{-i\omega t} \, . \tag{2.2}$$

If $\Gamma(\omega)$ is the reflection coefficient at the frequency $\omega$, then the reflected current is

$$I^{ref}(z,t) = f^{ref}(t + z/c) \quad , \tag{2.3}$$

where

$$f^{ref}(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\omega\, \Gamma(\omega)\, F(\omega)\, e^{-i\omega t} \quad . \tag{2.4}$$

Moreover, at $z \to -\infty$, the total current on the inner conductor of the coaxial line approaches $I^{inc} + I^{ref}$. It has been assumed that only the TEM mode propagates.

Let $Y(\omega)$ and $Z(\omega) = [Y(\omega)]^{-1}$ be the apparent terminal admittance and impedance respectively. Let

$$R_c = (2\pi)^{-1} \zeta_0 \, \ln (b/a') \tag{2.5}$$

be the characteristic resistance of the coaxial line, where $\zeta_0$ is the characteristic impedance of free space. Then

$$\Gamma(\omega) = [R_c - Z(\omega)]/[R_c + Z(\omega)] \quad . \tag{2.6}$$

If the frequency-independent capacitance is neglected, then the apparent terminal admittance is given by[3]

$$Y(\omega) = 2ik\zeta_0^{-1} \int_{C_0} d\zeta\,(\zeta^2 - k^2)^{-1}[\gamma + \ln\tfrac{1}{2}(k^2-\zeta^2)^{\frac{1}{2}}a - \tfrac{1}{2}\pi i]^{-1} \quad , \tag{2.7}$$

where $k = \omega/c$, $\gamma$ is Euler's constant, and the contour of integration $C_0$ is shown in Fig. 2. As pointed out before,[3] there is an ambiguity in (2.7) due to the zeroes of the quantity in the bracket at $\pm[k^2 + 4e^{-2\gamma}a^{-2}]^{\frac{1}{2}}$; this ambiguity is of no consequence here. For the present purpose, it is sufficient to expand the integrand of (2.7) in powers of $\ln$ (ka). The two leading terms are

$$Y(\omega) = 2\pi\zeta_0^{-1}\left\{-[\ln(\tfrac{1}{2}ka) + \gamma - \tfrac{1}{2}\pi i]^{-1} + \ln 2[\ln(\tfrac{1}{2}ka) + \gamma - \tfrac{1}{2}\pi i]^{-2}\right\} .$$

Therefore

$$Z(\omega) = (2\pi)^{-1}\zeta_0[\ln(ka)^{-1} - \gamma + \tfrac{1}{2}\pi i] \quad . \tag{2.8}$$

This expression is also used by King and Schmitt.[2]

The following property of the right-hand side of (2.8) is of paramount importance here. So far $\omega$ has been considered to be real and (2.8) is obtained on this basis. However, the right-hand side of (2.8) can be analytically continued into the entire complex $\omega$-plane except for a branch cut along the negative imaginary axis, and moreover this analytic continuation in the cut plane satisfies the relation

$$Z^*(-\omega^*) = Z(\omega) , \tag{2.9}$$

where the asterisk denotes complex conjugation. In general, the real part of a microwave impedance must be non-negative in the upper half plane. This is not satisfied by the right-hand side of (2.8) when

$$|\omega| > c\, e^{-\gamma}/a . \tag{2.10}$$

For many reasons (2.8) must break down when (2.10) holds. The present consideration is meaningful only when the contribution from the region (2.10) is not of importance.

In the next two sections, the reflected current is to be obtained approximately on the basis of (2.3) - (2.6) and (2.8) for two different cases of incident current.

## 3. Case of a Sharp Stop

In this section, the case is considered where

$$f^{inc}(t) = \begin{cases} 1 & \text{for } t > 0 , \\ 0 & \text{for } t < 0 . \end{cases} \tag{3.1}$$

Hence

$$F(\omega) = i/\omega . \tag{3.2}$$

Here the reflected current is given in terms of

$$f^{ref}(t) = -i(2\pi)^{-1} \int_{C_1} \omega^{-1} d\omega\, e^{-i\omega t} \frac{\ln(ka)^{-1} - \ln(b/a') - \gamma + \frac{1}{2}\pi i}{\ln(ka)^{-1} + \ln(b/a') - \gamma + \frac{1}{2}\pi i} , \tag{3.3}$$

where the contour of integration $C_1$ is shown in Fig. 3 . Accordingly

$$f^{ref}(t) = 0 \tag{3.4}$$

for $t < 0$ , and

$$f^{ref}(t) = -1 + \frac{-i}{2\pi} \int_0^\infty \frac{dx}{x} e^{-xt} \left[ \frac{\ln c(ax)^{-1} - \ln(b/a') - \gamma + \pi i}{\ln c(ax)^{-1} + \ln(b/a') - \gamma + \pi i} - \frac{\ln c(ax)^{-1} - \ln(b/a') - \gamma - \pi i}{\ln c(ax)^{-1} + \ln(b/a') - \gamma - \pi i} \right] , \tag{3.5}$$

for $t > 0$ . It is convenient to define

$$t_0 = (a'/b)(a/c) , \tag{3.6}$$

which plays the role of a characteristic time for the present problem. If

$$\tau = t/t_0 , \tag{3.7}$$

then, after a change of variable, (3.5) may be simplified to the form

$$f^{ref}(t) = -1 + 2\, \ln(b/a') \int_0^\infty x^{-1}\, dx\, e^{-x\tau} \left[ (\ln x + \gamma)^2 + \pi^2 \right]^{-1} . \qquad (3.8)$$

Equation (3.8) is valid only for $\tau >> 1$. In this range, the right-hand side of (3.8) may be evaluated approximately as follows. Since it is known that

$$\int_0^{e^{-\gamma}} x^{-1}\, dx\, (1 - e^{-x}) = \int_{e^{-\gamma}}^\infty x^{-1}\, dx\, e^{-x} , \qquad (3.9)$$

and since the factor $[(\ln x\ \gamma)^2 + \pi^2]^{-1}$ in the integrand of (3.8) varies relatively slowly, (3.8) may be approximated by

$$f^{ref}(t) = -1 + 2\, \ln(b/a') \int_0^{\tau^{-1} e^{-\gamma}} x^{-1}\, dx \left[ (\ln x + \gamma)^2 + \pi^2 \right]^{-1} . \qquad (3.10)$$

An elementary integration gives, for $t > 0$,

$$f^{ref}(t) = -1 + 2\, \pi^{-1} \ln(b/a') \left[ \tfrac{1}{2}\pi - \tan^{-1}(\pi^{-1} \ln \tau) \right] . \qquad (3.11)$$

Equations (3.4) and (3.11) furnish the desired answer to the special case under consideration.

Equation (3.11) is valid only when $\tau >> 1$. Hence no information has been obtained when $t$ is small or comparable to the characteristic time $t_0$. The behavior of the reflected current is expected to be rather complicated for such small $t$, since very high frequency components play an important role in this range. This is perhaps to be expected from a comparison with the idealized problem of an infinite dipole antenna driven by a delta-function generator.[1] On the one hand, there exists no known way to treat these very high-frequency components theoretically for a linear antenna driven from a coaxial line. On the other hand, in an actual experiment, there is a finite rise time for the incident pulse anyway, and hence these very high-frequency components are not present. Therefore, for the purpose of comparison with possible future experimental results, it is more interesting to study the effect of a finite rise time. A particularly simple model is treated in the next section.

## 4. Case of Gradual Rise

To distinguish the various quantities that appear in this section from the corresponding ones in Sec. 3, an additional subscript 1 is used here. In the present case, the incident current is given in terms of

$$f_1^{inc}(t) = \begin{cases} 1 - e^{-t/t_1} & \text{for } t > 0 , \\ 0 & \text{for } t < 0 . \end{cases} \tag{4.1}$$

Hence

$$F_1(\omega) = i\,[\omega^{-1} - (\omega + i t_1^{-1})^{-1}] . \tag{4.2}$$

Throughout this section, it is further assumed that

$$\tau_1 = t_1/t_0 >> 1 . \tag{4.3}$$

The reflected current is given in terms of

$$f_1^{ref}(t) = -i(2\pi)^{-1} \int_{C_1} d\omega\,[\frac{1}{\omega} - \frac{1}{\omega + i t_1^{-1}}]\, e^{-i\omega t}\, \frac{\ln(ka)^{-1} - \ln(b/a') - \gamma + \frac{1}{2}\pi i}{\ln(ka)^{-1} + \ln(b/a') - \gamma + \frac{1}{2}\pi i} . \tag{4.4}$$

Again, for $t \leq 0$,

$$f_1^{ref}(t) = 0 . \tag{4.5}$$

Since the integrand in (4.4) is unbounded near $\omega = -i t_1^{-1}$, it is more convenient to make use of (4.5) and rewrite (4.4) in the form

$$f_1^{ref}(t) = -i(2\pi)^{-1} \int_{C_1} d\omega\,[\frac{1}{\omega} - \frac{1}{\omega + i t_1^{-1}}]\, [e^{-i\omega t} - e^{-t/t_1}]\, \frac{\ln(ka)^{-1} - \ln(b/a') - \gamma + \frac{1}{2}\pi i}{\ln(ka)^{-1} + \ln(b/a') - \gamma + \frac{1}{2}\pi i} . \tag{4.6}$$

In this form, the procedure of Sec. 3 may be followed to yield successively, for $t \geq 0$,

$$f_1^{\mathrm{ref}}(t) = -(1-e^{-t/t_1}) - \frac{1}{2\pi}\int_0^\infty dx\,[\,\frac{1}{x} - \frac{1}{x - t_1^{-1}}\,]\,[\,e^{-xt} - e^{-t/t_1}\,]$$

$$\left[\frac{\ln c(ax)^{-1} - \ln(b/a') - \gamma + \pi i}{\ln c(ax)^{-1} + \ln(b/a') - \gamma + \pi i} - \frac{\ln c(ax)^{-1} - \ln(b/a') - \gamma - \pi i}{\ln c(ax)^{-1} + \ln(b/a') - \gamma - \pi i}\right], \quad (4.7)$$

and

$$f_1^{\mathrm{ref}}(t) = -(1-e^{-t/t_1}) + 2\ln(b/a')\int_0^\infty dx\,[x^{-1} - (x-\tau_1^{-1})^{-1}]\,[e^{-x\tau} - e^{-\tau/\tau_1}]\,[(\ln x + \gamma)^2 + \pi^2]^{-1}, \quad (4.8)$$

in complete analogy with (3.5) and (3.8).

The approximate evaluation of the right-hand side of (4.8) is based on the same idea as before, namely on the observation that the last factor in the integrand varies relatively slowly. Let $x_1 = x_1(t, t_1)$ be determined by the condition

$$\int_0^{x_1^{-1}e^{-\gamma}} dx\left\{[x^{-1} - (x-\tau_1^{-1})^{-1}]\,[e^{-x\tau} - e^{-\tau/\tau_1}] - x^{-1}[1 - e^{-\tau/\tau_1}]\right\}$$

$$= \int_{x_1^{-1}e^{-\gamma}}^\infty dx\,[x^{-1} - (x-\tau_1^{-1})^{-1}]\,[\,e^{-x\tau} - e^{-\tau/\tau_1}\,]. \quad (4.9)$$

Then, for $t \geq 0$, (4.8) is reduced approximately to

$$f_1^{ref}(t) = (1 - e^{-\tau/\tau_1}) \left\{ -1 + 2\pi^{-1} \ln (b/a') \left[ \tfrac{1}{2}\pi - \tan^{-1}(\pi^{-1} \ln x_1) \right] \right\}. \qquad (4.10)$$

Equations (4.5) and (4.10) furnish the desired answer; it only remains to find $x_1$.

In order to simplify (4.9), some properties of the exponential integral are needed. Generalizing the notation of Jahnke and Emde[4], let

$$\mathrm{Ei}(z) = \int_{-\infty}^{z} \xi^{-1} e^{\xi} d\xi$$

be an analytic function of the complex variable $z$ with a branch cut coinciding with the positive real axis. Also let

$$\overline{\mathrm{Ei}}(x) = \mathrm{Re}\,\mathrm{Ei}(x)$$

for $x > 0$, then

$$\overline{\mathrm{Ei}}(x) = (P) \int_{-\infty}^{x} \xi^{-1} e^{\xi} d\xi ,$$

where (P) denotes principal value at $\xi = 0$. With this notation, (4.9) can be written as

$$\ln (x_1/\tau) - e^{-\tau/\tau_1} \ln (x_1/\tau_1) + e^{-\tau/\tau_1} [\overline{\mathrm{Ei}} (\tau/\tau_1) - \gamma] = 0 ,$$

and hence

$$x_1 = \tau \exp \left\{ -(e^{\tau/\tau_1} - 1)^{-1} [\overline{\mathrm{Ei}} (\tau/\tau_1) - \ln (\tau/\tau_1) - \gamma] \right\} . \qquad (4.11)$$

This completes the solution.

It may be of some interest to consider briefly two extreme cases. $\mathrm{Ei}(x)$ may be expressed alternatively as

$$\mathrm{Ei}(x) = e^x \ln x + \gamma - \int_0^x d\xi \, e^{\xi} \ln \xi \; .$$

Hence for $x \ll 1$, $\mathrm{Ei}(x)$ is approximately

$$\mathrm{Ei}(x) = \ln x + \gamma + x \; .$$

Accordingly, when $\tau \ll \tau_1$, (4.11) reduces to

$$x_1 \sim \tau/e \quad . \tag{4.12}$$

On the other hand, when $\tau \gg \tau_1$,

$$x_1 \sim \tau \, e^{-\tau_1/\tau} \quad . \tag{4.13}$$

Thus, for $t \gg t_1$, the reflected current is essentially independent of $t_1$, as may be expected.

In view of (4.12) and (4.13), a rough approximation to the reflected current is given very simply by putting $x_1 = \tau$, or

$$f_{10}^{\,ref}(t) = (1 - e^{-\tau/\tau_1}) f^{ref}(t) \quad . \tag{4.14}$$

## 5. Numerical Results and Discussion

A quantitative study of the properties of a pulse reflected back into a perfectly conducting coaxial line from the junction between the line and a monopole antenna erected vertically over a ground screen involves at least five parameters. . Of these three, the radius $a$ of the antenna, the radius $a'$ of the inner conductor of the coaxial line, and the inner radius $b$ of the coaxial shield are contained in the characteristic time $t_0 = aa'/bc$ where $c$ is the velocity of light. The fourth parameter is the rise and decay time $t_1$ of the pulse if it is assumed that the rise and decay are identical functions of the time. (If this assumption is not made, an additional parameter is required.) The fifth parameter is the duration of the pulse , $T$ .

A convenient radius for antennas in a variety of measurements is $a \sim 3\text{mm}$ . If the ratio $b/a'$ for the coaxial line is taken to be 2 — corresponding to a characteristic resistance $R_c = 41$ ohms — the characteristic time $t_0$ as defined in (3.6) is only about $5 \times 10^{-12}$ seconds. This is an extremely short time for microwave measurements. Indeed, a pulse with a rise time much less than $10^{-10}$ seconds is at present difficult to obtain. This means that $t_1/t_0 = \tau_1 \gtrsim 20$ unless an antenna with a radius that is much greater than 3mm is used in the experiment. Therefore, the conclusion is reached that the pulse shapes and amplitudes determined in Sec. 4 for finite rise times should correspond much more closely to possible experimental situations than those obtained in Sec. 3 where a sharp step is assumed. For comparison with measurable results, it is certain that a pulse with a finite rise time, a finite duration, and a finite decay time must be used. Since the problem is linear, the incident current is of the form

$$I_1^{inc}(z,t) = f_1^{inc}(t-z/c) - f_1^{inc}(t-T-z/c) \; , \tag{5.1}$$

where $f_1^{inc}(t)$ is given by (4.1) and where it has been assumed that the decay of the pulse is the same as the rise. If mercury switches are used, the particular form (4.1) is probably a better approximation for the rise than for the decay.

For purposes of comparison graphs of the incident sharp step $f^{inc}(t)$ as defined in (3.1) and of the gradual step $f_1^{inc}(t)$ with rise time $t_1$ as defined in (4.1) are shown together in Fig. 4. In the same figure are also shown sharp rectangular pulses $I^{inc}(t)$ and pulses with finite and equal rise and decay times $t_1$ as given by $I_1^{inc}(t)$. These pulses have the forms

$$I^{inc}(t) = f^{inc}(t) - f^{inc}(t-T) \tag{5.2}$$

and

$$I_1^{inc}(t) = f_1^{inc}(t) - f_1^{inc}(t-T) \quad , \tag{5.3}$$

where $T$ is the duration of the pulse. This is assigned the values $T/t_1 = 1,2,5$ and 10 for the gradual pulse. The same values of $T$ are used for the sharp pulse to facilitate comparison. It is seen from the figure that with a finite rise time the incident pulse is not flat on top unless $T/t_1$ exceeds about 5.

Physically such an incident pulse consists of a concentration of positive (or negative) charges that moves outward on the coaxial line with the velocity of light $c$, if the dielectric is air and perfect conductors are assumed. Such a moving concentration of charges constitutes a current $I(z,t)$. In a sharp pulse the charges are confined to a physical pulse length $s = c/T$; in a gradual pulse of the same duration, the charge concentration actually extends over a greater distance if the region of decay is included.

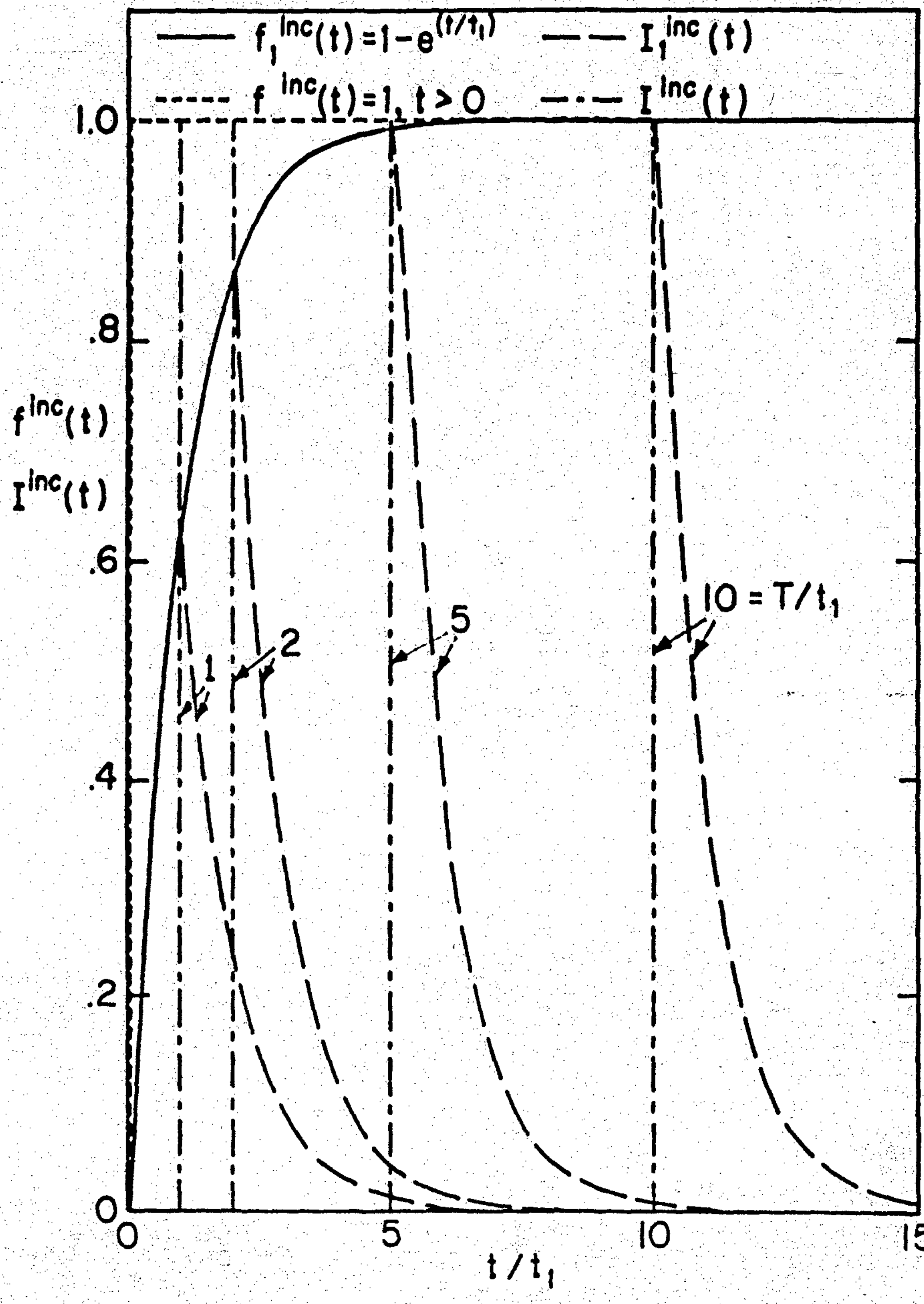



FIG. 4 INCIDENT STEP AND PULSE WITH INFINITE AND FINITE RISE TIME $t_1$, AND PULSE LENGTH T

When an incident concentration of positive charges reaches the junction between the coaxial line and the antenna, a transmitted concentration of charges travels outward on the antenna, a reflected concentration travels back along the coaxial line. It is this latter that is given by $f^{ref}(t)$ in (3.11) or $f_1^{ref}(t)$ in (4.10) when the incident signal is a sharp or a gradual step, and by

$$I^{ref}(t) = f^{ref}(t) - f^{ref}(t-T) \tag{5.4}$$

or

$$I_1^{ref}(t) = f_1^{ref}(t) - f_1^{ref}(t-T) \tag{5.5}$$

when the incident signal is a sharp pulse or a gradual pulse with identical rise and decay.

Consider first the reflection for an incident signal consisting of the sharp step $f^{inc}(t)$ as defined in (3.1). Physically this corresponds to a uniform concentration of positive charge per unit length advancing along an uncharged coaxial line with the velocity of light. When the sudden rise in charge per unit length from zero to a final value reaches the end of the line, the transmission of an outward-traveling signal along the antenna and a reflected signal back along the line begins. The reflected disturbance is given by (3.11). However, since the restriction $t \gg t_0$ has been imposed in the derivation, this formula is not a good approximation when $\tau$ is small. The reflected pulse must, of course, be observed at a great distance from the antenna. Since the nature of the reflected disturbance depends critically on the characteristic resistance $R_c = \frac{\zeta_0}{2\pi} \ln(b/a')$ of the coaxial line, it is convenient to use this as the parameter in the numerical evaluation of (3.11) in the alternative form

$$f^{ref}(\tau) = -1 + \frac{R_c}{30}\left[\frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left(\frac{\ln \tau}{\pi}\right)\right]. \tag{5.6}$$

Curves computed from (5.6) are shown in Fig. 5. The range of $R_c$ is from 50 to 285 ohms; the radius $a$ of the antenna is taken to be 3/16 in or 2.38 mm. When $R_c = 50$ ohms, the characteristic time is $t_0 = 3.45 \times 10^{-12}$ seconds.

When a sharp step of charges is incident upon a termination such as an antenna, the reflection observed back on the line in an interval $T$ must be the same as for an incident rectangular pulse of duration $T$. This is a consequence of causality together with the linearity characteristic of the problem. Since a shorter pulse contains relatively more high-frequency components in its spectrum, the initial behavior of a reflected pulse is determined primarily by the characteristics of the junction at high frequencies. This is of particular significance in determining the shape of the reflected signal, since the impedance of the antenna terminating the line is a function of frequency.

The impedance of an infinitely long antenna as given in (2.8) decreases as the frequency is increased.[2] If such an antenna (or an antenna of finite length insofar as the first reflection from its base is concerned) terminates a coaxial line with a given characteristic resistance $R_c = 50$ ohms, the current-reflection coefficient tends to be positive for the high frequencies and negative for the low frequencies. Hence, unless $R_c$ is very low, the reflection of a sharp step as it arrives at a point along the line begins with a positive peak. This decreases in amplitude for later times, and eventually reduces to zero and becomes negative as the low-frequency end of the signal becomes dominant.

As the incident concentration of positive charges reaches the end of the line, it continues out onto the antenna with an <u>increased</u> magnitude, while a compensating concentration of <u>negative</u> charges travels back along the line to reduce somewhat the constant positive charge left by the incident step. Note that negative

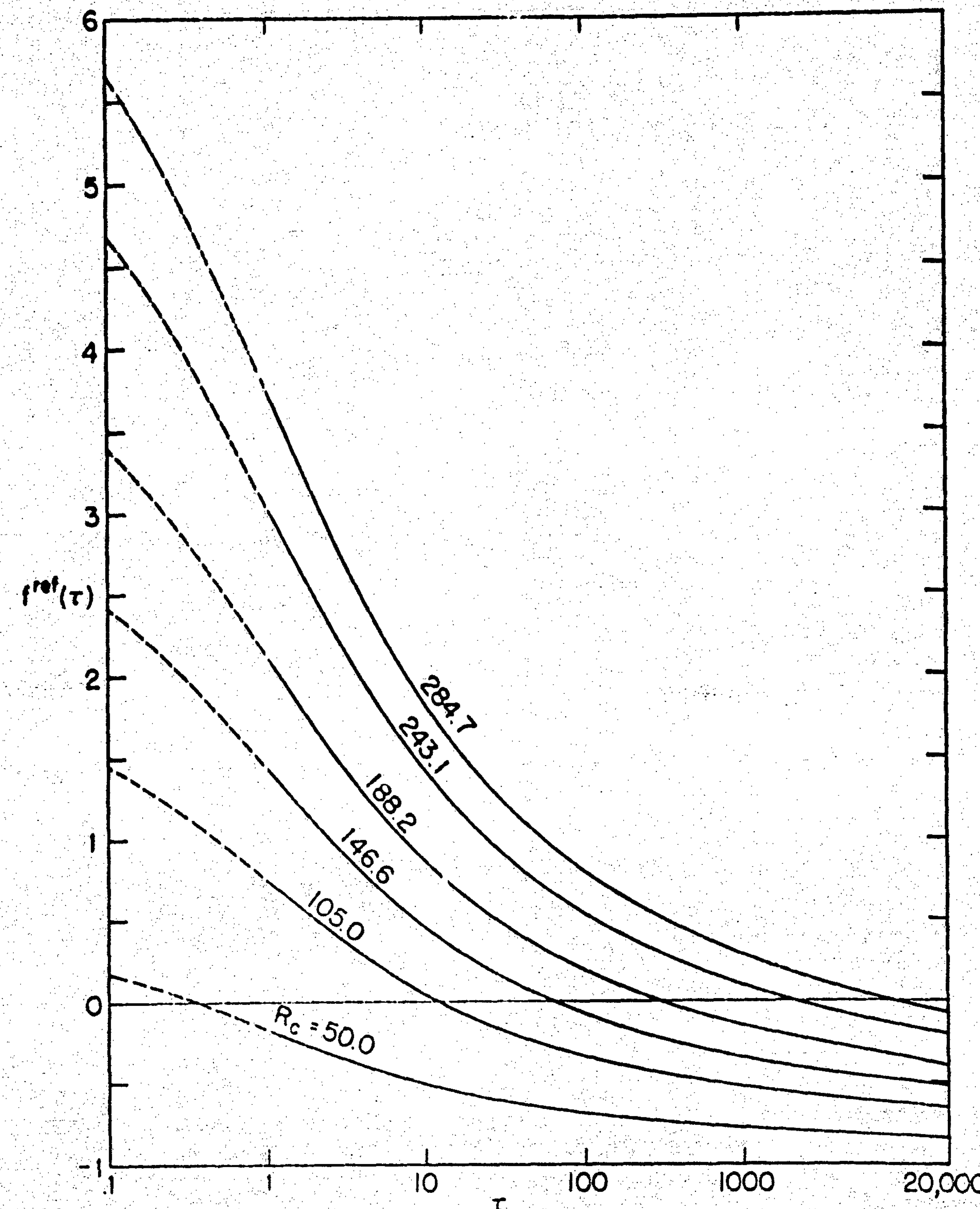



FIG. 5 REFLECTION $f^{ref}(\tau)$ OF INCIDENT SHARP STEP. $\tau = t/t_0$

charges moving back along the line constitute a positive current. It is seen in Fig. 5 that the reflected current is positive for all values of $R_c \geq 50$ during a sufficiently short initial period. The smaller the $R_c$, the shorter, of course, is this period.

Owing to the uncertainty in the initial value of $f^{ref}(\tau)$, it is not convenient to construct a reflected pulse $I^{ref}(\tau)$ by superimposing the reflections $f^{ref}(\tau)$ and $-f^{ref}(\tau - T/t_0)$ from successive incident steps.

Computations to determine the reflected signal $f_1^{ref}(t)$ for the gradual incident step and $I_1^{ref}(t) = f_1^{ref}(t) - f_1^{ref}(t-T)$ for the gradual incident pulse have been carried out for two somewhat different sets of conditions. For the first set the following data (corresponding approximately to an experiment already reported)[2] were used: $a = 3/16$ in $= 2.38$ mm, and $R_c = 50$ ohms; these correspond to $t_0 = 3.45 \times 10^{-12}$ seconds.* For the rise time, the value $t_1 = 1.035 \times 10^{-9}$ seconds for $\tau_1 = t_1/t_0 = 300$ approximates that used in the experiment. Curves of $f_1^{ref}(t)$ and four different reflected pulses $f_1^{ref}(t) - f_1^{ref}(t-T)$ with the pulse durations $T/t_1 = 1, 2, 5$ and $10$ are shown in Fig. 6. Note that $f_1^{ref}(t)$ and $I_1^{ref}(t)$ resemble, respectively, $f_1^{inc}(t)$ and $I_1^{inc}(t)$ in Fig. 4. Owing to its rather long rise time, the pulse does not contain many very high-frequency components, so that with the

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* The coaxial line used in the experiment had polystyrene ($\epsilon_r = 2.45$) as dielectric instead of air as assumed in the theoretical derivation. It is readily verified that if the characteristic resistance of the line is the same, there is no change in $t_0$ which is now defined by $t_0 = \frac{a}{c}\left(\frac{a'}{b}\right)^u$ where $u = \sqrt{\epsilon_r}$. Since $R_c = \frac{\zeta_0}{\sqrt{\epsilon_r}} \ln \frac{b}{a'} = \zeta_0 \ln \left(\frac{b}{a'}\right)^{u^{-1}}$, it is clear that for any given $R_c$, the quantity $\left(\frac{b}{a'}\right)^{u^{-1}}$ is a constant independent of the dielectric constant.

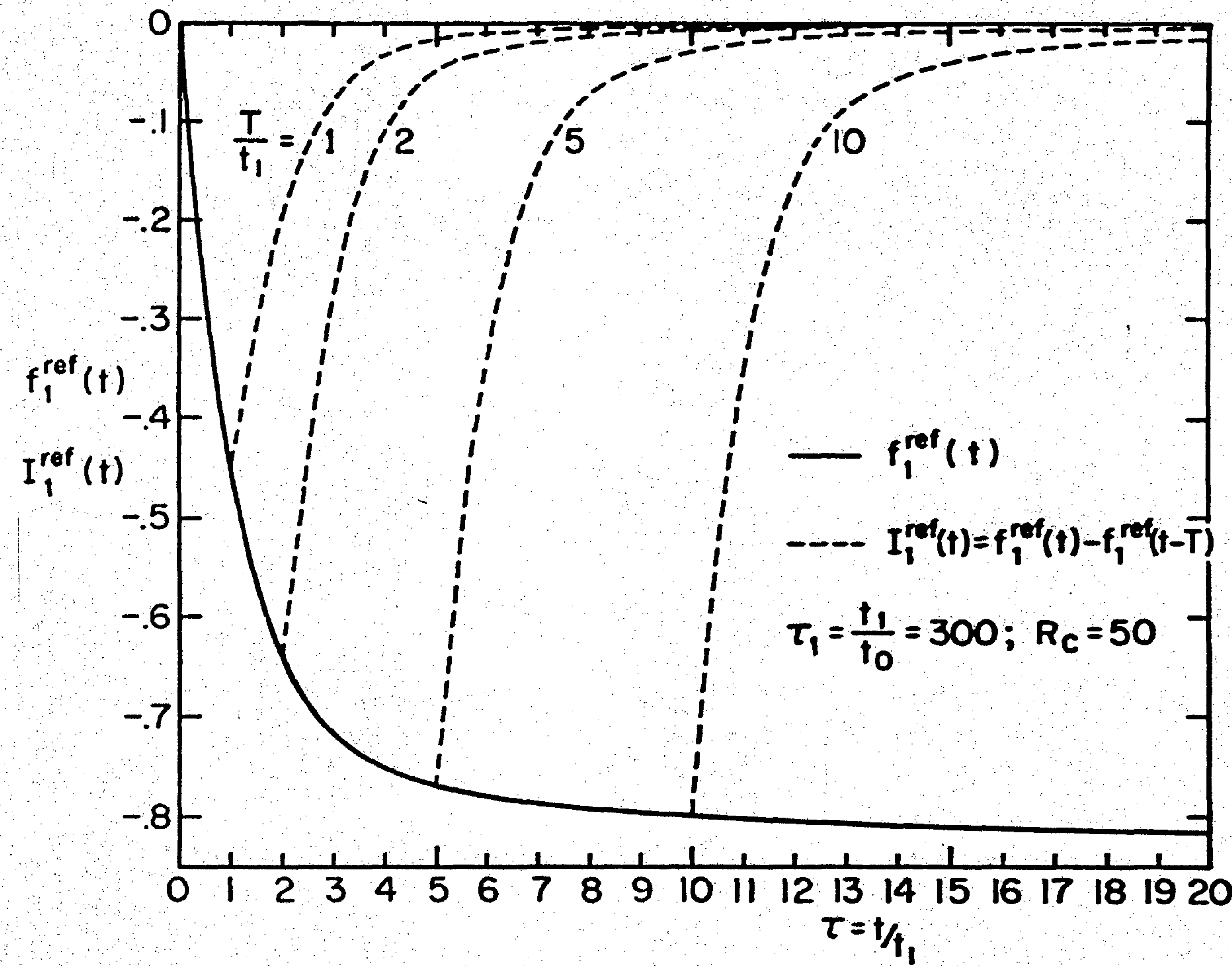


FIG. 6 REFLECTED STEP AND PULSES WITH FINITE RISE TIME $t_1$ AND PULSE LENGTH T.

characteristic resistance $R_c = 50$ ohms of the coaxial line the incident current is reflected with a negative sign. The maximum value of the measured reflected pulse[2] with $t_1 \doteq 10^{-9}$ sec, $T = 3 \times 10^{-9}$ sec, $a = 3/16$ in and $R_c = 50$ ohms, was 0.72. Note that this value is in good agreement with the maximum for $T/t_1 \sim 3$ as seen from Fig. 6.

Very interesting results are obtained with a second set of conditions in which the effect of changing the radius $a'$ of the inner conductor (and hence the characteristic resistance $R_c$) of the coaxial line is studied for an extremely short rise time. For the antenna of radius 3/16 in = 2.38 mm used in the previous case, the rise time used is $t_1 = 0.69 \times 10^{-10}$ second. (For a somewhat thicker antenna with $a = 3.45$ mm the same results apply to a rise time of $t_1 = 10^{-10}$ second.) Let $a'$ and, hence, $t_0$ be varied in such a manner that with $t_1$, $a$, and $b$ fixed, $\tau_1 = t_1/t_0$ has the value $\tau_1 = 20, 50, 100, 500, 1000$. These correspond to $R_c = 50, 105.2, 146.6, 187, 243, 265$ ohms. Curves showing $f_1^{ref}(t)$ and $I_1^{ref}(t)$ with $T/t_1 = 1, 2, 5$ and 10 are given in Fig. 7 for each of the five values of $\tau_1$ or $R_c$. A composite superposition of all of these curves in Fig. 8 is useful for overall comparisons.

A study of the five sets of curves in Fig. 7 and Fig. 8 shows that for the low characteristic resistance $R_c = 50$ ohms, the shapes of the reflected pulses are quite similar to those in Fig. 6 where the rise time is 15 times as long. They are also very much like the incident pulses shown in Fig. 4. The reflected current pulse is negative for $R_c = 50$ ohms. However, as the value of $R_c$ is increased, the initial fairly rapid rise begins with a positive rather than negative peak that corresponds to a concentration of negative charges traveling back along the line.

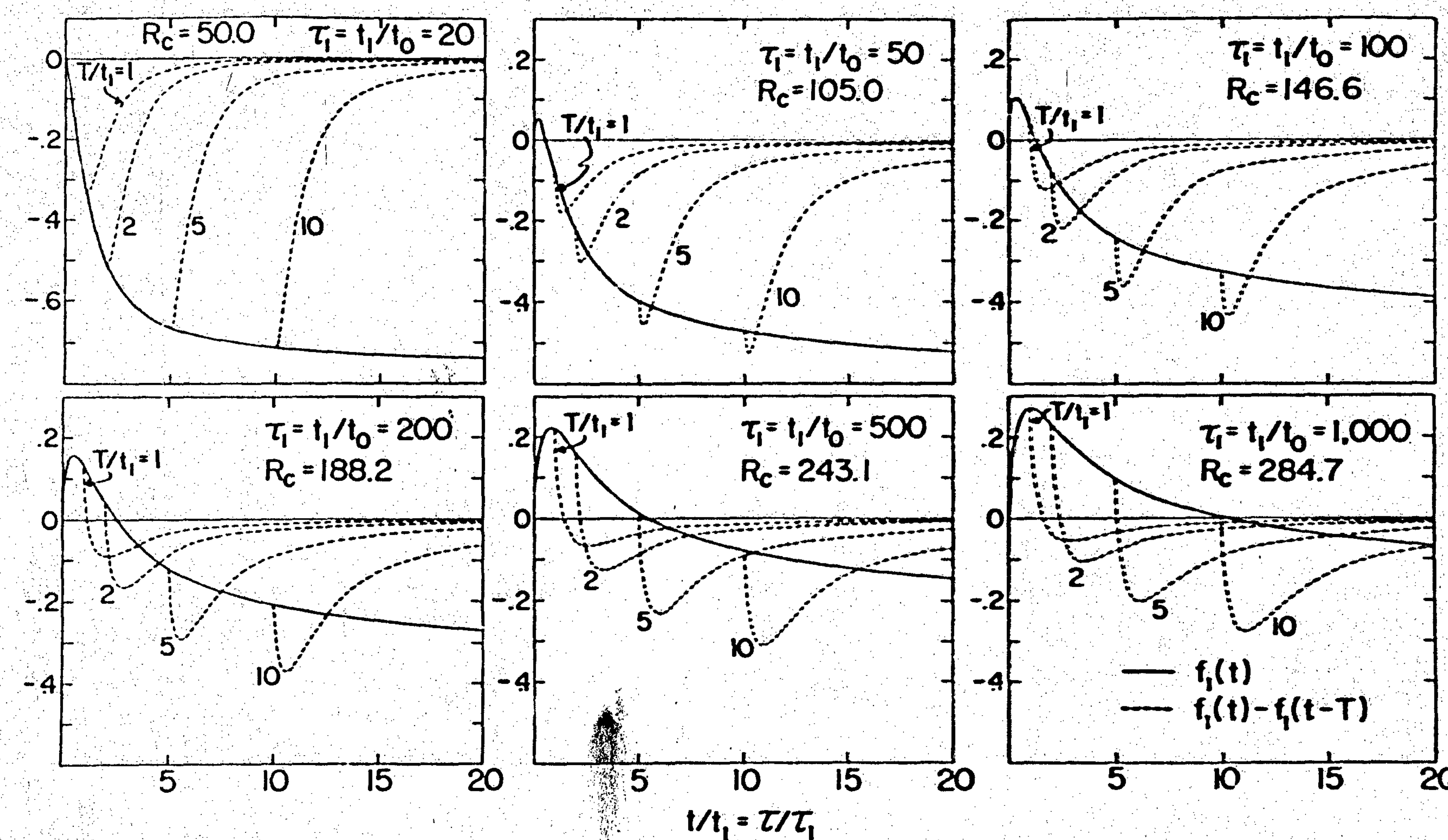


FIG. 7 REFLECTED STEPS AND PULSES WITH FINITE RISE TIME $t_1$ AND PULSE LENGTH $T$

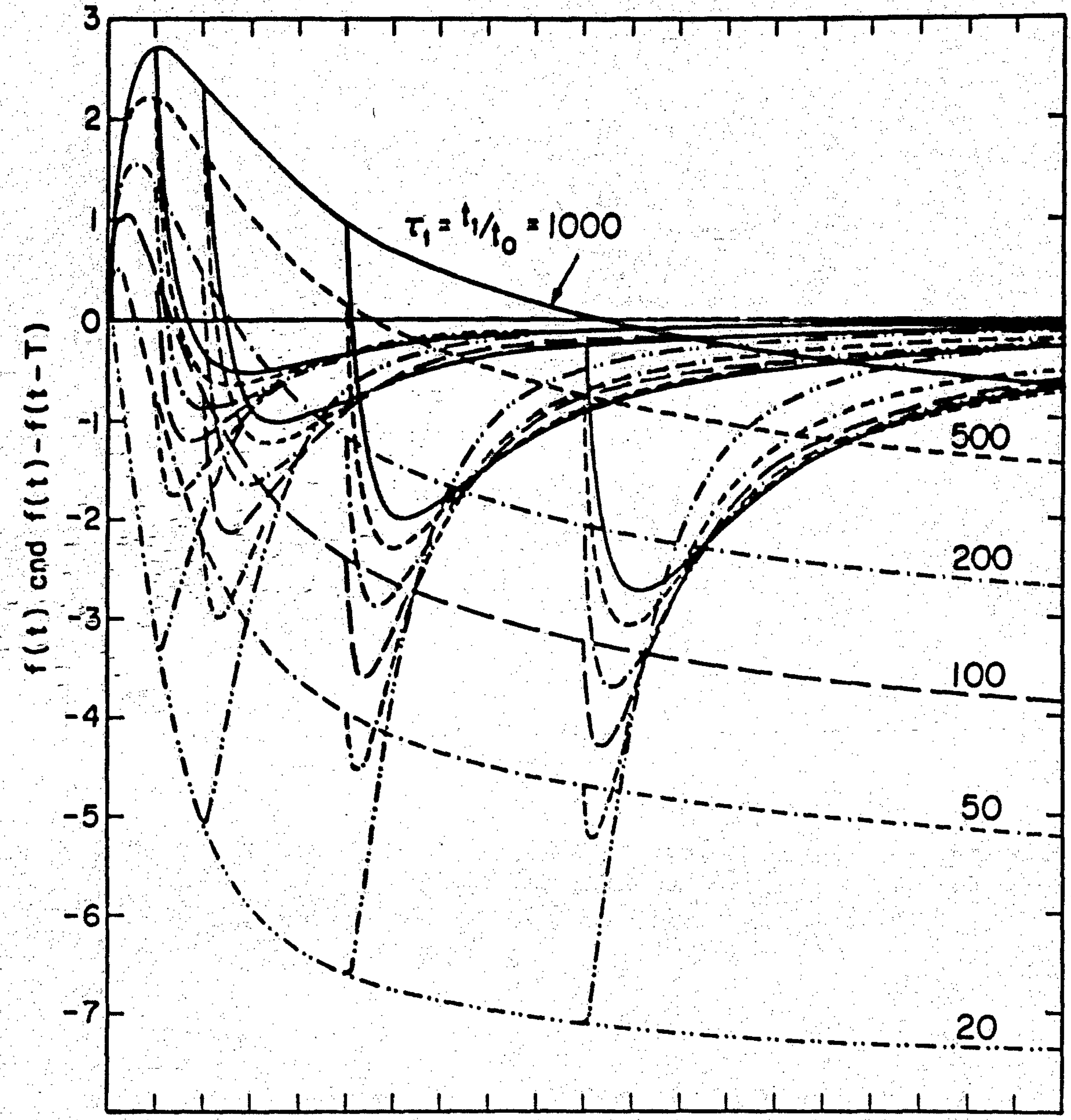


FIG. 8 SUPERPOSITION OF THE CURVES OF FIG. 7

After the positive peak comes a negative peak which indicates that the concentration of negative charges is followed by a concentration of positive charges. This behavior is in complete agreement with the earlier discussion in conjunction with Fig. 5. In order to observe a positive reflected current experimentally a pulse with a very short rise time and a line with a fairly large value of $R_c$ are required.

## Acknowledgments

The authors are indebted to Prof. H. J. Schmitt for valuable discussions and to Miss Dilla Gooch for carrying out the computations.

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