NOLR 1241

WHITE OAK, MARYLAND

A compilation of AERODYNAMIC NOMENCLATURE AND AXES SYSTEMS

UNITED STATES NAVAL ORDNANCE LABORATORY

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A compilation of AERODYNAMIC NOMENCLATURE AND AXES SYSTEMS

UNITED STATES NAVAL ORDNANCE LABORATORY, WHITE OAK, SILVER SPRING, MARYLAND

A compilation of aerodynamic nomenclature and axes systems is presented. The primary and secondary (subscript and superscript) symbols and aerodynamic coefficients are treated in detail. The body-, stability-, wind-, aeroballistic-, and tunnel-axes systems are defined and presented pictorially. Transfer equations from the body-axes system to the other systems together with their derivations are included.

by John Wright

Of this book, an initial edition of four hundred copies was printed for distribution by the United States Naval Ordnance Laboratory, White Oak, Maryland. The text is IBM Executive, titles and headings are Typo Roman, and the formulas and symbols have been lettered especially for this edition. The cover and the text were printed by the Navy Printing and Publications Service, Potomac River Naval Command, on paper approved by the Joint Committee on printing, and supplied by the Government Printing Office, Washington, D.C.

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FOREWORD

It has long been recognized that standardization of symbols used in the aeronautical sciences would aid in the dissemination of technical information among various agencies and companies. The Applied Aerodynamics Division of the Aerodynamics Department of the United States Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland, has analyzed its needs in this respect. The standards to be used by this division are set forth in this publication. Letter symbols, system of axes and coefficients contained herein are, for the most part, in agreement with reference (1). Some portions have been taken almost entirely from this reference.

This work was sponsored under Task No. RMMO-42-003 and was reviewed by a committee from the Applied Aerodynamics Division consisting of Dr. R. Lehnert, J. E. Greene, I. Shantz, J. A. Darling, R. T. Groves, and E. J. Redman.

The author gratefully acknowledges the outstanding work of Mr. Mark Ash II of the Publications Division of NOL for the preparation and pictorial presentation herein.

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Captain, USN
Commander

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G. K. HARTMANN
By direction

CONTENTS

ILLUSTRATIONS

INTRODUCTION

It is intended that this publication act primarily as a guide to the standardization of symbols, coefficients, and systems of axes used by the Applied Aerodynamics Division of the Aerodynamics Department of the United States Naval Ordnance Laboratory, White Oak, Maryland. However, the information contained herein is applicable to aerodynamics in general and its use is in no way confined to the Applied Aerodynamics Division. It is recognized that no one system of axes or nomenclature can cover every possible case and it is not presumed that this report is a complete treatise on the subject.

The information is divided into three sections.

Section I: Letter symbols

They are first presented in alphabetical order according to symbols, both the primary and secondary (subscript and superscript). Then for convenience they are presented in alphabetical order according to concept. Alternate symbols are shown in parentheses if not otherwise stated. If no dimensions are given, the item is dimensionless or the dimensions must be defined on a per-case basis.

Section II: Forces, moments, and coefficients.

Their symbols and their definitions are presented. The presentation is categorized according to application.

Section III: Systems of axes, angles, and transfer equations.

The axes systems, forces, moments, and angular displacements are defined and then presented pictorially in Figures 1 through 12. The transfer equations from the body-axes system to the stability-, wind-, aeroballistic-, and tunnel-axes systems are presented.

Appendix A contains the derivation of the transfer equations presented in Section III. Appendix B presents the derivation of the relationships between the angles ψ, θ, and ϕ when the sequence is changed.

SECTION I

LETTER SYMBOLS

Section Ia Principal symbols

SECTION Ia

PRINCIPAL SYMBOLS

	Symbol	Definition	Dimensions
A			
	a (c)	Speed of sound	LT^{-1}
	a.c.	Aerodynamic center	See (1)
	A	Area; Reference area	L^2
	AR	Aspect ratio b^2/S	
B			
	b	Span	L
C			
	c	Local coefficient	
	c	Gas velocity; rockets	LT^{-1}
	c	Chord; of an airfoil	L
	c.g.	Center of gravity	See (1)
	c.p.	Center of pressure	See (1)
	c_p	Specific heat at constant pressure	$L^2T^{-2}\theta^{-1}$
	c_f	Local skin-friction coefficient $\frac{2\tau_w}{\rho_e u_e^2}$	
	c_v	Specific heat at constant volume	$L^2T^{-2}\theta^{-1}$
	C	Coefficient	

(1) In stating the a.c., c.g., or c.p. location with respect to some reference, the word 'location', 'position', or an equivalent must be used in connection with a.c., c.g. or c.p. . The dimension is then L .

Symbol	Definition	Dimensions
D		
d (D)	Diameter	L
E		
E	Energy	ML^2T^{-2}
F		
f	Frequency	T^{-1}
F	Force	MLT^{-2}
G		
g	Acceleration due to gravity	LT^{-2}
G	Mass velocity mass flow per unit cross sectional area per unit time, weight flow per unit cross sectional area per unit time	$ML^{-2}T^{-1}$ Mass flow $ML^{-1}T^{-3}$ Weight flow
H		
h	Enthalpy per unit mass or per unit weight	L^2T^{-2} Per unit mass L Per unit weight
h	Altitude	L
h (q)	Heat-flow rate; per unit area per degree across a boundary surface	$MT^{-3}\theta^{-1}$
H	Boundary-layer shape parameter $\frac{\delta^*}{\theta}$	
H	Enthalpy; total heat content	ML^2T^{-2}
H	Angular momentum	ML^2T^{-1}

Symbol	Definition	Dimensions
I		
i	Angle of incidence	
I	Mass moment of inertia	ML^2
I	Area moment of inertia; second moment of area	L^4
I	Impulse	MLT^{-1}
J		
J	Mechanical equivalent of heat; Joule's constant	
K		
k	Factor for comparison purposes	To be defined
k	Gage constant	To be defined
k (ρ)	Radius of gyration	L
k	Thermal conductivity	$MLT^{-3}\theta^{-1}$
L		
L	Absolute length (used in dimensional analysis)	L
l	Length; Distance	L
M		
m	Mass	M
M	Mass (used in dimensional analysis)	M
M	Moment	ML^2T^{-2}
Ma	Mach number	

N

Symbol	Definition	Dimensions
n	Denominator in exponent of boundary layer power profile	
n	Number of items	
n	Load factor	
Nu	Nusselt number	

P

Symbol	Definition	Dimensions
p (ω_X)	Angular velocity of the body-axes system about the X-axis (angular velocity of the body about its longitudinal axis)	T^{-1}
P (p)	Pressure	$ML^{-1}T^{-2}$
P.E.	Probable error	
Pe	Peclet number	
Pr	Prandtl number	

Q

Symbol	Definition	Dimensions
q (ω_Y)	Angular velocity of the body-axes system about the Y-axis	T^{-1}
q (Q)	Dynamic Pressure	$ML^{-1}T^{-2}$
q	Quantity of heat per unit time	$ML^{2}T^{-3}$
q	Quantity of heat per unit mass or per unit weight	$L^{2}T^{-2}$ Per unit mass L Per unit weight
Q	Quantity of heat	$ML^{2}T^{-2}$

Symbol	Definition	Dimensions
R		
r (ω_z)	Angular velocity of the body-axes system about the Z-axis	T^{-1}
r	Radius	L
R	Gas constant	$L^2T^{-2}\theta^{-1}$
R	Turbulence correlation coefficient	
Re	Reynolds number	
S		
s	Entropy per unit mass or per unit weight	$L^2T^{-2}\theta^{-1}$ Per unit Mass $L\theta^{-1}$ Per unit weight
s	Length along a body surface	L
S	Entropy	$ML^2T^{-2}\theta^{-1}$
S	Surface or projected area	L^2
St	Stanton number	
T		
t	Thickness	L
t	Temperature; general	θ
t (τ)	Time	T
T	Time; when used in dimensional analysis	T
T	Temperature; absolute	θ
T	Period of oscillation	T

U

Symbol	Definition	Dimensions
u	Internal energy per unit mass or per unit weight	L^2T^{-2} Per unit mass; L Per unit weight
u	Velocity component along the X-axis	LT^{-1}
u	Velocity component along the body surface	LT^{-1}
U (E)	Internal energy	ML^2T^{-2}
U (H)	Heat-transfer coefficient; overall	$MT^{-3}\theta^{-1}$

V

Symbol	Definition	Dimensions
v	Velocity component along the Y-axis	LT^{-1}
v	Volume per unit mass or per unit weight	$M^{-1}L^3$ Per unit mass; $M^{-1}L^2T^2$ Per unit weight
V (U)	Velocity	LT^{-1}
V	Volume	L^3

W

Symbol	Definition	Dimensions
w	Velocity component along the Z-axis	LT^{-1}
w	Weight flow per unit time	MLT^{-3} Weight flow
	Mass flow per unit time	MT^{-1} Mass flow
W	Weight	MLT^{-2}

Symbol	Definition	Dimensions
X		
x	Coordinate along the X-axis	L
Y		
y	Coordinate along the Y-axis	L
y	Distance measured perpendicular to a surface	L
Z		
z	Coordinate along the Z-axis	L
α		
α	Angle of attack $\left[\tan^{-1}\frac{w}{u}\right]$	
α'	Complex or total angle of attack $(\alpha + i\beta)$ $\left[\cos^{-1}\frac{u}{V}\right]$	
α	Nozzle divergence half angle	
β		
β	Angle of sideslip in the stability-axes system $\left[\sin^{-1}\frac{v}{V}\right]$	
β'	Angle of sideslip in the body-axes system $\left[\tan^{-1}\frac{v}{u}\right]$	
β	Mach number relation $\left\|Ma^2 - 1\right\|^{\frac{1}{2}}$	
β	Nozzle convergence half angle	

Symbol	Definition	Dimensions
γ		
γ (k)	Ratio of specific heats	
Γ		
Γ	Circulation	$L^2 T^{-1}$
Γ	Dihedral angle	
δ		
δ	Boundary-layer thickness	L
δ	Angular displacement of control surface or tab	
δ^*	Displacement thickness of boundary-layer	
ϵ		
ϵ	Angle of downwash	
ϵ	Emissivity (for radiant heat)	
η		
η (R)	Temperature-recovery factor	
η	Efficiency	
θ		
θ	Angle of pitch	
θ	Momentum thickness of boundary-layer	L
θ	Absolute temperature when used in dimensional analysis	θ

	Symbol	Definition	Dimensions
λ			
	λ	Damping rate	T^{-1}
	λ	Taper ratio	
	λ	Mean free path	L
	λ	Wave length	L
Λ			
	Λ	Sweepback angle	
μ			
	μ	Mach angle	
	μ	Damping constant	$ML^2 T^{-1}$
	μ	Absolute viscosity	$ML^{-1}T^{-1}$
ν			
	ν	Kinematic viscosity $= \frac{\mu}{\rho}$	$L^2 T^{-1}$
ρ			
	ρ	Mass density	ML^{-3}
σ			
	σ	Angle of sidewash	
	σ (s)	Gyroscopic stability factor	
	σ	Relative density	
	σ	Standard deviation	To be defined

Symbol	Definition	Dimensions
τ		
τ	Shear stress	$ML^{-1}T^{-2}$
ϕ		
ϕ	Angle of roll	
ϕ'	Aerodynamic roll angle $\left[\tan^{-1} \frac{v}{w}\right]$	
ϕ	Potential function	To be defined
ψ		
ψ	Stream function	To be defined
ψ	Angle of yaw	
ω		
ω (Ω)	Angular velocity	T^{-1}
ω	Exponent in viscosity temperature relationship	

Section Ib Subscript symbols

SECTION Ib

SUBSCRIPT SYMBOLS

INTRODUCTION

Symbols recommended for use as subscripts are presented in the following pages. In general, any of the primary symbols may be used as subscripts. Therefore, it is not intended that the following list comprise all possible subscript symbols.

If the use of multiple subscripts becomes necessary, the following rules should be observed:

Rule a. Adjacent subscripts imply that the latter subscript modifies the preceding subscript, or that adjacent subscripts together represent an abbreviation.

Example 1. M_{x_s} -Moment about the stability -axis

Example 2. p_{st} -Static pressure

Rule b. Two adjacent subscripts separated by a comma imply that the second modifies the quantity represented by the principal symbol or the principal symbol together with preceding subscripts.

Example 1. $M_{x,\delta}$ -Rolling moment due to angular displacement of a control surface

Rule c. A subscript to a subscript denotes the derivative of the quantity, represented by the principal symbol, with respect to the quantity related to the second subscript.

Example 1. $C_{l_p} = \dfrac{\partial C_l}{\partial\left(\dfrac{pd}{2V}\right)}$

SECTION Ib

SUBSCRIPT SYMBOLS

	Symbol	Concept	Remarks
A	a	Absolute	
	a	Adiabatic	Use ad as alternate
	a	Aeroballistic	
	a	Aileron	
	ac	Aerodynamic center	
	av	Average	
	A	Axial	
B	b	Base	
	b	Basic	
	b	Burner; Burnt	
	b	Burning	
	bar	Barometric	
	B	Body	
C	c	Calibrated	Use cal as alternate
	c	Chord	
	c	Combustion	
	c	Combustion chamber	
	c	Compressible	
	c	Compression	Use comp as alternate
	c	Compressor	
	c	Critical	Use cr as alternate
	calc	Calculated	
	cg	Center of gravity	
	corr	Corrected	
	cp	Center of pressure	
	C	Crosswind	

Symbol	Concept	Remarks
D		
d	Diffuser	Use diff as alternate
d	Duct	
D	Drag	
E		
e	Boundary layer; outer edge of	
e	Earthbound	
e	Effective	Use eff as alternate
e	Elevator	
e	Engine	
e	Equilibrium	
e	Equivalent	
e	Exhaust	
e	Exit	
F		
f	Fin	
f	Flap	
f	Friction	
f	Fuselage	
F	Frontal	
G		
g	Gage	
H		
h	Horizontal	
H	Hinge	
I		
i	Ideal	
i	Incidence (angle)	
i	Indicated	
i	Induced	
i	Initial	
i	Inlet; Intake; Input	
i	Interference	
inc	Incompressible	
J		
j	Jet	

	Symbol	Concept	Remarks
L			
	l	Distance	
	l	Lower surface	
	l	Rolling moment	
	lam	Laminar	
	lat	Lateral	
	le	Leading edge	
	L	Left	
	L	Lift	
	L	Lower	
M			
	m	Mean	See also av for average
	m	Pitching moment	
	M	Moment	
	max	Maximum	
	min	Minimum	
N			
	n	Net	
	n	Normal to a surface	
	n	Nozzle	
	n	Yawing moment	
	np	Neutral point	
	N	Normal, perpendicular to XY - plane	
	N	Normal force	
O			
	o	Standard or reference conditions; standard sea-level conditions	Use sl as alternate for sea-level conditions
	o	Supply conditions	Use t as alternate
	o	Zero lift	

	Symbol	Concept	Remarks
P			
	p	Angular velocity of the body-axes system about the X-axis	
	p	Parasite	Use para as alternate
	p	Perturbation	
	p	Polar	
	p	Potential	Use pot as alternate
	p	Propellant; propulsive	
	P	Power	
Q			
	q	Angular velocity of the body-axes system about the Y-axis	
R			
	r	Angular velocity of the body-axes system about the Z-axis	
	r	Radial	
	r	Ram	
	r	Recovery	
	r	Relative	
	r	Root	
	r	Rudder	
	R	Resultant	
	R	Right	
S			
	s	Slipstream	
	s	Stability	
	s	Stabilizer	
	sp	Specific	
	st	Static	
	S	Stall conditions	
T			
	t	Tab	Use tab as alternate
	t	Tail	
	t	Tangential	
	t	Throat	Use th as alternate
	t	Tip	
	t	Trim	

	Symbol	Concept	Remarks
T			
	te	Trailing edge	
	th	Theoretical	
	turb	Turbulent	
	T	Terminal velocity conditions	
	T	Thrust	Use th as alternate
	T	Tunnel	Use t as alternate
U			
	u	Upper surface	
V			
	v	Vertical	
	vol	Volumetric	
W			
	w	Wake	
	w	Wall; surface	
	w	Wave	
	w	Wind	Use W as alternate
	w	Wing	
X			
	xs	Excess	
	X	Component parallel to X-axis	Positive in positive X-direction
Y			
	Y	Component parallel to Y-axis	Positive in positive Y-direction
Z			
	Z	Component parallel to Z-axis	Positive in positive Z-direction
∞			
	∞	Undisturbed free-stream; Ambient conditions	

Superscript symbols

Section Ic Superscript symbols

SECTION Ic

SUPERSCRIPT SYMBOLS

INTRODUCTION

The following page presents the superscript symbols usually accepted. When necessary, subscript symbols may be used for superscripts.

Multiple superscript symbols may be used if necessary and their use is governed by rules a and b governing the use of multiple subscripts. If a principal symbol with a superscript is to be raised to a power, the symbol including superscript should be placed in parentheses before indicating the power.

SECTION Ic

SUPERSCRIPT SYMBOLS

Symbol	Concept	Remarks
$\dot{}$ (Dot)	First derivative with respect to time	These symbols are written over, not after the principal symbol
$\ddot{}$ (Double Dot)	Second derivative with respect to time	
' (Prime)	First derivative with respect to distance Effective or precise value To designate a second set of axes or quantities related to such axes Stagnation conditions behind a normal shock	Alternate for subscript
" (Double Prime)	Second derivative with respect to distance To designate a third set of axes or quantities related to such axes	
— (Bar)	Mean value	This symbol is written over, not after the principal symbol
o (Zero)	Total, supply conditions	See also subscript (t) or (o)
* (Asterisk)	Characteristic or reference value; critical conditions at Ma=1	

Section Id Alphabetical index according to concept

SECTION Id

ALPHABETICAL INDEX ACCORDING TO CONCEPT

Summarizing the lists of symbols, an alphabetical index according to concept is presented in the following pages.

SECTION Id

ALPHABETICAL INDEX ACCORDING TO CONCEPT

A

CONCEPT	SYMBOL Principal	Subscript	Superscript
Absolute		a	
Absolute temperature	T		
Acceleration due to gravity	g		
Adiabatic		a (ad)	
Aeroballistic		a	
Aerodynamic center	.a.c.	ac	
Aileron		a	
Angle;			
dihedral	Γ		
Mach	μ		
Angle of			
attack	α		
downwash	ϵ		
incidence	i	i	
pitch	θ		
roll	ϕ		
sideslip	β		
sidewash	σ		
sweepback	Λ		
yaw	ψ		
Angular			
displacement of control surface or tab	δ		
momentum	H		
velocity	ω		
Angular velocity of body-axes system			
about the X-axis	p (ω_X)	p	
about the Y-axis	q (ω_Y)	q	
about the Z-axis	r (ω_Z)	r	

CONCEPT	SYMBOL Principal	Subscript	Superscript
A			
Area			
reference	A		
surface or projected	S (A)		
Aspect ratio	AR (Æ)		
Average		av	
Axial		A	
B			
Barometric		bar	
Base		b	
Basic		b	
Body		B	
Boundary layer			
displacement thickness of	δ^*	δ^*	
momentum thickness of	θ	θ	
thickness of	δ	δ	
Burner		b	
Burning		b	
Burnt		b	
C			
Calculated		calc	
Calibrated		c (cal)	
Center of			
gravity	c.g.	cg	
pressure	c.p.	cp	
Characteristic or reference value			*
Chord	c	c	
Circulation	Γ		

CONCEPT	SYMBOL Principal	Subscript	Superscript
C			
Coefficient;	C		
heat-transfer; overall	U (H)		
local	c		
turbulence correlation	R		
turbulence exchange	ϵ		
Combustion		c	
Combustion chamber		c	
Compressible		c (comp)	
Components parallel to X-, Y-, Z-axis respectively		X, Y, Z	
Convergence half angle; nozzle	β		
Coordinate system	X, Y, Z		
Corrected		corr	
Critical conditions; at Ma=1		c (cr)	
Crosswind		C	
D			
Density			
mass	ρ	ρ	
relative	σ		
Derivative			
first, with respect to distance			' (prime)
first, with respect to time			˙ (dot)
second, with respect to distance			" (double prime)
second, with respect to time			¨ (double dot)
Diameter	d (D)		
Diffuser		d	
Distance	l	l	

CONCEPT	SYMBOL Principal	Subscript	Superscript
D			
Divergence half angle; nozzle	α		
Drag		D	
E			
Earthbound		e	
Elevator		e	
Energy,	E		
internal	U (E)		
internal, per unit mass or weight	u (e) u (e)		
Engine		e	
Enthalpy			
per unit mass or weight	h		
total heat content	H		
Entropy			
per unit mass or weight	s		
Equilibrium		e	
Equivalent		e	
Excess		xs	
Exhaust		e	
Exit		e	
F			
Fin		f	
Flap		f	
Force	F		
Free-stream; undisturbed		∞	
Frequency	f		
Fuselage; body		f	

CONCEPT	SYMBOL Principal	Subscript	Superscript
G			
Gage		g	
Gas		g	
Gas constant	R		
Gas velocity (rockets)	c		
H			
Heat;		h	
exchanger			
flow rate, per unit area, per degree across a boundary surface	h		
Heat			
quantity of	Q		
specific quantity, per mass or weight	q		
Heat			
transfer coefficient			
overall	u (H)		
local	h		
Horizontal		h	
I			
Ideal		i	
Indicated		i	
Induced		i	
Initial		i	
Inlet; Intake; Input		i	
Interference		i	
Isentropic		isen	

CONCEPT	SYMBOL Principal	Subscript	Superscript
J			
Jet		j	
K			
Kinematic viscosity	ν		
L			
Laminar		lam	
Lateral		lat	
Leading edge		le	
Length	l	l	
Lift		L	
Load factor	n		
Lower		L	
Lower surface		l	
M			
Mach number	Ma		
Mach number relation $\lvert Ma^2-1\rvert^{\frac{1}{2}}$	β		
Mass	m		
Mass flow, per unit cross-sectional area per unit time	G		
Mass flow; per unit time	w		
Maximum		max	
Mean		m	— (bar)
Mean free path	λ		
Mechanical equivalent of heat; Joule's constant	J		

	CONCEPT	SYMBOL Principal	Subscript	Superscript
M				
	Minimum		min	
	Moment	M	M	
	Moment of inertia area	I		
	Moment of inertia mass	I		
N				
	Net		n	
	Neutral point		np	
	Normal;		N	
	perpendicular to the XY-plane		N	
	perpendicular to a surface		n	
	Nozzle;		n	
	convergence half angle	β		
	divergence half angle	α		
	exit		e	
	Nusselt number	Nu		
P				
	Parasitic		p (para)	
	Peclet number	Pe		
	Potential function	ϕ		
	Power		P	
	Prandtl number	Pr		
	Pressure			
	general	P (p)	p (P)	
	center of	c.p.	cp	
	dynamic	q (Q)		
	total (measured by Pitot tube in supersonic flow)	P (p)	o (t)	' (prime)

CONCEPT	SYMBOL Principal	Subscript	Superscript
R			
Radial		r	
Radius;	r		
of gyration	k (ρ)		
Ram		r	
Recovery		r	
Recovery factor; temperature	η		
Reference (or standard conditions)		o	
Reference (or characteristic value)			*
Relative		r	
Resultant		R	
Reynolds number	Re		
Root		r	
Rudder		r	
S			
Span	b		
Specific		sp	
Specific heat			
at constant pressure	c_p		
at constant volume	c_v		
ratio of $\frac{c_p}{c_v}$	γ (k)		
Speed of sound	a (c)		
Stability		s (S)	
Stabilizer		s	
Stagnation conditions			
isentropic		o (t)	
Supply conditions		o (t)	o (zero)
Stall conditions		s	

	CONCEPT	SYMBOL Principal	Subscript	Superscript
S				
	Static		st	
	Stanton number	St		
	Stream function	ψ		
T				
	Tab		t	
	Tail		t	
	Tangential		t	
	Taper ratio	λ		
	Temperature			
	absolute	T		
	general	t		
	recovery factor	η		
	Theoretical		th	
	Thermal conductivity	k		
	Thickness	t		
	Throat	t (th)		
	Time	t (τ)		
	Tip		t	
	Total (supply conditions)		o (t)	o (zero)
	Trailing edge		te	
	Tunnel		T (t)	
	Turbine		t	
	Turbulent		turb	
U				
	Upper surface		u	

CONCEPT	SYMBOL Principal	Subscript	Superscript
V			
Velocity	V (U)		
Velocity; angular	ω (Ω)		
Velocity component;			
along the X-axis	u		
along the Y-axis	v		
along the Z-axis	w		
Vertical		v	
Viscosity			
absolute	μ		
kinematic	ν		
Volume;	V		
per unit mass or per unit weight	v		
Volumetric		vol	
W			
Wall; surface		w	
Wave		w	
Wave length	λ		
Weight	W		
Weight flow per unit time	w		
Wind		w (W)	
Wing		w	

SECTION II

Section IIa Static forces & moments & their coefficients referred to body axes

Section IIb Static forces & moments & their coefficients referred to stability axes

Section IIc Static forces & moments & their coefficients referred to wind axes

Section IId Magnus forces & moments & their coefficients referred to aeroballistic axes

Section IIe Damping forces & moments & their coefficients referred to body axes

SECTION II

FORCES, MOMENTS, AND COEFFICIENTS

INTRODUCTION

In the following pages, the symbols and definitions for the forces, moments, and their respective coefficients are presented. The axes used in these definitions are those discussed in Section III. All coefficients are defined in terms used for missiles or projectiles. For airplanes, the coefficients are essentially the same except for the following differences.

a. The reference area* is either surface or projected area 'S', usually the wing area

b. The reference length* is either

   1. the span 'b', for rolling and yawing moment coefficients, or

   2. the chord 'c', usually the geometric or aerodynamic chord, for pitching moment coefficient.

....

*The reference area and length used for airplanes is sometimes used for finned missiles. Usually the choice of the reference area and length depends on the effectiveness of the fins with respect to the coefficient under consideration. For instance, for roll damping, the span 'b' is usually used in preference to the diameter 'd' .

SECTION II a

STATIC FORCES AND MOMENTS AND THEIR COEFFICIENTS REFERRED TO BODY AXES

SYMBOL	Definition	Coefficient symbol and definition	Positive direction
F_A	Axial force: force along the X-axis	$C_A = \frac{F_A}{qA}$	- X
F_Y	Side force: force along the Y-axis	$C_Y = \frac{F_Y}{qA}$	Y
F_N (N)	Normal force: force along the Z-axis	$C_N = \frac{F_N}{qA}$	- Z
M_X	Rolling moment: moment about the X-axis	$C_l = \frac{M_X}{qAd}$	Tends to rotate the +Y-axis into the +Z-axis
M_Y	Pitching moment: moment about the Y-axis	$C_m = \frac{M_Y}{qAd}$	Tends to rotate the +Z-axis into the +X-axis
M_Z	Yawing moment: moment about the Z-axis	$C_n = \frac{M_Z}{qAd}$	Tends to rotate the +X-axis into the +Y-axis

SECTION II b

STATIC FORCES AND MOMENTS AND THEIR COEFFICIENTS REFERRED TO STABILITY AXES

SYMBOL	Definition	Coefficient symbol and definition	Positive direction
F'_D	Component of the drag force along the X_s-axis	$C'_D = \frac{F'_D}{qA}$	$-X_s$
F_Y	Side force: force along the Y_s-axis	$C_Y = \frac{F_Y}{qA}$	Y_s
F_L (L)	Lift force: force along the Z_s-axis	$C_L = \frac{F_L}{qA}$	$-Z_s$
M_{Xs}	Rolling moment: moment about the X_s-axis	$C_{l_s} = \frac{M_{Xs}}{qAd}$	Tends to rotate the $+Y_s$-axis into the $+Z_s$-axis
M_{Ys} (M_Y)	Pitching moment: moment about the Y_s-axis	$C_{m_s} = \frac{M_{Ys}}{qAd}$	Tends to rotate the $+Z_s$-axis into the $+X_s$-axis
M_{Zs}	Yawing moment: moment about the Z_s-axis	$C_{n_s} = \frac{M_{Zs}}{qAd}$	Tends to rotate the $+X_s$-axis into the $+Y_s$-axis

SECTION II c

STATIC FORCES AND MOMENTS AND THEIR COEFFICIENTS REFERRED TO WIND AXES

SYMBOL	Definition	Coefficient symbol and definition	Positive direction
F_D (D)	Drag force: force along the X_w-axis	$C_D = \frac{F_D}{qA}$	$-X_w$
F_C (C)	Crosswind force: force along the Y_w-axis	$C_C = \frac{F_C}{qA}$	Y_w
F_L (L)	Lift force: force along the Z_w-axis	$C_L = \frac{F_L}{qA}$	$-Z_w$
M_{X_w}	Rolling moment: moment about the X_w-axis	$C_{l_w} = \frac{M_{X_w}}{qAd}$	Tends to rotate the $+Y_w$-axis into the $+Z_w$-axis
M_{Y_w}	Pitching: moment about the Y_w-axis	$C_{m_w} = \frac{M_{Y_w}}{qAd}$	Tends to rotate the $+Z_w$-axis into the $+X_w$-axis
M_{Z_w}	Yawing moment: moment about the Z_w-axis	$C_{n_w} = \frac{M_{Z_w}}{qAd}$	Tends to rotate the $+X_w$-axis into the $+Y_w$-axis

SECTION IId

MAGNUS FORCES AND MOMENTS AND THEIR COEFFICIENTS REFERRED TO AEROBALLISTIC AXES

SYMBOL	Definition	Coefficient symbol and definition	Positive direction
$F_{Ya,p}$	Magnus force along the Y_a-axis: force due to spin of the body while at a total angle of attack (α')	$C_{Ya,p} = \frac{F_{Ya,p}}{qA}$; $C_{Ya,p} = \frac{\partial C_{Ya,p}}{\partial\left(\frac{pd}{2V}\right)}\left(\frac{pd}{2V}\right)$ Slope Coefficient $C_{Ya_p} = \frac{\partial C_{Ya,p}}{\partial\left(\frac{pd}{2V}\right)}$ or $C_{Ya_p} = C_{Ya,p}\left(\frac{2V}{pd}\right)$	$+Y_a$
$F_{Na,p}$	Magnus force along the Z_a-axis: force due to spin of the body while at a total angle of attack (α')	$C_{Na,p} = \frac{F_{Na,p}}{qA}$; $C_{Na,p} = \frac{\partial C_{Na,p}}{\partial\left(\frac{pd}{2V}\right)}\left(\frac{pd}{2V}\right)$ Slope Coefficient $C_{Na_p} = \frac{\partial C_{Na,p}}{\partial\left(\frac{pd}{2V}\right)}$ or $C_{Na_p} = C_{Na,p}\left(\frac{2V}{pd}\right)$	$-Z_a$
$M_{Ya,p}$	Magnus moment about the Y_a-axis: moment due to spin of the body while at a total angle of attack (α')	$C_{ma,p} = \frac{M_{Ya,p}}{qAd}$; $C_{ma,p} = \frac{\partial C_{ma,p}}{\partial\left(\frac{pd}{2V}\right)}\left(\frac{pd}{2V}\right)$ Slope Coefficient $C_{ma_p} = \frac{\partial C_{ma,p}}{\partial\left(\frac{pd}{2V}\right)}$ or $C_{ma_p} = C_{ma,p}\left(\frac{2V}{pd}\right)$	Tends to rotate the $+Z_a$-axis into the $+X_a$-axis

SECTION IId CONTINUED

SYMBOL	Definition	Coefficient symbol and definition	Positive direction
$M_{Za,p}$	Magnus moment about the Z_a-axis: moment due to spin of the body while at a total angle of attack (α')	$C_{na,p}=\frac{M_{Za,p}}{qAd}$; $C_{na,p}=\frac{\partial C_{na,p}}{\partial\left(\frac{pd}{2V}\right)}\left(\frac{pd}{2V}\right)$ Slope Coefficient $C_{na_p}=\frac{\partial C_{na,p}}{\partial\left(\frac{pd}{2V}\right)}$ or $C_{na_p}=C_{na,p}\left(\frac{2V}{pd}\right)$	Tends to rotate the $+X_a$-axis into the $+Y_a$-axis

NOTES:

1. The subscript 'a' may be dropped from the symbol when it is clearly stated that the axes system is the aeroballistic-axes system and no other axes system is used in the same publication.

2. Following the rules for multiple subscripts C_{na_p}, C_{ma_p}, etc. will be defined as

$$\frac{\partial C_{na}}{\partial\left(\frac{pd}{2V}\right)} \quad \text{and} \quad \frac{\partial C_{ma}}{\partial\left(\frac{pd}{2V}\right)}$$

However, the only contribution to C_{na} or C_{ma} that is changing with the quantity $\frac{pd}{2V}$ is $C_{na,p}$ and $C_{ma,p}$, respectively. Therefore, the manner in which C_{na} and C_{ma} are defined in the above table is equally correct and aids in showing the relationship between $C_{na,p}$ and C_{na_p} etc. .

SECTION IIe

DAMPING FORCES AND MOMENTS AND THEIR COEFFICIENTS REFERRED TO BODY AXES

SYMBOL	Definition	Coefficient symbol and definition	Positive direction
$F_{N,q}$	Damping force: normal force due to angular velocity about the body Y-axis	$C_{N,q} = \frac{F_{N,q}}{QA}$; $C_{N,q} = \frac{\partial C_{N,q}}{\partial\left(\frac{qd}{2V}\right)}\left(\frac{qd}{2V}\right)$ Slope Coefficient $C_{N_q} = \frac{\partial C_{N,q}}{\partial\left(\frac{qd}{2V}\right)}$	-Z
$F_{N,\dot{\alpha}}$	Damping force: normal force due to failure of the flow to instantaneously respond to change in angle of attack (α) (Sometimes referred to as a lag force)	$C_{N,\dot{\alpha}} = \frac{F_{N,\dot{\alpha}}}{QA}$; $C_{N,\dot{\alpha}} = \frac{\partial C_{N,\dot{\alpha}}}{\partial\left(\frac{\dot{\alpha}d}{2V}\right)}\left(\frac{\dot{\alpha}d}{2V}\right)$ Slope Coefficient $C_{N_{\dot{\alpha}}} = \frac{\partial C_{N,\dot{\alpha}}}{\partial\left(\frac{\dot{\alpha}d}{2V}\right)}$	-Z
$F_{N,q}+F_{N,\dot{\alpha}}$	Total damping force due to rate of change of angle of attack (α)	$C_{N,q}+C_{N,\dot{\alpha}}$ Slope Coefficient $C_{N_q}+C_{N_{\dot{\alpha}}}$	-Z

NOTE: In this section, the symbol 'Q' is used for dynamic pressure in the equations instead of 'q' to avoid confusion with angular velocity about the Y-axis 'q'.

SYMBOL	Definition	Coefficient symbol and definition	Positive direction
$F_{Y,r}$	Damping force: side force due to angular velocity about the body Z-axis	$C_{Y,r} = \frac{F_{Y,r}}{QA}$; $C_{Y,r} = \frac{\partial C_{Y,r}}{\partial\left(\frac{rd}{2V}\right)} \left(\frac{rd}{2V}\right)$ Slope Coefficient $C_{Y_r} = \frac{\partial C_{Y,r}}{\partial\left(\frac{rd}{2V}\right)}$	+Y
$F_{Y,\dot{\beta}'}$	Damping force: side force due to failure of the flow to instantaneously respond to change in angle of sideslip (β') (Sometimes referred to as a lag force)	$C_{Y,\dot{\beta}'} = \frac{F_{Y,\dot{\beta}}}{QA}$; $C_{Y,\dot{\beta}'} = \frac{\partial C_{Y,\dot{\beta}'}}{\partial\left(\frac{\dot{\beta}'d}{2V}\right)} \left(\frac{\dot{\beta}'d}{2V}\right)$ Slope Coefficient $C_{Y_{\dot{\beta}}} = \frac{C_{Y,\dot{\beta}'}}{\left(\frac{\dot{\beta}'d}{2V}\right)}$	+Y
$F_{Y,r} + F_{Y,\dot{\beta}'}$	Total damping force due to rate of change of angle of sideslip ($\dot{\beta}'$)	$C_{Y,r} + C_{Y,\dot{\beta}'}$ Slope Coefficient $C_{Y_r} + C_{Y_{\dot{\beta}'}}$	+Y

SYMBOL	Definition	Coefficient symbol and definition	Positive direction
$M_{Y,q}$	Damping moment: pitching moment due to angular velocity about the body Y -axis	$C_{m,q} = \frac{M_{Y,q}}{QAd}$; $C_{m,q} = \frac{\partial C_{m,q}}{\partial\left(\frac{qd}{2V}\right)}\left(\frac{qd}{2V}\right)$ Slope Coefficient $C_{m_q} = \frac{\partial C_{m,q}}{\partial\left(\frac{qd}{2V}\right)}$	Tends to rotate the +Z-axis into the +X-axis
$M_{Y,\dot{\alpha}}$	Damping moment: pitching moment due to failure of the flow to instantaneously respond to change in angle of attack (α) (Sometimes referred to as a lag moment)	$C_{m,\dot{\alpha}} = \frac{M_{Y,\dot{\alpha}}}{QAd}$; $C_{m,\dot{\alpha}} = \frac{\partial C_{m,\dot{\alpha}}}{\partial\left(\frac{\dot{\alpha}d}{2V}\right)}\left(\frac{\dot{\alpha}d}{2V}\right)$ Slope Coefficient $C_{m_{\dot{\alpha}}} = \frac{\partial C_{m,\dot{\alpha}}}{\partial\left(\frac{\dot{\alpha}d}{2V}\right)}$	Tends to rotate the +Z-axis into the +X-axis
$M_{Y,q}+M_{Y,\dot{\alpha}}$	Total damping moment due to rate of change of angle of attack (α)	$C_{m,q} + C_{m,\dot{\alpha}}$ Slope Coefficient $C_{m_q} + C_{m_{\dot{\alpha}}}$	Tends to rotate the +Z-axis into the +X-axis

SYMBOL	Definition	Coefficient symbol and definition	Positive direction
$M_{Z,r}$	Damping moment: yawing moment due to angular velocity about the body Z-axis	$C_{n,r} = \frac{M_{Z,r}}{QAd}$; $C_{n,r} = \frac{\partial C_{n,r}}{\partial\left(\frac{rd}{2V}\right)} \left(\frac{rd}{2V}\right)$ Slope Coefficient $C_{n_r} = \frac{\partial C_{n,r}}{\partial\left(\frac{rd}{2V}\right)}$	Tends to rotate the +X-axis into the +Y-axis
$M_{Z,\dot{\beta}}$	Damping moment: yawing moment due to failure of the flow to instantaneously respond to change in angle of sideslip (β') (Sometimes referred to as a lag moment)	$C_{n,\dot{\beta}'} = \frac{M_{Z,\dot{\beta}'}}{QAd}$; $C_{n,\dot{\beta}'} = \frac{\partial C_{n,\dot{\beta}'}}{\partial\left(\frac{\dot{\beta}'d}{2V}\right)} \left(\frac{\dot{\beta}'d}{2V}\right)$ Slope Coefficient $C_{n_{\dot{\beta}'}} = \frac{\partial C_{n,\dot{\beta}'}}{\partial\left(\frac{\dot{\beta}'d}{2V}\right)}$	Tends to rotate the +X-axis into the +Y-axis
$M_{Z,r}+M_{Z,\dot{\beta}'}$	Total damping moment due to rate of change of angle of sideslip (β')	$C_{n,r} + C_{n,\dot{\beta}'}$ Slope Coefficient $C_{n_r} + C_{n_{\dot{\beta}'}}$	Tends to rotate the +X-axis into the +Y-axis
$M_{X,p}$	Damping moment: rolling moment due to angular velocity about the body X-axis	$C_{l,p} = \frac{M_{X,p}}{QAd}$; $C_{l,p} = \frac{\partial C_{l,p}}{\partial\left(\frac{pd}{2V}\right)} \left(\frac{pd}{2V}\right)$ Slope Coefficient $C_{l_p} = \frac{\partial C_{l,p}}{\partial\left(\frac{pd}{2V}\right)}$	Tends to rotate the +Y-axis into the +Z-axis

SECTION III

Section IIIa Systems of axes and angles

Section IIIb Transfer equations

Systems of axes, angles, & transfer equations

SECTION III

SYSTEMS OF AXES, ANGLES, AND TRANSFER EQUATIONS

This section is divided into two parts, **a** and **b**. In part **a** the axes systems and angles are discussed. In part **b** the equations to transfer the forces and moments from the body-axes system to the other axes systems are discussed.

SECTION IIIa

SYSTEMS OF AXES AND ANGLES

Several axes systems are used in aerodynamics. Each system is an orthogonal right-hand system, i.e., when the YZ -plane (perpendicular to the X-axis) is viewed looking in the +X -direction, the +Z -axis is rotated clockwise 90 degrees from the +Y-axis. The axes systems are further defined below and presented pictorially in Figures 1 through 12.

Tunnel axes

X_T longitudinal tunnel axis, positive upstream.

Y_T lateral tunnel axis, perpendicular to the longitudinal tunnel axis, usually taken as horizontal and positive to the right when looking in the $+X_T$-direction.

Z_T third tunnel axis, perpendicular to the longitudinal and lateral tunnel axes.

Body axes

X longitudinal body axis, positive forward.

Y lateral body axis, perpendicular to the longitudinal body axis, usually taken in the plane of the wings or of a set of fins and positive to the right when the roll angle is zero.

Z third body axis, perpendicular to the longitudinal and lateral body axes.

Non-rolling body axes

X' — longitudinal non-rolling body axis, positive forward, coincident with the body X-axis.

Y' — lateral non-rolling body axis, perpendicular to the longitudinal non-rolling body axis, coincident with and positive in the same direction as the body and wind Y-axis when the yaw and roll angles are zero.

Z' — third non-rolling body axis, perpendicular to the longitudinal and lateral non-rolling body axes.

The non-rolling body-axes system, sometimes known as the missile-axes system, is frequently used in studies of missile-flight dynamics. It is free to rotate through the angles of yaw and pitch but never rotates through an angle of roll. It is always identical with the body-axes system before the body is rotated through an angle of roll.

Wind axes

X_w — longitudinal wind axis, parallel to the relative wind, positive into the wind.

Y_w — lateral wind axis, perpendicular to the longitudinal wind axis, horizontal and positive to the right when the roll angle is zero.

Z_w — third wind axis, perpendicular to the longitudinal and lateral wind axes and always lies in the XZ-plane of the body-axes system.

Stability axes

X_s — longitudinal stability axis, parallel to the projection of the velocity (V) on the plane containing the X- and Z-body axes, positive forward.

Y_s — lateral stability axis, perpendicular to the longitudinal stability axis, coincident with and positive in the same direction as the lateral body axis (Y).

Z_s — third stability axis, perpendicular to the longitudinal and lateral stability axes, coincident with the Z_w-axis.

Aeroballistic axes

X_a : longitudinal aeroballistic axis, coincident with and positive in the same direction as the longitudinal body axis (X).

Y_a : lateral aeroballistic axis, perpendicular to the longitudinal aeroballistic axis.

Z_a : third aeroballistic axis, perpendicular to the longitudinal and lateral aeroballistic axes and in the plane containing the complex or total angle of attack (α'); positive in the direction of w_a, the component of the velocity of the body with respect to the wind along the Z_a-axis.

In aerodynamics, there are several angles frequently used. These angles fall into two categories: aerodynamic angles and position or orientation angles.

Aerodynamic angles

α : angle of attack, angle between the projection of the wind X-axis (X_w) on the XZ-plane and the body X-axis (X); positive rotates the $+Z$-axis into the $+X$-axis:

$$\alpha = \tan^{-1} \frac{w}{u}$$

α' : total or complex angle of attack, angle between the wind X-axis (X_w) and the body X-axis (X); always positive:

$$\alpha' = \cos^{-1} \frac{u}{V} = \tan^{-1} \frac{\sqrt{v^2 + w^2}}{u}$$

β : angle of sideslip in the stability-axes system, angle between the wind X-axis (X_w) and the projection of this axis on the XZ-plane; positive rotates the Y_s-axis into the $+X_s$-axis:

$$\beta = \sin^{-1} \frac{v}{V} = \tan^{-1} \frac{v}{\sqrt{V^2 - v^2}}$$

β' : angle of sideslip in the body-axes system, angle between the projection of the wind X-axis (X_w) on the XY-plane and the body X-axis (X); positive rotates the $+Y$-axis into the $+X$-axis:

$$\beta' = \tan^{-1} \frac{v}{u}$$

ϕ' : aerodynamic roll angle, angle between the aeroballistic Y-axis (Y_a) and the body Y-axis (Y); positive clockwise when looking in the $+X$-direction:

$$\phi' = \tan^{-1} \frac{v}{w}$$

Position or orientation angles ....

ψ : angle of yaw, positive rotates the +X-axis into the +Y-axis.

θ : angle of pitch, positive rotates the +Z -axis into the +X -axis.

ϕ : angle of roll, positive rotates the +Y -axis into the +Z -axis.

ψ , θ, and ϕ constitute a system of three angles which uniquely defines the orientation of the body-axes system with respect to a fixed reference set of axes. Any orientation of the body-axes system is obtained by rotating the reference-axes system through each of the three angles in turn. The sequence in which these rotations are taken is important. If the sequence is changed the orientation of the body-axes system with respect to the fixed reference set of axes will change.

$\phi_{\alpha'}$: roll angle which gives the orientation of the plane containing the total angle of attack with respect to a vertical plane containing the velocity (V). It is the angle between these two planes measured from a vertical (up) position in a plane that is perpendicular to the velocity (V), positive clockwise when looking in the direction of the velocity.

The velocity and velocity components used in these definitions are the velocity of the body with respect to the air and the components thereof.

Pictorial presentations of the various axes and angles, where the axes are considered as diameters of a sphere and all angles are represented as arcs of great circles, are shown in Figures 1 through 12. The fixed reference-axes system used in connection with the orientation angles ψ, θ, and ϕ in this presentation is the tunnel-axes system. The tunnel-axes system is shown alone in Figure 1 with the three planes containing its axes. These three basic planes are shown, similarly shaded, in all subsequent figures and the angles ψ, θ, and ϕ are measured in or from these planes. As was pointed out in the definition of the angles ψ, θ, and ϕ, the orientation of the body-axes system with respect to the fixed reference-axes system (tunnel axes) depends upon the sequence in which the rotation through these angles takes place. (This point will be discussed in detail in Section III b.) To be more comprehensive, the axes systems are presented for two sequences of these angular rotations, ψ, θ, ϕ and θ, ψ, ϕ.

Figures 2 through 10 show the axes and angles for the sequence ψ, θ, and ϕ. The body-, wind-, stability-, and tunnel-axes systems are presented in Figures 2 through 6. Each successive figure presents a case of increasing complexity with Figure 6 presenting the case where the body is rotated through angles of yaw, pitch, and roll. For clarity the aeroballistic-axes system is omitted from these figures. Figures 7 through 10 present a comparison of the body- and aeroballistic-axes systems and the angles associated with these axes systems. For the case where the body is rotated through an angle of pitch only, the body- and aeroballistic-axes systems are coincident and therefore, no comparison is presented for this case.

Figures 11 and 12 show the body-, wind-, stability-, and tunnel-axes systems and the angles associated with these axes systems when the sequence of the angular rotations is θ, ψ, and ϕ. Figure 11 presents the case for pitch and yaw and Figure 12 presents the case for pitch, yaw, and roll. For the simpler cases of pitch only, yaw only, or pitch and roll, the order in which the rotations occur will be the same whether the overall sequence is θ, ψ, ϕ or ψ, θ, ϕ. These cases are covered in the preceding figures and are therefore not repeated.

SECTION IIIb

TRANSFER EQUATIONS

With internal balances, which are in general use in wind-tunnel testing, data are usually obtained in the body-axes system. It is sometimes desirable to use the data in one of the other axes systems. This part of Section III presents the equations used to transfer data from the body-axes system to the other axes systems. The equations for the general case (see Figures 6 and 10) where the body is rotated through an angle of yaw (ψ), an angle of pitch (θ), and an angle of roll (ϕ), are listed in terms of the aerodynamic angles (α , α' , β , β' , and ϕ'). The expressions for the aerodynamic angles in terms of the orientation angles ψ , θ , and ϕ , which are physically set in the tunnel, are then presented. The transfer equations are then expressed in terms of these orientation angles. The transfer equations to the tunnel-axes system are included for completeness. The derivation of these equations is presented in Appendix A. The special cases shown in Figures 2 through 5, and 7 through 9 are then considered. These transfer equations are obtained from those for the general case by considering one or more of the orientation angles to be zero.

The angles ψ , θ , and ϕ , by definition, uniquely define the orientation of the body-axes system with respect to a fixed set of axes. The fixed reference-axes system used herein is an orthogonal right-hand system (see the definition of orthogonal right-hand system, page 53) in which the +X-axis is coincident with and in the direction of the velocity (velocity of the body with respect to the free-air stream), the +Y-axis is to the right when looking in the + X-direction, and the Z-axis lies in a vertical plane. It is seen that in the case of a wind tunnel, this axes system is the tunnel-axes system.

In the wind tunnel, the body, and therefore the body-axes system, can be rotated through the angles ψ , θ , and ϕ . The rotation of the body through these angles causes any or all of the aerodynamic angles α , α' , β , β', and ϕ' to exist. The value of these aerodynamic angles is dependent upon the orientation of the body-axes system with respect to the velocity.

If the sequence is changed in which the rotation through the angles ψ, θ, and ϕ occurs, then the orientation of the body-axes system with respect to the tunnel-axes system is also changed. (The rotations through the angles ψ, θ, and ϕ are about the body axes which have been rotated in turn.) Therefore, the orientation of the body-axes system with respect to the velocity is changed and consequently the values of the aerodynamic angles α, α', β, β', and ϕ' are changed. This is true even though values of the angles ψ and θ are selected for the different sequences such that the body X-axis is located in the same position. Figure 13 shows that if this is done the orientation of the body-axes system with respect to the tunnel-axes system for the two cases differs by a roll angle only.

The equations presented herein are for the case where the sequence is ψ, θ, ϕ. The mechanism which rotates the body-axes system in the wind tunnel is sometimes limited to the sequence θ, ψ, ϕ.[1] If this sequence is taken for a particular set of values of the angles θ, ψ, ϕ, there exists a second set of values for these angles, which, if taken in the sequence ψ, θ, ϕ, will orient the body axes in the identical position in respect to the fixed reference-axes system. The relationships between these two sets of angles are given in Table I and their derivation is presented in Appendix B. Should the sequence θ, ψ, ϕ be used, the transfer equations presented herein would not be directly applicable but would have to be converted using the relationships given in Table I.

For any given values of the angles ψ, θ, and ϕ, (provided the sequence in which the rotations take place is unchanged), certain definite values of the aerodynamic angles α, α', β, β', and ϕ' exist. The values of these aerodynamic angles depend upon the orientation of the body-axes system with respect to the velocity and can be measured in terms of the velocity components along the body axes. The sign convention for these angles depends upon the direction of the velocity components with respect to the body axes. The body-axes system is attached to the body and rolls with the body. It is apparent that if the roll angle (ϕ) is changed by $\pm 180°$, the resulting aerodynamic angles will differ by sign only. (The one exception is α' which is always positive.) The absolute values will be the same in each case, with the exception of the angle ϕ' which is discussed in a subsequent paragraph.

[1] This sequence results when the mechanism for rotating the body through an angle of pitch is limited to a rotation in the plane of the tunnel X_T-, Z_T-axes and a rotation in yaw is accomplished with a dog-leg. With this system the rotation through an angle of yaw always occurs in an axes system that has already been rotated through an angle of pitch.

It is apparent that the expressions for the aerodynamic angles in terms of ψ, θ, and ϕ will not always yield the proper sign. To determine the correct sign, it is necessary to analyze the physical problem. The cases in which ψ, θ, and ϕ are within the limits of $-90° \leq \psi \leq 90°$, $-90° \leq \theta \leq 90°$, and $-360° \leq \phi \leq 360°$ respectively, have been analyzed and the results are presented in Table II. With this table, the proper sign for the aerodynamic angles can easily be determined when ψ, θ, and ϕ are within the stated limits. If these limits for ψ and θ are exceeded the individual physical problem should be analyzed. Because of the relatively infrequent occurrence of tunnel tests in which these limits are exceeded, these cases are considered outside the scope of this report and an analysis of these cases is not presented.

The aerodynamic roll angle (ϕ') is actually the sum of a roll angle which is induced by a rotation in yaw (ψ) and a rotation in pitch (θ) and the actual roll angle (ϕ) of the body-axes system. Therefore, the expression given herein for ϕ' in terms of the orientation angles ψ, θ, and ϕ for the general case where the body-axes system is rotated through an angle of yaw (ψ), an angle of pitch (θ), and an angle of roll (ϕ) can be divided into two additive parts. The first part is the expression for ϕ' when the body-axes system is only rotated through an angle of yaw and an angle of pitch. The second part is simply the roll angle (ϕ). If the first part of the expression is called ϕ_1' then the expression for ϕ' becomes

$$\phi' = \phi_1' + \phi$$

The expression given herein for ϕ_1' is in the form $\cos^{-1}(X)$,

$$\text{where } X = \frac{\sin\theta \cos\psi}{\sqrt{1-(\cos\theta \cos\psi)^2}}$$

The angle ϕ_1' can assume values from 0 to 360 degrees depending on the values of the angles ψ and θ. There are several $\cos^{-1}$ relationships through this range of angular values, and the proper one must be used in order to obtain the correct value of ϕ_1'. Table II gives the proper relationship depending on the values of ψ and θ.

There are certain convenient relationships between the orientation angles measured in the tunnel and the aerodynamic angles for the various special cases presented. Even though these relationships are evident in the equations, it is considered desirable to elaborate on them at this point.

a : If the body is rotated through an angle of yaw, ψ, only, then the measured angle, ψ, is equal to the numerical value of the aerodynamic angles α', β and β' but carries the opposite sign to that of β and β'. (The sign of α' is always positive.)

b : If the body is rotated through an angle of pitch, θ, only, then the measured angle θ, is equal to the aerodynamic angles α and α'. (The sign of α' is always positive.)

c : If the body is rotated through a positive angle of pitch, θ, and an angle of roll, ϕ, then the measured angles θ and ϕ are equal to the aerodynamic angles α' and ϕ' respectively.

d : If the body is rotated through a negative angle of pitch, θ, and an angle of roll, ϕ, then the measured angle θ is equal to the numerical value of the aerodynamic angle α' and the angle ϕ' is equal to the angle $\phi + 180°$.

e : If the body is rotated through an angle of yaw, ψ, and an angle of pitch, θ, then the measured angle, ψ, is equal to the numerical value of the aerodynamic angle β but carries the opposite sign. The measured angle, θ, is equal to the aerodynamic angle α.

f : If the body is rotated through an angle of yaw, ψ, and an angle of pitch, θ, then the aerodynamic roll angle, ϕ', will exist. (This fact is utilized in establishing Table II.)

g : If the conditions of (*f*) above exist and the body is then rotated through a roll angle, ϕ, equal to $(-\phi')$, the aerodynamic angles β, β', and ϕ' will equal zero and α will equal α'. The forces and moments in and about the body-axes system would then be the same as those in a pure pitch case with θ equal to α'.

h : If the body is rotated through a roll angle equal to $(-\phi')$, the body-axes system will become coincident with the aeroballistic-axes system. Stated differently, the orientation of the body-axes system and the aeroballistic-axes system with respect to the reference-axes system used herein (tunnel axes) differs by a roll angle only.

The equations to transfer the force and moment coefficients from the body-axes system to the other axes systems and the expressions for the aerodynamic angles in terms of the orientation angles are given in the following pages.

YAW, PITCH, AND ROLL

If the body is rotated through an angle of yaw (ψ), an angle of pitch (θ), and an angle of roll (ϕ), the transfer equations in terms of the aerodynamic angles are:

Stability axes

$$C_{X_s} = -C_D' = -C_A \cos\alpha - C_N \sin\alpha$$

$$C_{Y_s} = C_Y$$

$$C_{Z_s} = -C_L = C_A \sin\alpha - C_N \cos\alpha$$

$$C_{l_s} = C_l \cos\alpha + C_n \sin\alpha$$

$$C_{m_s} = C_m$$

$$C_{n_s} = -C_l \sin\alpha + C_n \cos\alpha$$

Wind axes

$$C_{X_w} = -C_D = -C_A \cos\alpha \cos\beta + C_Y \sin\beta - C_N \sin\alpha \cos\beta$$

$$C_{Y_w} = C_C = C_A \cos\alpha \sin\beta + C_Y \cos\beta + C_N \sin\alpha \sin\beta$$

$$C_{Z_w} = -C_L = C_A \sin\alpha - C_N \cos\alpha$$

$$C_{l_w} = C_l \cos\alpha \cos\beta + C_m \sin\beta + C_n \sin\alpha \cos\beta$$

$$C_{m_w} = -C_l \cos\alpha \sin\beta + C_m \cos\beta - C_n \sin\alpha \sin\beta$$

$$C_{n_w} = -C_l \sin\alpha + C_n \cos\alpha$$

Aeroballistic axes

$$C_{X_a} = -C_A$$

$$C_{Y_a} = C_Y \cos\phi' + C_N \sin\phi'$$

$$C_{Z_a} = -C_{N_a} = C_Y \sin\phi' - C_N \cos\phi'$$

$$C_{l_a} = C_l$$

$$C_{m_a} = C_m \cos\phi' - C_n \sin\phi'$$

$$C_{n_a} = C_m \sin\phi' + C_n \cos\phi'$$

The expressions for the aerodynamic angles in terms of the orientation angles ψ, θ, and ϕ are:

$$\alpha = \cos^{-1}\left[\frac{\cos\theta \cos\psi}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}\right]$$

$$\alpha' = \cos^{-1}(\cos\theta \cos\psi)$$

$$\beta = \sin^{-1}(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)$$

$$\beta' = \tan^{-1}\left(\sin\phi \tan\theta - \frac{\cos\phi \tan\psi}{\cos\theta}\right)$$

$$\phi' = \cos^{-1}\left[\frac{(\sin\phi \cos\psi + \cos\phi \sin\theta \cos\psi)}{\sqrt{1-(\cos\theta \cos\psi)^2}}\right] = \cos^{-1}\left[\frac{\sin\theta \cos\psi}{\sqrt{1-(\cos\theta \cos\psi)^2}}\right] + \phi = \phi_l' + \phi$$

Note: Wherever a radical is involved, whether in the above equations or in the transfer equations, the positive root should always be used.

The transfer equations expanded in terms of the orientation angles ψ, θ, and ϕ are:

Stability axes

$$C_{X_s} = -C_D' = -C_A\left[\frac{\cos\theta\ \cos\psi}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right] - C_N\left[\frac{\sin\phi\ \sin\psi+\cos\phi\ \cos\psi\ \sin\theta}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right]$$

$$C_{Y_s} = C_Y$$

$$C_{Z_s} = -C_L = C_A\left[\frac{\sin\phi\ \sin\psi+\cos\phi\ \cos\psi\ \sin\theta}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right] - C_N\left[\frac{\cos\theta\ \cos\psi}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right]$$

$$C_{l_s} = C_l\left[\frac{\cos\theta\ \cos\psi}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right] + C_n\left[\frac{\sin\phi\ \sin\psi+\cos\phi\ \cos\psi\ \sin\theta}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right]$$

$$C_{m_s} = C_m$$

$$C_{n_s} = -C_l\left[\frac{\sin\phi\ \sin\psi+\cos\phi\ \cos\psi\ \sin\theta}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right] + C_n\left[\frac{\cos\theta\ \cos\psi}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right]$$

Wind axes

$$C_{X_w} = -C_D = -C_A(\cos\theta\ \cos\psi) + C_Y(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi) - C_N(\sin\phi\ \sin\psi+\cos\phi\ \cos\psi\ \sin\theta)$$

$$C_{Y_w} = C_C = C_A\left[\frac{(\cos\theta\ \cos\psi)(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right] + C_Y\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}$$
$$+ C_N\left[\frac{(\sin\phi\ \sin\psi+\cos\phi\ \cos\psi\ \sin\theta)(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right]$$

$$C_{Z_w} = -C_L = C_A\left[\frac{(\sin\phi\ \sin\psi)+(\cos\phi\ \cos\psi\ \sin\theta)}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right] - C_N\left[\frac{\cos\theta\ \cos\psi}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right]$$

$$C_{l_w} = C_l(\cos\theta\ \cos\psi) + C_m(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi) + C_n(\sin\phi\ \sin\psi+\cos\phi\ \cos\psi\ \sin\theta)$$

$$C_{m_w} = -C_l\left[\frac{(\cos\theta\ \cos\psi)(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right] + C_m\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}$$
$$- C_n\left[\frac{(\sin\phi\ \sin\psi+\cos\phi\ \cos\psi\ \sin\theta)(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right]$$

$$C_{n_w} = -C_l\left[\frac{\sin\phi\ \sin\psi+\cos\phi\ \cos\psi\ \sin\theta}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right] + C_n\left[\frac{\cos\theta\ \cos\psi}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta-\cos\phi\ \sin\psi)^2}}\right]$$

Aeroballistic axes

$$C_{Xa} = -C_A$$

$$C_{Ya} = C_Y \left[\frac{(\sin\phi\ \sin\psi + \cos\phi\ \sin\theta\ \cos\psi)}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right] + C_N \left[\frac{(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right]$$

$$C_{Za} = -C_{Na} = C_Y \left[\frac{(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right] - C_N \left[\frac{(\sin\phi\ \sin\psi + \cos\phi\ \sin\theta\ \cos\psi)}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right]$$

$$C_{la} = C_l$$

$$C_{ma} = C_m \left[\frac{(\sin\phi\ \sin\psi + \cos\phi\ \sin\theta\ \cos\psi)}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right] - C_n \left[\frac{(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right]$$

$$C_{na} = C_m \left[\frac{(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right] + C_n \left[\frac{(\sin\phi\ \sin\psi + \cos\phi\ \sin\theta\ \cos\psi)}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right]$$

The transfer equations for the tunnel-axes system in terms of the orientation angles are included for completeness.

Tunnel axes

$$C_{XT} = -C_A(\cos\theta\ \cos\psi) + C_Y(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi) - C_N(\sin\phi\ \sin\psi + \cos\phi\ \sin\theta\ \cos\psi)$$

$$C_{YT} = -C_A(\cos\theta\ \sin\psi) + C_Y(\cos\phi\ \cos\psi + \sin\phi\ \sin\theta\ \sin\psi) - C_N(\cos\phi\ \sin\theta\ \sin\psi - \sin\phi\ \cos\psi)$$

$$C_{ZT} = C_A \sin\theta + C_Y(\sin\phi\ \cos\theta) - C_N(\cos\phi\ \cos\theta)$$

$$C_{lT} = C_l(\cos\theta\ \cos\psi) + C_m(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi) + C_n(\sin\phi\ \sin\psi + \cos\phi\ \sin\theta\ \cos\psi)$$

$$C_{mT} = C_l(\cos\theta\ \sin\psi) + C_m(\cos\phi\ \cos\psi + \sin\phi\ \sin\theta\ \sin\psi) + C_n(\cos\phi\ \sin\theta\ \sin\psi - \sin\phi\ \cos\psi)$$

$$C_{nT} = -C_l \sin\theta + C_m(\sin\phi\ \cos\theta) + C_n(\cos\phi\ \cos\theta)$$

YAW ONLY

If the body is rotated through an angle of yaw (ψ) only, the expressions for the aerodynamic angles reduce to:

$$\alpha = 0$$

$$\alpha' = |\psi|$$

$$\beta = -\psi$$

$$\beta' = -\psi$$

$$\phi' = 90^\circ \text{ (1)}$$

and the transfer equations reduce to:

Stability axes

$$C_{X_s} = -C_D' = -C_A$$

$$C_{Y_s} = C_Y$$

$$C_{Z_s} = -C_L = -C_N$$

$$C_{l_s} = C_l$$

$$C_{m_s} = C_m$$

$$C_{n_s} = C_n$$

(1) The sign of ϕ' is the opposite to that of ψ .

Wind axes

$C_{X_w} = -C_D = -C_A \cos\psi - C_Y \sin\psi$

$C_{Y_w} = C_C = C_Y \cos\psi - C_A \sin\psi$

$C_{Z_w} = -C_L = -C_N$

$C_{l_w} = C_l \cos\psi - C_m \sin\psi$

$C_{m_w} = C_m \cos\psi + C_l \sin\psi$

$C_{n_w} = C_n$

Aeroballistic axes

$C_{X_a} = -C_A$

$C_{Y_a} = \pm C_N$ (1)

$C_{Z_a} = \pm C_Y$ (2)

$C_{l_a} = C_l$

$C_{m_a} = \pm C_n$ (2)

$C_{n_a} = \pm C_m$ (1)

Tunnel axes

$C_{X_T} = -C_A \cos\psi - C_Y \sin\psi$

$C_{Y_T} = -C_A \sin\psi + C_Y \cos\psi$

$C_{Z_T} = -C_N$

$C_{l_T} = C_l \cos\psi - C_m \sin\psi$

$C_{m_T} = C_l \sin\psi + C_m \cos\psi$

$C_{n_T} = C_n$

(1) The sign preceding C_N or C_m is opposite to that of ψ.

(2) The sign preceding C_Y or C_n is the same as that of ψ.

PITCH ONLY

If the body is rotated through an angle of pitch (θ) only, the expressions for the aerodynamic angles reduce to:

$$\alpha = \theta$$

$$\alpha' = |\theta|$$

$$\beta = 0$$

$$\beta' = 0$$

$$\phi' = 0$$

and the transfer equations reduce to:

Stability axes

$$C_{X_s} = -C_{D'} = -C_A \cos\theta - C_N \sin\theta$$

$$C_{Y_s} = C_Y$$

$$C_{Z_s} = -C_L = C_A \sin\theta - C_N \cos\theta$$

$$C_{l_s} = C_l \cos\theta + C_n \sin\theta$$

$$C_{m_s} = C_m$$

$$C_{n_s} = -C_l \sin\theta + C_n \cos\theta$$

Wind axes

$$C_{X_w} = -C_D = -C_A \cos\theta - C_N \sin\theta$$

$$C_{Y_w} = C_C = C_Y$$

$$C_{Z_w} = -C_L = C_A \sin\theta - C_N \cos\theta$$

$$C_{l_w} = C_l \cos\theta + C_n \sin\theta$$

$$C_{m_w} = C_m$$

$$C_{n_w} = -C_l \sin\theta + C_n \cos\theta$$

Aeroballistic axes

$$C_{X_a} = -C_A$$

$$C_{Y_a} = \pm C_Y \ ^{(1)}$$

$$C_{Z_a} = \pm C_N \ ^{(1)}$$

$$C_{l_a} = C_l$$

$$C_{m_a} = \pm C_m \ ^{(1)}$$

$$C_{n_a} = \pm C_n \ ^{(1)}$$

Tunnel axes

$$C_{X_T} = -C_A \cos\theta - C_N \sin\theta$$

$$C_{Y_T} = C_Y$$

$$C_{Z_T} = C_A \sin\theta - C_N \cos\theta$$

$$C_{l_T} = C_l \cos\theta + C_n \sin\theta$$

$$C_{m_T} = C_m$$

$$C_{n_T} = -C_l \sin\theta + C_n \cos\theta$$

....

(1) The sign preceding C_Y, C_N, C_m, C_n, is the same as that of θ.

YAW AND PITCH

If the body is rotated through an angle of yaw (ψ) and an angle of pitch (θ), the expressions for the aerodynamic angles reduce to:

$$\alpha = \theta$$

$$\alpha' = \left| \cos^{-1} (\cos\theta \; \cos\psi) \right|$$

$$\beta = -\psi$$

$$\beta' = \tan^{-1} \left(-\frac{\tan\psi}{\cos\theta} \right)$$

$$\phi' = \cos^{-1} \left(\frac{\sin\theta \; \cos\psi}{\sqrt{1-(\cos\theta \; \cos\psi)^2}} \right)$$

and the transfer equations reduce to:

Stability axes

$$C_{X_s} = -C_D' = -C_A \cos\theta - C_N \sin\theta$$

$$C_{Y_s} = C_Y$$

$$C_{Z_s} = -C_L = C_A \sin\theta - C_N \cos\theta$$

$$C_{l_s} = C_l \cos\theta + C_n \sin\theta$$

$$C_{m_s} = C_m$$

$$C_{n_s} = -C_l \sin\theta + C_n \cos\theta$$

Wind axes

$$C_{X_w} = -C_D = -C_A(\cos\theta\ \cos\psi) - C_Y \sin\psi - C_N(\sin\theta \cos\psi)$$

$$C_{Y_w} = C_C = -C_A(\sin\psi\ \cos\theta) + C_Y \cos\psi - C_N(\sin\psi \sin\theta)$$

$$C_{Z_w} = -C_L = C_A \sin\theta - C_N \cos\theta$$

$$C_{l_w} = C_l\ (\cos\theta \cos\psi) - C_m \sin\psi + C_n(\sin\theta \cos\psi)$$

$$C_{m_w} = C_l\ (\cos\theta \sin\psi) + C_m \cos\psi + C_n(\sin\theta \sin\psi)$$

$$C_{n_w} = -C_l \sin\theta + C_n \cos\theta$$

Aeroballistic axes

$$C_{X_a} = -C_A$$

$$C_{Y_a} = C_Y \left[\frac{\sin\theta\ \cos\psi}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right] - C_N \left[\frac{\sin\psi}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right]$$

$$C_{Z_a} = -C_{N_a} = -C_Y \left[\frac{\sin\psi}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right] - C_N \left[\frac{\sin\theta\ \cos\psi}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right]$$

$$C_{l_a} = C_l$$

$$C_{m_a} = C_m \left[\frac{\sin\theta\ \cos\psi}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right] + C_n \left[\frac{\sin\psi}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right]$$

$$C_{n_a} = -C_m \left[\frac{\sin\psi}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right] + C_n \left[\frac{\sin\theta\ \cos\psi}{\sqrt{1-(\cos\theta\ \cos\psi)^2}}\right]$$

Tunnel axes

$$C_{X_T} = -C_A\ (\cos\theta\ \cos\psi) - C_Y\ \sin\psi - C_N\ (\sin\theta\ \cos\psi)$$

$$C_{Y_T} = -C_A\ (\cos\theta\ \sin\psi) + C_Y\ \cos\psi - C_N\ (\sin\theta\ \sin\psi)$$

$$C_{Z_T} = C_A \sin\theta - C_N \cos\theta$$

$$C_{l_T} = C_l\ (\cos\theta\ \cos\psi) - C_m\ \sin\psi + C_n\ (\sin\theta\ \cos\psi)$$

$$C_{m_T} = C_l\ (\cos\theta\ \sin\psi) + C_m\ \cos\psi + C_n\ (\sin\theta\ \sin\psi)$$

$$C_{n_T} = -C_l\ \sin\theta + C_n\ \cos\theta$$

PITCH AND ROLL

If the body is rotated through an angle of pitch (θ) and an angle of roll (ϕ), the expressions for the aerodynamic angles reduce to:

$$\alpha = \cos^{-1} \frac{\cos\theta}{\sqrt{1-(\sin\phi\ \sin\theta)^2}}$$

$$\alpha' = |\theta|$$

$$\beta = \sin^{-1}(\sin\phi\ \sin\theta)$$

$$\beta' = \tan^{-1}(\sin\phi\ \tan\theta)$$

$$\phi' = \phi$$

and the transfer equations reduce to:

Stability axes

$$C_{X_s} = -C_D' = -C_A\left[\frac{\cos\theta}{\sqrt{1-(\sin\phi\ \sin\theta)^2}}\right] - C_N\left[\frac{\cos\phi\ \sin\theta}{\sqrt{1-(\sin\phi\ \sin\theta)^2}}\right]$$

$$C_{Y_s} = C_Y$$

$$C_{Z_s} = -C_L = C_A\left[\frac{\cos\phi\ \sin\theta}{\sqrt{1-(\sin\phi\ \sin\theta)^2}}\right] - C_N\left[\frac{\cos\theta}{\sqrt{1-(\sin\phi\ \sin\theta)^2}}\right]$$

$$C_{l_s} = C_l\left[\frac{\cos\theta}{\sqrt{1-(\sin\phi\ \sin\theta)^2}}\right] + C_n\left[\frac{\cos\phi\ \sin\theta}{\sqrt{1-(\sin\phi\ \sin\theta)^2}}\right]$$

$$C_{m_s} = C_m$$

$$C_{n_s} = -C_l\left[\frac{\cos\phi\ \sin\theta}{\sqrt{1-(\sin\phi\ \sin\theta)^2}}\right] + C_n\left[\frac{\cos\theta}{\sqrt{1-(\sin\phi\ \sin\theta)^2}}\right]$$

Wind axes

$$C_{X_w} = -C_D = -C_A \cos\theta + C_Y (\sin\phi \sin\theta) - C_N (\cos\phi \sin\theta)$$

$$C_{Y_w} = C_C = C_A \left[\frac{1/2 \sin 2\theta \sin\phi}{\sqrt{1-(\sin\phi \sin\theta)^2}} \right] + C_Y \sqrt{1-(\sin\phi \sin\theta)^2} + C_N \left[\frac{1/2 \sin 2\phi \sin^2\theta}{\sqrt{1-(\sin\phi \sin\theta)^2}} \right]$$

$$C_{Z_w} = -C_L = C_A \left[\frac{\cos\phi \sin\theta}{\sqrt{1-(\sin\phi \sin\theta)^2}} \right] - C_N \left[\frac{\cos\theta}{\sqrt{1-(\sin\phi \sin\theta)^2}} \right]$$

$$C_{l_w} = C_l \cos\theta + C_m (\sin\phi \sin\theta) + C_n (\cos\phi \sin\theta)$$

$$C_{m_w} = -C_l \left[\frac{1/2 \sin 2\theta \sin\phi}{\sqrt{1-(\sin\phi \sin\theta)^2}} \right] + C_m \sqrt{1-(\sin\phi \sin\theta)^2} - C_n \left[\frac{1/2 \sin 2\phi \sin^2\theta}{\sqrt{1-(\sin\phi \sin\theta)^2}} \right]$$

$$C_{n_w} = -C_l \left[\frac{\cos\phi \sin\theta}{\sqrt{1-(\sin\phi \sin\theta)^2}} \right] + C_n \left[\frac{\cos\theta}{\sqrt{1-(\sin\phi \sin\theta)^2}} \right]$$

Aeroballistic axes

$$C_{X_a} = -C_A$$

$$C_{Y_a} = C_Y \cos\phi + C_N \sin\phi$$

$$C_{Z_a} = C_Y \sin\phi - C_N \cos\phi$$

$$C_{l_a} = C_l$$

$$C_{m_a} = C_m \cos\phi - C_n \sin\phi$$

$$C_{n_a} = C_m \sin\phi + C_n \cos\phi$$

Tunnel axes

$$C_{X_T} = -C_A \cos\theta + C_Y (\sin\phi \sin\theta) - C_N (\cos\phi \sin\theta)$$

$$C_{Y_T} = C_Y \cos\phi + C_N \sin\phi$$

$$C_{Z_T} = C_A \sin\theta + C_Y (\sin\phi \cos\theta) - C_N (\cos\phi \cos\theta)$$

$$C_{l_T} = C_l \cos\theta + C_m (\sin\phi \sin\theta) + C_n (\cos\phi \sin\theta)$$

$$C_{m_T} = C_m \cos\phi - C_n \sin\phi$$

$$C_{n_T} = -C_l \sin\theta + C_m (\sin\phi \cos\theta) + C_n (\cos\phi \cos\theta)$$

REFERENCE

(1) "Letter Symbols for Aeronautical Sciences"
(ASA Y10.7 - 1954)

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Table I	Relationship between the orientation angles for two sequences of rotations
Table II	Determination of sign and value of aerodynamic angles

TABLE I

RELATIONSHIP BETWEEN THE ORIENTATION ANGLES FOR TWO SEQUENCES OF ROTATIONS

If the body is rotated through the orientation angles in the order ψ, θ, ϕ or θ, ψ, ϕ such that the body X-axis is positioned in the same location, the following equations present the relationships existing between the orientation angles for the two sequences.

Sequence 1 ψ_1 θ_1 ϕ_1		Sequence 2 θ_2 ψ_2 ϕ_2
θ_1	=	$\sin^{-1}(\sin\theta_2 \cos\psi_2)$
ψ_1	=	$\tan^{-1}\left(\frac{\tan\psi_2}{\cos\theta_2}\right)$
ϕ_1	=	$\tan^{-1}(\tan\theta_2 \sin\psi_2) + \phi_2$

TABLE II

DETERMINATION OF SIGN AND VALUE OF AERODYNAMIC ANGLES

Table to determine the value of the aerodynamic angle ϕ' and the sign of the other aerodynamic angles when the values of the orientation angles ψ, θ and ϕ are within the limits $90° \geq \psi \geq -90°$; $90° \geq \theta \geq -90°$; $360° \geq \phi \geq -360°$ and the rotations through these angles occur in the order ψ, θ and ϕ.

Part 1

conditions	$\cos^{-1}$ relationship
$\psi \geq 0$; $\theta \geq 0$	$\cos^{-1}(X) = \cos^{-1}(X)$
$\psi \geq 0$; $\theta < 0$	$\cos^{-1}(-X) = 180 - \cos^{-1}(X)$
$\psi < 0$; $\theta \leq 0$	$\cos^{-1}(-X) = 180 + \cos^{-1}(X)$
$\psi < 0$; $\theta > 0$	$\cos^{-1}(X) = 360 - \cos^{-1}(X)$

Part 2

conditions	sign of aerodynamic angles
$270 - \phi_1' < \phi < 90 - \phi_1'$	α is (+)
$90 - \phi_1' < \phi < 270 - \phi_1'$	α is (-)
$-\phi_1' < \phi < 180 - \phi_1'$	β and β' are (+)
$180 - \phi_1' < \phi < -\phi_1'$	β and β' are (-)

When the value of ϕ is equal to these limits, the value of the respective aerodynamic angle is zero.

INSTRUCTIONS FOR USING TABLE II

Use the known values of the angles ψ and θ from the problem in question to determine the value of the $\cos \phi_1'$ from the equation

$$X = \cos \phi_1' = \frac{\sin\theta \cos\psi}{\sqrt{1-(\cos\theta \cos\psi)^2}}$$ (Use positive root of the radical)

Also use these values of ψ and θ to enter Table II, Part 1, and select the proper $\cos^{-}$ relationship to use in determining the value of ϕ_1'.

Use the value of ϕ from the problem in question and determine the value ϕ' from the equation.

$$\phi' = \phi_1' + \phi$$

Use these values of ϕ_1' and ϕ to enter Table II, Part 2, and determine the sign of the other aerodynamic angles.

ILLUSTRATIONS

Figures 1 through 13

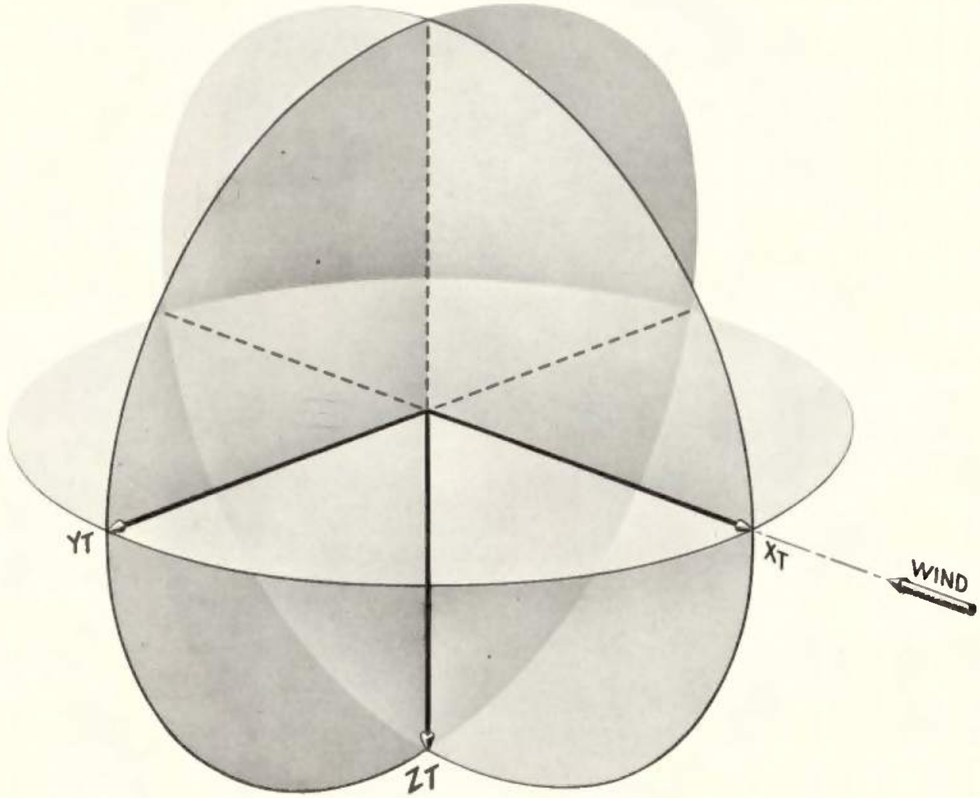

Figure 1

TUNNEL–AXES SYSTEM

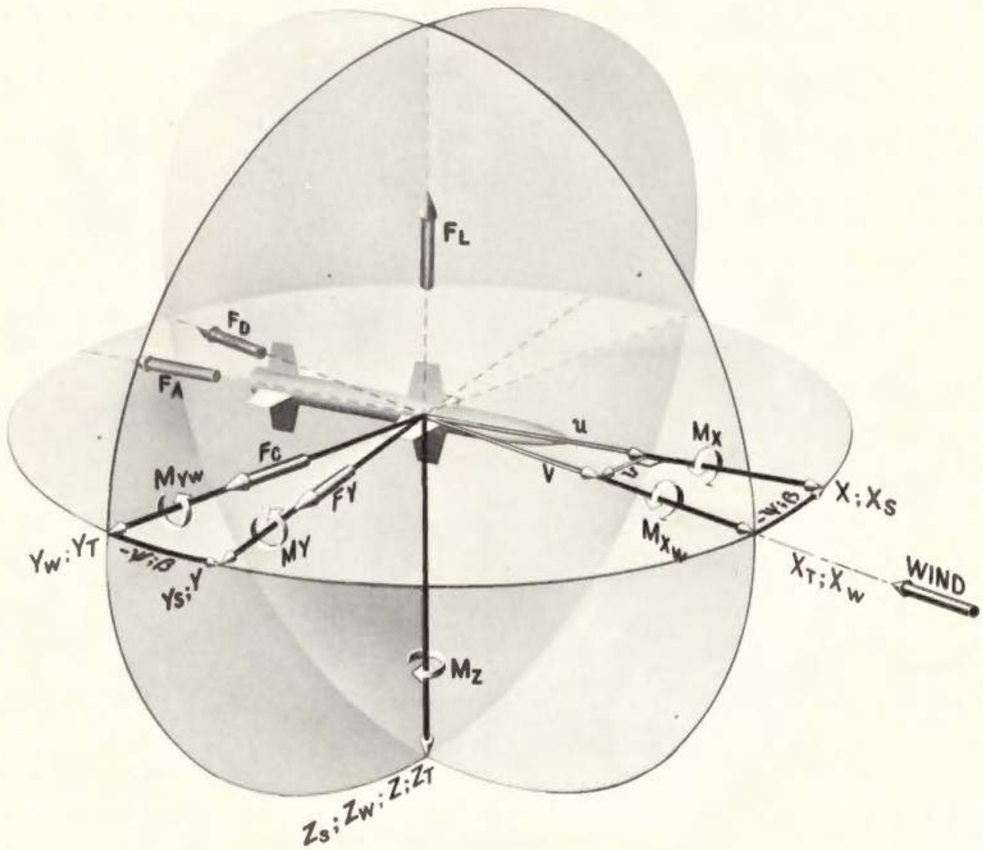

Figure 2

BODY-, STABILITY-, WIND-, AND TUNNEL-AXES SYSTEMS WHEN THE BODY IS ROTATED THROUGH AN ANGLE OF YAW ONLY

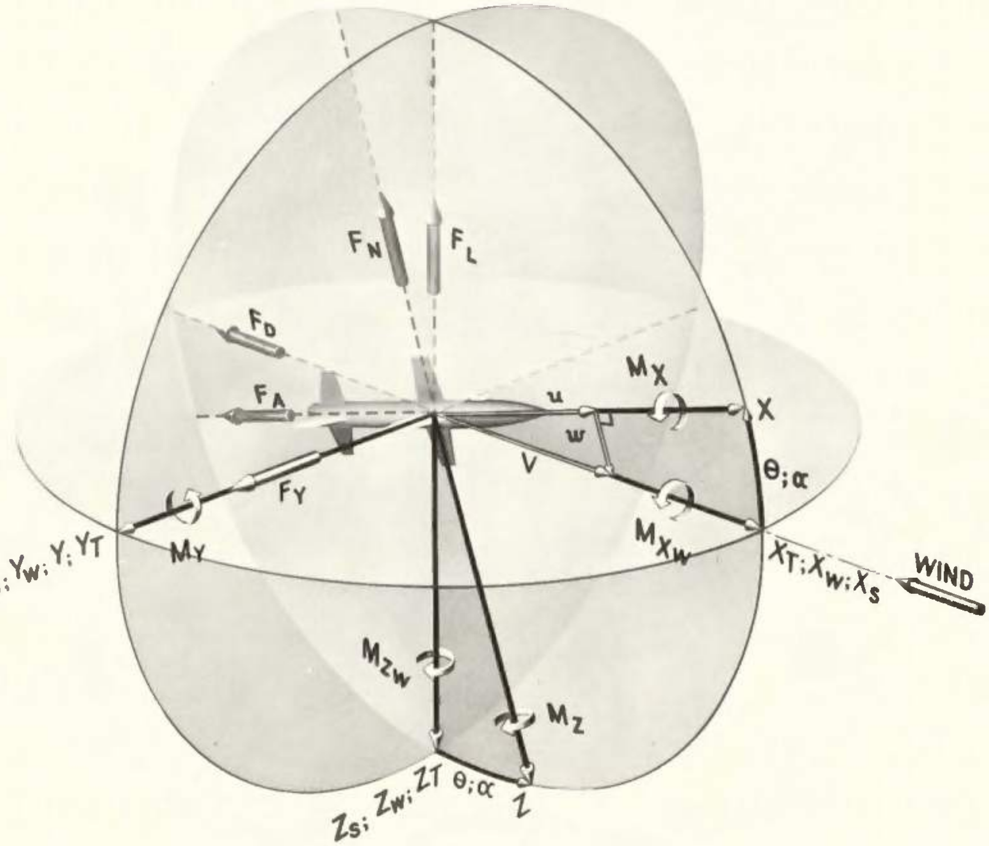

Figure 3

BODY–, STABILITY–, WIND–, AND TUNNEL–AXES SYSTEMS WHEN THE BODY IS ROTATED THROUGH AN ANGLE OF PITCH ONLY

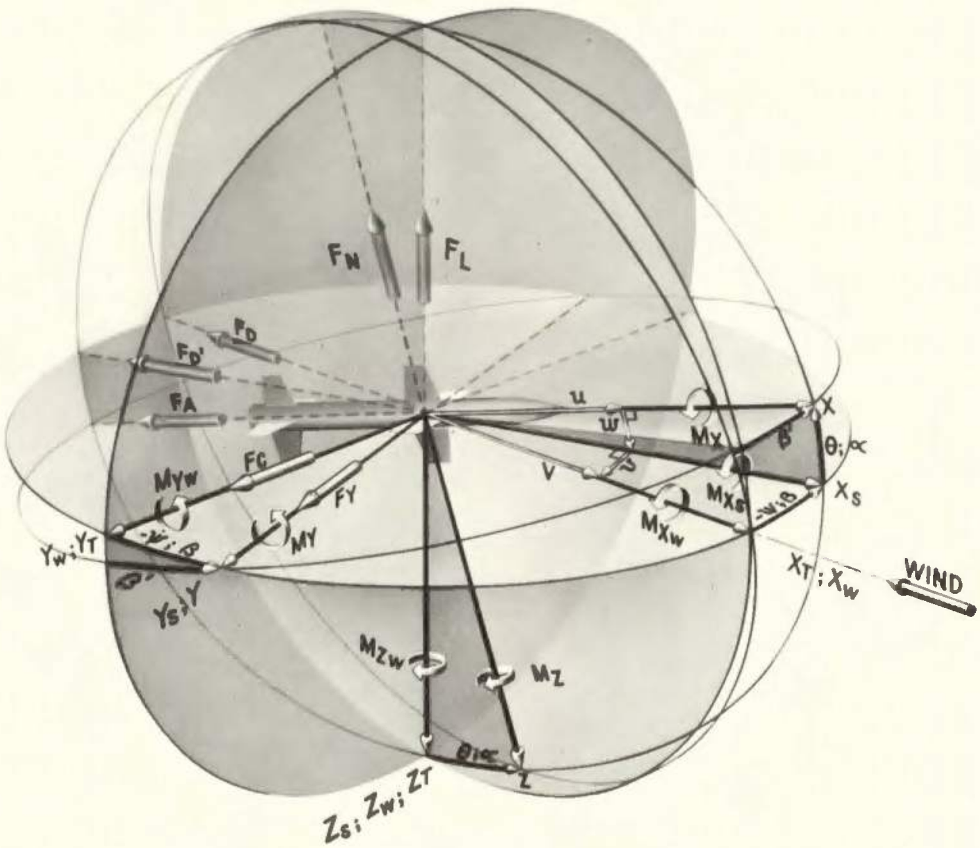

Figure 4

BODY–, STABILITY–, WIND–, AND TUNNEL–AXES SYSTEMS WHEN THE BODY IS ROTATED THROUGH AN ANGLE OF YAW AND AN ANGLE OF PITCH IN THAT ORDER

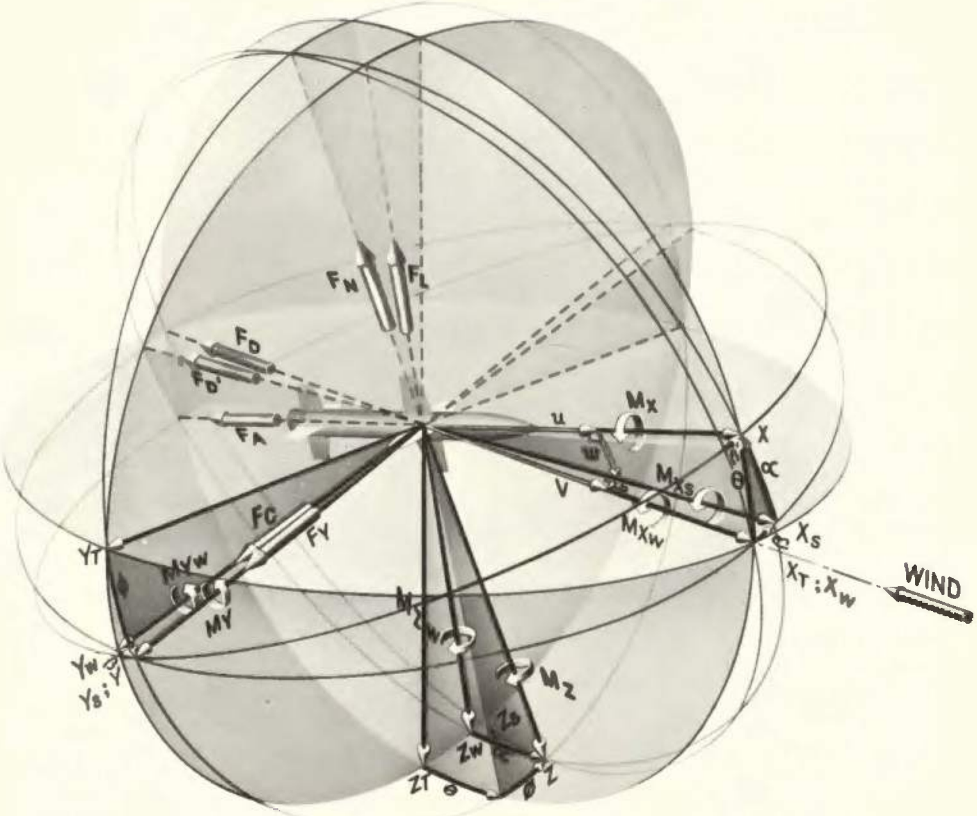

Figure 5

BODY-, STABILITY-, WIND-, AND TUNNEL-AXES SYSTEMS WHEN THE BODY IS ROTATED THROUGH AN ANGLE OF PITCH AND AN ANGLE OF ROLL IN THAT ORDER

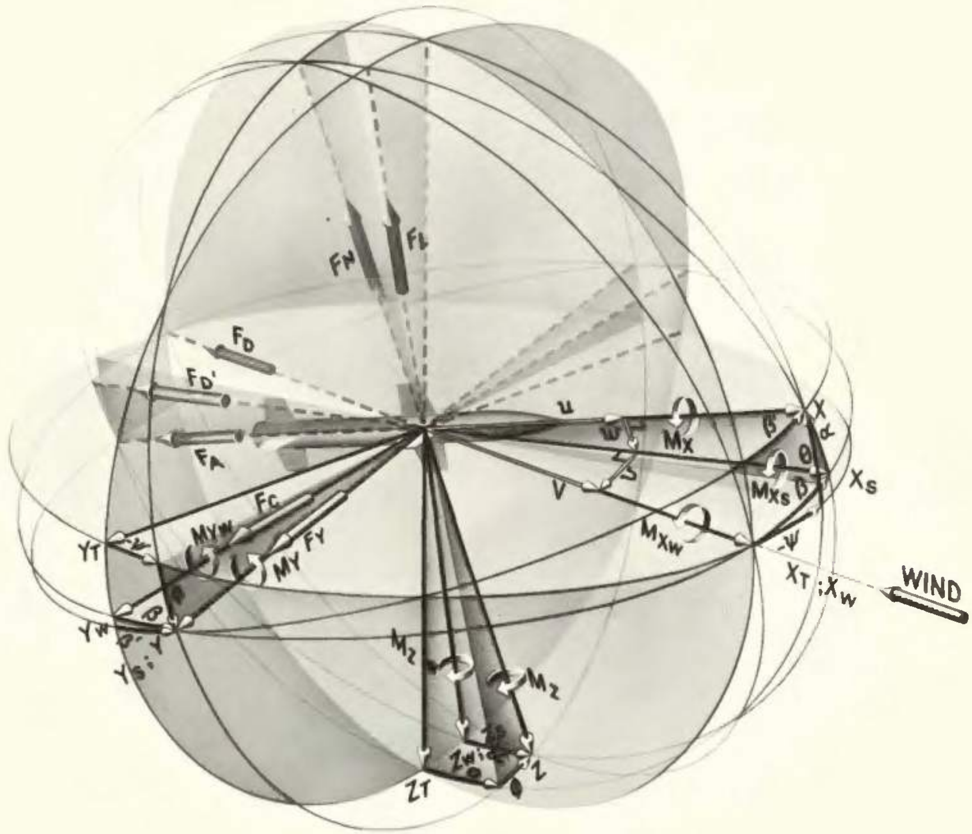

Figure 6

BODY–, STABILITY–, WIND–, AND TUNNEL–AXES SYSTEMS WHEN THE BODY IS ROTATED THROUGH AN ANGLE OF YAW, AN ANGLE OF PITCH, AND AN ANGLE OF ROLL IN THAT ORDER

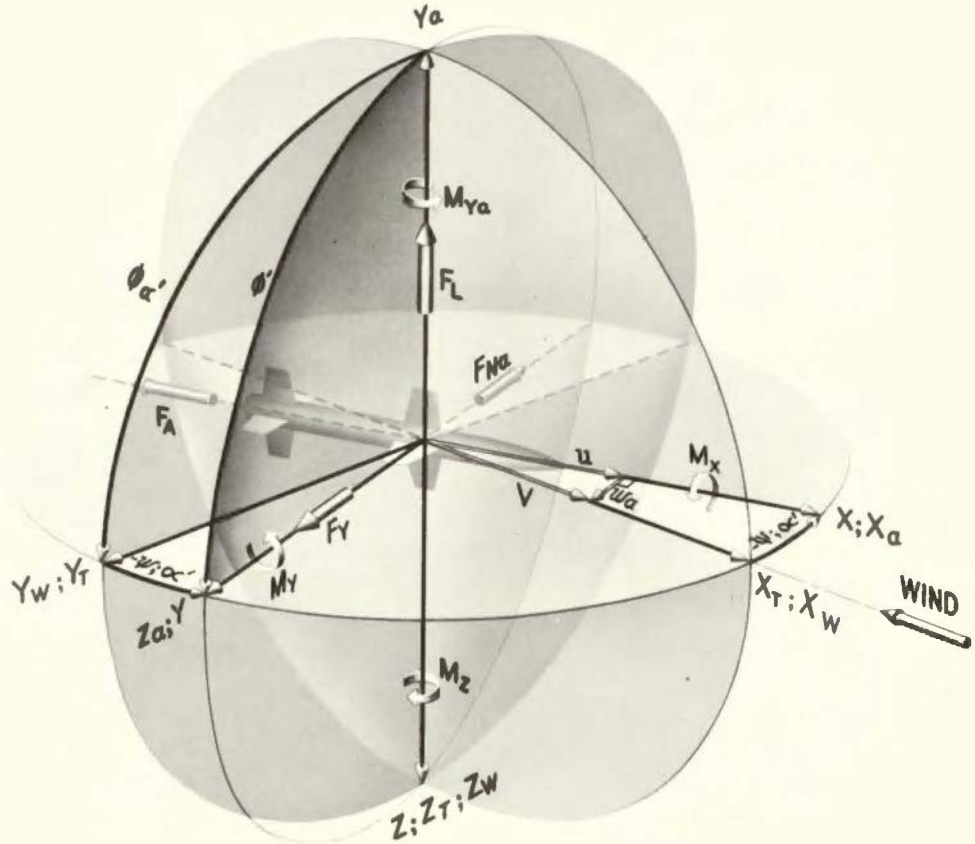

Figure 7

BODY-, AEROBALLISTIC-, WIND-, AND TUNNEL-AXES SYSTEMS WHEN THE BODY IS ROTATED THROUGH AN ANGLE OF YAW ONLY

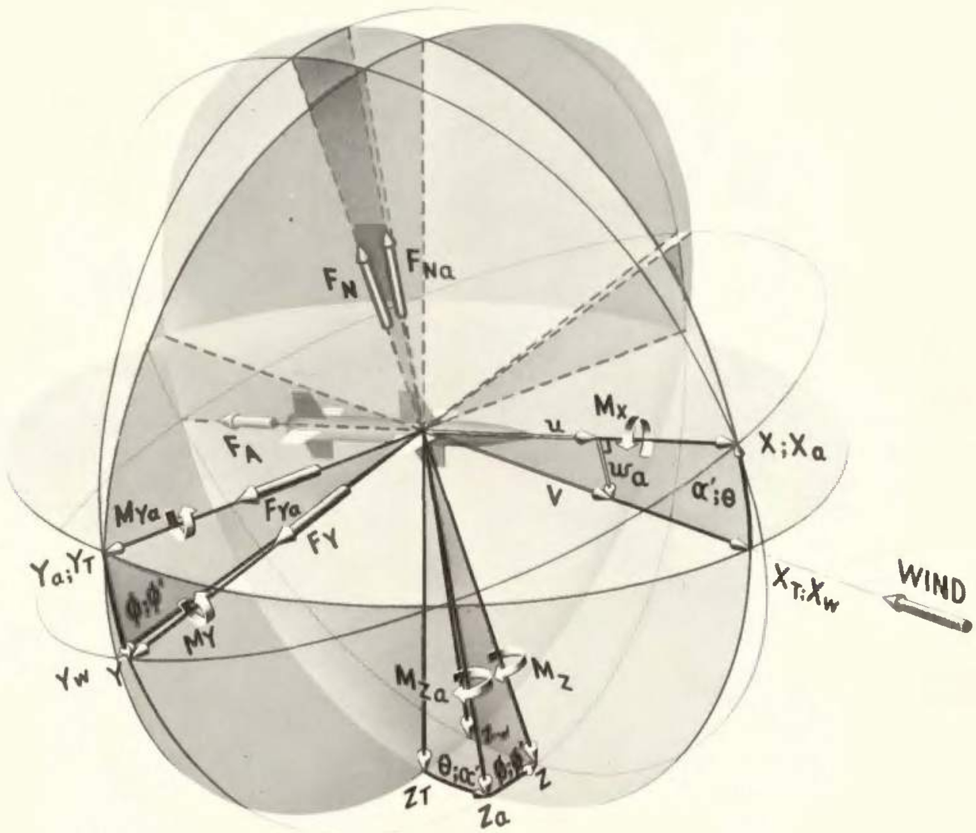

Figure 8

BODY–, AEROBALLISTIC–, WIND–, AND TUNNEL–AXES SYSTEMS WHEN THE BODY IS ROTATED THROUGH AN ANGLE OF PITCH AND AN ANGLE OF ROLL IN THAT ORDER

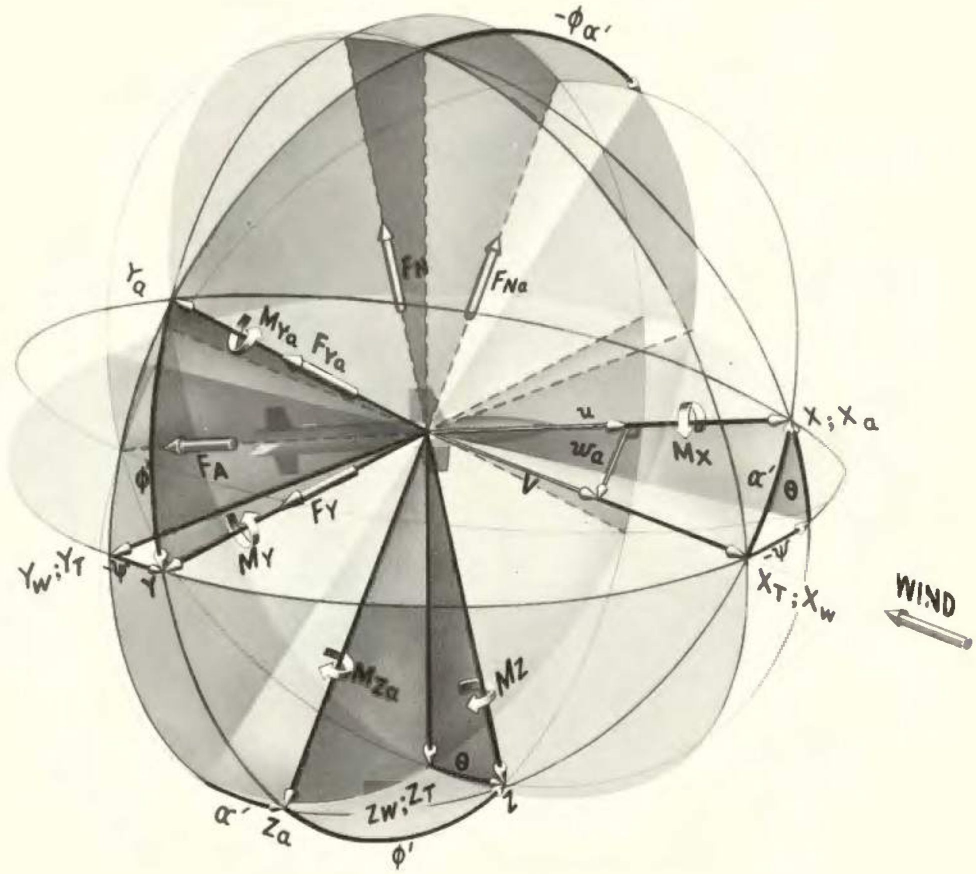

Figure 9

BODY-, AEROBALLISTIC-, WIND-, AND TUNNEL-AXES SYSTEMS WHEN THE BODY IS ROTATED THROUGH AN ANGLE OF YAW AND AN ANGLE OF PITCH IN THAT ORDER

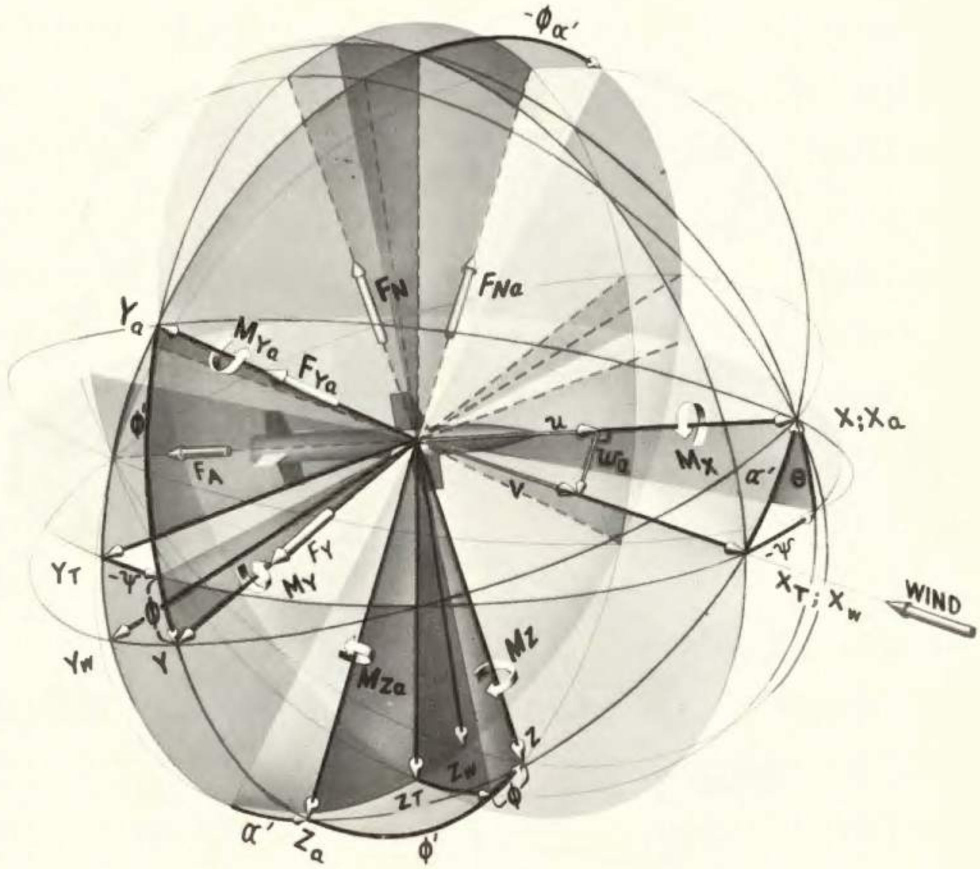

Figure 10

BODY–, AEROBALLISTIC–, WIND–, AND TUNNEL–AXES SYSTEMS WHEN THE BODY IS ROTATED THROUGH AN ANGLE OF YAW, AN ANGLE OF PITCH, AND AN ANGLE OF ROLL IN THAT ORDER

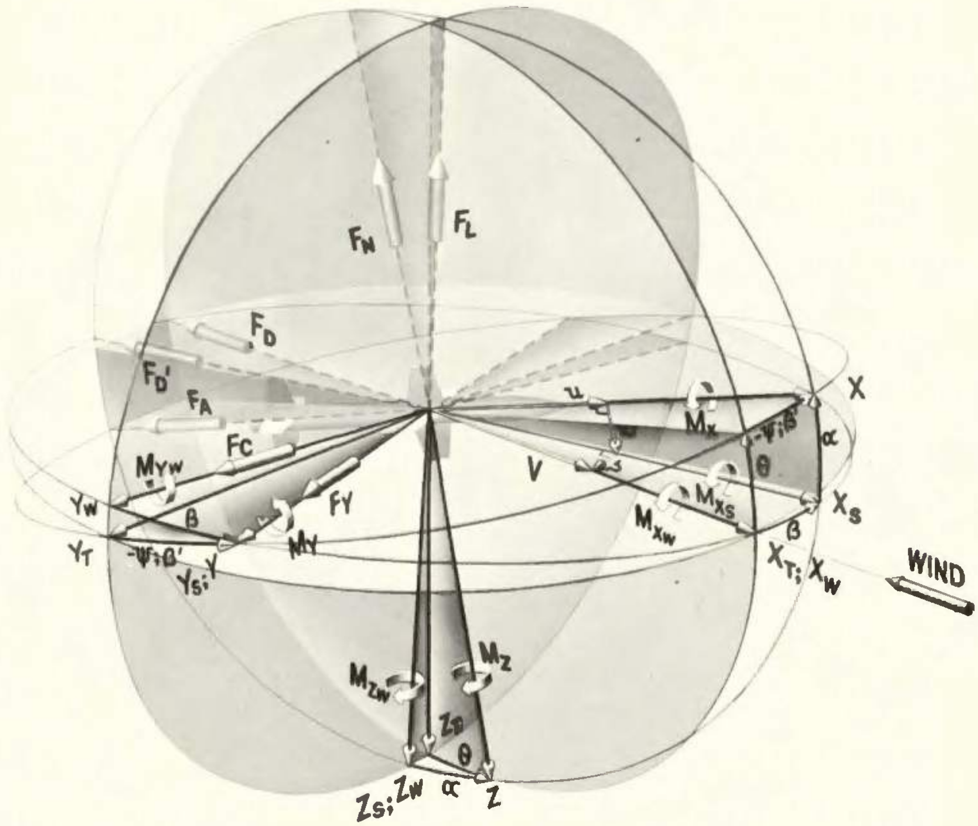

Figure 11

BODY–, STABILITY–, WIND–, AND TUNNEL–AXES SYSTEMS WHEN THE BODY IS ROTATED THROUGH AN ANGLE OF PITCH AND AN ANGLE OF YAW IN THAT ORDER

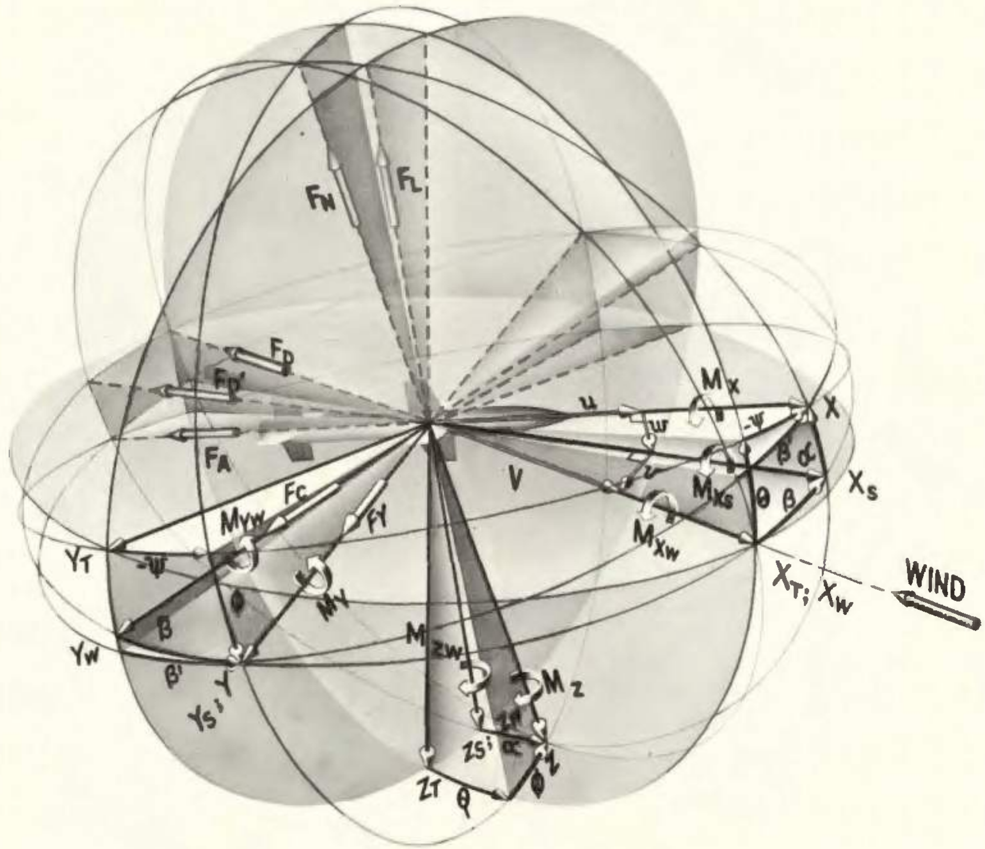

Figure 12

BODY-, STABILITY-, WIND-, AND TUNNEL-AXES SYSTEMS WHEN THE BODY IS ROTATED THROUGH AN ANGLE OF PITCH, AN ANGLE OF YAW, AND AN ANGLE OF ROLL IN THAT ORDER

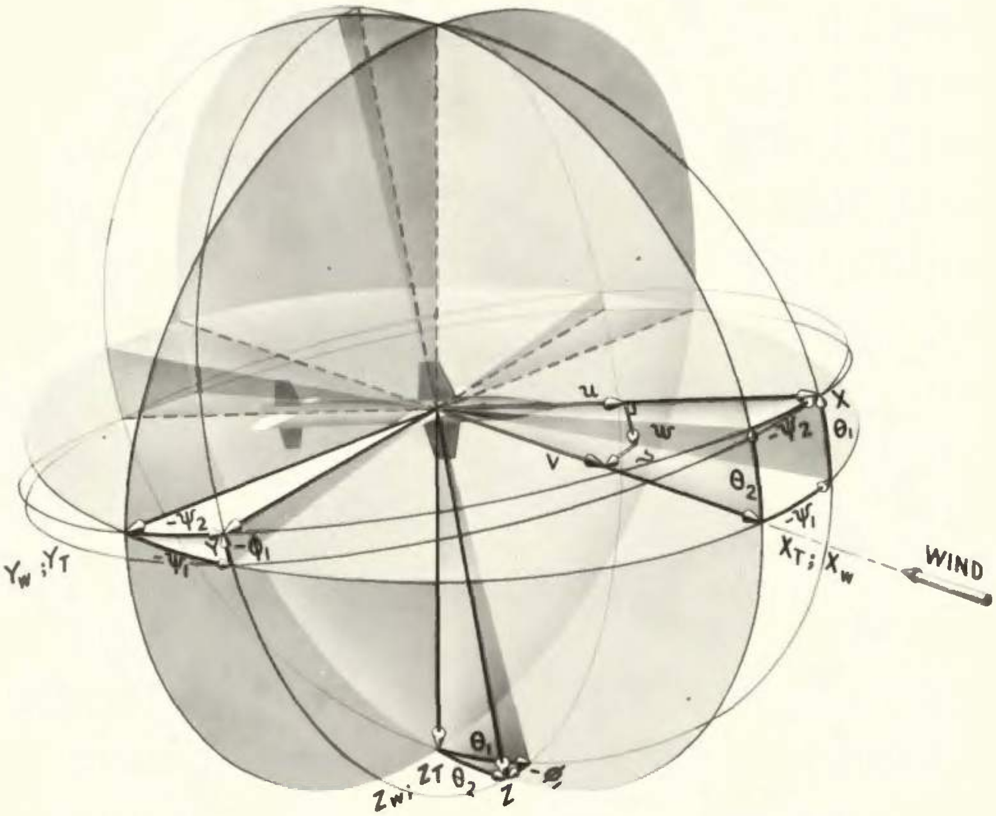

Figure 13

COMPARISON OF THE ANGLES OF YAW, PITCH, AND ROLL REQUIRED TO ORIENT THE BODY–AXES SYSTEM IN THE SAME MANNER WITH RESPECT TO THE TUNNEL–AXES SYSTEM WHEN THE SEQUENCE IS CHANGED

Appendices

Appendix A Derivation of transfer equations and expressions for the aerodynamic angles in terms of the orientation angles

Appendix B Derivation of the relationships between the angles ψ, θ, and ϕ when the sequence is changed

APPENDIX A

DERIVATION OF TRANSFER EQUATIONS AND EXPRESSIONS FOR THE AERODYNAMIC ANGLES IN TERMS OF THE ORIENTATION ANGLES

In the following pages the mathematical expressions for the aerodynamic angles α, α', β, β' and ϕ', and equations to transfer the forces and moments from the body-axes system to the other axes systems in terms of the orientation angles ψ, θ, and ϕ measured in the wind tunnel are derived.

Starting with the body-axes system,

Axes

X, Y, Z

Forces

F_X, F_Y, F_Z

and referring to Figure 6, let this axes system rotate about the Y-axis back through the angle α. The transfer equations for the stability-axes system in terms of the aerodynamic angle α are then obtained.

Axes

X_s, Y_s, Z_s

Forces

$$F_{X_s} = F_X \cos\alpha + F_Z \sin\alpha$$

$$F_{Y_s} = F_Y$$

$$F_{Z_s} = -F_X \sin\alpha + F_Z \cos\alpha$$

Let the stability-axes system rotate about the Z_s -axis back through the angle β, referring again to Figure 6, and the transfer equations for the wind-axes system in terms of the aerodynamic angles are obtained.

Axes

X_w, Y_w, Z_w

Forces

$$F_{X_w} = F_{X_s} \cos\beta + F_{Y_s} \sin\beta$$

$$F_{Y_w} = -F_{X_s} \sin\beta + F_{Y_s} \cos\beta$$

$$F_{Z_w} = F_{Z_s}$$

$$F_{X_w} = F_X \cos\alpha \cos\beta + F_Y \sin\beta + F_Z \sin\alpha \cos\beta$$

$$F_{Y_w} = -F_X \cos\alpha \sin\beta + F_Y \cos\beta - F_Z \sin\alpha \sin\beta$$

$$F_{Z_w} = -F_X \sin\alpha + F_Z \cos\alpha$$

Since the X-axis of the tunnel-axes system (X_T) and the X-axis of the wind-axes system (X_w) are identical it follows that

$$F_{XT} = F_X \cos\alpha \cos\beta + F_Y \sin\beta + F_Z \sin\alpha \cos\beta \qquad \text{(AI)}$$

Starting with the body-axes system, again work to the tunnel-axes system in a different manner. Let the body-axes system rotate about the X-axis back through the angle ϕ. Call this new working axes system the $(\ _1)$-axes system.

This axes system can be found in Figure 6 although it is not labeled. The X_1-axis is the X-axis. The Y_1 - and Z_1 -axes are shown as plane intersections.

Axes

X_1, X_1, Z_1

Forces

$$F_{X1} = F_X$$

$$F_{Y1} = F_Y \cos\phi - F_Z \sin\phi$$

$$F_{Z1} = F_Y \sin\phi + F_Z \cos\phi$$

Now let the $(\ _1)$-axes system rotate,about the Y_1 -axis back through the angle θ. Call this new working system the $(\ _2)$-axes system.

Axes

X_2, Y_2, Z_2

Forces

$$F_{X2} = F_{X1} \cos\theta + F_{Z1} \sin\theta$$

$$F_{Y2} = F_{Y1}$$

$$F_{Z2} = -F_{X1} \sin\theta + F_{Z1} \cos\theta$$

$$F_{X2} = F_X \cos\theta + F_Y \sin\phi \sin\theta + F_Z \cos\phi \sin\theta$$

$$F_{Y2} = F_Y \cos\phi - F_Z \sin\phi$$

$$F_{Z2} = -F_X \sin\theta + F_Y \sin\phi \cos\theta + F_Z \cos\phi \cos\theta$$

The ($_2$)-axes system can also be found in Figure 6, but it is not labeled. The Z_2 -axis is the Z_T -axis. The X_2 - and Y_2 -axes are shown as plane intersections.

Now let the ($_2$)-axes system rotate about the Z_2 -axis back through the angle $(-\psi)$ and the transfer equations for the tunnel-axes system in terms of the orientation angles ψ , θ and ϕ are obtained.

Axes

X_T, Y_T, Z_T

Forces

$$F_{XT} = F_{X2} \cos\psi + F_{Y2} \sin\psi$$

$$F_{YT} = -F_{X2} \sin\psi + F_{Y2} \cos\psi$$

$$F_{ZT} = F_{Z2}$$

$$F_{XT} = F_X \cos\theta \cos\psi + F_Z \cos\phi \sin\theta \cos\psi + F_Y \sin\phi \sin\theta \cos\psi + F_Y \cos\phi \sin\psi - F_Z \sin\phi \sin\psi$$

$$F_{YT} = F_Y \cos\phi \cos\psi - F_Z \sin\phi \cos\psi - F_X \cos\theta \sin\psi - F_Z \cos\phi \sin\theta \sin\psi - F_Y \sin\phi \sin\theta \sin\psi$$

$$F_{ZT} = F_Z \cos\phi \cos\theta + F_Y \sin\phi \cos\theta - F_X \sin\theta$$

Recalling that the angle ψ used in the derivation is negative, and therefore

$$\sin(-\psi) = -\sin\psi$$

$$\cos(-\psi) = \cos\psi$$

and gathering terms the following equations for the tunnel-axes system are obtained.

$$\text{(A2)} \quad F_{XT} = F_X \cos\theta \cos\psi + F_Y (\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi) + F_Z (\sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi)$$

$$F_{YT} = F_X \cos\theta \sin\psi + F_Y (\cos\phi \cos\psi + \sin\phi \sin\theta \sin\psi) + F_Z (\cos\phi \sin\theta \sin\psi - \sin\phi \cos\psi)$$

$$F_{ZT} = -F_X \sin\theta + F_Y \sin\phi \cos\theta + F_Z \cos\phi \cos\theta$$

Since equation (AI) equals equation (A2) the following equations are true

$$\text{(A3)} \quad \cos\alpha \cos\beta = \cos\theta \cos\psi$$

$$\text{(A4)} \quad \sin\beta = \sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi$$

$$\text{(A5)} \quad \sin\alpha \cos\beta = \sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi$$

Earlier in the derivation it was shown that the transfer equations for the stability axes and the wind axes take the following form;

Stability axes

$$F_{X_s} = F_Z \sin\alpha + F_X \cos\alpha$$

$$F_{Y_s} = F_Y$$

$$F_{Z_s} = -F_X \sin\alpha + F_Z \cos\alpha$$

Wind axes

$$F_{X_w} = F_X \cos\alpha \cos\beta + F_Y \sin\beta + F_Z \sin\alpha \cos\beta$$

$$F_{Y_w} = -F_X \cos\alpha \sin\beta + F_Y \cos\beta - F_Z \sin\alpha \sin\beta$$

$$F_{Z_w} = -F_X \sin\alpha + F_Z \cos\alpha$$

Therefore, it is apparent that in addition to equations (A3), (A4), and (A5), expressions for $\cos\beta$, $\sin\alpha$, $\cos\alpha$, $(\cos\alpha \sin\beta)$ and $(\sin\alpha \sin\beta)$ in terms of ψ, θ and ϕ are needed.

From the relation

$$\cos\beta = \sqrt{1-\sin^2\beta}$$

and equation (A4)

(A6) $$\cos\beta = \sqrt{1-(\sin\phi\cos\psi\sin\theta - \cos\phi\sin\psi)^2}$$

and from equation (A5)

$$\sin\alpha = \frac{\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi}{\cos\beta}$$

therefore,

(A7) $$\sin\alpha = \frac{\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi}{\sqrt{1-(\sin\phi\cos\psi\sin\theta - \cos\phi\sin\psi)^2}}$$

From equation (A3)

$$\cos\alpha = \frac{\cos\theta\cos\psi}{\cos\beta}$$

therefore,

(A8) $$\cos\alpha = \frac{\cos\theta\cos\psi}{\sqrt{1-(\sin\phi\cos\psi\sin\theta - \cos\phi\sin\psi)^2}}$$

If equations (A4) and (A8) are combined, then

$$\text{(A9)} \quad \cos\alpha \sin\beta = \frac{(\cos\theta \cos\psi)(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}$$

If equations (A4) and (A7) are combined, then

$$\text{(A10)} \quad \sin\alpha \sin\beta = \frac{(\sin\phi \sin\psi + \cos\phi \cos\psi \sin\theta)(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}$$

By substituting equations (A3) through (A10) into the transfer equations the desired equations for the stability and wind axes containing functions of ψ, θ and ϕ only are obtained.

Stability axes

$$F_{X_s} = F_X\left[\frac{\cos\theta \cos\psi}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}\right] + F_Z\left[\frac{\sin\phi \sin\psi + \cos\phi \cos\psi \sin\theta}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}\right]$$

$$F_{Y_s} = F_Y$$

$$F_{Z_s} = -F_X\left[\frac{\sin\phi \sin\psi + \cos\phi \cos\psi \sin\theta}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}\right] + F_Z\left[\frac{\cos\theta \cos\psi}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}\right]$$

Wind axes

$$F_{X_w} = F_X(\cos\theta\cos\psi) + F_Y(\sin\phi\cos\psi\sin\theta - \cos\phi\sin\psi) + F_Z(\sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta)$$

$$F_{Y_w} = -F_X\left[\frac{(\cos\theta\cos\psi)(\sin\phi\cos\psi\sin\theta - \cos\phi\sin\psi)}{\sqrt{1-(\sin\phi\cos\psi\sin\theta - \cos\phi\sin\psi)^2}}\right]$$

$$+ F_Y\sqrt{1-(\sin\phi\cos\psi\sin\theta - \cos\phi\sin\psi)^2}$$

$$- F_Z\left[\frac{(\sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta)(\sin\phi\cos\psi\sin\theta - \cos\phi\sin\psi)}{\sqrt{1-(\sin\phi\cos\psi\sin\theta - \cos\phi\sin\psi)^2}}\right]$$

$$F_{Z_w} = -F_X\left[\frac{\sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta}{\sqrt{1-(\sin\phi\cos\psi\sin\theta - \cos\phi\sin\psi)^2}}\right]$$

$$+ F_Z\left[\frac{\cos\theta\cos\psi}{\sqrt{1-(\sin\phi\cos\psi\sin\theta - \cos\phi\sin\psi)^2}}\right]$$

The following is the derivation of the equations for α' and ϕ' and the transfer equations from the body-axes system to the aeroballistic-axes system in terms of ψ, θ, and ϕ.

Referring to Figure 10, start with the body-axes system

Axes

X, Y, Z

Forces

F_X, F_Y, F_Z

Let this axes system rotate about the X-axis back through the angle ϕ' and the transfer equations for the aeroballistic-axes system in terms of the aerodynamic angle ϕ' are obtained.

Axes

X_a, Y_a, Z_a

Forces

$$F_{Xa} = F_X$$

$$F_{Ya} = F_Y \cos\phi' - F_Z \sin\phi'$$

$$F_{Za} = F_Y \sin\phi' + F_Z \cos\phi'$$

Now let the aeroballistic-axes system rotate about the Y_a -axis back through the angle α' The X-axis of the resulting axes system is the X_T -axis and therefore

$$F_{XT} = F_{Xa} \cos\alpha' + F_{Za} \sin\alpha'$$

$$F_{XT} = F_X \cos\alpha' + F_Y \sin\alpha' \sin\phi' + F_Z \sin\alpha' \cos\phi'$$

As was shown earlier in the derivation

$$F_{XT} = F_X \cos\theta \cos\psi + F_Y (\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi) + F_Z (\sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi)$$

therefore it follows that

(A11) $$\cos\alpha' = \cos\theta \cos\psi$$

(A12) $$\sin\alpha' \sin\phi' = \sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi$$

(A13) $$\sin\alpha' \cos\phi' = \sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi$$

From equation (A11)

$$\sin\alpha' = \sqrt{1-(\cos\theta \cos\psi)^2}$$

therefore from equation (A13) it follows that

(A14) $$\cos\phi' = \frac{\sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi}{\sqrt{1-(\cos\theta \cos\psi)^2}}$$

and from (A12) it follows that

(A15) $$\sin\phi' = \frac{\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi}{\sqrt{1-(\cos\theta \cos\psi)^2}}$$

When equations (A14) and (A15) are substituted into the transfer equations, the transfer equations take the following form for the aeroballistic-axes system.

Aeroballistic axes

$$F_{Xa} = F_X$$

$$F_{Ya} = F_Y\left[\frac{\sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi}{\sqrt{1-(\cos\theta \cos\psi)^2}}\right] - F_Z\left[\frac{\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi}{\sqrt{1-(\cos\theta \cos\psi)^2}}\right]$$

$$F_{Za} = F_Y\left[\frac{\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi}{\sqrt{1-(\cos\theta \cos\psi)^2}}\right] + F_Z\left[\frac{\sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi}{\sqrt{1-(\cos\theta \cos\psi)^2}}\right]$$

In the derivation thus far, expressions for α , α' , β and ϕ' in terms of the ψ , θ , and ϕ have been obtained. The following is the derivation of the equation for β' in terms of ψ , θ , and ϕ .

Referring again to Figure 6, start with the body-axes system,

Axes

X , Y , Z

Forces

F_X , F_Y , F_Z

Let this axes system rotate about the Z-axis back through the angle β'. Call this working-axes system the (')-axes system. It can be found on Figure 6 although not labeled. The X'-axis and the Y'-axis are shown as plane intersections. The Z'-axis is the Z-axis.

Axes

X', Y', Z'

Forces

$$F_X' = F_X \cos\beta' + F_Y \sin\beta'$$

$$F_Y' = -F_X \sin\beta' + F_Y \cos\beta'$$

$$F_Z' = F_Z$$

Now let the (')-axes system rotate about the Y'-axis through the angle θ'. The angle θ' can be found on Figure 6 but is not labeled. It is the angle between the X'-axis and the X_T-axis. Call this new working-axes system the (")-axes system. It can be found on Figure 6 but is not labeled. The X''-axis is the X_T-axis. The Y''-axis and the Z''-axis are shown as plane intersections.

Axes

X'', Y'', Z''

Forces

$$F_X'' = F_X' \cos\theta' + F_Z' \sin\theta'$$

$$F_Y'' = F_Y'$$

$$F_Z'' = -F_X' \sin\theta' + F_Z' \cos\theta'$$

$F_{X''}$ is the same as F_{XT} since X_T and X'' are the same axes. Therefore, it follows that

$$F_{X''} = F_{XT} = F_X \cos\beta' \cos\theta' + F_Y \sin\beta' \cos\theta' + F_Z \sin\theta'$$

From previous derivation it was shown that

$$F_{XT} = F_X \cos\theta \cos\psi + F_Y (\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi) + F_Z (\sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi)$$

therefore,

(A16) $\cos\beta' \cos\theta' = \cos\theta \cos\psi$

(A17) $\sin\beta' \cos\theta' = \sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi$

(A18) $\sin\theta' = \sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi$

Division of equation (A17) by equation (A16) results in

$$\frac{\sin\beta' \cos\theta'}{\cos\beta' \cos\theta'} = \frac{\sin\phi \cos\psi \sin\theta}{\cos\theta \cos\psi} - \frac{\cos\phi \sin\psi}{\cos\theta \cos\psi}$$

$$\tan\beta' = \sin\phi \tan\theta - \frac{\cos\phi \tan\psi}{\cos\theta}$$

This is the desired equation for β' in terms of ψ, θ, and ϕ.

In the preceding derivations, the forces used were those in the positive direction of the axes. The vector representation of a positive moment, in keeping with the established convention of the right-hand rule, would also be in the positive direction of the axes. Therefore, the transfer equations for the moments would take the same form as those derived for the forces. In aerodynamics, forces in the negative direction of the axes are often used in preference to those in the positive direction. The following relationships are listed for convenience.

Body axes

$$F_A = -F_X$$

$$F_Y \equiv F_Y$$

$$F_N = -F_Z$$

Stability axes

$$F_D' = -F_{Xs}$$

$$F_Y = F_{Ys}$$

$$F_L = -F_{Zs}$$

Wind axes

$$F_D = -F_{Xw}$$

$$F_C = F_{Yw}$$

$$F_L = -F_{Zw}$$

Aeroballistic axes

$$F_A = -F_{Xa}$$

$$F_{Ya}' \equiv F_{Ya}$$

$$F_{Na}' = -F_{Za}$$

The forces and moments are usually used in coefficient form in aerodynamics. (See Section II of this publication.) Considering the above relationships, the transfer equations in terms of coefficients and aerodynamic angles take the following form:

Stability axes

$$C_{Zs} = -C_D' = -C_A \cos\alpha - C_N \sin\alpha$$

$$C_{Ys} = C_Y$$

$$C_{Zs} = -C_L = C_A \sin\alpha - C_N \cos\alpha$$

$$C_{ls} = C_l \cos\alpha + C_n \sin\alpha$$

$$C_{ms} = C_m$$

$$C_{ns} = -C_l \sin\alpha + C_n \cos\alpha$$

Wind axes

$$C_{Zw} = -C_D = -C_A \cos\alpha \cos\beta + C_Y \sin\beta - C_N \sin\alpha \cos\beta$$

$$C_{Yw} = C_C = C_A \cos\alpha \sin\beta + C_Y \cos\beta + C_N \sin\alpha \sin\beta$$

$$C_{Zw} = -C_L = C_A \sin\alpha - C_N \cos\alpha$$

$$C_{lw} = C_l \cos\alpha \cos\beta + C_m \sin\beta + C_n \sin\alpha \cos\beta$$

$$C_{mw} = -C_l \cos\alpha \sin\beta - C_m \cos\beta - C_n \sin\alpha \sin\beta$$

$$C_{nw} = -C_l \sin\alpha + C_n \cos\alpha$$

Aeroballistic axes

$$C_{Xa} = -C_A$$

$$C_{Ya} = C_Y \cos\phi' + C_N \sin\phi'$$

$$C_{Za} = -C_{Na} = +C_Y \sin\phi' - C_N \cos\phi'$$

$$C_{la} = C_l$$

$$C_{ma} = C_m \cos\phi' - C_n \sin\phi'$$

$$C_{na} = C_m \sin\phi' + C_n \cos\phi'$$

In terms of the three orientation angles ψ, θ, and ϕ measured in the wind tunnel, these transfer equations expand to the following.

Stability axes

$$C_{X_s} = -C_D' = -C_A \left[\frac{\cos\theta \cos\psi}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}\right]$$

$$-C_N \left[\frac{\sin\phi \sin\psi + \cos\phi \cos\psi \sin\theta}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}\right]$$

$$C_{Y_s} = C_Y$$

$$C_{Z_s} = -C_L = C_A \left[\frac{\sin\phi \sin\psi + \cos\phi \cos\psi \sin\theta}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}\right]$$

$$-C_N \left[\frac{\cos\theta \cos\psi}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}\right]$$

$$C_{l_s} = C_l \left[\frac{\cos\theta \cos\psi}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}\right]$$

$$+C_n \left[\frac{\sin\phi \sin\psi + \cos\phi \cos\psi \sin\theta}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}\right]$$

$$C_{m_s} = C_m$$

$$C_{n_s} = -C_l \left[\frac{\sin\phi \sin\psi + \cos\phi \cos\psi \sin\theta}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}\right]$$

$$+C_n \left[\frac{\cos\theta \cos\psi}{\sqrt{1-(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)^2}}\right]$$

Wind axes

$$C_{X_w} = C_D = -C_A(\cos\theta\ \cos\psi) + C_Y(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)$$
$$- C_N(\sin\phi\ \sin\psi + \cos\phi\ \cos\psi\ \sin\theta)$$

$$C_{Y_w} = C_C = C_A\left[\frac{(\cos\theta\ \cos\psi)(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)^2}}\right]$$

$$+ C_Y\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)^2}$$

$$+ C_N\left[\frac{(\sin\phi\ \sin\psi + \cos\phi\ \cos\psi\ \sin\theta)(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)^2}}\right]$$

$$C_{Z_w} = C_L = C_A\left[\frac{(\sin\phi\ \sin\psi) + (\cos\phi\ \cos\psi\ \sin\theta)}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)^2}}\right]$$

$$- C_N\left[\frac{\cos\theta\ \cos\psi}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)^2}}\right]$$

$$C_{l_w} = C_l(\cos\theta\ \cos\psi) + C_m(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)$$
$$+ C_n(\sin\phi\ \sin\psi + \cos\phi\ \cos\psi\ \sin\theta)$$

$$C_{mw} = -C_l\left[\frac{(\cos\theta\ \cos\psi)(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)^2}}\right]$$

$$+ C_m\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)^2} -$$

$$- C_n\left[\frac{(\sin\phi\ \sin\psi + \cos\phi\ \cos\psi\ \sin\theta)(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)^2}}\right]$$

$$C_{nw} = -C_l\left[\frac{\sin\phi\ \sin\psi + \cos\phi\ \cos\psi\ \sin\theta}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)^2}}\right]$$

$$+ C_n\left[\frac{\cos\theta\ \cos\psi}{\sqrt{1-(\sin\phi\ \cos\psi\ \sin\theta - \cos\phi\ \sin\psi)^2}}\right]$$

Aeroballistic axes

$$C_{Xa} = -C_A$$

$$C_{Ya} = C_Y \left[\frac{(\sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi)}{\sqrt{1-(\cos\theta \cos\psi)^2}} \right] + C_N \left[\frac{(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)}{\sqrt{1-(\cos\theta \cos\psi)^2}} \right]$$

$$C_{Za} = -C_{Na} = C_Y \left[\frac{(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)}{\sqrt{1-(\cos\theta \cos\psi)^2}} \right] - C_N \left[\frac{(\sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi)}{\sqrt{1-(\cos\theta \cos\psi)^2}} \right]$$

$$C_{la} = C_l$$

$$C_{ma} = C_m \left[\frac{(\sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi)}{\sqrt{1-(\cos\theta \cos\psi)^2}} \right] - C_n \left[\frac{(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)}{\sqrt{1-(\cos\theta \cos\psi)^2}} \right]$$

$$C_{na} = C_m \left[\frac{(\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi)}{\sqrt{1-(\cos\theta \cos\psi)^2}} \right] + C_n \left[\frac{(\sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi)}{\sqrt{1-(\cos\theta \cos\psi)^2}} \right]$$

The transfer equations for the tunnel-axes system which were derived earlier are listed below in coefficient form.

Tunnel axes

$$C_{XT} = -C_A (\cos\theta \cos\psi) + C_Y (\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi) - C_N (\sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi)$$

$$C_{YT} = -C_A (\cos\theta \sin\psi) + C_Y (\cos\phi \cos\psi + \sin\phi \sin\theta \sin\psi) - C_N (\cos\phi \sin\theta \sin\psi - \sin\phi \cos\psi)$$

$$C_{ZT} = C_A \sin\theta + C_Y (\sin\phi \cos\theta) - C_N (\cos\phi \cos\theta)$$

$$C_{lT} = C_l (\cos\theta \cos\psi) + C_m (\sin\phi \cos\psi \sin\theta - \cos\phi \sin\psi) + C_n (\sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi)$$

$$C_{mT} = C_l (\cos\theta \sin\psi) + C_m (\cos\phi \cos\psi + \sin\phi \sin\theta \sin\psi) + C_n (\cos\phi \sin\theta \sin\psi - \sin\phi \cos\psi)$$

$$C_{nT} = -C_l \sin\theta + C_m (\sin\phi \cos\theta) + C_n (\cos\phi \cos\theta)$$

APPENDIX B

DERIVATION OF THE RELATIONSHIPS BETWEEN THE ANGLES ψ, θ, AND ϕ WHEN THE SEQUENCE IS CHANGED

When the rotations of the body-axes system through the orientation angles are taken in the sequence θ , ψ , ϕ there exists a set of values for the angles ψ , θ , and ϕ which, if taken in this sequence, would orient the body-axes system in the same manner with respect to the tunnel-axes system. The relationship between these angles is derived herein. To avoid confusion, when the sequence is ψ , θ , ϕ the angles are designated ψ_1 , θ_1 , and ϕ_1 ; when the sequence is θ , ψ , ϕ the angles are designated θ_2 , ψ_2 , and ϕ_2 . If the values of the angles ψ and θ are selected for the two sequences ψ , θ and θ , ψ such that the body X-axis is located in the same position, then the orientation of the body-axes system with respect to the tunnel axes, for the two sequences will differ only by a roll angle. This is evident in Figure 13 which shows the body-axes system oriented as it would be if the sequence were θ , ψ .

Referring to Figure 13, let the body-axes system rotate about the Z-axis back through the angle ($-\psi_2$). Call this new working-axes system the ($_1$)-axes system. The X_1-axis is shown as a plane intersection, but is not labeled, the Y_1-axis is the Y_T-axis and the Z_1-axis is the Z-axis.

Then:

Axes

X , Y , Z

Forces

$$F_{X1} = F_X \cos\psi_2 + F_Y \sin\psi_2$$

$$F_{Y1} = -F_X \sin\psi_2 + F_Y \cos\psi_2$$

$$F_{Z1} = F_Z$$

Since the angle ψ_2 used in the derivation is negative and since

$$\sin(-\psi_2) = -\sin\psi_2$$

$$\cos(-\psi_2) = \cos\psi_2$$

the above equations become

Axes

X_I, Y_I, Z_I

Forces

$$F_{XI} = F_X \cos\psi_2 - F_Y \sin\psi_2$$

$$F_{YI} = F_X \sin\psi_2 + F_Y \cos\psi_2$$

$$F_{ZI} = F_Z$$

Now let the ($_I$)-axes system rotate about the Y_I -axis back through the angle θ_2 and the transfer equations for the tunnel-axes system in terms of θ_2 and ψ_2 are obtained.

Axes

X_T, Y_T, Z_T

Forces

$$F_{XT} = F_{XI} \cos\theta_2 + F_{ZI} \sin\theta_2$$

$$F_{YT} = F_{YI} = F_X \sin\psi_2 + F_Y \cos\psi_2$$

$$F_{ZT} = -F_{XI} \sin\theta_2 + F_{ZI} \cos\theta_2$$

or

$$F_{XT} = F_X \cos\theta_2 \cos\psi_2 - F_Y \cos\theta_2 \sin\psi_2 + F_Z \sin\theta_2$$

$$F_{YT} = F_X \sin\psi_2 + F_Y \cos\psi_2$$

$$F_{ZT} = -F_X \sin\theta_2 \cos\psi_2 + F_Y \sin\theta_2 \sin\psi_2 + F_Z \cos\theta_2$$

Starting again with the body-axes system and referring to Figure 13, work to the tunnel-axes system through the angles ϕ_1 , θ_1 , and ψ_1 . Let the body-axes system rotate about the X-axis back through the angle $(-\phi_1)$. Call this working-axes system the ($_2$)-axes system. The X_2-axis is the X-axis. The X_2- and Z_2-axes are shown as plane intersections but are not labeled.

Then;

Axes

X_2, Y_2, Z_2

Forces

$$F_{X2} = F_X$$

$$F_{Y2} = F_Y \cos\phi_1 + F_Z \sin\phi_1$$

$$F_{Z2} = -F_Y \sin\phi_1 + F_Z \cos\phi_1$$

Since the angle ϕ_1 , used in the derivation is negative and therefore

$$\sin(-\phi_1) = -\sin\phi$$

$$\cos(-\phi_1) = \cos\phi_1$$

the above equations become

Axes

X_2, Y_2, Z_2

Forces

$$F_{X2} = F_X$$

$$F_{Y2} = F_Y \cos\phi_1 - F_Z \sin\phi_1$$

$$F_{Z2} = F_Y \sin\phi_1 + F_Z \cos\phi_1$$

Let the ($_2$)-axes system rotate about the Y_2 -axis back through the angle θ_1 . Call this working-axes system the ($_3$)-axes system. The X_3 - and Y_3 -axes are shown as plane intersections but are not labeled. The Z_3 -axis is the Z_T -axis.

Then;

Axes

X_3, Y_3, Z_3

Forces

$$F_{X3} = F_{X2} \cos\theta_1 + F_{Z2} \sin\theta_1$$

$$F_{Y3} = F_{Y2}$$

$$F_{Z3} = -F_{X2} \sin\theta_1 + F_{Z2} \cos\theta_1$$

$$F_{X3} = F_X \cos\theta_1 + F_Y \sin\phi_1 \sin\theta_1 + F_Z \cos\phi_1 \sin\theta_1$$

$$F_{Y3} = F_Y \cos\phi_1 - F_Z \sin\phi_1$$

$$F_{Z3} = -F_X \sin\theta_1 + F_Y \sin\phi_1 \cos\theta_1 + F_Z \cos\phi_1 \cos\theta_1$$

Let the ($_3$)-axes system rotate about the Z_3 -axis back through the angle $(-\psi_1)$, and the transfer equations for the tunnel-axes system in terms of ψ_1 and θ_1 are obtained.

Axes

X_T, Y_T, Z_T

Forces

$$F_{XT} = F_{X3} \cos\psi_1 + F_{Y3} \sin\psi_1$$

$$F_{YT} = -F_{X3} \sin\psi_1 + F_{Y3} \cos\psi_1$$

$$F_{ZT} = F_{Z3}$$

Since the angle ψ_1 used in this derivation is negative, the above equations become

Axes

X_T, Y_T, Z_T

Forces

$$F_{XT} = F_{X3}\cos\psi_1 - F_{Y3}\sin\psi$$

$$F_{YT} = F_{Y3}\cos\psi_1 + F_{X3}\sin\psi_1$$

$$F_{ZT} = F_{Z3}$$

or

$$F_{XT} = F_X\cos\theta_1\cos\psi_1 + F_Z\cos\phi_1\sin\theta_1\cos\psi_1 + F_Y\sin\phi_1\sin\theta_1\cos\psi_1 - F_Y\cos\phi_1\sin\psi_1 + F_Z\sin\phi_1\sin\psi_1$$

$$F_{YT} = F_Y\cos\phi_1\cos\psi_1 - F_Z\sin\phi_1\cos\psi_1 + F_X\cos\theta_1\sin\psi_1 + F_Z\cos\phi_1\sin\theta_1\sin\psi + F_Y\sin\phi_1\sin\theta_1\sin\psi_1$$

$$F_{ZT} = F_Z\cos\phi_1\cos\theta_1 + F_Y\sin\phi_1\cos\theta_1 - F_X\sin\theta_1$$

After collecting terms

$$F_{XT} = F_X(\cos\theta_1\cos\psi_1) + F_Y(-\cos\phi_1\sin\psi_1 + \sin\phi_1\sin\theta_1\cos\psi_1) + F_Z(+\sin\phi_1\sin\psi_1 + \cos\phi_1\sin\theta_1\cos\psi_1)$$

$$F_{YT} = F_X(\cos\theta_1\sin\psi_1) + F_Y(\cos\phi_1\cos\psi_1 + \sin\phi_1\sin\theta_1\sin\psi_1) + F_Z(-\sin\phi_1\cos\psi_1 + \cos\phi_1\sin\theta_1\sin\psi_1)$$

$$F_{ZT} = -F_X\sin\theta_1 + F_Y\sin\phi_1\cos\theta_1 + F_Z\cos\phi_1\cos\theta_1$$

It was shown earlier in the derivation that

$$F_{XT} = F_X \cos\theta_2 \cos\psi_2 - F_Y \cos\theta_2 \sin\psi_2 + F_Z \sin\theta_2$$

$$F_{YT} = F_X \sin\psi_2 + F_Y \cos\psi_2$$

$$F_{ZT} = -F_X \sin\theta_2 \cos\psi_2 + F_Y \sin\theta_2 \sin\psi_2 + F_Z \cos\theta_2$$

therefore the following relationships are true

(B1) $\cos\theta_2 \cos\psi_2 = \cos\theta_1 \cos\psi_1$

(B2) $\sin\psi_2 = \cos\theta_1 \sin\psi_1$

(B3) $\cos\theta_2 = \cos\phi_1 \cos\theta_1$

(B4) $\sin\theta_2 \sin\psi_2 = \sin\phi_1 \cos\theta_1$

(B5) $\sin\theta_2 \cos\psi_2 = \sin\theta$

If equation (B2) is divided by equation (B1) then

$$\frac{\sin\psi_2}{\cos\theta_2 \cos\psi_2} = \frac{\cos\theta_1 \sin\psi_1}{\cos\theta_1 \cos\psi_1}$$

which simplifies to

(B6) $$\tan\psi_1 = \frac{\tan\psi_2}{\cos\theta_2}$$

If equation (B4) is divided by equation (B3) then

$$\frac{\sin\theta_2 \sin\psi_2}{\cos\theta_2} = \frac{\sin\phi_1 \cos\theta_1}{\cos\phi_1 \cos\theta_1}$$

which simplifies to

$$\tan\phi_1 = \tan\theta_2 \sin\psi_2 \tag{B7}$$

The desired relationships are given by equations (B5), (B6), and (B7) which are restated below.

$$\theta_1 = \sin^{-1}(\sin\theta_2 \cos\psi_2)$$

$$\psi_1 = \tan^{-1}\left(\frac{\tan\psi_2}{\cos\theta_2}\right)$$

$$\phi_1 = \tan^{-1}(\tan\theta_2 \sin\psi_2)$$

If in the sequence θ, ψ, ϕ, the value of the angle ϕ_2 is other than zero, then the equation for ϕ_1, becomes;

$$\phi_1 = \tan^{-1}(\tan\theta_2 \sin\psi_2) + \phi_2$$