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INTERIM RESEARCH MEMORANDUM

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A GAME THEORETIC MODEL OF A SUBMARINE BARRIER TO DETECT TRANSITOR SUBMARINES

By

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IRM-6

ABSTRACT

This memorandum formulates and solves a barrier submarine-transitor submarine conflict as a two-person game. When described as a game, the conflict becomes a problem of obtaining distributions along the barrier of the locations of (1) transit lanes and (2) locations of the barrier submarine for which <u>on the average</u> a game theoretic optimum probability of detection by the barrier submarine is to be obtained. The game theoretic optimum or minimax solution gives (if it exists) at the least one distribution for the transit lanes having the property that, if the barrier submarine behaves optimally, no other distribution of lanes can decrease the probability of detection. In addition, the solution will give (if it exists) at least one distribution of barrier submarine positions, such that no other barrier submarine distribution will increase the probability of detecting an optimal transitor. Optimal solutions are derived in the appendix for games in which the barrier submarine detection equipment has (1) a definite range law and (2) a trapezoidal lateral range probability of detection law. The geometric method used there can be extended to other detection laws with an attendant increase in mathematical difficulties.

1. An Anti-transitor Submarine Barrier Game: In a segment of an anti-transitor submarine barrier assigned to a single barrier submarine, it is desired to find an "optimum" distribution of: (1) locations on the barrier to be taken by the barrier submarine, and (2) straight courses (or lanes) on which the transitor submarine adversary may cross the barrier when the same tactical situation is expected to occur a large number of times. For each transit the barrier submarine may take any location on its barrier, and the transitor may cross perpendicularly at any point on the barrier. When a large number of replications of a transit are made the transitor and the barrier submarine must choose locations with a frequency given by their optimum distributions. The length of the barrier segment is taken to be unity for convenience.

The game theoretic optimum distributions can be explained as follows: A payoff function $K = K(\xi, \eta)$ of two variables ξ and η is given. For the anti-transitor barrier, K is the probability that the barrier submarine detects the transitor. The variable ξ is the location of the barrier submarine; and η is the location of the intersection of the transitor track and the barrier. Both locations, by assumption, are between zero and 1. For each transit (or play of the game), ξ can be chosen by the barrier submarine and η can be chosen by the transitor. Of course the barrier submarine wishes to maximize K, while the transitor wishes for a minimum K Because of the interacting choices, the problem is not one of maximizing or minimizing; but rather one of the barrier submarine adjusting the weaknesses in its tactics so that any exploitation by the transitor will not decrease the probability of detection below a minimum, known to the barrier submarine This minimum probability is a function of the lateral range curve of the barrier submarine and the distribution for ξ chosen by the barrier submarine. On the other hand the transitor cannot (with an optimum or best barrier submarine distribution) reduce this minimum probability. The transitor, however, must choose a distribution for η which does not permit the barrier submarine to increase the probability of detection by exploiting any weaknesses in the transitor choice. Moreover, the transitor has to choose a distribution which avoids the potential strengths of the barrier submarine.

The interaction between the two adversaries makes visualization of the effects of potential tactics difficult. One or two examples of opposing tactics may give more insight into the conflict. Suppose, for example, that for each play of the game (i.e., each transit) the barrier submarine chooses to stay at the center of the barrier; and suppose that the transitor chooses to cross the barrier at the center every time. If the detection law for the barrier submarine diminishes at the edges of its barrier segment, then the transitor tactic is a foolish one for it. Exploiting the tapering of the detection law by the transitor would decrease its probability of being detected.

On the other hand, if the transitor stayed on the edge always and the barrier submarine persisted in choosing the center, the barrier submarine would not be acting in its own best interests: it could increase the probability of detecting the transitor by moving over to (or near) the proper edge. But then the transitor should move in its own best interest. And so on.

The problem of getting a game theoretic solution becomes one of getting distributions for selecting locations, such that even though an adversary knows the other's distribution, the former can do nothing to improve its payoff on the average.

The choices for the barrier submarine and the transitor will be formalized; and distributions will be constructed for lateral range curves which are rectangular or trapezoidal (including triangular) in appendix A. The methods used in the examples of appendix A are adaptable to any symmetric lateral range probability of detection curve. Although there are some more or less general results available for continuous games (e.g., reference (a)), the games of this memorandum generally do not fit these theories. Rather a step-by-step procedure, starting from the fundamentals of game theory is employed, which constructs a solution for each game.

2. Limitations on Game Theoretic Solutions: There are two types of limitations in game theoretic models: (1) the limitations on the game theoretic or minimax solution and (2) the limitations inherent in the assumptions. The latter will be discussed first.

a. The games solved assume that any transitor crossing the barrier line is detectable, which is certainly the case for a passive barrier submarine and a continuously noisy transitor. Hence the games only apply to an active, silent barrier submarine and a continuously noisy transitor. By a modification of the lateral range probability of detection curve to account for the average probability of snorkelling, the same geometric construction should give a solution when the transitor is not continuously noisy. Generally, the value of the game is lower if the transitor snorkles intermittently.

b. It has been assumed that only a portion of a barrier is to be investigated, that this portion has a single barrier submarine and a single transitor. Thus the segment is assumed to be isolated from the rest of the barrier and that there are no edge effects from adjacent segments of the barrier. Again by modifying the lateral range detection curve the effect of barrier submarines in adjacent segments can be subsumed. Such an adjustment will permit an extension to a multiple submarine barrier.

The limitations (or better the meaning) of the minimax solutions should be clearly understood.

a. First the minimax solution gives little information about the outcome of a single play of the game (i.e., single barrier transit). Rather it gives an average probability of detection for a situation in which a very large number of transits are made when the adversaries choose the strategies for the individual plays from their minimax distributions. For each transit a selection of transit location and barrier submarine position is to be made in accordance with the minimax distribution of each. There is no shifting of transverse location during a transit. Crudely, the expected a priori probability of detection for any play of the game is the minimax value of the payoff function K.

b. The game being played (i.e., the payoff function and the distributions from which the plays are chosen) must be the same for both. Thus the payoff function used to describe the game must be the same for the transitor and for the barrier submarine. Since, as will be shown, the payoff function depends on the lateral probability of detection curve, both transitor and barrier submarine must know the shape of the barrier submarine detection curve. Unless the distributions are independent of the detection curve (or at least insensitive to the possible shapes), the game theoretic model is more applicable to exercises than to an anti-enemy barrier.

c. The minimax distributions are conservative. Each adversary knows that a "best" choice has been made: for no matter what the other does, on the average, his probability of detection cannot be adversely affected.

d. It is to be emphasized that these distributions are derived in the complete <u>lack</u> of knowledge of either player about the other on any specific play. If intelligence indicates some preferential strategy for one adversary, then obviously the minimax distribution should be abandoned by the other to exploit any potential advantage.

3. <u>Results from the Game Theoretic Model</u>: Figure 1 shows the average probability of detection (i.e., the value of the game) when the barrier submarine has (1) a definite detection law, (2) an isosceles trapezoidal law with a 60° base angle, and (3) an isosceles trapezoidal law with a 75° base angle. The average probability of detection is to be expected when both barrier submarine and transitor use their minimax distributions for locations. However, figure 1, although indicating the number of positions required for the barrier submarine's minimax distribution does not indicate where the positions might be. Moreover, the transit lanes for the transitor are required for a complete solution for the barrier as a game. Figures 2, 3, and 4 show the points or intervals which must be occupied by the barrier submarine and by the transitor for one minimax solution of the game. Before making several explanatory comments about the latter figures, two examples of the use of the figures will be given:

Example 1. Suppose that the lateral range probability of detection curve is a definite range law whose width is 0.4 of the width of the barrier segment. From figure 1 the average probability of detection is 0.33. The locations of the barrier submarine and the transitor are shown on figure 2.

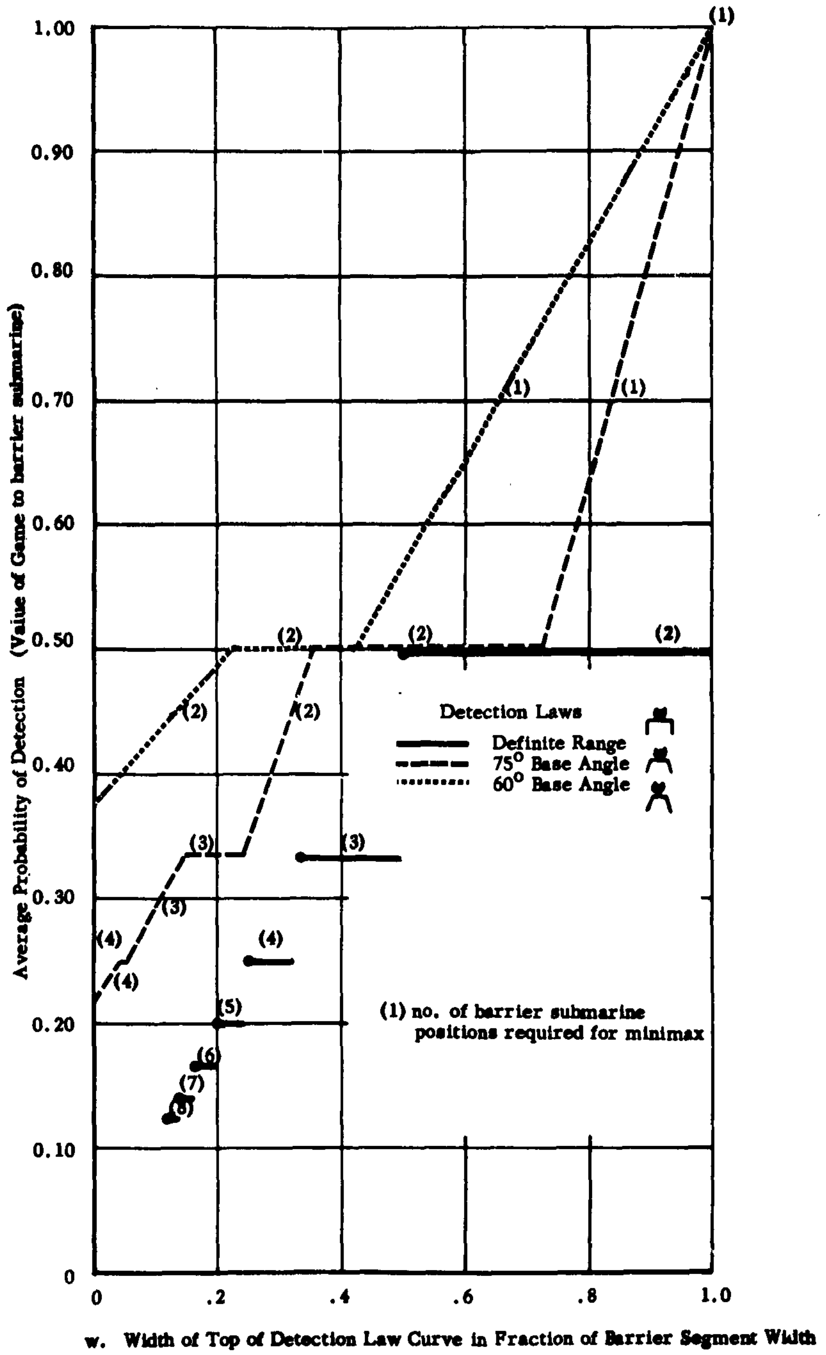


FIG. 1: AVERAGE PROBABILITY OF DETECTION AS A FUNCTION OF DETECTION LAW

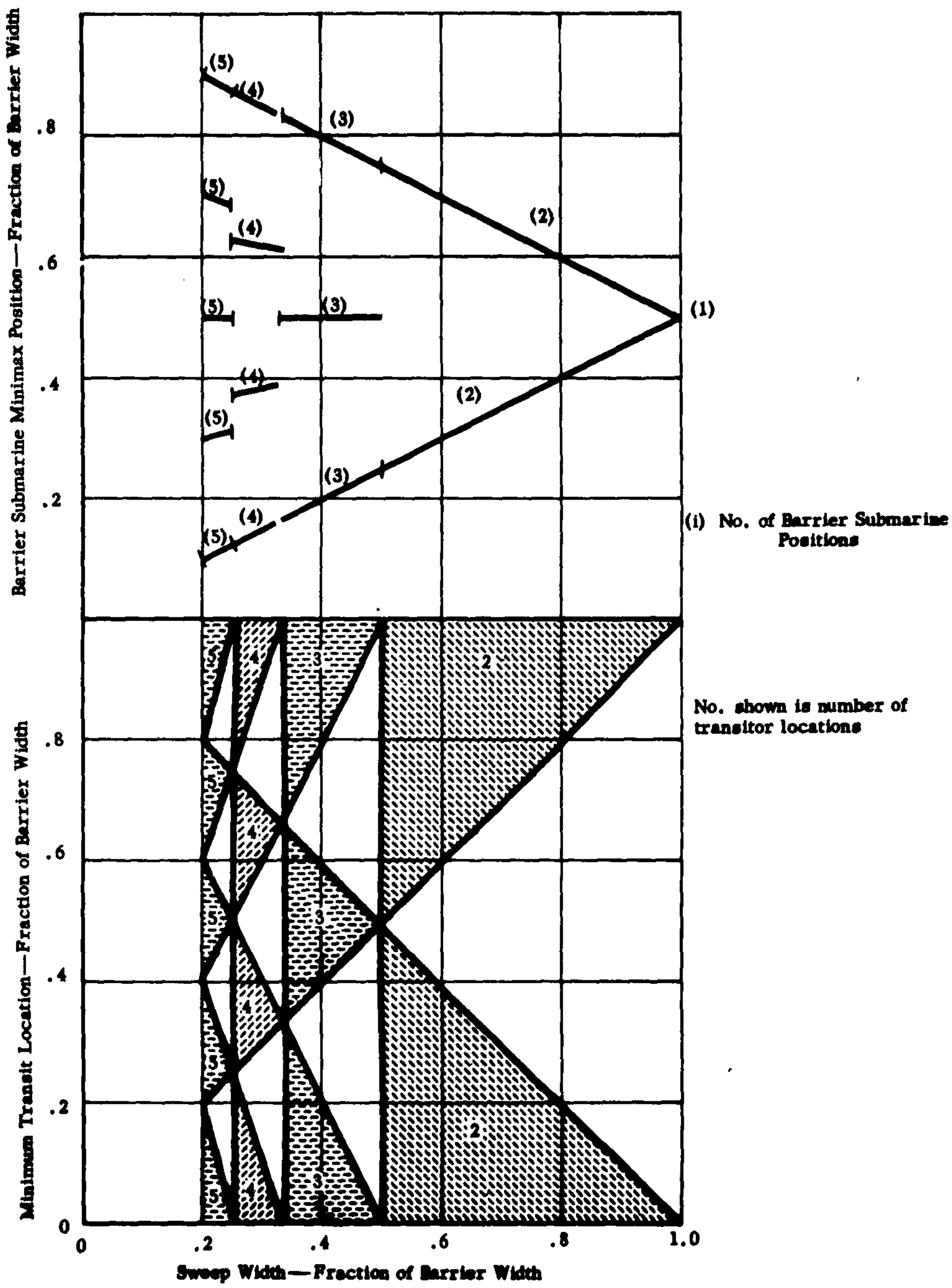


FIG. 2: MINAMAX LOCATIONS FOR A SINGLE BARRIER SUBMARINE BARRIER SEGMENT DEFINITE RANGE LAW

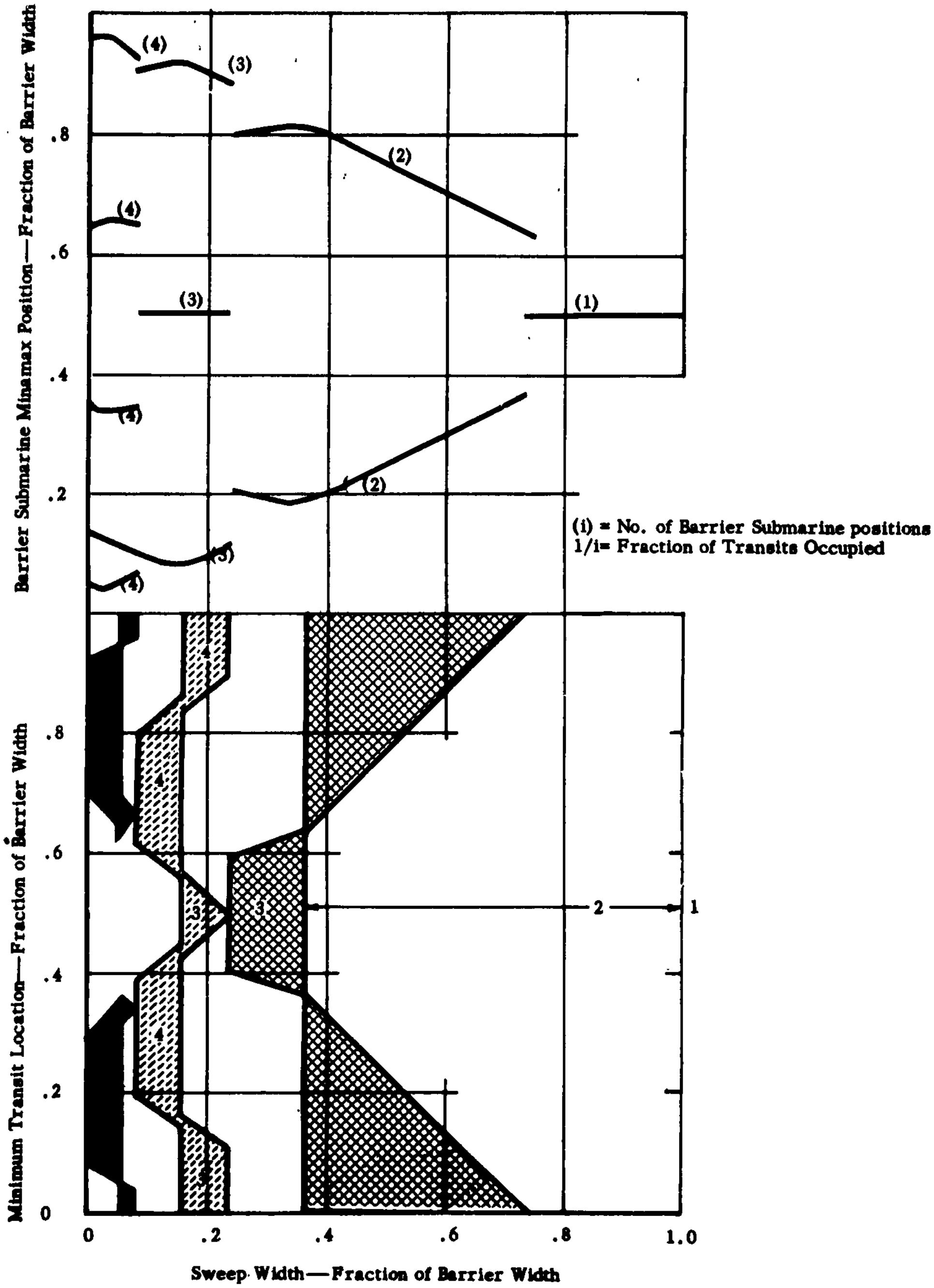


FIG. 3: MINIMAX LOCATIONS FOR A SINGLE BARRIER SUBMARINE BARRIER SEGMENT TRAPEZOIDAL RANGE LAW

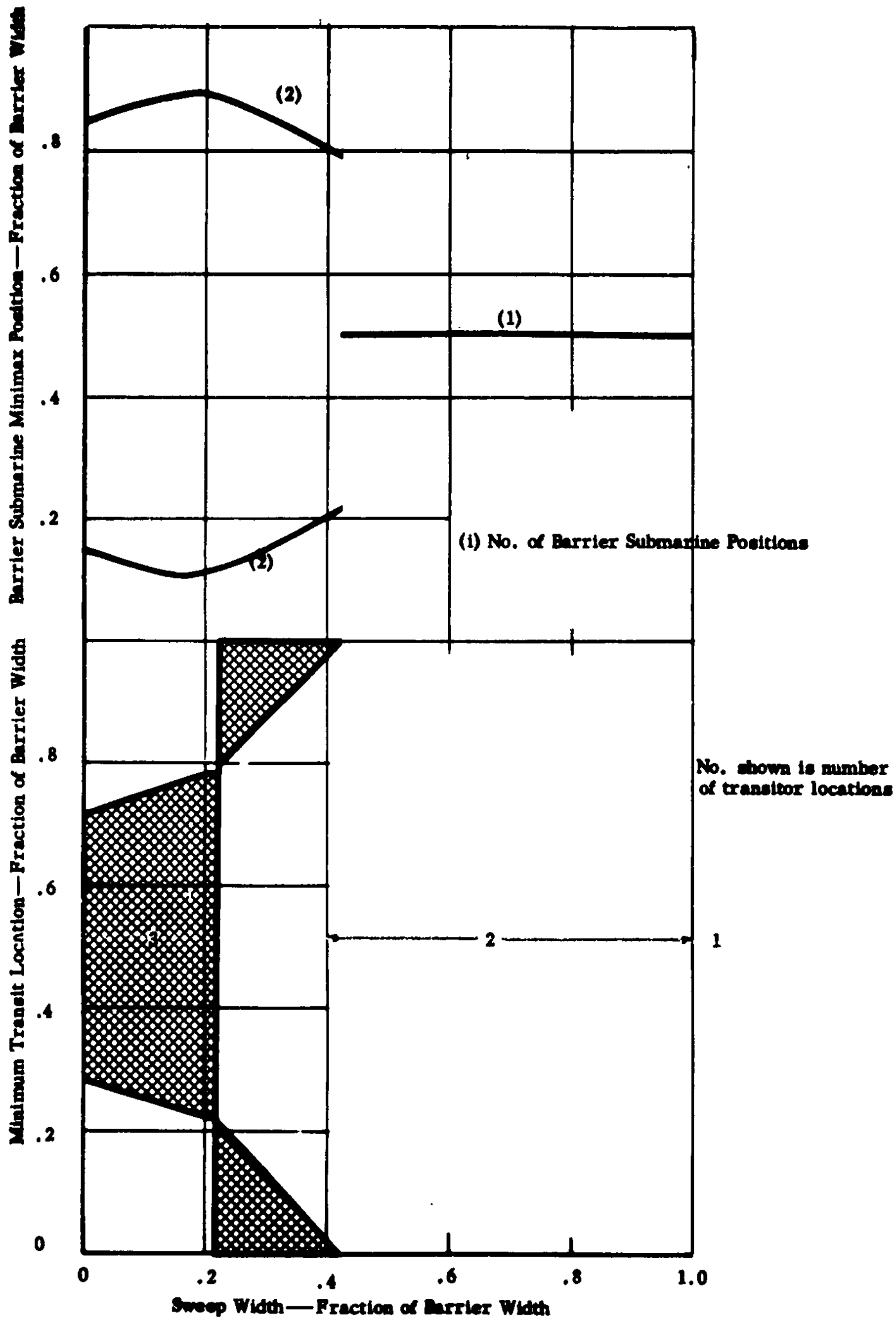


FIG. 4: MINAMAX LOCATIONS FOR A SINGLE BARRIER SUBMARINE BARRIER SEGMENT TRAPEZOIDAL RANGE LAW

The points 0.2, 0.5, and 0.8 of the width of the barrier must each be occupied by the barrier submarine one-third of the time. On the other hand, the transitor has three intervals, from 0.0 to 0.2, 0.4 to 0.6 and 0.8 to 1.0, each of which must be occupied on one-third of the plays. When an interval is selected, any point of that interval may be occupied. Different geometrical points in one interval are mathematically equivalent for the game.

Example 2: Suppose that the detection curve has a 60° base and a width of top of 0.2 the width of the barrier segment. Figure 1 shows the average probability of detection to be 0.49. The barrier submarine is to spend one-half time at 0.11 and one-half the time at 0.89 of the barrier width. The situation for the transitor is somewhat different. One-third of the plays, the transitor is to occupy each boundary. The transitor is to spend the remaining third on the interval between 0.22 and 0.78. It is important that the boundaries not be neglected; the boundaries are always suitable locations for the transitor. Sometimes, as in example 1, the boundaries are connected to intervals; whereas in example 2, the boundaries are isolated points.

Example 3: To illustrate still another kind of location, suppose that the base angle is 90°--a definite detection law-- and the width of the top is 0.5 of the barrier width. The probability and the locations of the barrier submarine are obtained as before. However, the entire width of the barrier is shown for the transitor locations. The proper interpretation must be made. This full width is a boundary between the triangles for 2 locations and the triangles for 3 locations of the transitor. But only one solution is possible; note that figure 1 shows that for a width of 0.5, the upper solid line gives the value of the game. Hence the barrier submarine must occupy 2 positions. Those 2 locations are shown on figure 2 as at 0.25 and 0.75. The transitor must spend half of its time in each half of the barrier. The segments shown in the lower part of figure 2 are from 0.0 to 0.5 and 0.5 to 1.0.

It should be noted that the combinations of locations of figures 2, 3, and 4 are not the only combinations which give the minimax value for the game. The ones shown, in which only distinct points are permissible for the barrier submarine are those which provide the biggest bonus for the barrier submarine, <u>on the average</u> when the transitor persistently fails to use its minimax strategy <u>in an unknown way</u>. This is explained diagrammatically in appendix A. Other distributions would permit intervals for the barrier submarine and corresponding changes for the transitor locations. These others have not been considered here.

Reference: (a) Theory of Infinite Games, Samuel Karlin, New York: Addison-Wesley, 1959

APPENDIX A

1. Introduction:

This appendix contains a geometric construction of a game theoretic solution for an anti-transitor barrier. The barrier game model considered is one for which the segment assigned to one barrier submarine has unit length. The barrier submarine must choose its location for each play from a distribution to be determined. The opposition to the single barrier submarine is a single transitor submarine which must transit the barrier segment (of unit length). For each play the transitor must select a transit lane (perpendicular to the barrier line) from a distribution to be determined. This formulation is one of an infinite game (i.e., has a continuum of choices for both adversaries) on the unit square (each distribution is to be determined over the unit interval). (See reference (a)). The payoff function is the probability of the barrier submarine detecting the transitor. Location of transit lane, location of barrier submarine and shape of the barrier submarine's lateral range probability of detection curve are the essential variables in the payoff function.

The geometric method will be displayed for trapezoidal lateral range probability of detection curves, including limiting cases of rectangular and triangular detection laws. Other convex shapes (at least) are amenable to the method.

2. Some Game Theoretic Fundamentals:

What follows is a brief statement of the fundamentals of game theory. The notation is that of reference (a), where a complete, readable, and understandable statement of game theory is given. The description here is just sufficient to make the appendix self-contained, and is limited to the requirements of the appendix.

Abstractly, a game may be defined as a triple

$$(K, X, Y),$$

where K is a function, the payoff, of two variables

$$K = K(\xi, \eta)$$

and X and Y are suitable classes of distribution functions. In practice there are two players, usually designated I and II; I makes choices ξ

from a distribution X, and II makes choices η from a distribution Y. K, the payoff to player I, may be a probability, a financial profit, a score etc. for one of the players. The problem of game theory is to find a "best" choice for each player; where, as indicated by the functional form K (ξ, η), each choice influences the value of the payoff. The "best" choices are the selection C of one distribution function x for ξ from X and one y for η from Y. It might be noted, parenthetically, that in general the "best" distributions are not unique.

For distributions $x \in X$ and $y \in Y$ the payoff is

$$K(x,y),$$

where without risk of confusion the distributions are used rather than the individual choices as before. This latter payoff is an average probability, an average profit, or average score. In the new notation the game theory problem is to find, if possible, a distribution x_o in X and y_o in Y which have the following properties: (1) if y is any distribution in Y

$$K(x_o,y) \geq K(x_o,y_o), \tag{1}$$

and (2) if x is any distribution in X

$$K(x_o,y_o) \geq K(x,y_o). \tag{2}$$

These two conditions are symbolic transcriptions of the literary statements in the introduction concerning optimal game theoretic behaviors of the barrier submarine and the transitor. The value of the game (to player I who chooses x_o) is defined to be

$$K(x_o,y_o).$$

Reference (a) contains an excellent survey of methods for various infinite games on the unit square. Such games are choices of locations on the unit interval for each player (i.e., a choice of distributions over the unit interval). The methods are usually tied to a specific form of the payoff function $K(\xi, \eta)$ such as convex, polynomial, analytic, etc. Generally the payoff functions used below do not fit the specific theories; hence for each game an optimal choice for player I will be obtained by a geometric construction, from which an optimal choice for II can be obtained. It is to be emphasized that the optimal choices obtained are those satisfying (1) and (2). They are not the maximum payoff (say) for player I, if player II does not use his optimal tactic. But in the absence of information about the choice made by player II, player I's choice guarantees him a certain minimum payoff, which minimum cannot be increased by any other choice of his.

3. Definite Range Game:

Suppose that the function $K(\xi, \eta)$ has the definite range form:

$$K(\xi, \eta) = \begin{cases} 1 & |\xi - \eta| \leq W/2 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that the game is to be played over the unit interval. Then if $W \geq 1$, the optimum strategy for the defender is to take any location for which the definite range curve fills the unit interval. One such location is to occupy the center of the interval. For such relatively large values of w, any transit position is optimum for the transitor; for there is no transit path which will lower the probability of detection less than 1.

There are non-optimum positions for the barrier submarine--any position whose distance from the center is greater than W/2. For such positions a gap in coverage exists which can be exploited by transitor to decrease the minimum probability of detection from one to zero. Hence the value of this game is one, the minimax solution.

Now suppose that W satisfies

$$1/2 \leq W < 1.$$

There is no single location on which the barrier submarine can spend all its searching time, because a gap in coverage will always exist. Again this gap can be exploited by proper choice of the transitor location to reduce the maximum probability of its being detected. Thus the barrier submarine must select at least two points on which to spend part of its search effort.

The next distribution to investigate is one in which the barrier submarine spends one half of its time on each of two points selected in such a way that each point of the unit interval is covered part of the time. One pair of locations is shown in figure A-1. This pair is also an optimum for the barrier submarine.

To demonstrate this optimality, three variations are possible:

(1) Variation in position of the two locations

(2) Variation in number of locations

(3) Variation in time at each location.

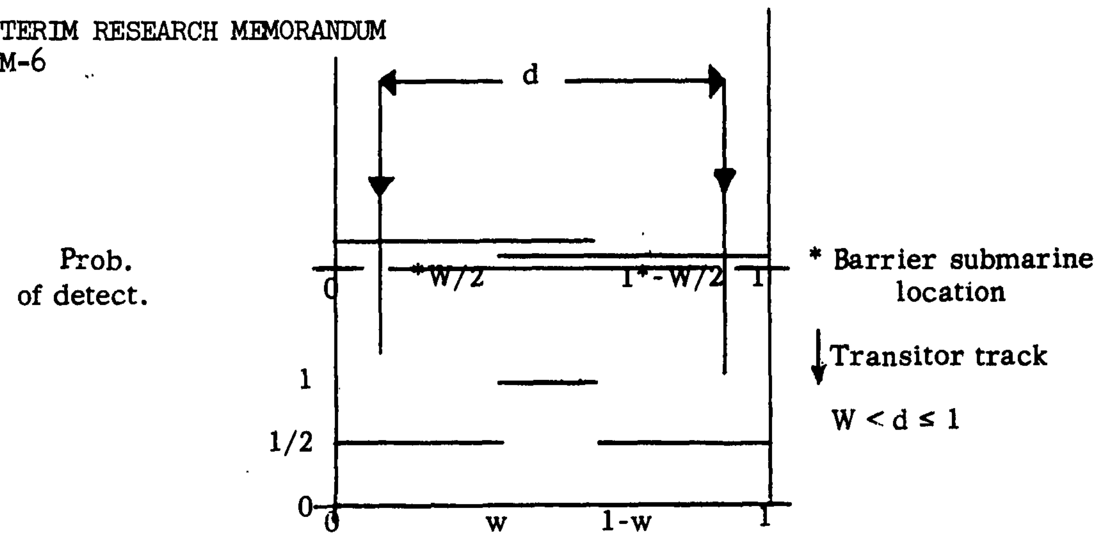


FIG. A-1: AN OPTIMUM LOCATION OF SEARCH EFFORT AND TRANSITS FOR THE DEFINITE RANGE GAME WHEN $1/2 \leq W < 1$

If the left position is moved to the right, or if the right position is moved to the left, a gap will be left at one boundary (or both). The gap could be exploited by the transitor to reduce the probability of being detected (to zero for these gaps).

Moving either or both of the locations outward towards the nearer boundary has a different effect. Note that the composite probability curve has a section in the center (where the overlap occurs) where the probability is one. As the points move closer to the boundaries, the length of the overlap decreases. Until the overlap decreases to zero, there is no decrease in minimum probability. Hence there is no loss in the minimax value of the game, and some shifting of the barrier submarine is possible. However, the positions shown in figure A-1 give the largest "bonus" as mentioned in the body. The area under the probability of detection curve is the largest, which implies that there is a greater probability of detecting (on the average) a transitor who does not use its minimax distribution, while exploitation of the barrier submarine weaknesses (other than those of the minimax distribution) is prevented. Note, that there is no implication here that the minimax distribution for the barrier submarine is its "best" distribution to counter a persistent non minimax selection by the barrier submarine. The positions shown in figure A-1 are chosen for the barrier submarine because (1) they form a set for which a minimax distribution can be found and (2) they permit the largest area under the probability curve--the biggest bonus.

The optimal distribution of time at each position is 1/2. If any other time-sharing is used, then the transitor can exploit the side with the smaller barrier submarine effort (time) to decrease the average probability of detection. For example, if the time-sharing is 1/3 and 2/3, then the transitor can transit the "weak" side everytime to reduce the average probability from 1/2 to 1/3.

Finally three or more barrier submarine locations can be used. Three locations are shown in figure A-2. Note that gaps occur if the end positions are moved inward; hence

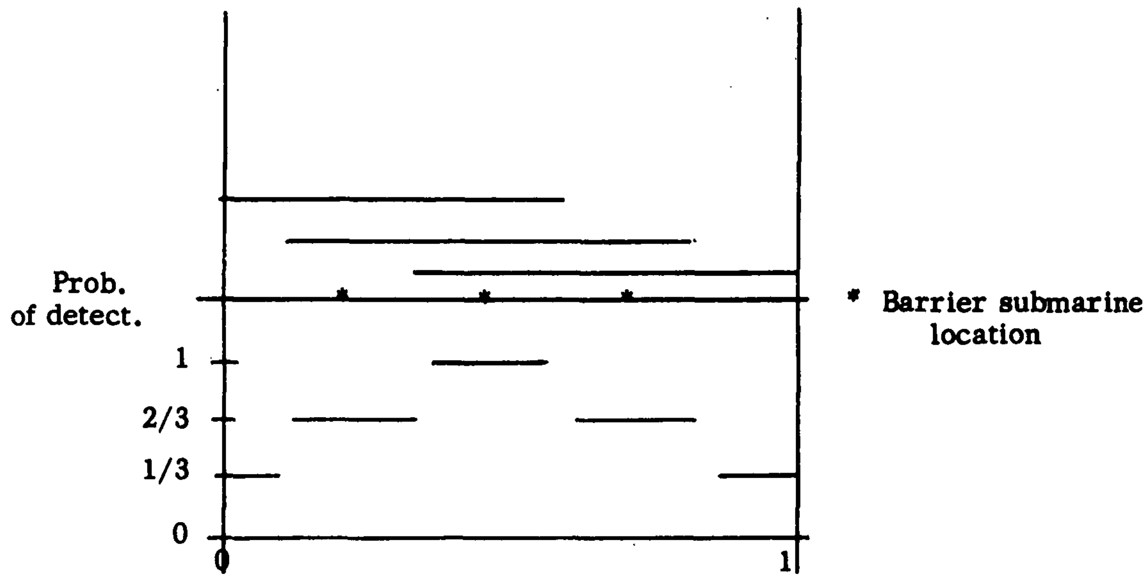


FIG. A-2: POSSIBLE SEARCH EFFORT FOR THREE LOCATIONS WHEN $1/2 \leq W < 1$

the outer locations are fixed. (Again no effort is assumed outside the segment shown.) The "middle" location can be moved around. If there is any transiting position which is defended less than half the time, this position can be exploited by the transitor to its advantage. The only way to prevent this exploitation is for the roving location to be placed on the other two positions for 1/2 of <u>its</u> time. But this is just the previous distribution. Similar arguments can be adduced for other numbers of locations and distributions of effort. Hence a distribution which maximizes the minimum probability of detection by the barrier submarine is that shown in figure 1.

The distribution of transitting effort can be varied as can the barrier submarine's effort. An optimum distribution for the transitor is equal time spent in two transitting lanes (normal to the barrier line , the only type considered) whose distances are at least W apart. If unequal times are spent, then by a corresponding shift of searching time, the barrier submarine can increase the average probability of detection. If the distance between the lanes is less than W, the barrier submarine can spend all its time on one properly selected location and again get an increase. Moreover, when the transits are greater than W apart, no gain accrues to the barrier submarine by decreasing the spacing of its effort (i.e., the transitor forces the barrier submarine to search in two locations).

If the transitor uses three or more locations, all pairs of tracks cannot be separated by W; and hence such a tactic is similar to the unequal time sharing tactic of the transitor, allowing the average probability of detection to be raised by a proper selection of barrier submarine effort.

When $1/3 \leq W < 1/2$, three positions each are required for the barrier submarine and the transitor. Any fewer locations leave gaps in the barrier to be exploited. Three locations are shown in figure A-3. Again these locations can be moved without leaving gaps or without decreasing the minimum probability of detection. However, no sliding will increase the area under the composite probability curve. Hence the points $w/2$, $1/2$, and $1-w/2$ are the three points which will be used for the minimax barrier submarine positions. No more than three points are possible. To prevent the decrease of the value of the game below $1/3$, $1/3$ of the time must be spent protecting each boundary. This requires $1/3$ of the time on $w/2$ or an equivalent point closer to the boundary and $1/3$ of the time on $1-w/2$ (or an equivalent closer to the boundary). Thus the remaining $1/3$ of the time must be spent at the center (or an equivalent) to protect the center interval not covered by other locations.

Three transit lanes are required. With the barrier submarine locations fixed at points above, the lanes can be drawn from the three intervals of minimum probability of detection, shown in figure A-3. Argument, similar to those before, show that each interval must be occupied $1/3$ of the time.

4. Trapezoidal Game: The trapezoidal game is one having an isosceles trapezoid for a separate lateral range probability of detection curve. Basically, such a curve depends on two parameters: (e.g., the width of the smaller base, w, and the angle at the larger base, D). To restrict the range all results were obtained for $D = \pi/3$ radians; but the construction can be executed for any D, $0 \leq D < \pi/2$. If $w \geq 1$, then one optimum barrier submarine position is $1/2$, and all transitors positions are equally advantageous for the transitor. As w becomes less than 1, then the edges of the barrier are the minimum probabilities of detection. Because of the symmetry two transit locations are required; the optimum position for the barrier submarine is in the center. The two optimum transits are each boundary. (NOTE: These positions neglect the edge effects from any adjoining submarines.) The average probability of detection is just the probability of detection at the edges of the barrier.

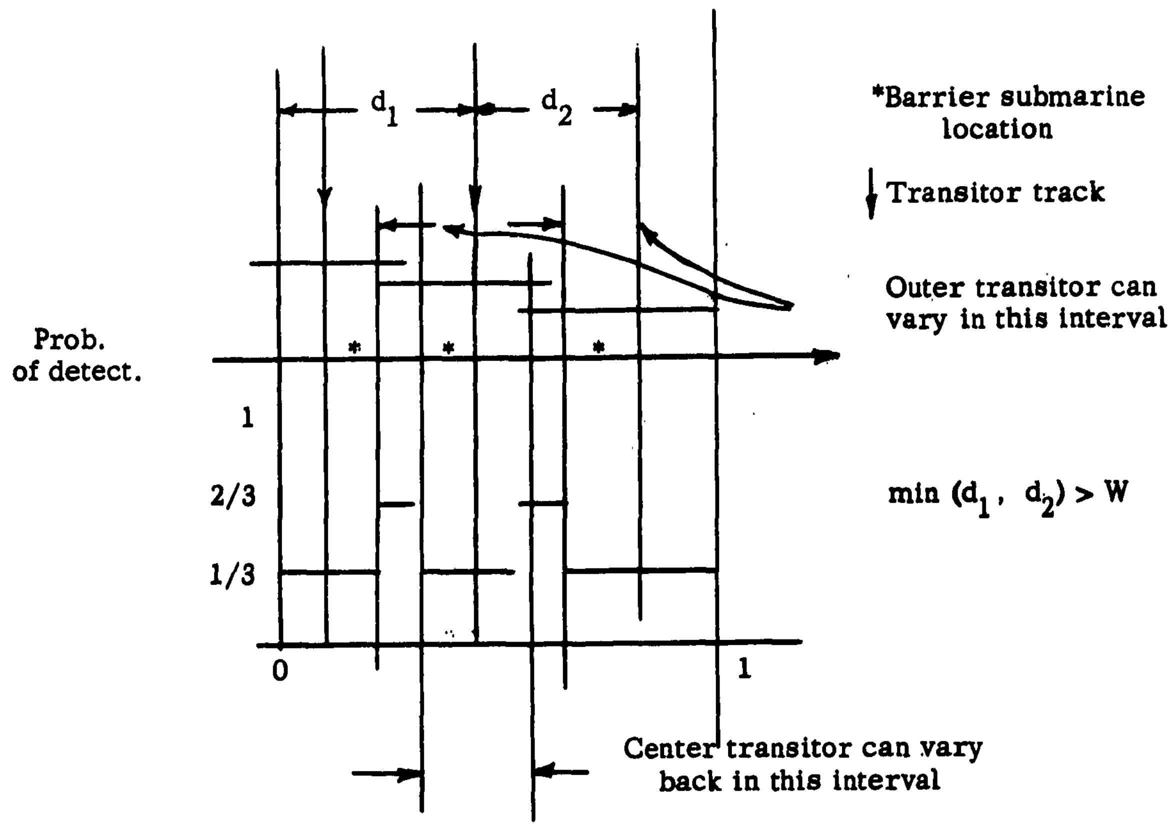
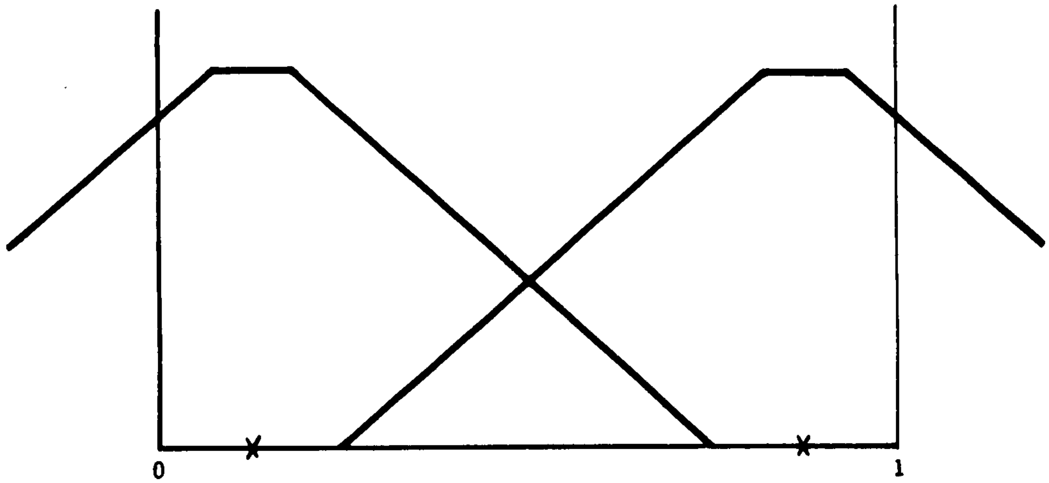


FIG. A-3: AN OPTIMUM STRATEGY FOR $1/3 \leq W < 1/2$

As w continues to decrease, the value of the game--determined by the height of the intersection of the sides of the trapezoid and the edge of the barrier--continues to decrease. When the value of the game for a single barrier submarine location decreases to the value which can be obtained by using two locations for the barrier submarine, a shift in optimum strategy to the use of the two locations is required.

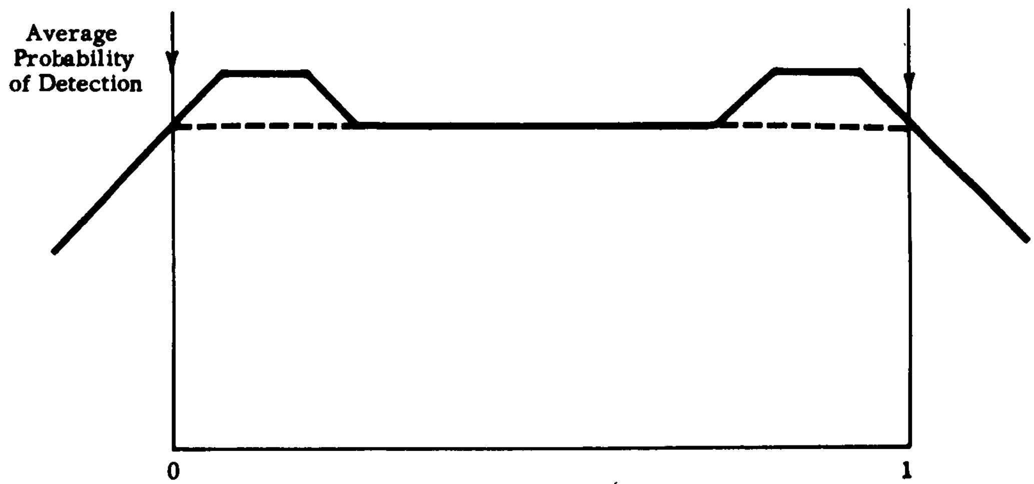
There are three locations for minimum values when the barrier submarine must occupy two locations: each side and the center. In general the location of the two barrier submarine positions is fixed to make these three minima equal. Figure A-4 shows the average probability of detection for two locations. The optimum positions for the barrier submarine are selected to make the obvious potential minima equal. It should be noted that each ordinate of the trapezoid must be reduced by a factor of N, changing the shape of the trapezoid. This is shown in figure A-5 for N = 2.



Position in Barrier Element

* Barrier Submarine location

OPTIMAL LOCATIONS OF BARRIER SUBMARINE WHEN TWO LOCATIONS ARE REQUIRED



Position in Barrier Element

FIG. A-4: AVERAGE PROBABILITY OF DETECTION WHEN TWO POSITIONS ARE TIME SHARED EQUALLY

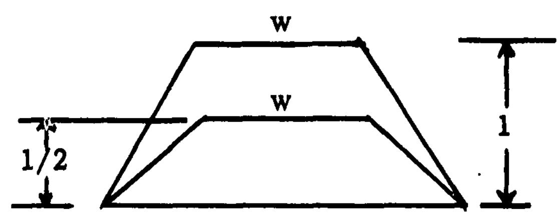


FIG. A-5: PROBABILITY TRAPEZOID FOR 50% TIME AT EACH LOCATION

A figure different from figure A-4 is possible with a trapezoidal probability curve. If the top of the trapezoid is broad enough, an increase in probability is possible in the center as shown in figure A-6. Here a triangular area in the center will permit some shifting the barrier submarine locations, with an attendant decrease in size of the triangle, but inducing no gaps. However, to keep the largest "bonus" the positions of the barrier submarine are assumed to be points and the intervals corresponding to them can be obtained for the transitor.

That these locations yield a minimax follows from arguments similar to those given previously. Again if equal time is not spent at each location, the transitor can exploit the position with the smallest effort. If more than the minimum points are used, then additional overlap is obtained with a lowering of probability at one or more points, which can then be exploited by the transitor. The transitor must exploit each weakness (valley in the average probability curve) equally. Otherwise the barrier submarine can change its tactic to lessen the probability in one or more values, which increases the minimum probability elsewhere.

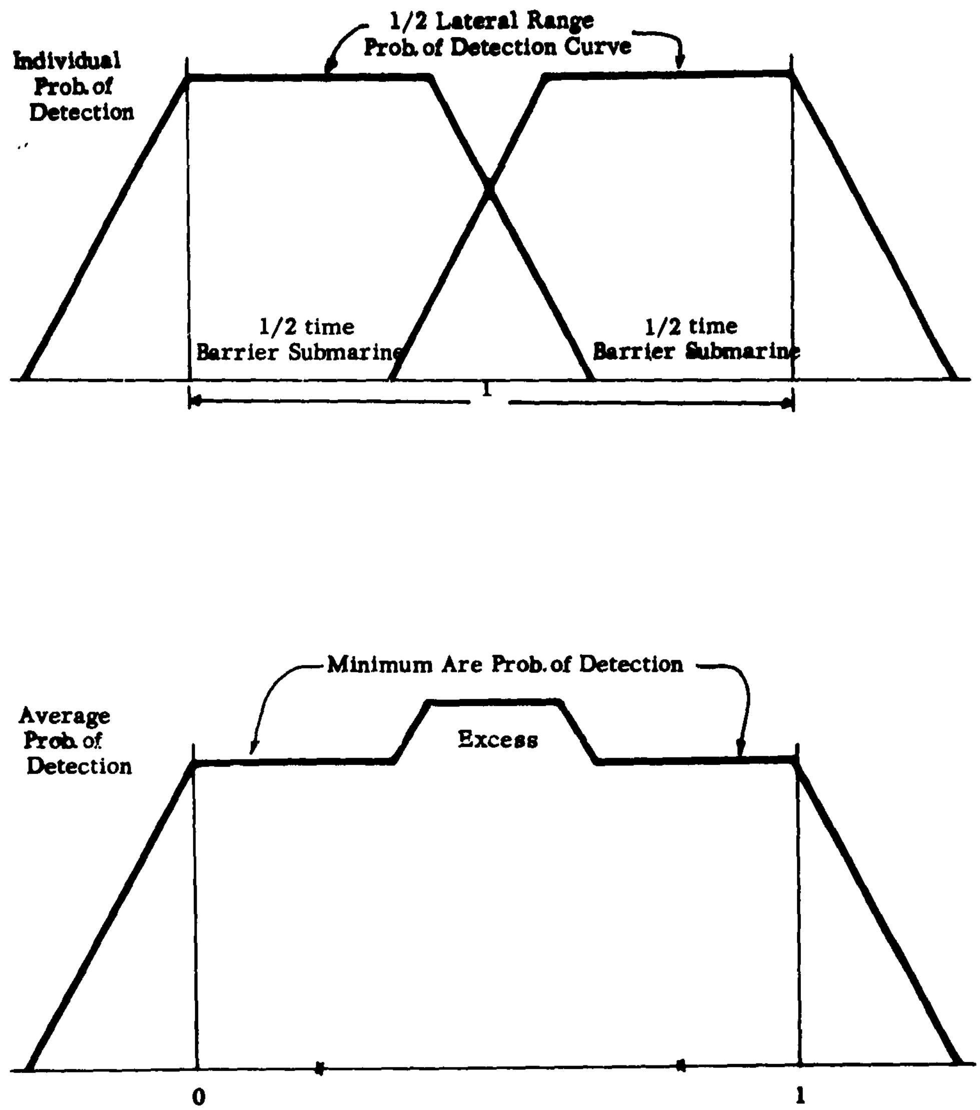


FIG. A-6: EXCESS AVERAGE PROBABILITY OF DETECTION FOR FIXED BARRIER SUBMARINE LOCATIONS

5. Other games: Games with payoff functions

$$\exp\left(-(\xi - \eta)^2/2\sigma^2\right) \text{ and } \frac{1}{1 + \lambda(\xi - \eta)^2}$$

which have more usually shaped lateral range probability of detection curves, can be approximately solved graphically by the above procedure. The first function fits the theory of bell shaped kernels in chapter 7 of reference (a). No solutions are shown, but several theorems are stated (and proved) which give some properties of the minimax locations. (No intervals are possible, and there are only a finite number of points for each adversary.) The second function is known to have similar solutions to the exponential payoff, but there is no known theory to provide the basis.