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NON-UNIFORM MAGNETOHYDRODYNAMIC SHOCK PROPAGATION, WITH SPECIAL REFERENCE TO CYLINDRICAL AND SPHERICAL SHOCK WAVES

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Summary

Previous work on non-uniform shock propagation in monatomic conducting gases is generalized to an arbitrary value of the adiabatic index. Specifically, the perturbation generated when an initially uniform hydromagnetic shock of arbitrary strength impinges on an area variation is determined, the problem being linearized on the basis of small area variations. When the shock encounters the area change, the shock strength is altered, and the subsequent flow is non-isentropic. There are two distinct contributions to the perturbation, namely, a permanent perturbation due to the area change and a transient reflected disturbance, and expressions for these are obtained.

A first order relation between area change and shock strength is obtained and integrated numerically to give an area-shock strength relationship valid for channels with finite continuous area variation. Particular area distributions are utilized to discuss converging cylindrical and spherical hydromagnetic shocks.

The present theory includes results on non-uniform gas dynamic shock propagation as a special case.

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1. Introduction

In recent years, there has been a great deal of work on shock propagation in channels with small non-uniform cross-sectional area distributions. By the use of a linearization based on small area variations, Chester [1] found that the pressure perturbation produced when an initially plane shock wave passed through a non-uniform transition section, which joined two channels of constant but unequal cross-sectional area, was given by:

$$-K(P_2 - P_1)[\Delta A]/A ,$$

where $P_2 - P_1$ was the initial pressure discontinuity across the shock, $[\Delta A]$ the net change in area and the parameter K a monotonically decreasing function of the shock strength with only a small total variation.

Chester started with the full three-dimensional equations of motion and then carried out an averaging process by considering only the average pressure. This final restriction means, however, that the same results must be obtainable by investigating the problem from the very beginning by a one-dimensional approach. This very important simplification, which permits of great extension, was first observed by Paul Germain, and Chester's results obtained through a one-dimensional analysis by Gundersen [6], extracts from which appeared in [7], the analysis being based on characteristic perturbations presented by Germain and Gundersen [5].

It is surprising that the simple one-dimensional approach led to an improvement, namely, the term $[\Delta A]/A$ in Chester's work was replaced by $\bar{A}/A$ where $\bar{A}$ was the perturbed area distribution. This showed clearly the first order differential relation between the area change and shock strength at any point in the non-uniform section. The significance of such a relation was first realized by Chisnell [2], who was led to it in a different way, essentially by assuming that Chester's steady state solution, valid for large time only, could be utilized. Chisnell used an integrated form (in closed-form) of the shock strength-area relationship to give an approximate description of the motion of the shock in terms of the area of the duct. By suitable choices of the area distribution, a description of converging cylindrical and spherical shocks was given, and the results checked by comparison with previous similarity solutions, valid in the neighborhood of the points of collapse of the shocks. The comparison showed the remarkable accuracy of Chisnell's work.

Extensions of Chisnell's work, including closed-form shock strength-area relations, for piston-driven shocks, where reflected waves from the piston come back to interact with the shock and modify its strength, were given for linear and quadratic area distributions, the appropriate ones for cylindrical and spherical shocks, in [8] and [9]. Those results are, of course, applicable to reflections from a contact layer.

Piston-driven shocks propagating through ducts with arbitrary, though small, area variations were considered by Mirels [15] by a source distribution method and in [10] by the small perturbation method, and the solution given in infinite series form.

The problem originally considered by Chester has been extended to the case where the fluid in front of the shock was not at rest in [11] and to the case that the area perturbation was also time-dependent in [12].

Recently, many others have worked on similar or related problems, e.g., in [16] and [17], Chester's results were rederived. Recently, it was shown [13] that in a monatomic conducting gas subjected to a transverse magnetic field, the perturbation produced when an initially uniform plane hydromagnetic shock of arbitrary strength encountered an area variation could be determined, the problem being linearized on the basis of small area variations, and the solution presented in a form which included the usual gas dynamic results, described in the foregoing, as a special case. In (14), a theory rather parallel to that of Chisnell [2] was developed and converging cylindrical and spherical hydromagnetic shocks discussed. The limitation of a monatomic gas was due to the fact that analogs of the usual Riemann invariants could then be determined explicitly. In the present paper, it is shown that this restriction is unnecessary in spite of the fact that analogs of the Riemann invariants cannot be determined explicitly, at least in terms of elementary functions.

The problem considered is that of an initially uniform plane hydromagnetic shock travelling with constant speed into an ideal gas at rest in a duct which has a section of non-uniform area. To ensure that all changes in the motion of the shock are due to the area variations, it is assumed that these are confined to the region $x > 0$, whereas to the left of the cross-section $x = 0$, the tube is of constant cross-sectional area, and the shock propagates with initially constant speed in this portion of the tube.

When the shock encounters the area variation, it is perturbed, the shock strength altered and the subsequent flow non-isentropic. There are two distinct contributions to the disturbance, viz., a permanent perturbation due directly to the area variation and a transient disturbance, due to reflections from the shock of the permanent disturbance, which propagates with velocity

$$\omega = [c^2 + b^2]^{\frac{1}{2}} ,$$

where c is the local speed of sound and b the Alfvèn speed, with respect to the flow behind the shock.

In the neighborhood of the shock, the ultimate effect is an altered shock strength and concomitant pressure change behind the shock. When the main flow behind the shock has speed $< \omega$, the transient reflected disturbance is convected to the left with speed ω relative to the fluid while for flow speed $> \omega$, this disturbance is convected to the right with speed ω relative to the main flow. This latter must be added to the steady flow solution. Expressions for these various contributions are obtained.

Specifically, it is shown that the pressure perturbation immediately behind the incident shock is given by:

$$\overline{P}_2 = - K_2(\sigma, m_1)(P_2 - P_1)\overline{A}_2/A_2$$

where perturbations are denoted by a bar, $P_2 - P_1$ is the original pressure discontinuity across the shock, A_2 the cross-sectional area, σ the shock strength defined as a density ratio and $m_1 = b_1/c_1$, a measure of the applied transverse field.

For all m_1, $\lim_{\sigma \to 1+} K_2 = 0.5$ and $\lim_{\sigma \to 6-} K_2 = 0.39414$ for $\gamma = 7/5$.

But these are precisely the limits for the corresponding parameter in the gas dynamic case which corresponds to $m_1 = 0$ so there is the result: <u>For very weak or very strong shocks [in an asymptotic sense], the results are independent of the applied field and agree with the usual gas dynamic results.</u>

In magnetohydrodynamics, strong shocks occur for σ close to $(\gamma+1)/(\gamma-1)$, where γ is the adiabatic index, or for a very strong applied field for any $\sigma > 1$, i.e., m_1 is then large.

For $m_1 = 0$, K_2 is a monotonically decreasing function of the shock strength and agrees exactly with the corresponding parameter in the aforementioned publications. For any $m_1 \neq 0$, there no longer is a monotonic variation with σ, but each curve is concave upward with curves for greater m_1 lying beneath those for lesser m_1 and all curves pass through the points $(\sigma, K_2) = (1, 0.5)$ and $(6, 0.39414)$ for $\gamma = 7/5$. Further, for fixed incident shock strength, K_2 is a monotonically decreasing function of m_1, i.e., in a diverging (converging) channel, the pressure decrement (increment) is decreased (decreased) by increasing the applied field.

If the area variations are confined to a transition section of finite extent joining two portions of constant cross-sectional area, the total effect of the passage through the transition section is an altered shock strength with the shock asymptotically becoming again uniform. Qualitatively, the motion of the shock is independent of whether the main flow behind the shock has speed $< \omega$ or $> \omega$.

Finally, the first order differential relation between shock strength and area variation is integrated numerically, and particular area distributions are used to consider converging cylindrical and spherical magnetohydrodynamic shocks which, in contrast to plane shocks, are inherently unstable and ultimately become strong. This section generalizes the results of [14] to arbitrary values of the adiabatic index γ, though numerical tables are presented only for $\gamma = 7/5$.

Specifically, it is shown that the strengths of converging cylindrical and spherical magnetohydrodynamic shocks near the centers of these fronts are proportional to D^{-K^*} and D^{-2K^*}, respectively, <u>independent of the applied field</u>, where D is the distance from the centers and $K^* = 0.39414$ for $\gamma = 7/5$.

The theory of gas dynamic shock propagation in non-uniform ducts is contained as a special case of the theory presented in this paper, i.e., it is obtained simply by setting $m_1 = 0$, so that there is a check on the theory presented herein.

2. General Theory

The quasi-one-dimensional non-steady motion of an ideal, inviscid, perfectly conducting compressible fluid subjected to a transverse magnetic field, i.e., the induction $\vec{B} = (0, 0, B)$, is governed by the system of equations:

$$2c_t/(\gamma - 1) + 2uc_x/(\gamma - 1) + cu_x + ucA_x/A = 0 \tag{2.1}$$

$$u_t + uu_x + 2cc_x/(\gamma - 1) + b^2 B_x/B - c^2 s_x/\gamma(\gamma - 1)c_v = 0 \tag{2.2}$$

$$B_t + uB_x + Bu_x = 0 \tag{2.3}$$

$$s_t + us_x = 0 \tag{2.4}$$

where u, c, s, ρ, $b^2 = B^2/\mu\rho$, μ, γ and A are, respectively, the particle velocity, local speed of sound, specific entropy, density, square of the Alfvèn speed, permeability, ratio of specific heat at constant pressure c_p and at constant volume c_v and cross-sectional area of the channel. Partial derivatives are denoted by subscripts, and all dependent variables are function of x and t alone save A which is considered time independent.

The characteristics of this system are:

$$\frac{dx}{dt} = u,\ u + \omega,\ u - \omega$$

where $\omega^2 = c^2 + b^2$.

For an arbitrary isentropic constant area flow, the base flow in the neighborhood of which perturbations will be considered, the basic system of equations (2.1), (2.2) and (2.3) may be written in the following characteristic form [3]:

$$x_\beta - (u + \omega)t_\beta = 0 \tag{2.5}$$

$$x_\alpha - (u - \omega)t_\alpha = 0 \tag{2.6}$$

$$x_\xi - ut_\xi = 0 \tag{2.7}$$

$$(\omega^2 - c^2)B_\beta/B + \omega u_\beta + 2cc_\beta/(\gamma - 1) = 0 \tag{2.8}$$

$$(\omega^2 - c^2)B_\alpha/B - \omega u_\alpha + 2cc_\alpha/(\gamma - 1) = 0 \tag{2.9}$$

$$B_\xi/B - 2c_\xi/(\gamma - 1)c = 0 \tag{2.10}$$

with characteristic parameters (α, β, ξ).

From equation (2.10), it is clear that the quantity $B/c^{2(\gamma - 1)}$ is constant along each particle path, which is a well-known consequence of the assumption of infinite electrical conductivity, i.e., the magnetic field is "frozen" into the fluid, and for a constant state or a simple wave flow, $B/c^{2(\gamma - 1)}$ is constant throughout the flow so that equations (2.8) and (2.9) may be written as:

$$u_\beta + 2\omega c_\beta/(\gamma - 1)c = 0 \tag{2.11}$$

$$-u_\alpha + 2\omega c_\alpha/(\gamma - 1)c = 0 \tag{2.12}$$

Since $\rho = r_2 c^{2/(\gamma-1)}$, $B = r_1 c^{2/(\gamma-1)}$, with constants r_1 and r_2,

$$b^2 = r_1^2 c^{2/(\gamma-1)}/\mu r_2 \equiv kc^{2/(\gamma-1)}$$

so that
$$\omega^2 = c^2[1 + kc^\lambda]$$

where $\lambda = 2(2-\gamma)/(\gamma-1)$.

For the case of a monatomic gas $\gamma = 5/3$, and equations (2.11) and (2.12) may be integrated explicitly to yield:

$$u/2 + [1+kc]^{3/2}/k = u/2 + (\omega/c)^3/k = \alpha \tag{2.13}$$

$$-u/2 + [1+kc]^{3/2}/k = -u/2 + (\omega/c)^3/k = \beta \tag{2.14}$$

where (α, β) may be considered as generalizations of the usual Riemann invariants.

For an arbitrary value of γ, the solutions of equations (10) and (11) may formally be written as:

$$u/2 + \int [1 + kc^\lambda]^{\frac{1}{2}}\, dc/(\gamma - 1) = \alpha \tag{2.15}$$

$$-u/2 + \int [1 + kc^\lambda]^{\frac{1}{2}}\, dc/(\gamma - 1) = \beta \tag{2.16}$$

The solution of the present problem for a monatomic gas was obtained by a transformation of dependent variables which made (2.13) and (2.14) linear relations. This same technique is employed for an arbitrary value of γ. Although the integrals appearing in equations (2.15) and (2.16) may be evaluated in terms of hypergeometric functions, an explicit evaluation is unnecessary. The introduction of the new dependent variable $w = \int [1+kc^{\lambda}]^{\frac{1}{2}} dc$ transforms equations (2.1) and (2.2) to:

$$w_t/(\gamma-1) + uw_x/(\gamma-1) + \omega u_x/2 + u\omega A_x/2A = 0 \tag{2.17}$$

$$u_t/2 + uu_x/2 + \omega w_x/(\gamma-1) = c^2 s_x/2\gamma(\gamma-1)c_v \tag{2.18}$$

where use has been made of the relation $B_x/B = 2c_x/(\gamma-1)c$ and (2.3) omitted. Adding and subtracting equations (2.17) and (2.18) gives the system to be utilized in the sequel:

$$w_t/(\gamma-1) + u_t/2 + (u+\omega)[u_x/2 + w_x/(\gamma-1)] = -u\omega A_x/2A + c^2 s_x/2\gamma(\gamma-1)c_v \tag{2.19}$$

$$w_t/(\gamma-1) - u_t/2 + (u-\omega)[-u_x/2 + w_x/(\gamma-1)] = -u\omega A_x/2A - c^2 s_x/2\gamma(\gamma-1)c_v \tag{2.20}$$

$$s_t + us_x = 0 \tag{2.21}$$

The close analogy of this system with the basic equations of gas dynamics should be noted, viz., a hyperbolic system of three non-linear first order partial differential equations with Riemann invariants which are linear relations between the dependent variables. The system is, in fact, the magnetohydrodynamic generalization of such and includes the gas dynamic equations as a special case. For $k = 0$, $\omega = c$, $w = c$ and the system (2.19) - (2.21) reduces exactly to that used by Gundersen [6], [7] to obtain the solution of the propagation of a shock in a tube of varying cross-section and perturbations of simple waves. The above system has also been utilized to obtain the perturbation of a magnetohydrodynamic simple wave, and the results will be presented in a forthcoming paper.

For the present problem, it is only necessary to perturb in the neighborhood of a constant state (isentropic), and a formal linearization of equations (2.19) - (2.21) in the neighborhood of this known state, denoted by the subscript zero, leads to the following system of linear equations for the terms of first order, denoted by the subscript one:

$$R_t + (u_0 + \omega_0) R_x = -u_0 \omega_0 A_{1x}/2A_0 + c_0^2 s_{1x}/2\gamma(\gamma-1)c_v \tag{2.22}$$

$$S_t + (u_0 - \omega_0) S_x = -u_0 \omega_0 A_{1x}/2A_0 - c_0^2 s_{1x}/2\gamma(\gamma-1)c_v \tag{2.23}$$

$$s_{1t} + u_0 s_{1x} = 0 \tag{2.24}$$

where $R = u_1/2 + w_1/(\gamma-1)$, $S = -u_1/2 + w_1/(\gamma-1)$.

According to (2.24), s_1 remains constant along the particle paths of the given flow, i.e., along $dx/dt = u_0$. Since $\rho_0(dx - u_0 dt)$ is the exact differential of a function ψ_0 which, when equated to a constant defines the particle paths, the solution of (2.24) may be written as:

$$s_1 = \Omega(\psi_0)$$

with Ω an arbitrary differentiable function.

It is convenient to define a new function $T_0(x, t)$ by

$$c_0^2 s_{1x} = c_0^2 \rho_0 \Omega'(\psi_0) = \gamma(\gamma-1) c_v T_0 \quad .$$

Then, the general solution to (2.22) - (2.24) may be written as:

$$R = F[x - (u_0 + \omega_0)t] + E[x - u_0 t]/\omega_0 - u_0 \omega_0 A_1 / 2A_0(u_0 + \omega_0) \tag{2.25}$$

$$S = G[x - (u_0 - \omega_0)t] + E[x - u_0 t]/\omega_0 - u_0 \omega_0 A_1 / 2A_0(u_0 - \omega_0) \tag{2.26}$$

$$T_0 = 2E'[x - u_0 t] \tag{2.27}$$

in terms of three arbitrary functions of one argument. From the general solution, the four distinct contributions to the perturbation may be noted, namely, one due to the entropy variations, which travels along the particle paths and is measured by E, a disturbance due directly to the area variations, a perturbation propagating with velocity ω_0, the true speed of sound, with respect to the fluid along the family of characteristics $x - (u_0 + \omega_0)t$ = constant and measured by F and a perturbation propagating with velocity ω_0 with respect to the fluid along the family of characteristics $x - (u_0 - \omega_0)t$ = constant and measured by G.

3. Solution in the Vicinity of the Incident Shock

Jump conditions across normal hydromagnetic shocks have been considered by several authors, e.g., Friedrichs [4]. Let U be the shock velocity, $v = U - u$ and the subscripts one and two designate flow quantities in the regions in front of and behind the shock. Then the analogs of the Rankine-Hugoniot relations are:

$$\rho_1 v_1 = \rho_2 v_2$$

$$B_1 v_1 = B_2 v_2$$

$$\rho_1 v_1^2 + P_1 + B_1^2/2\mu = \rho_2 v_2^2 + P_2 + B_2^2/2\mu$$

$$\gamma P_1/(\gamma-1)\rho_1 + v_1^2/2 + B_1^2/\rho_1\mu = \gamma P_2/(\gamma-1)\rho_2 + v_2^2/2 + B_2^2/\rho_2\mu$$

For gas-dynamical shocks, knowledge of the flow in front of and one parameter behind the shock suffices to give a complete solution. For magneto-hydrodynamic shocks, all parameters behind the shock may be expressed in terms of those in front and two parameters, e.g., the shock strength and one which gives a measure of the applied field.

Let $\sigma = \rho_2/\rho_1$, $\tau = P_2/P_1$, $m = b/c$, $n = u/c$, $M = v/c$ and $q^2 = \omega^2/c^2 = 1 + m^2$. Then the following relations hold, where n_1 is assumed zero and $\theta = (\gamma+1)/(\gamma-1)$:

$$v_1/v_2 = \rho_2/\rho_1 = B_2/B_1 = b_2^2/b_1^2 = \sigma$$

$$\tau = [\theta\sigma - 1 - \gamma m_1^2(1-\sigma)^3/2]/(\theta-\sigma)$$

$$c_2^2/c_1^2 = \tau/\sigma$$

$$m_2^2/m_1^2 = \sigma^2/\tau \tag{3.1}$$

$$M_1^2 = \{m_1^2[\gamma/(\gamma-1) + (2-\gamma)\sigma/(\gamma-1)] + 2/(\gamma-1)\}\sigma/(\theta-\sigma)$$

$$M_2^2 = M_1^2/\sigma\tau$$

$$n_2 = (1-\sigma^{-1})M_1(\sigma/\tau)^{\frac{1}{2}}$$

which express the flow parameters behind the shock in terms of σ and m_1.

The effect of an area variation on the motion of an initially uniform hydromagnetic shock propagating with constant speed into a fluid at rest will now be determined by the use of the general solution for the non-isentropic perturbation of a constant state as given by equations (2.25) - (2.27). The particular method of generation of the shock is left open save that what ever that may be, the mechanism is sufficiently far removed so that no reflections come back to interfere with the basic interaction considered herein. From the formulation of the problem, there is no mechanism downstream of the shock which could give rise to the term $F[x-(u_2-\omega_2)t]$ in (2.25) so that the pressure perturbation behind the shock may be obtained by setting $F = 0$. This gives:

$$\bar{u}_2/2 + \bar{w}_2/(\gamma-1) - c_2^2\bar{s}_2/2\omega_2\gamma(\gamma-1)c_v = -u_2\omega_2\bar{A}_2/2A_2(u_2+\omega_2) \qquad (3.2)$$

where perturbations of a base quantity are denoted by a bar. Since

$$\bar{w}_2 = \omega_2\bar{c}_2/c_2 \; ; \qquad -\bar{s}_2/2\gamma(\gamma-1)c_v = \bar{P}_2/2\gamma P_2 - \bar{c}_2/(\gamma-1)c_2 \; ,$$

the latter from the equation of state, equation (3.2) may be written as:

$$\bar{u}_2/2 + m_2^2\bar{c}_2/(\gamma-1)q_2 + c_2^2\bar{P}_2/2\omega_2\gamma P_2 = -u_2\omega_2\bar{A}_2/2A_2(u_2+\omega_2)$$

or

$$\bar{u}_2/c_2 + [\gamma(m_2^2+1)-1]\bar{P}_2/\gamma(\gamma-1)P_2q_2 - m_2^2\bar{\rho}_2/\rho_2q_2(\gamma-1) = -n_2q_2\bar{A}_2/(n_2+q_2)A_2 \qquad (3.3)$$

From the jump conditions, equations (3.1):

$$\frac{\bar{T}}{T} = \left\{ \frac{(\theta^2-1)\sigma + m_1^2\gamma\sigma[3\theta-1-6\theta\sigma+3(\theta+1)\sigma^2-2\sigma^3]/2}{(\theta-\sigma)[\theta\sigma-1-\gamma m_1^2(1-\sigma)^3/2]} \right\} \frac{\bar{\sigma}}{\sigma}$$

$$\frac{\bar{u}_2}{c_2} = n_2\left(\frac{\bar{\sigma}}{\sigma}\right)\left\{ \frac{2(\theta+1)\theta/\gamma + 2(\theta+1)(\theta-2)\sigma/\gamma + m_1^2[\theta(1+\theta)+(1+\theta)(\theta-2)\sigma+(\theta-3)(2\theta-1)\sigma^2+(3-\theta)\sigma^3]}{2(\theta-\sigma)(\sigma-1)\{2(\theta+1)/\gamma + m_1^2[(1+\theta)+\sigma(\theta-3)]\}} \right\}$$

so that equation (3.3) gives:

$$\bar{P}_2/P_2 = - K_1[\sigma, m_1]\bar{A}_2/A_2 \tag{3.4}$$

where

$$K_1 = \frac{n_2 q_2}{n_2+q_2}\left[\frac{(\theta^2-1)\sigma + m_1^2\gamma\sigma[3\theta-1-6\theta\sigma + 3(\theta+1)\sigma^2 - 2\sigma^3]/2}{\theta\sigma - 1 - \gamma m_1^2(1-\sigma)^3/2}\right]\left\{-\frac{m_2^2(\theta-\sigma)}{q_2(\gamma-1)} + \right.$$

$$n_2^{\bullet}\left[\frac{2(\theta+1)\theta/\gamma + 2(\theta+1)(\theta-2)\sigma/\gamma + m_1^2\{\theta(1+\theta)+(1+\theta)(\theta-2)\sigma + (\theta-3)(2\theta-1)\sigma^2+(3-\theta)\sigma^3\}}{2(\sigma-1)\{2(\theta+1)/\gamma + m_1^2[(1+\theta) + \sigma(\theta-3)]\}}\right] +$$

$$\left.\left[\frac{\gamma(m_2^2+1)-1}{\gamma(\gamma-1)q_2}\right]\left[\frac{(\theta^2-1)\sigma + m_1^2\gamma\sigma\{3\theta-1-6\theta\sigma + 3(\theta+1)\sigma^2-2\sigma^3\}/2}{\theta\sigma - 1 - \gamma m_1^2(1-\sigma)^3/2}\right]\right\}^{-1}$$

or

$$\bar{P}_2 = - K_2(\sigma, m_1)(P_2 - P_1)\bar{A}_2/A_2 \tag{3.5}$$

where

$$K_2 = \left[\frac{\gamma m_1^2(1-\sigma)^3/2 + 1 - \theta\sigma}{(\theta+1)(1-\sigma) + \gamma m_1^2(1-\sigma)^3/2}\right] K_1$$

Strong shocks can occur in hydromagnetics in two ways, namely, for σ close to $(\gamma+1)/(\gamma-1)$, e.g., 6 for $\gamma = 7/5$, or for a very strong applied field for any $\sigma > 1$, i.e., m_1 is then large.

For all m_1 and $\gamma = 7/5$, $\lim_{\sigma \to 1+} K_2 = 0.5$ and $\lim_{\sigma \to 6-} K_2 = 0.39414$; but these are precisely the limits for the corresponding parameter in the gas dynamic case which corresponds to $m_1 = 0$ so there is the result:

For very weak or very strong shocks [in an asymptotic sense], the results are independent of the applied field and agree with the usual gas dynamic results.

For $m_1 = 0$, K_2 is a monotonically decreasing function of the shock strength and agrees precisely with the corresponding parameter in the gas dynamic theory. For any $m_1 \neq 0$, the monotonicity is lost, but each curve is concave upward with curves for greater m_1 lying below those for lesser m_1, and all curves pass through the points $(\sigma, K_2) = (1, 0.5)$ and $(6, 0.39414)$ for $\gamma = 7/5$. Further, for fixed incident shock strength, K_2 is a monotonically decreasing function of m_1 so that in a diverging (converging) channel, the pressure decrement (increment) is decreased (decreased) by increasing the applied field. Qualitively, the motion of the shock is independent of whether the main flow speed behind the shock is $< \omega$ or $> \omega$.

Graphs of K_2 are presented in Figures 1, 2, 3 and 4 for $\gamma = 7/5$.

4. The Reflected Disturbance

To complete the solution, an expression for the reflected disturbance must be obtained and this is done through equation (2.26). The evaluation of the arbitrary function G is most readily accomplished by noting that the system (2.25) - (2.27) may be written as:

$$\bar{u}_2/c_2 + 2m_2^2\bar{c}_2/(\gamma-1)q_2c_2 + \bar{P}_2/\gamma P_2 q_2 = -n_2 q_2 \bar{A}_2/(n_2+q_2)A_2$$

$$-\bar{u}_2/c_2 + 2m_2^2\bar{c}_2/(\gamma-1)q_2c_2 + \bar{P}_2/\gamma P_2 q_2 = -n_2 q_2 \bar{A}_2/(n_2-q_2)A_2 + 2G[x-(u_2-\omega_2)t]/c_2$$

Thus on addition, which serves to eliminate $\bar{u}_2$:

$$\left[\frac{(\gamma-1)+\gamma m_2^2}{q_2 A_2(\gamma-1)} - \frac{m_2^2}{q_2(\gamma-1)}\left\{\frac{(\theta-\sigma)[\theta\sigma-1-\gamma m_1^2(1-\sigma)^3/2]}{(\theta^2-1)\sigma+m_1^2\gamma\sigma[3\theta-1-6\theta\sigma+3(\theta+1)\sigma^2-2\sigma^3]/2}\right\}\right]\frac{\bar{P}_2}{P_2}$$

$$+\frac{n_2^2 q_2\bar{A}_2}{(n_2^2-q_2^2)A_2} = G[x-(u_2-\omega_2)t]/c_2$$

Evaluating this on the shock, $x = Ut$, and replacing $\bar{P}_2/P_2$ by its value from (3.4) gives:

$$G[\lambda]/c_2 = K_3\bar{A}_2[\delta\lambda]/A_2$$

where

$$\lambda = x - (u_2 - \omega_2)t$$

$$\delta = (M_2 + n_2)/(M_2 + q_2)$$

$$K_3 = \frac{n_2^2 q_2}{n_2^2 - q_2^2} - \frac{K_1}{(\gamma-1)q_2}\left[\frac{(\gamma-1)+\gamma m_2^2}{\gamma} - \frac{m_2^2(\theta-\sigma)[\theta\sigma - 1 - \gamma m_1^2(1-\sigma)^3/2]}{(\theta^2-1)\sigma + m_1^2\gamma\sigma[3\theta - 1 - 6\theta\sigma + 3(\theta+1)\sigma^2 - 2\sigma^3]/2}\right]$$

Thus:

$$\overline{P}_2/(P_2 - P_1) = -\xi_1 \overline{A}_2[x]/A_2 - \xi_2 \overline{A}_2[\delta\lambda]/A_2 \qquad (4.1)$$

where:

$$\xi_1 = \frac{\dfrac{n_2^2 q_2}{n_2^2 - q_2^2}\left[\dfrac{\theta\sigma - 1 - \gamma m_1^2(1-\sigma)^3/2}{(\theta+1)(\sigma-1) + \gamma m_1^2(\sigma-1)^3/2}\right]}{\dfrac{\gamma - 1 + \gamma m_2^2}{q_2\gamma(\gamma-1)} - \dfrac{m_2^2}{(\gamma-1)q_2}\left\{\dfrac{(\theta-\sigma)[\theta\sigma - 1 - \gamma m_1^2(1-\sigma)^3/2]}{(\theta^2-1)\sigma + m_1^2\gamma\sigma[3\theta - 1 - 6\theta\sigma + 3(\theta+1)\sigma^2 - 2\sigma^3]/2}\right\}}$$

$$\xi_2 = K_2 - \xi_1$$

The parameter δ is a monotonic increasing function of σ (for all m_1) and varies between the limits

$$0.5 \leq \delta < 1.6429$$

for all m_1 and $\gamma = 7/5$. Graphs are presented in Figure 5. It might be noted that Chester's graph of the corresponding parameter is incorrect, and, apparently, the reciprocal of the true value has been plotted.

When the flow speed behind the shock is $< \omega_2$, a disturbance will be reflected to the left downstream of the shock, and, from (4.1), the pressure perturbation will ultimately be given by:

$$\bar{P}_2 = -\xi_2 \bar{A}_2 [\delta\lambda]/A_2$$

since $\bar{A}_2 = 0$ for $x < 0$.

For flow speed $> \omega_2$ behind the shock, the reflected disturbance will be convected to the right and the pressure perturbation is given by (4.1). Graphs of the parameter ξ_2 are presented in Figure 6 for $\gamma = 7/5$.

5. The Shock Strength Area Relation

Equation (3.5) is a first order differential relation between area and shock strength, namely,

$$dA/A + d\tau/(\tau-1)K_2(\sigma, m_1) = 0 \tag{5.1}$$

where

$$\tau = [\theta\sigma - 1 - \gamma m_1^2 (1-\sigma)^3/2]/(\theta - \sigma) ,$$

and the solution may be written as:

$$A = \nu \exp\left[- \int \frac{d\tau}{(\tau-1)K_2(\sigma, m_1)} \right] \tag{5.2}$$

or

$$A = \nu F(\sigma, m_1) \tag{5.3}$$

where

$$\log_e F(\sigma, m_1) = - \int \left[\frac{\theta^2 - 1 + m_1^2 \gamma[3\theta - 1 - 6\theta\sigma + 3(1+\theta)\sigma^2 - 2\sigma^3]/2}{(\theta-\sigma)(\sigma-1)[\theta + 1 + \gamma m_1^2 (\sigma-1)^2/2] K_2(\sigma, m_1)} \right] d\sigma$$

and ν is a constant of integration. The integral is given in terms of σ since K_2 is known in terms of σ.

Equation (5.2) contains the results of Chisnell as a special case, viz., $m_1 = 0$ and may be considered as the magnetohydrodynamic generalization of such. Chisnell showed that an integrated form of the shock strength-area relation could be utilized to discuss channels with finite continuous area variations and obtained a check on his theory by choosing particular area distributions and showed that converging cylindrical and spherical shocks could be discussed with remarkable accuracy. Equation (5.2) also includes as a special case previous work on monatomic conducting fluids presented in [14].

Some comments can be made immediately about converging cylindrical and spherical hydromagnetic shocks which can be discussed as particular cases by choosing linear and quadratic area distributions. The cylindrical shock corresponds to a wedge-shaped channel with cross-sectional area proportional to the distance from the center of the shock front while the spherical shock corresponds to a conical channel with area proportional to the square of the distance from the center. Although plane shocks are stable, converging cylindrical and spherical shocks are unstable and as the shock converges, its strength increases and ultimately becomes singular at the center. In the neighborhood of the singular point, (5.2) shows that A is proportional to τ^{-1/K^*}, where K^* is the asymptotic limit of K_2, so that the shock strengths of converging cylindrical and spherical hydromagnetic shocks near the center are proportional, respectively, to D^{-K^*} and D^{-2K^*}, where D is the distance of the shock from its axis or point of symmetry. But $K^* = 0.394141$, independent of the applied field, and this is exactly the value Chisnell obtained for gas dynamic shocks! Thus the result:

Near the axis or point of symmetry, the strengths of converging cylindrical and spherical magnetohydrodynamic shocks are independent of the applied field and are given by the usual gas dynamic theory.

6. Evaluation of $F(\sigma, m_1)$

Since K_2 is given in terms of σ, the integrated form of the shock strength-area relation in (5.3) is that chosen for numerical evaluation. F defines a one parameter family of functions with m_1 as parameter and tables of this function for various values of m_1 are presented in the Appendix. The tables refer to the evaluation of

$$- \log_e F(\sigma, m_1) = \int_{1.01}^{\sigma} \left[\frac{\theta^2 - 1 + m_1^2 \gamma[3\theta - 1 - 6\theta x + 3(1+\theta)x^2 - 2x^3]/2}{(\theta - x)(x-1)[\theta + 1 + \gamma m_1^2 (x-1)^2/2] K_2(x, m_1)} \right] dx$$

The tables may be used in several ways, e.g., suppose there is a section of variable area connecting two channels of constant but unequal area, say A_1 and A_2, then for a given m_1,

$$\frac{F(\sigma_2, m_1)}{F(\sigma_1, m_1)} = \frac{A_2}{A_1} \tag{6.1}$$

If A_1, A_2 and σ_1 are known, σ_2 may be obtained by interpolating inversely from the tables.

After passage through the transition section, the shock travels with this altered strength, σ_2, and ultimately becomes uniform. Conversely, given σ_1, σ_2, m_1 and A_1, the tables may be utilized to determine what A_2 would give the desired σ_2. Equation (6.1) could also be considered as one for m_1 with σ_1, σ_2, A_1 and A_2 prescribed.

Converging cylindrical and spherical shocks may be treated by choosing channels with areas proportional to D and D^2, respectively, where D is the distance from the center.

The author wishes to thank Mr. J. Al-Abdulla for carrying out the computations employed in this paper.

APPENDIX

The tables refer to the evaluation of $Z(\sigma, m_1) \equiv$

$$- \log_e F(\sigma, m_1) = \int_{1.01}^{\sigma} \left[\frac{(\theta^2-1) + m_1^2 \gamma [3\theta - 1 - 6\theta x + 3(1+\theta)x^2 - 2x^3]/2}{(\theta-x)(x-1)[\theta+1+\gamma m_1^2 (x-1)^2/2]\, K_2(\sigma, m_1)} \right] dx$$

The numbers in parentheses refer to the power of ten by which each entry is to be multiplied.

σ	$Z(\sigma, 0)$	$Z(\sigma, 1)$
1.2	6.65000 (0)	6.66275 (0)
1.4	1.36500 (1)	1.37038 (1)
1.6	2.06500 (1)	2.07765 (1)
1.8	2.76500 (1)	2.78834 (1)
2.0	3.46500 (1)	3.50263 (1)
2.2	4.16500 (1)	4.22054 (1)
2.4	4.86500 (1)	4.94203 (1)
2.6	5.56500 (1)	5.66694 (1)
2.8	6.26500 (1)	6.39504 (1)
3.0	6.96500 (1)	7.12610 (1)
3.2	7.66500 (1)	7.85988 (1)
3.4	8.36500 (1)	8.59614 (1)
3.6	9.06500 (1)	9.33473 (1)
3.8	9.76500 (1)	1.00755 (2)
4.0	1.04650 (2)	1.08185 (2)
4.2	1.11650 (2)	1.15638 (2)
4.4	1.18650 (2)	1.23116 (2)
4.6	1.25650 (2)	1.30623 (2)
4.8	1.32650 (2)	1.38167 (2)
5.0	1.39650 (2)	1.45761 (2)
5.2	1.46500 (2)	1.53426 (2)
5.4	1.53650 (2)	1.61206 (2)
5.6	1.60650 (2)	1.69194 (2)
5.8	1.67650 (2)	1.77700 (2)

σ	$Z(\sigma, 2)$	$Z(\sigma, 3)$
1.2	6.70180 (0)	6.76868 (0)
1.4	1.38736 (1)	1.41736 (1)
1.6	2.11842 (1)	2.18986 (1)
1.8	2.86366 (1)	2.98974 (1)
2.0	3.62160 (1)	3.80758 (1)
2.2	4.38964 (1)	4.63410 (1)
2.4	5.16488 (1)	5.46244 (1)
2.6	5.94468 (1)	6.28849 (1)
2.8	6.72696 (1)	7.11011 (1)
3.0	7.51020 (1)	7.92647 (1)
3.2	8.29343 (1)	8.73751 (1)
3.4	9.07609 (1)	9.54358 (1)
3.6	9.85792 (1)	1.03452 (2)
3.8	1.06390 (2)	1.11432 (2)
4.0	1.14194 (2)	1.19383 (2)
4.2	1.22000 (2)	1.27314 (2)
4.4	1.29803 (2)	1.35234 (2)
4.6	1.37623 (2)	1.43155 (2)
4.8	1.45467 (2)	1.51093 (2)
5.0	1.53355 (2)	1.59070 (2)
5.2	1.61318 (2)	1.67117 (2)
5.4	1.69410 (2)	1.75300 (2)
5.6	1.77755 (2)	1.83738 (2)
5.8	1.86748 (2)	1.92857 (2)

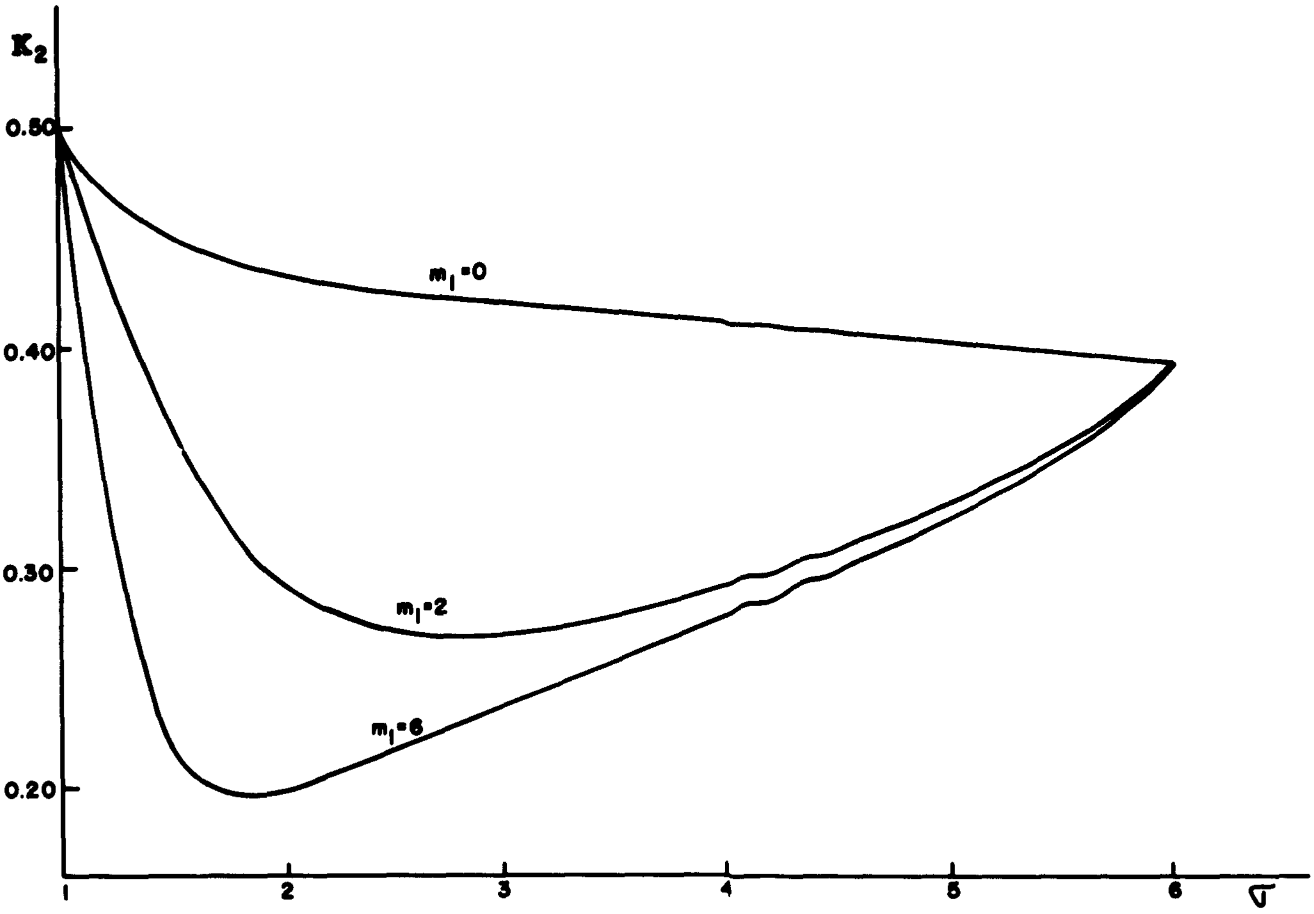

Figure 1

A graph of the parameter K_2 vs. σ for $m_1 = 0$, the ordinary gas dynamic case, and for $m_1 = 2$ and 6.

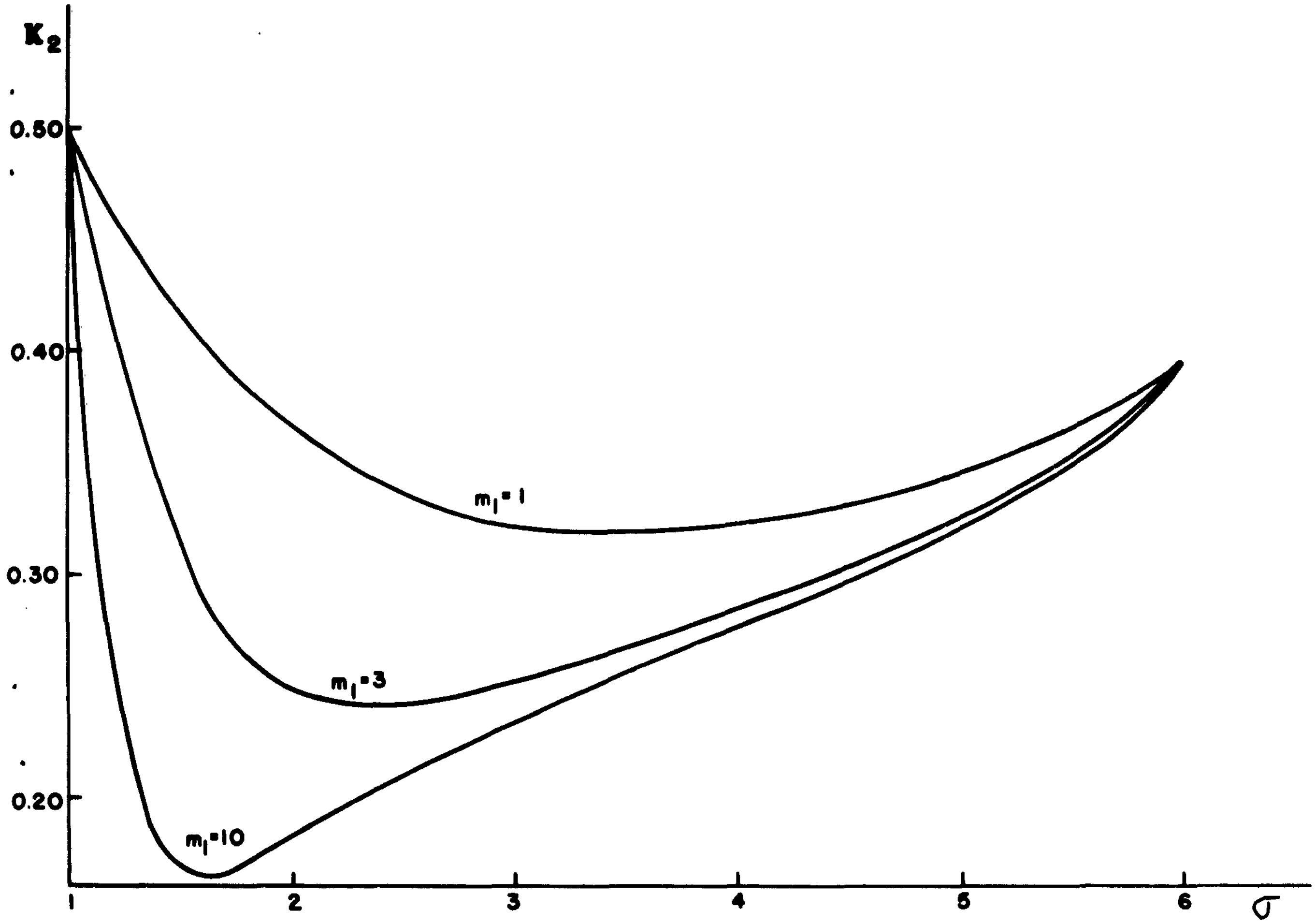

Figure 2

A graph of the parameter K_2 vs. σ for $m_1 = 1$, 3 and 10.

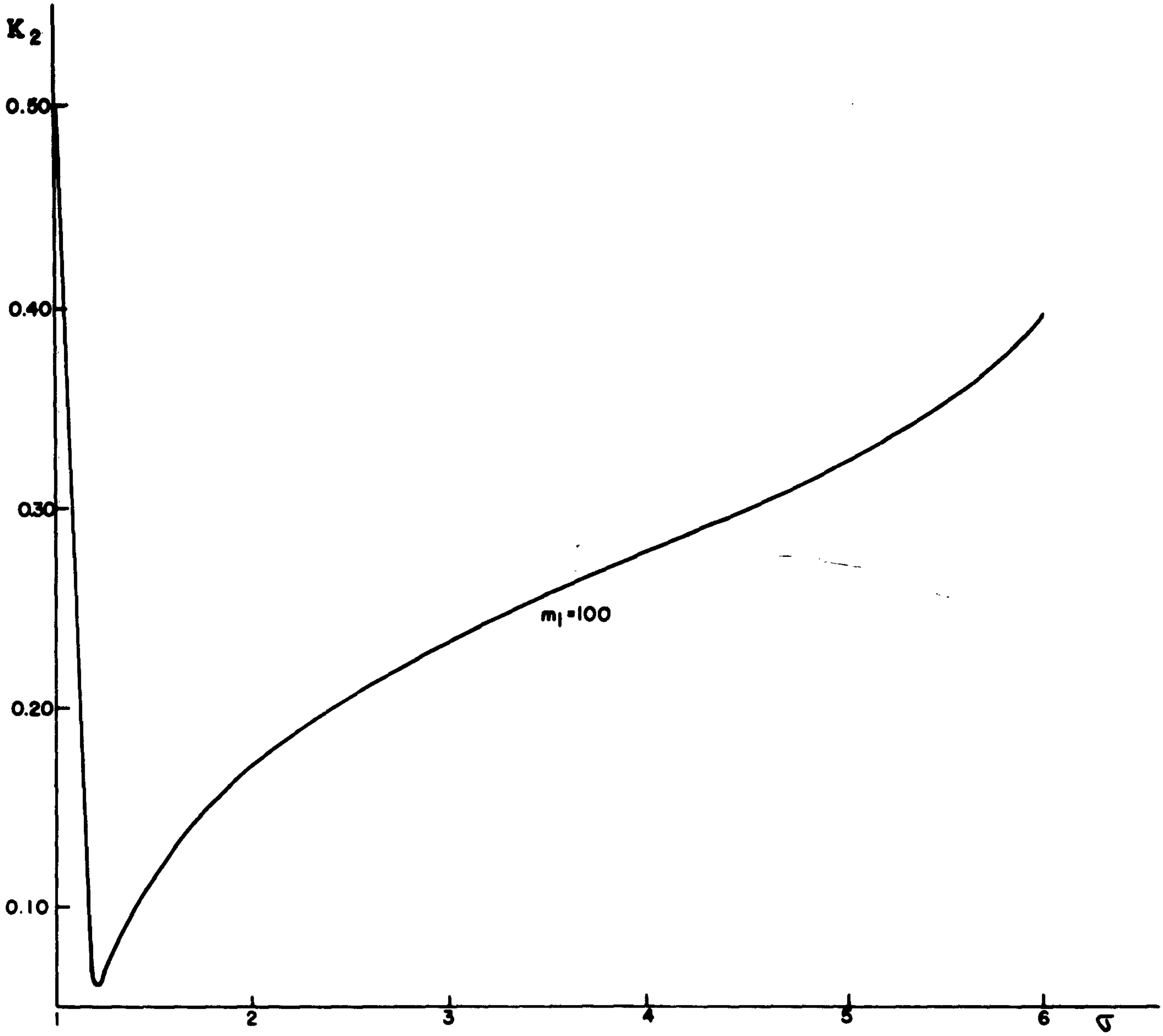

Figure 3

A graph of the parameter K_2 vs. σ for $m_1 = 100$.

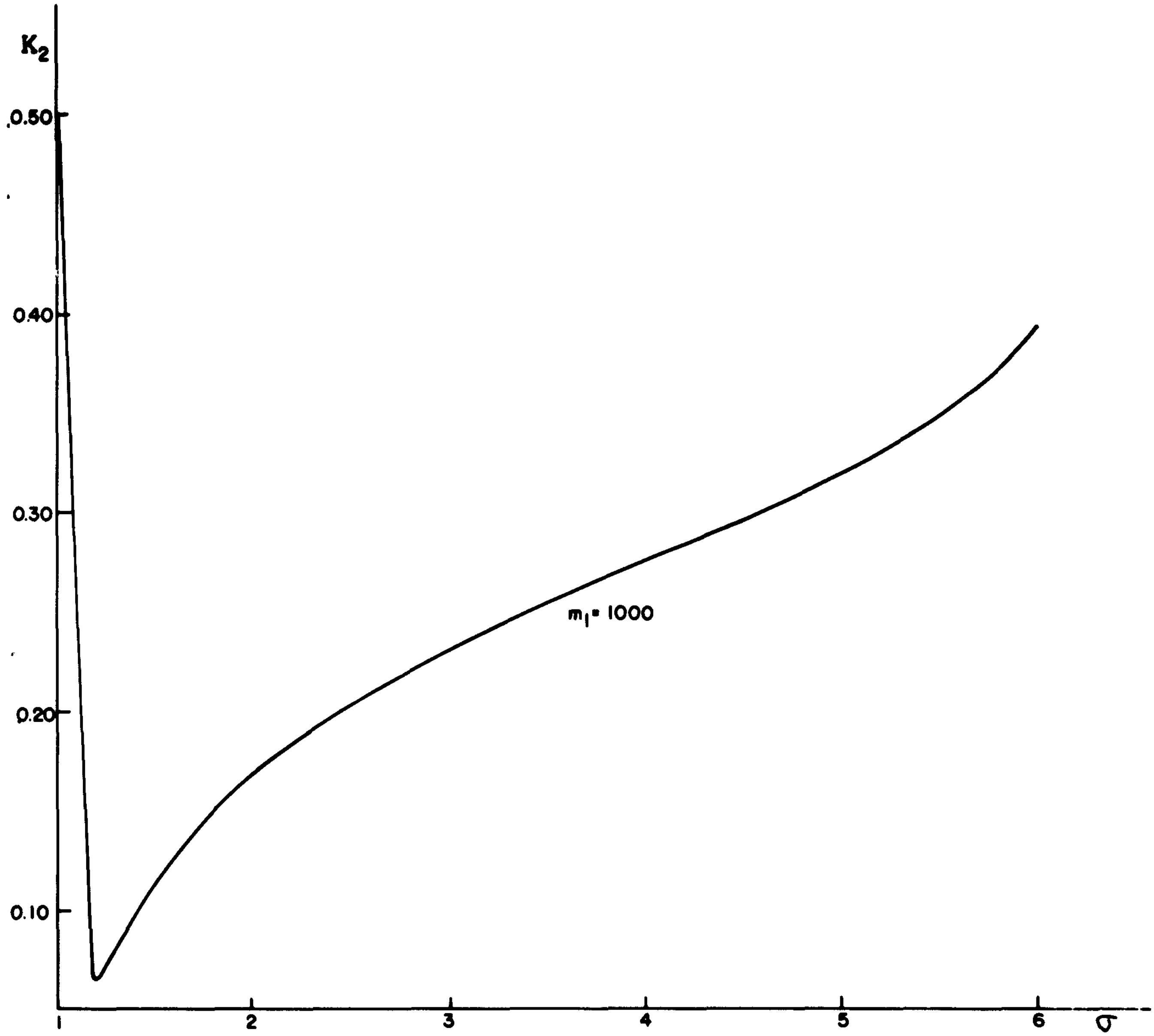

Figure 4

A graph of the parameter K_2 vs. σ for $m_1 = 1000$.

Near the minimum point of the graph, this curve crosses that for $m_1 = 100$, though it lies beneath the latter for all other σ. This indicates a possible reversal of effects for very large applied fields, i.e., an increase in m_1 may lead to an increase in K_2. More extensive numerical investigations are thus indicated.

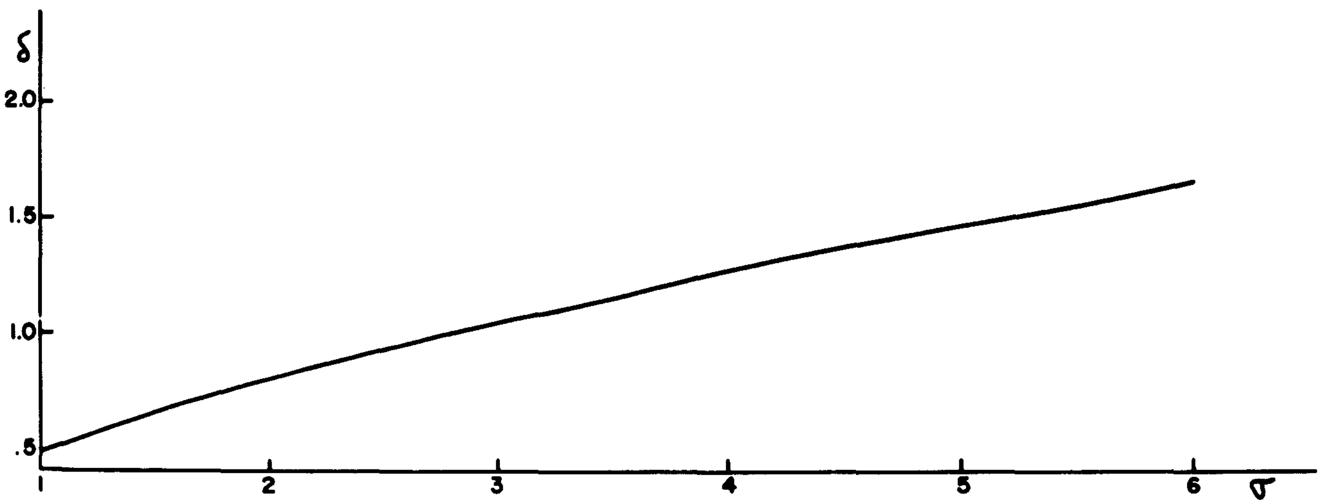

Figure 5

A graph of the parameter δ vs. σ.

Since there is a small total variation with m_1, only the case $m_1 = 0$ is given.

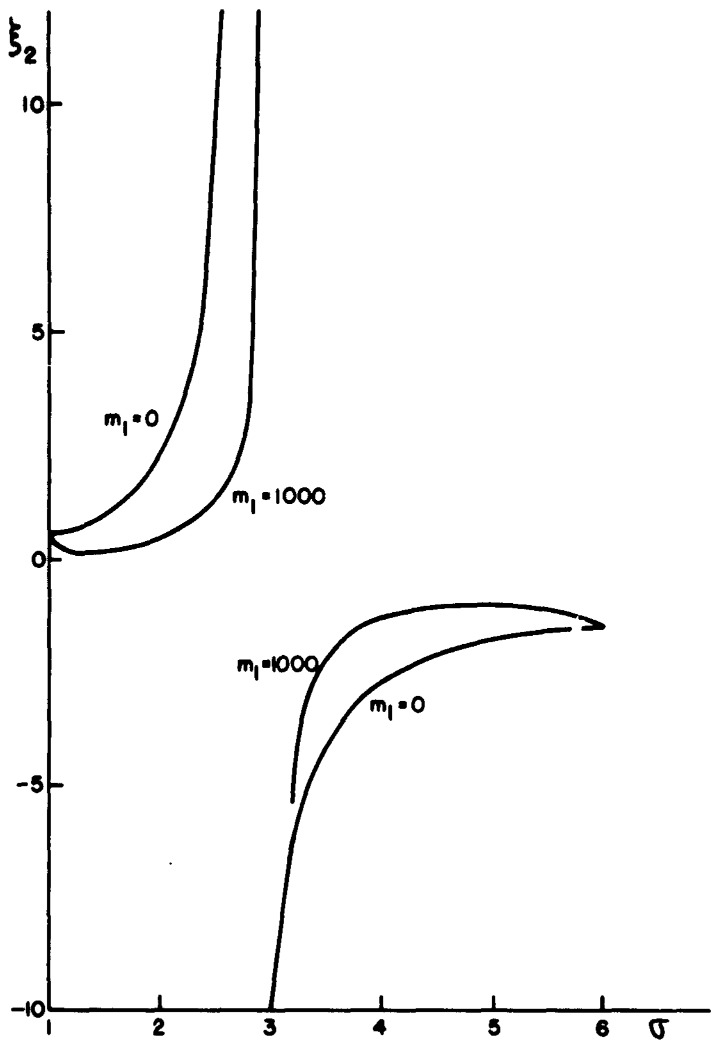

Figure 6

A graph of the parameter ξ_2 vs. σ.

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