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SOME METHODS FOR ESTABLISHING INTERPLANETARY TRANSFER ORBITS

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PREFACE

This Memorandum is intended to be a fundamental aid in studies of orbital mechanics. The relatively simple, straightforward computational procedures described herein can be used to establish approximate heliocentric transfer orbits and their elements. On the basis of these approximate orbits, exact interplanetary orbits can be calculated.

These computational procedures will be of use to persons engaged in space-surveillance activities and in interplanetary-mission planning.
This Memorandum presents and discusses some methods for establishing heliocentric interplanetary transfer orbits. The four basic methods and their variations can be used to establish orbits having specified transfer angles, transfer times, hyperbolic excess velocities, or heliocentric departure velocities. Each method consists of a step-by-step computation procedure which utilizes the equations of two-body motion and appropriate trigonometric relations to establish the desired transfer orbit.

Each of the methods for establishing a desired transfer orbit requires an iterative process. Thus, the methods are best applied by using a large-scale digital computer. In this way numerous orbits can be established and the orbit which is optimum for some requirement can be selected. None of the methods permits a direct analytical determination of an optimum orbit.
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LIST OF SYMBOLS

\( a = \) semimajor axis

\( a_e, e_e = \) semimajor axis and eccentricity of Earth's orbit

\( a_s, e_s = \) semimajor axis and eccentricity of the transfer orbit

\( E = \) eccentric anomaly

\( \mathbf{E}_n, \mathbf{S}_n, \mathbf{P}_n = \) \( n \)-th positions of the Earth, vehicle, and destination planet

\( e = \) eccentricity

\( i_p = \) inclination of destination planet orbit plane to the ecliptic

\( i_s = \) inclinations of the transfer orbit to the ecliptic plane

\( K = \) arbitrary constant

\( \mu = \) gravitational parameter

\( \eta_s = \) mean angular rate of the vehicle in orbit

\( P_p, P_e = \) semilatus rectum of destination planet orbit and Earth orbit

\( r = \) distance from center of force to body

\( r_{ed} = \) distance from the Sun to the Earth at departure time

\( r_{pa} = \) distance from the destination to the Sun at arrival time

\( r_{pp}, r_{ep} = \) perihelion distances of the orbits of the destination planet and Earth

\( t_a = \) arrival time

\( t_d = \) departure time

\( t_s = \) transfer time
$V_{sd}$ = heliocentric departure velocity of the vehicle
$V_{e}$ = hyperbolic excess velocity
$v$ = true anomaly

$v_s - v_{sd}$ = vehicle transfer angle

$v_{sd}, v_s$ = true anomaly of the vehicle at departure and arrival

$\alpha$ = angle between the velocity vectors of Earth and the vehicle at departure time

$\gamma_{ed}$ = angle between the heliocentric velocity vector of Earth and normal to the Earth-Sun line, measured in the ecliptic plane

$\gamma_{sd}$ = angle between the vehicle's velocity vector and normal to the vehicle-Sun line, measured in the transfer-orbit plane

$\omega_p$ = argument of perigee of the destination planet

$\eta$ = mean angular rate

$\Lambda_{cd}$ = heliocentric longitude of the Earth at departure time

$\Lambda_{pa}, \lambda_{pa}$ = heliocentric longitude and latitude of the destination planet at arrival time

$\Lambda_{sd}$ = heliocentric longitude of the vehicle at departure time
I. INTRODUCTION

Interplanetary transfer orbits may be established subject to one or more of several possible constraints. For example, one may want to find all orbits which have the same characteristic departure velocity, or a fixed time of transfer may be the most important consideration.\(^{(1,2)}\) There are several other considerations such as arrival date, true anomaly of the departure point, etc.

This memorandum presents and discusses some computation procedures for determining the elements of elliptical interplanetary transfer orbits subject to some of the above considerations. Specifically, several different approaches are presented and the computation procedure for each is given. For all the computation procedures the transfer orbit is assumed to extend between two massless points which coincide with the center of the Earth and the center of the destination planet.

The three-dimensional transfer orbits obtained using massless departure and destination planets are more realistic than those obtained using a simplified two-dimensional model of the solar system. The former are accurate enough to give a good approximation to the true values of the transfer angle, transfer time, required velocities, etc.

Four different computation procedures, with additional variants of each, to establish the heliocentric transfer orbital elements are presented and discussed. The four different computation procedures may be employed to establish transfer orbits having, respectively, fixed transfer angles, fixed transfer times, fixed characteristic departure velocities, and, finally, fixed initial heliocentric velocities. In each case only the two-body problem is
considered with the Sun as the central body. All of the methods presented require an ephemeris of planets for the determination of planetary positions. However, these positions could be computed accurately enough using the mean orbital elements of the planetary orbits.

The above requirements are only a few of the more common ones that could be considered when establishing a transfer orbit. None of the suggested computational procedures permits a direct analytical determination of a transfer which is optimum for any one of the requirements. They are intended for use in preparing a program for a large-scale digital computer which would compute numerous orbits. Then, these orbits could be studied in order to select the optimum one.
II. GEOMETRY

Because the planets rotate about the sun in mutually inclined elliptical orbits and at different angular rates, their relative orientations continuously change. The rate of change of orientation of any two planets depends on the relative size, shape, and mutual inclination of the orbits. In general, for direct transfer to an outer planet (larger semimajor axis than the departure planet) the transfer orbit will start near perihelion for transfer angles less than 360°. At the time of departure the destination planet will, in general, lead the departure planet in its motion around the Sun. For direct transfer to an inner planet the destination planet will lag and the transfer orbit will begin near aphelion. For a given transfer angle it turns out that the departure date can vary over a relatively short period of time without incurring severe penalties. In other words, the amount of the variation in the departure date depends on the allowable variation in the magnitude and direction of the velocity vector of the vehicle at both the departure and arrival dates.

In the simplified problem which treats circular, coplanar orbits, it is possible to compute heliocentric transfer orbits using the closed analytical expressions of the two-body problem. For some purposes, the orbits established in this way are satisfactory since they give a fairly good estimate of the departure and arrival conditions. However, because the planetary orbits are eccentric and inclined the orbital data obtained using the simplified model can differ greatly from that obtained using the more realistic three-dimensional model of the solar system. For example, in the two-dimensional problem the required cutoff velocity will decrease as the transfer angle approaches 180°.
In the more realistic three-dimensional model the cutoff velocity will increase rapidly as the transfer angle approaches 180° unless the arrival planet approaches the ecliptic plane at the arrival date (this will be a rare event).
III. EQUATIONS OF MOTION

All of the methods presented and discussed here utilize approximately the same set of equations which describe simple two-body motion. Only the sequence of use and emphasis of the various equations vary.

The equations listed below are in a general form. The sequence in which equations should be used and the symbols used are indicated in each method. Some of the basic equations are given here in order to keep the equations and comments in the step-by-step computation procedures to a minimum. Thus, the methods are easier to understand and easier to compare. Equations based on two-body motion which may be used to decrease the number of iterations are also given. In some of the methods, however, the use of these equations may require more computation time than is saved by reducing the number of iterations.

The pertinent equations of motion and definition of symbols are as follows:

The distance from the central body to the vehicle is

\[ r = \frac{a(1-e^2)}{1 + e \cos (v - v_0)} \] (1)

where

- \( a \) = semimajor axis
- \( e \) = eccentricity
- \( v_0 \) = initial true anomaly
- \( v \) = instantaneous true anomaly

The time required to traverse a segment of an elliptical orbit is,
according to Kepler's equation

\[ t - t_0 = \frac{1}{n} \left[ E - E_0 - e (\sin E - \sin E_0) \right] \]  \hspace{1cm} (2)

where \( E \) is the eccentric anomaly and \( n \) is the mean angular rate in the orbit. The quantities \( E \) and \( n \) are obtained from

\[ E = \sin^{-1} \left( \frac{\sqrt{1 - e^2} \sin \nu}{1 + e \cos \nu} \right) \]  \hspace{1cm} (3)

and

\[ n = \frac{k^1}{a^{3/2}} \]  \hspace{1cm} (4)

The transfer time may be computed using Lambert's Theorem. There are four possible elliptical paths, namely, direct, aphelion, perihelion, and indirect. (1) The four equations which are used for computing the transfer time are

\[ t(\text{direct}) = \frac{1}{n} \left[ \eta - \eta_1 - (\sin \eta - \sin \eta_1) \right] \]  \hspace{1cm} (5a)

\[ t(\text{aphelion}) = \frac{2\pi}{n} - \frac{1}{n} \left[ \eta + \eta_1 - (\sin \eta + \sin \eta_1) \right] \]  \hspace{1cm} (5b)

\[ t(\text{perihelion}) = \frac{1}{n} \left[ \eta + \eta_1 - (\sin \eta + \sin \eta_1) \right] \]  \hspace{1cm} (5c)

\[ t(\text{indirect}) = \frac{2\pi}{n} - \frac{1}{n} \left[ \eta - \eta_1 - (\sin \eta - \sin \eta_1) \right] \]  \hspace{1cm} (5d)

where, in general

\[ \sin \eta/2 = \frac{1}{2} \left( \frac{r_1 + r_2 + c}{a} \right)^{1/2} \]

\[ \sin \eta_1/2 = \frac{1}{2} \left( \frac{r_1 + r_2 - c}{a} \right)^{1/2} \]
and

\[ c = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos (v - v_o)} \]  \hspace{1cm} (6)

For transfer angles less than or equal to 180° either Eq. (5a) or (5b) applies. For angles greater than or equal to 180° Eq. (5c) or (5d) applies.

The transfer angle can be expressed in terms of the heliocentric latitude at the arrival date and the difference in the heliocentric longitudes of departure and arrival. The equation for transfer angle is

\[ v - v_o = \cos^{-1} \left[ \cos \lambda \cos (\Lambda - \Lambda_o) \right] \]  \hspace{1cm} (7)

where \( \lambda \) and \( \Lambda \) are the instantaneous heliocentric latitude and longitude of the vehicle in the heliocentric transfer orbit.
IV. OUTLINE OF METHODS OF COMPUTATION

For the planning of manned or unmanned space missions it will probably be necessary to establish the transfer orbit subject to certain constraints such as transfer time, departure date, arrival date, guidance and control, available thrust energy, etc. The constraints will depend in part on whether the mission is manned or unmanned and on the mission requirement. For example, a manned mission may require that the transfer time be limited and specified fairly accurately in advance of departure. Therefore, it will be necessary to establish a transfer orbit which will permit transfer in the specified time. Also, if the mission requires that the vehicle pass near the destination planet it may be desirable to either minimize the distance from vehicle to Earth or to minimize the velocity of the vehicle relative to the destination planet. Again, this can be done if the transfer orbit is established subject to the proper constraints.

The computation procedures are presented in four groups according to the various constraints. Within each group there are variations which are indicated by a quantity called the parameter. The groups are outlined as follows:

A. Constraint - fixed transfer angle, \((v_a - v_{sd})\), where \(v_{sd}\) and \(v_a\) are the true anomalies of the vehicle at departure and arrival

1. Parameter - \(t_a\)
2. Parameter - \(t_d\)
3. Parameter - \(v_{sd}\)
B. Constraint - fixed transfer time, $t_s$
   1. Parameter - $t_d$
   2. Parameter - $t_a$
   3. Parameter - $v_{sd}$

C. Constraint - fixed heliocentric departure velocity, $V_{sd}$
   1. Parameter - $t_d$
   2. Parameter - $t_a$

D. Constraint - fixed hyperbolic excess velocity, $V_o$
   1. Parameter - $t_d$
   2. Parameter - $t_a$
V. PRESENTATION AND DISCUSSION OF METHODS

A.1 FIXED \((v_s - v_{sd})\), PARAMETER \(t_a\)

1. Choose \(t_a\) and obtain \(\Lambda_{pa}, \lambda_{pa}\), and \(r_{pa}\) from an ephemeris or by computation. \(\Lambda_{pa}\) and \(\lambda_{pa}\) are the heliocentric longitude and latitude of the destination planet and \(r_{pa}\) is the distance between the Sun and destination planet at the arrival time, \(t_a\).

2. Compute \(\Lambda_{sd} = \Lambda_{pa} - \cos^{-1}\left[\frac{\cos (v_s - v_{sd})/\cos \lambda_{pa}}{P_e}\right]\)

3. Assume \(\Lambda_{ed} = \Lambda_{sd}\) and compute \(r_{ed}\) from

\[
    r_{ed} = \frac{P_e}{1 + e_e \cos (\Lambda_{ed} - \Lambda_{ep})}
\]

where \(P_e\) and \(e_e\) are the semilatus rectum and the eccentricity of the Earth's orbit. The longitude of Earth's perihelion is \(\Lambda_{ep}\).

4. Assume that \(r_{sd} = r_{ed}\) and compute the minimum value of \(a_s\), the semimajor axis of the transfer ellipse, from

\[
    a_s(\text{min}) = \frac{r_{sd} + r_{pa} + c}{4}
\]

where

\[
    c = \sqrt{r_{sd}^2 + r_{pa}^2 - 2r_{sd}r_{pa} \cos (v_s - v_{sd})}
\]

5. Compute \(t_s\) from Eq. (5a) if \((v_s - v_{sd}) \leq 180^\circ\) or from Eq. (5c) if \((v_s - v_{sd}) \geq 180^\circ\).

6. Compute \(t_d = t_a - t_s\) (step 5) and obtain \(\Lambda_{ed}\) and \(r_{ed}\).
7. Compute $\Delta = \Lambda_{sd} - \Lambda_{ed}$ (step 6).

If $\Delta > 0$, decrease $t_s$. Use Eq. (5a) or (5c) and $a_s = a_s (\text{min}) + \Delta a_s$ to compute a new value of $t_s$. (Note: For $a_s (\text{min})$, the derivative $da_s/dt_s = 0$. A $\Delta a_s = 0,1 a_s (\text{min})$ will yield a value for $da_s/dt_s$ which represents approximately the change of $t_s$ with $a_s$.)

If $\Delta < 0$, increase $t_s$. Use Eq. (5b) or (5d) and $a_s = a_s (\text{min}) + \Delta a_s$ to compute a new $t_s$.

8. Compute $t_d = t_a - t_s$ (step 7) and obtain $\Lambda_{ed}$ and $r_{ed}$.

9. Compute $\Delta = \Lambda_{sd} - \Lambda_{ed}$ (step 8).

If $|\Delta| > K$ (the size of $K$ depends on the accuracy required), change $a_s$ (step 7) by an amount $\Delta a_s$, compute $t_s'$ and return to step 8. The approximate change in $a_s$ may be obtained numerically or analytically from

$$\Delta a_s = \Delta t_s / X$$

where

$$X = \sqrt{\frac{a_s}{u_s}} \left[ \frac{3}{2} t_s \eta_1 \left( 1 - \cos \eta \right) \tan \eta/2 \right] \left( 1 - \cos \eta_1 \right) \tan \eta_1/2$$

(8)

The multiple signs in the brackets depend on the type of transfer path as follows:

- Direct = - , +
- Perihelion = - , -
- Aphelion = + , +
- Indirect = + , -

The equation for $X$ is derived in the Appendix.

If $|\Delta| \leq K$, the transfer orbit is established.

In A.1 the arrival point is determined by $t_a$ and the heliocentric longitude
of the departure point is then determined by using the coordinates of the
destination planet at \( t_a \) and the transfer angle \((v_s - v_{sd})\).

By assuming that \( \Lambda_{ed} = \Lambda_{sd} \) and computing \( r_{ed} \), the departure point, \( S_d \),
of the transfer orbit is fixed when we assume that \( r_{sd} = r_{ed} \).

The time of transfer, \( t_s \), is computed for different values of \( a_s \) until
the longitude of the Earth at the computed time of departure, \( \Lambda_{ed} \), is ap-
proximately equal to \( \Lambda_{sd} \).

Figure 1 illustrates the geometry of the motion for a typical case as
the iterations are performed.

![Diagram of the geometry of method A.1](image)

**Fig. 1 — Geometry of method A.1**
By starting at the time of arrival, $t_a^*$, and moving backwards in time, the motion of the vehicle and the Earth is as indicated in Fig. 1. As an example, assume that at $t_a$ the planet is at $P_a$ and the Earth is at $E_a$. During the $t_s^*$, which is computed using the initial value of $a_s^*$, i.e., $a_s^* (\text{min})$, the Earth moves from $E_a$ to $E_1$ and the vehicle moves from $P_a$ to $S_d^*$. By changing $a_s^*$ so as to decrease $t_s^*$ the second position of the Earth may be $E_2$ when the vehicle is at $S_d^*$. Additional changes in $a_s^*$ will cause the $n$th position of the Earth, $E_n^*$, to coincide with $S_d^*$. When this is accomplished the transfer orbit is established.

A.2 FIXED ($v_s - v_{sd}$), PARAMETER $t_d$

1. Choose $t_d$ and obtain $\lambda_{ed}$ and $r_{ed}$, the heliocentric longitude and distance of the Earth relative to the Sun at departure.

2. Compute $\lambda_{pa}$, the heliocentric longitude of the destination planet at arrival, from

$$\lambda_{pa} = \Omega_p + \tan^{-1} (\cos \Theta_p \tan u_{pa})$$

where $\Omega_p$ and $\Theta_p$ are, respectively, the heliocentric longitude of the ascending node between the destination planet's orbit plane and the ecliptic and the inclination of the destination planet's orbit plane to the ecliptic.

The angle $u_{pa} = v_{pa} + \omega_p$ is the angle between the radius to the planet at arrival and the line of nodes, i.e., the argument of latitude, and it is obtained from

$$u_{pa} = \sin^{-1} \left[ \frac{\cos (v_s - v_{sd}) \cos \beta - \cot \Theta_p \cos (\Omega_p - \lambda_{sd}) \sin \beta}{-\sin (\Omega_p - \lambda_{sd}) \cos \Theta_p \cos \beta + \cot (v_s - v_{sd}) \cos (\Omega_p - \lambda_{sd})} \right]$$
where

\[ \beta = \sin^{-1} \left[ \frac{\sin I_p \sin (\Omega_p - \Lambda_{sd})}{\sin (\nu_s - \nu_{sd})} \right] \]

and

\[ \Lambda_{sd} = \Lambda_{ed} \]

The equation for \( v_{pa} \) is obtained using standard spherical trigonometric formulae and therefore is not derived here.

3. Compute

\[ r_{pa} = \frac{P_p}{1 + e_p \cos (u_{pa} - \omega_p)} \]

where \( P_p, e_p, \) and \( \omega_p \) are known quantities and are the semilatus rectum, the eccentricity, and the argument of perihelion of the destination planet's orbit.

4. Compute the minimum value of \( a_s \) from

\[ a_s (\text{min}) = \frac{r_{sd} + r_{pa} + c}{k} \]

where \( r_{sd} = r_{ed} \) and \( c \) is obtained as in A.1, step 4.

5. Compute \( t_s \) as in A.1, step 5.

6. Compute \( t_a = t_d + t_s \) and obtain \( \Lambda_{pa}, \lambda_{pa}, \) and \( r_{pa}. \)

7. Compute

\[ (v_s - v_{sd}) = \cos^{-1} \left[ \cos \lambda_{pa} \cos (\Lambda_{pa} - \Omega_{sd}) \right] \]

8. Compute \( \Delta v_s = (v_s - v_{sd})(\text{step 7}) - (v_s - v_{sd})(\text{fixed}). \)

If \( \Delta v_s > 0, \) decrease \( t_s; \) if \( \Delta v_s < 0, \) increase \( t_s. \)
The new value for $t_s$ is computed as in A.1, step 7.

9. Compute $t_a = t_d + t_s$ (step 8) and obtain $\lambda_{pa}$, $\lambda_{pa}$, and $\rho_{pa}$.

10. Compute $(v_s - v_{sd})$ as in step 7.

11. Compute $\Delta v = (v_s - v_{sd})(\text{step 10}) - (v_s - v_{sd})(\text{fixed})$.

If $|\Delta v_s| > K$, change $a_s$ (step 7), compute $t_s$, and return to step 8.

The amount that $a_s$ should be changed, $\Delta a_s$, in order to make $|\Delta v_s| \leq K$ can be determined numerically or analytically by solving for $\Delta a_s$ as follows and returning to step 8.

$$\Delta a_s = \Delta v/X$$

where

$$X = \frac{d(v_s - v_{sd})}{da_s}$$

The expression for $X$, which is derived in the Appendix, is as follows:

$$X = \frac{a_s^2 \rho_{pa}^2}{2} \left[ \frac{3}{2} t_s n_s - (1 - \cos \eta) \tan \eta/2 + (1 - \cos \eta_1) \tan \eta_1/2 \right]$$

where the multiple signs in the brackets depend on the type of transfer orbit as follows:

- Direct = -, +
- Perihelion = -, -
- Apohelion = +, +
- Indirect = +, -

In this method the departure point of the vehicle coincides with the Earth, and thus the position of the vehicle at departure is determined by choosing a $t_d$.

The first three steps are used to determine the vehicle arrival point on the orbit of the destination planet, $S_a$. In general, the point will
represent a past or future position of the destination planet depending on the time required for the vehicle to traverse the specified transfer angle.

The remaining steps are used to compute different values of \((v_s - v_{sd})_{c}\), using values of \(t_a\) which depend on the transfer orbit, until the computed \((v_s - v_{sd})\) matches the fixed \((v_s - v_{sd})_{f}\) -- in other words, until the position of the destination planet coincides with the position of the vehicle at arrival time.

Figure 2 shows the geometry of the motion for a typical configuration of the Earth, vehicle, and destination planet. At the departure date assume that the Earth and vehicle are at position \(E_d\), \(S_d\) and the destination planet is at \(P_d\). A point on the destination planet’s orbit, \(S_a\), is found which makes the angle between the radius to the point, \(r_{pa}\), and the radius to the Earth and vehicle equal to the specified \((v_s - v_{sd})_{f}\). Next, an \(a_s\)
is chosen, in this case the minimum $a_s$, and the $t_s$ is computed for transfer path 1. During the $t_s$ the planet moves from position $P_d$ to position $P_{al}$. From Fig. 2 it is clear that the $(v_s - v_{sd})_c$ between a radius to $P_{al}$ and $S_d$ does not equal $(v_s - v_{sd})_f$. Thus, another $t_s$ is computed using a modified $a_s$, which results in transfer path number 2. When the vehicle reaches $S_a$ using this path the planet may be at $P_{a2}$ and the difference between the specified and computed $(v_s - v_{sd})$ has decreased. The process continues until the nth position of the planet, $P_{an}$, and $S_a$ are approximately coincident, i.e., when the specified and computed $(v_s - v_{sd})$ are approximately equal.

A.3 FIXED $(v_s - v_{sd})$, PARAMETER $v_{sd}$

1. Choose $v_{sd}$.

2. Choose $t_a$ and obtain $\Lambda_{pa}$, $\lambda_{pa}$, and $r_{pa}$.

3. Compute $\Lambda_{sd} = \Lambda_{pa} - \cos^{-1}\left[\cos(v_s - v_{sd})/\cos \lambda_{pa}\right]$, where $\Lambda_{sd}$ is the heliocentric longitude of the velocity at departure.

4. Assume that $\Lambda_{ed} = \Lambda_{sd}$ and compute $r_{ed}$ from Eq. (1).

5. Compute $a_s$ and $e_s$, the eccentricity of the transfer ellipse, using the equations for $r_{pa}$ and $r_{ed}$, i.e., Eq. (1).

6. Compute $t_s$ using Eq. (2).

7. Compute $t_d = t_a - t_s$ and obtain $\Lambda_{ed}$ and $r_{ed}$.

8. Compute $\Delta\Lambda = \Lambda_{sd} - \Lambda_{ed}$.

If $|\Delta\Lambda| > K$, change $t_a$ by an amount $\Delta t_a$ where
\[ \Delta t_a = \frac{\Delta (\Delta \lambda)}{\sqrt{\frac{p_s^2}{r_{ed}^2} - \frac{p_d^2}{r_{sd}^2}}} \] (10)

and return to step 2.

If \( \Delta \lambda \leq K \), the transfer orbit is established.

In the above procedure, the time of arrival, \( t_a \), is varied until the departure longitudes of the vehicle and the Earth are approximately equal.

By assuming that \( \lambda_{ed} = \lambda_{sd} \) and that \( r_{sd} = r_{ed} \), the position of the vehicle at each departure date, which is the start of the heliocentric transfer orbit, will be on the Earth's orbit. Since \( (v_s - v_{sd}) \) is constant, a change in the \( t_a \) will change not only the arrival position but also the departure position of the vehicle. Since the angular rate of the Earth around the Sun differs from that of vehicle departure position, a change in \( t_a \) will result in a change in the difference of the longitudes of the Earth and vehicle at departure.

Figure 3 shows how the positions of the Earth and vehicle at departure approach coincidence as the number of iterations increases. The angle between the radii from the Sun to \( E_n \) and \( P_{an} \) is a fixed value, namely \( (v_s - v_{sd}) \). The letters \( E, S, \) and \( P \) denote positions of the Earth, vehicle, and the destination planet, respectively.

Initially, the required position of the destination planet at arrival is \( P_{al} \). The vehicle and Earth are found to be at positions \( S_{dl} \) and \( E_{dl} \) at departure.

The second departure position of the vehicle, \( S_{d2} \), is computed using the second arrival position of the destination planet, \( P_{a2} \). The corresponding
position of the Earth at departure is $E_d^2$.

The time of arrival is varied until the difference in the departure longitudes of the Earth and vehicle is approximately zero. Assume that this condition is met when the third value of $t_a$ is computed. Then, the vehicle and Earth will be at point $E_{d3}', S_{d3}$ at departure, the vehicle will intercept the destination planet at position $P_{a3}'$ and the transfer orbit is established.

The usefulness of $v_{sd}$ as a parameter is not obvious as in the case of $t_d$ or $t_a$; however, it is a useful tool in that a transfer orbit can be established with a relatively short computation procedure. Also, the elevation angle of the vehicle's velocity vector at departure, $\gamma_{sd}$, changes directly with $v_{sd}$. In general, as $v_{sd}$ increases, $\gamma_{sd}$ will increase, and as a consequence, required hyperbolic excess or cutoff velocity will increase.

![Diagram](Fig. 3 — Geometry of method A.3)
B.1 FIXED $t_d$, PARAMETER $t_d$

1. Choose $t_d$ and obtain $\lambda_{ed}$ and $r_{ed}$.

2. Compute $t_a = t_d + t_s$ and obtain $\lambda_{pa}$, $\lambda_{pa}$, and $r_{pa}$.

3. Compute $(v_s - v_{sd})$ from

$$ (v_s - v_{sd}) = \cos^{-1} \left[ \cos \lambda_{pa} \cos (\lambda_{pa} - \lambda_{sd}) \right] $$

where

$$ \lambda_{sd} = \lambda_{ed} $$

4. Compute the minimum value of $a_s$ from

$$ a_s (\text{min}) = \frac{r_{sd} + r_{pa} + c}{4} $$

where $r_{sd} = r_{ed}$ and $c$ is obtained from Eq. (6).

5. Compute $t_s$ as in A.1, step 5.

6. Compute $\Delta t_s = t_s (\text{step 5}) - t_s (\text{fixed})$.

   If $\Delta t_s \neq 0$, compute a new $t_s$ as in A.1, step 7.

7. Compute $\Delta t_s = t_s (\text{step 6}) - t_s (\text{fixed})$.

   If $|\Delta t_s| > K$, change $a_s$ (step 6) by an amount $\Delta a_s$, compute $t_s$ and $\Delta t_s$.

   Repeat step 7 until the orbit is established, i.e., until $|\Delta t_s| \leq K$.

   The amount of change for $a_s$ may be computed as in A.1, step 9.

With $t_s$ a given constant, a choice of $t_d$ determines the required $t_a$.

The heliocentric coordinates of the Earth at $t_d$ and those of the destination planet at $t_a$ represent two fixed points separated by a constant $(v_s - v_{sd})$ which is determined by the coordinates.
In order for the vehicle to intercept the destination planet at $t_a$, the actual $t_s$, which depends on the transfer orbit used, must equal the given $t_s$. This is accomplished by changing $a_s$, which causes a change in the actual $t_s$.

In Fig. 4 the point $E_d$, $S_d$ represents the position of the Earth and vehicle at $t_d$. The point $S_a$ represents the position of the destination planet at the required $t_a$ which is fixed by the given $t_s$ and chosen $t_d$.

Assume that the semimajor axis of transfer orbit number 1 is $a_{\text{min}}$ and that the corresponding $t_s$ is larger than the given $t_s$. Then, it is clear that the vehicle will arrive at position $S_a$ too late and the destination planet will have moved to position $P_{a1}$. If the correct equation for $t_s$ is used with an increased $a_s$, the actual $t_s$ will decrease and the

![Fig. 4 — Geometry of method B.1](image-url)
vehicle may arrive on transfer orbit number 2 either too early or too late. According to Fig. 4, arrival is too late, i.e., \( t_s \) is too large, and the destination planet is at \( P_{a2} \) when the vehicle is at \( S_a \). Further changes in \( a_s \) will result in a \( t_s \) which will permit the vehicle to travel on transfer orbit number \( n \) and intercept the destination planet at \( S_a \).

### B.2 FIXED \( t_s \), PARAMETER \( t_d \)

1. Choose the \( t_a \) and obtain \( \lambda_{pa}, \lambda_{pa}', \) and \( r_{pa} \).
2. Compute \( t_d = t_a - t_s \) and obtain \( \lambda_{ed} \) and \( r_{ed} \).

3-7. These steps are the same as in B.1.

Since this method differs only slightly from B.1 where \( t_d \) was the parameter, the discussion of the method is similar to that of B.1, and Fig. 4 serves to illustrate the geometry.

A choice of the \( t_a \) fixes the rendezvous position of the vehicle and destination planet at point \( S_a \). Since the \( t_s \) is fixed, a choice of \( t_a \) determines uniquely the required \( t_d \). Thus, the required departure position of the vehicle, which must coincide with the Earth’s position at departure, is determined.

As in B.1, \( a_s \) is varied until the computed \( t_s \) matches the given \( t_s \). When this occurs, the Earth and vehicle are at the point \( E_d \), \( S_d \) at the required departure date and the transfer orbit is established.

### B.3 FIXED \( t_s \), PARAMETER \( v_{sd} \)

1. Choose \( v_{sd} \).
2. Choose \( t_d \) and compute

\[
  t_a = t_d + t_s
\]
3. Obtain $\lambda_{ed}$, $r_{ed}$, and $\lambda_{pa}$, $\lambda_{pa}$, $r_{pa}$ for $t_d$ and $t_a$, respectively.

4. Compute $v_s$ from

$$v_s = v_{sd} + \cos^{-1} \left[ \cos \lambda_{pa} \cos (\lambda_{pa} - \lambda_{sd}) \right]$$

where

$$\lambda_{sd} = \lambda_{ed}$$

5. Compute $e_s$ and $a_s$ using Eq. (1).

6. Compute the minimum value of $a_s$ from

$$a_s(\text{min}) = \frac{1}{4} \left( r_{sd} + r_{pa} + c \right)$$

where $r_{sd} = r_{ed}$ and $c$ is obtained using Eq. (6).

7. Compute $t_s$ for $a_s(\text{min})$ using Eq. (5a) or (5c) depending on the size of $(v_s - v_{sd})$.

8. Compute $\Delta t_s = t_s (\text{step 7}) - t_s (\text{fixed})$.

If $\Delta t_s \neq 0$ use either Eq. (5a) or (5b), depending on the type of transfer orbit desired, with $a_s$ (step 5) to compute $t_s$ if $(v_s - v_{sd}) \leq 180^0$. Similarly, if $(v_s - v_{sd}) > 180^0$, use either Eq. (5c) or (5d).

If $\Delta t_s > 0$, decrease $t_s$ (step 7) by using $a_s$ (step 5) with Eq. (5a) or (5c) depending on the size of $(v_s - v_{sd})$.

If $\Delta t_s < 0$, increase $t_s$ (step 7) by using $a_s$ (step 5) with Eq. (5b) or (5d) depending on the size of $(v_s - v_{sd})$.

9. Compute $\Delta t_s = t_s (\text{step 8}) - t_s (\text{fixed})$.

If $|\Delta t_s| > K$, change $t_d$ by an amount $\Delta t_d$ and return to step 2.
\[ \Delta t_d = \Delta t_s / X \]

where

\[
X = \sqrt{\frac{s_p}{r_{pa}}} \left[ \left( \frac{e_p [n_s (1 + e_s^2) - r_{sd}] \sin \nu_s}{(1 - e_s^2)(r_{sd} - r_{pa})} \right) \left[ \frac{3}{2} n_s^2 + (1 - \cos \eta) \tan \frac{\eta}{2} \right] + (1 - \cos \eta) \tan \frac{\eta}{2} \right] + \frac{r_{sd}}{2c} \sin (v_s - v_{sd}) \left[ + \tan \frac{\eta}{2} + \tan \frac{\eta}{2} \right] \right] \] (11)

The multiple signs in the brackets depend on the type of transfer path as follows:

- **Direct** = −, + and +, +
- **Perihelion** = −, − and +, −
- **Aphelion** = +, + and −, +
- **Indirect** = +, − and −, −

The equation for \( X \) is derived in the Appendix.

The discussion concerning the usefulness of \( v_{sd} \) as a parameter is given in Method A.3.

The geometry of this method is illustrated by Fig. 5.

A selection of \( t_d \) gives a position for the Earth at departure \( E_{dl} \) and because \( t_s \) is fixed, \( t_a \) is determined, and consequently the coordinates of the planet position, \( P_{al} \), at the arrival time are known.

The time required for the vehicle to travel from \( E_{dl} \) to \( P_{al} \) is computed using a prescribed \( v_{sd} \). If the computed \( t_s \) differs from the fixed \( t_s \), the vehicle will not intercept the planet at \( P_{al} \). Since the \( t_s \) will, in this case, vary directly with \( (v_s - v_{sd}) \), a change in \( (v_s - v_{sd}) \) will cause a change in \( t_s \).
A second value of $t_d$, and consequently $t_s$, will fix the Earth at point $E_{d2}$ and the planet at $P_{a2}$. Since the angular rates of the planets around the Sun are different, the angle between the radii to points $E_{d2}$ and $P_{a2}$ will not equal the angle between the radii to points $E_{d1}$ and $P_{a1}$. Consequently, the $t_s$ will be different.

By varying $t_d$ the correct relative orientation of the departure and destination planets can be found which will give the correct $t_s$.

Fig. 5 — Geometry of method B.3
C.1 FIXED $V_{sd}$ PARAMETER $t_d$

1. Choose $t_d$ and obtain $\Lambda_{ed}$ and $r_{ed}$.

2. Compute $a_s$ from

$$a_s = \frac{\mu_s r_{sd}}{2 \mu_s - r_{sd} v^2_{sd}}$$

where $r_{sd} = r_{ed}$.

3. Choose $t_a$ and obtain $\Lambda_{pa}$, $\lambda_{pa}$, and $r_{pa}$.

4. Compute $t_s = t_a - t_d$.

5. Compute $(v_s - v_{sd})$ from

$$(v_s - v_{sd}) = \cos^{-1} \left[ \cos \lambda_{pa} \cos (\Lambda_{pa} - \Lambda_{sd}) \right]$$

6. Compute the minimum value of $a_s$ from

$$a_s(\text{min}) = \frac{1}{4} (r_{sd} + r_{pa} + c)$$

where $c$ is obtained from Eq. (6).

7. Compute $\Delta a_s = a_s(\text{min}) - a_s$ (step 2).

If $\Delta a_s > 0$, change $t_a$ by $\Delta t_a$ where

$$\Delta t_a = \frac{\Delta a_s}{X}$$

where

$$X = \frac{\sqrt{\mu_s p r_{sd}}}{k c r_{pa}} \sin (\Lambda_{pa} - \Lambda_{sd})$$  (12)
and return to step 3. The equation for X is derived in the Appendix.

If $\Delta a_s \leq 0$, proceed to step 8.

8. Compute $t_s$ for $a_s$ (min) using Eq. (5a) or (5c) depending on the size of $(v_s - v_{sd})$.

9. Compute $\Delta t_s = t_s$ (step 8) - $t_s$ (step 4).

   If $\Delta t_s = 0$, either Eq. (5a) or (5b), depending on the type of transfer orbit desired, can be used with $a_s$ (step 2) to compute $t_s$ if $(v_s - v_{sd}) \leq 180^\circ$. Similarly, if $(v_s - v_{sd}) \geq 180^\circ$, either Eq. (5c) or (5d) would be used.

   If $\Delta t_s > 0$, decrease $t_s$ (step 8) by using $a_s$ (step 2) with Eq. (5a) or (5c) depending on the size of $(v_s - v_{sd})$.

   If $\Delta t_s < 0$, increase $t_s$ (step 8) by using $a_s$ (step 2) with Eq. (5b) or (5d) depending on the size of $(v_s - v_{sd})$.

10. Compute $\Delta t_s = t_s$ (step 9) - $t_s$ (step 4).

   If $|\Delta t_s| > K$, change $t_a$ by $\Delta t_a$ and return to step 3.

   $\Delta t_a = \Delta t_s / X$

where

$$X = \frac{\sqrt{a_s P r_{sd}} \sin (\lambda_{pa} - \lambda_{sd}) (\mp \tan \eta / 2 \mp \tan \eta_{1 / 2}) - 1}{2c r_{pa}}$$  \hspace{1cm} (13)

The multiple signs in the parentheses depend on the type of transfer path as follows:

Direct = +, +  \hspace{1cm} Perihelion = +, -

Aphelion = -, +  \hspace{1cm} Indirect = -, -

The equation for $X$ is derived in the Appendix.
If $|\Delta t_s| \leq K$ the transfer orbit is established.

As in A.1, step 9, the equation for $X$ will depend on the size of $(v_s - v_{sd})$ and the type of transfer path.

This computation procedure may be used to obtain transfer orbits for given heliocentric departure velocities for a specified departure date.

By using $t_d$ as a parameter and assuming that the Earth and vehicle are coincident at departure, a search for the correct trajectory is made by varying $t_a$ until the vehicle intercepts the destination planet. For $V_{sd}$ and $t_d$ some arrival times cannot be realized, i.e., the $a_s$ which is determined by $V_{sd}$ and $t_d$ will be less than the required minimum which takes into account the distance to the destination planet at arrival and the transfer angle. Thus, a change in $t_a$ will be required.

Figure 6 shows the geometry of motion for a typical case which utilizes this computation procedure.

The departure position of the Earth and vehicle, $E_d, S_d$, is determined by a choice of $t_d$. The time of arrival is arbitrarily chosen and the heliocentric coordinates of the destination planet are obtained. Assume that the planet is at point $S_a$ at the time of arrival. The coordinates of points $S_d$ and $S_a$ are used to compute the minimum value of $a_s$. This permits one to determine if the given $a_s$ is large enough to yield a transfer orbit. If the given $a_s$ is less than the minimum $a_s$ (see step 7) a new value of $t_a$ is determined. Otherwise a $t_s$ is computed, using the minimum $a_s$, which corresponds to transfer orbit number one. If the $t_s$ is too large the planet may be at $P_{al}$ when the vehicle arrives at $S_a$. The relative position of $P_{al}$ and $S_a$ permits one to select the correct equation (see Eq. 5) for $t_s$ when transfer orbit number two is determined using the given $a_s$. 
In general the first value of \( t_a \) will not be correct and several values of \( t_a \) will be used before the correct transfer orbit is established.

Fig. 6 — Geometry of method C.1
C.2 FIXED $V_{sd}$, PARAMETER $t_a$

1. Choose $t_a$ and obtain $\Lambda_{pa}$, $\Lambda_{pa'}$, and $r_{pa}$.

2. Choose $t_d$ and obtain $\Lambda_{ed}$ and $r_{ed}$.

3. Compute $t_s = t_a - t_d$.

4. Compute $a_s$ from

$$a_s = \frac{\mu r_{sd}}{2\mu - r_{sd} V_{sd}^2}$$

where $r_{sd} = r_{ed}$.

5. Compute $(v_s - v_{sd})$ from

$$(v_s - v_{sd}) = \cos^{-1} \left[ \cos \lambda_{pa} \cos (\Lambda_{pa} - \Lambda_{sd}) \right]$$

6. Compute the minimum value of $a_s$ from

$$a_s(\text{min}) = \frac{1}{4} (r_{sd} + r_{pa} + c)$$

where $c$ is obtained using Eq. (6).

7. Compute $\Delta a_s = a_s(\text{min}) - a_s(\text{step 4})$.

If $\Delta a_s > 0$, change $t_d$ by an amount $\Delta t_d$ and return to step 2.

$$\Delta t_d = \Delta a_s / \chi$$

where

$$\chi = \sqrt{\frac{\mu P e r_{pa}}{4c r_{sd}}} \sin (\Lambda_{pa} - \Lambda_{sd})$$

(14)

The derivation of the equation for $\chi$ is similar to the derivation of
the equation for $X$ in C.1, step 7 and therefore is not included in

the Appendix.

If $\Delta a_s \leq 0$, proceed to step 8.

8. Compute $t_s$ for $a_s$ (min) using Eq. (5a) or (5c) depending on the size

of $(v_s - v_{sd})$.

9. Compute $\Delta t_s = t_s$ (step 8) - $t_s$ (step 3).

If $\Delta t_s = 0$ either Eq. (5a) or (5b), depending on the type of transfer

orbit desired, can be used with $a_s$ (step 4) to compute $t_s$ if $(v_s - v_{sd})$

$\leq 180^\circ$. Similarly, if $(v_s - v_{sd}) > 180^\circ$, either Eq. (5c) or (5d) can

be used.

If $\Delta t_s > 0$, decrease $t_s$ (step 8) by using $a_s$ (step 4) with Eq. (5a)

or (5c) depending on the size of $(v_s - v_{sd})$.

If $\Delta t_s < 0$, increase $t_s$ (step 8) by using $a_s$ (step 4) with Eq. (5b) or

(5d) depending on the size of $(v_s - v_{sd})$.

10. Compute $\Delta t_s = t_s$ (step 9) - $t_s$ (step 3).

If $|\Delta t_s| > K$, change $t_d$ by $\Delta t_d$ and return to step 2.

$$\Delta t_d = \Delta t_s / X$$

where

$$X = -\sqrt{a_s \over 8 c} \frac{r_{pa}}{2 r_{sd}^c} \sin (\Lambda_{pa} - \Lambda_{sd}) \left[ \tan {\eta_s \over 2} + \tan {\eta_1 \over 2} \right] + 1 \quad (15)$$

The signs of the bracketed terms are determined as in C.1, step 10.

The derivation of the equation for $X$ is similar to that for $\Delta t_a$ in

C.1, step 10, and therefore is not given.

If $|\Delta t_s| \leq K$, the transfer orbit is established.
This method of establishing a transfer orbit is similar to C.1. The major difference is that the time of arrival is fixed and the departure date is used as a variable. Because $t_d$ is varied it is necessary to compute a new value of $a_s$ for each iteration, while in C.1 a choice of $t_d$ fixed the value of $a_s$ for all the computations.

The time of arrival is chosen and we assume that the vehicle is coincident with the destination planets at arrival. Next $t_d$ is varied until a transfer orbit is found which will make the departure coordinates of the Earth and vehicle equal.

Similar to C.1, some departure dates cannot be used for the given $V_{sd}$ and selected $t_a$. That is, the required $V_{sd}$ or $a_s$ may exceed the given values. In this event it is necessary to select another departure date so as to reduce the required $a_s$.

Figure 7 shows the geometry and relative positions of the planets and vehicle using this computation procedure.

The geometry of the motion is easy to understand if we start at $t_a$ and move the planets and vehicle backwards in time. At $t_a$ the destination planet and vehicle are at point $S_a$, $P_a$ on the destination planet's orbit and the Earth is at point $E_a$. Assume that transfer orbit 1 corresponds to the minimum value of $a_s$. During the corresponding time $t_s$ the vehicle moves to point $S_d$ while the Earth may move to point $E_{dl}$. In that case the $t_s$ is too large and consequently the computed $t_d$ is too small (early).

The relative positions of the Earth and vehicle at departure, i.e., $E_{dl}$ and $S_d$, indicate which part of Eq. (5) should be used with the given $a_s$ to compute a new value of $t_s$. If orbit 2 corresponds to the given $a_s$, then the Earth may be at $E_{d2}$ when the vehicle is at $S_d$. Since $E_{d2}$ and $S_d$ are
Fig. 7 — Geometry of method C.2

not coincident the selected \( t_d \) must be changed, i.e., point \( S_d \) must be at some other point on the Earth's orbit. The new value of \( t_d \) may be determined by iteration or by using the approximate equation for \( \Delta t_d \), step 7.

The \( t_d \) is varied until the positions of the Earth and vehicle coincide at \( t_d' \).

D.1 FIXED \( V_c \) PARAMETER \( t_d \)

1. Choose \( t_d \) and obtain \( \lambda_{ed} \) and \( r_{ed} \).
2. Choose $t_s$.  

3. Compute $t_a = t_d + t_s$ and obtain $\Lambda_{pa}$, $r_{pa}$, and $r_{pa}$. 

4. Compute $(v_s - v_{sd}) = \cos^{-1} \left[ \cos \lambda_{pa} \cos (\Lambda_{pa} - \Lambda_{sd}) \right] 

where 

$$\Lambda_{sd} = \Lambda_{ed}$$

5. Choose $v_{sd}$ and compute $v_s$ from 

$$v_s = (v_s - v_{sd}) \text{ (step 5)} + v_{sd}$$

6. Compute $e_s$ and $a_s$ using Eq. (1). 

7. Compute the minimum $a_s$ from 

$$a_s(\min) = \frac{1}{4} (r_{sd} + r_{pa} + c)$$

where $r_{sd} = r_{ed}$ and $c$ is obtained from Eq. (6). 

8. Compute $t_s$ using $a_s(\min)$ and Eq. (5a) or (5c) depending on the size of $(v_s - v_{sd})$. 

9. Compute $\Delta t_s = t_s \text{ (step 8)} - t_s \text{ (step 2)}$ and compute $t_s$ as in method C.2, step 9. 

10. Compute $\Delta t_s = t_s \text{ (step 9)} - t_s \text{ (step 2)}$. 

If $|\Delta t_s| > K$, change $v_{sd} \text{ (step 5)}$ by $\Delta v_{sd}$ and return to step 5. 

$$\Delta v_{sd} = \Delta t_s / X$$

where
\[ X = \sqrt{\frac{\mu_s}{a_s}} \left( \frac{e_s}{1 - e_s^2} \right) \left[ \frac{a_s(1 + e_s^2) - r_{sd}}{r_{sd} - r_{pa}} \left( r_{pa} \sin \nu_s - r_{sd} \sin \nu_{sd} \right) - r_{sd} \sin \nu_{sd} \right] \times \left[ + \frac{3}{2} \eta_s t_s \sin \nu_s - (1 - \cos \eta) \tan \eta/2 \right] \]

The multiple signs in the brackets depend on the type of transfer path as follows:

- **Direct**: = -, +
- **Perihelion**: = -, -
- **Aphelion**: = +, +
- **Indirect**: = +, -

The equation for \( X \) is derived in the Appendix.

If \( |\Delta t_s| \leq K \), proceed to step 11.

11. Compute the Earth's velocity vector at departure from

\[ V_{ed} = \sqrt{\mu_s} \left( \frac{2}{r_{ed} - \frac{1}{a_e}} \right) \]

where \( a_e \) is the semimajor axis of the Earth's orbit.

12. Compute the elevation angle of \( V_{ed} \) from

\[ \gamma_{ed} = \cos^{-1} \left[ \frac{\sqrt{\frac{a_e^2 (1 - e e^2)}{r_{ed} (2a_e - r_{ed})}}}{\sqrt{\frac{2}{r_{ed}}} - \frac{1}{a_e}} \right] \]

where the plus sign is used if \( r_{ed} > 0 \).

13. Compute the vehicle's velocity vector at departure from

\[ V_{sd} = \sqrt{\mu_s} \left( \frac{2}{r_{sd} - \frac{1}{a_s}} \right) \]
14. Compute the elevation angle of $\gamma_{sd}$ from

$$\gamma_{sd} = \cos^{-1} \left[ 1 + \sqrt{\frac{a_s^2 (1 - e_s^2)}{r_{sd} (2a_s - r_{sd})}} \right]$$

where the plus sign is used if $0 < \gamma_{sd} < \pi$.

15. Compute $i_s$, the inclination angle of the transfer orbit plane, from

$$i_s = \tan^{-1} \left[ \tan \lambda_{pa} / \sin (\lambda_{pa} - \lambda_{sd}) \right]$$

16. Compute $\alpha$, the angle between $\bar{V}_{ed}$ and $\bar{V}_{sd}$, from

$$\alpha = \cos^{-1} \left[ \sin \gamma_{ed} \sin \gamma_{sd} + \cos \gamma_{ed} \cos \gamma_{sd} \cos i_s \right]$$

17. Compute $V_\infty$ from

$$V_\infty = \sqrt{V_{sd}^2 + V_{ed}^2 - 2V_{sd}V_{ed} \cos \alpha}$$

18. Compute $\Delta V_\infty = V_\infty$ (step 17) - $V_\infty$ (fixed).

If $|\Delta V_\infty| > K$, change $t_s$ (step 2) by an amount

$$\Delta t_s = \Delta V_\infty / X$$

where

$$X = \frac{\partial V_\infty}{\partial V_{sd}} \frac{dV_{sd}}{dt_s} + \frac{\partial V_\infty}{\partial \alpha} \frac{d\alpha}{dt_s}$$

Note: no attempt is made here to express $X$ in terms of the orbital parameters since the equation is tedious even with several simplifying assumptions. A purely iterative procedure is suggested.
If $|\Delta V| \leq K$, the orbit is established.

In order for a transfer orbit to be possible, the specified $V_\infty$ must be equal to or greater than the minimum value of $V_\infty$. For transfer to inner or outer planets, $a_s \geq \frac{1}{2} \left( \frac{r_{pp}}{r_{ep}} + \frac{r_{ep}}{r_{pp}} \right)$ where the $r$'s are the perihelion distances of the orbits of Earth and the destination planet.

For transfer to outer planets, $V_\infty$ must satisfy the equation

$$V_\infty \geq \sqrt{\frac{2 \mu}{r_{pp}/r_{ep} \left( \frac{r_{pp}}{r_{ep}} + \frac{r_{ep}}{r_{pp}} \right)}} - \sqrt{\frac{\mu \left( \frac{2}{r_{ep}} - \frac{1}{a_e} \right)}{r_e}}$$

For transfer to inner planets

$$V_\infty \geq \sqrt{\frac{2 \mu}{r_{pa}/r_{ea} \left( \frac{r_{pa}}{r_{ea}} + \frac{r_{ea}}{r_{pa}} \right)}} - \sqrt{\frac{\mu \left( \frac{2}{r_{ea}} - \frac{1}{a_e} \right)}{r_e}}$$

where $r_{pa}$ and $r_{ea}$ are the aphelion distances of the planet and Earth.

The geometry of the motion for a special transfer is shown in Fig. 8.

The first ten steps of this method are used to establish an orbit between a known departure point and a known arrival point, which will permit the vehicle to intercept the destination planet. The remaining steps of the methods are used to determine the relative magnitudes of the computed $V_\infty$, which is based on the established transfer orbit, and the specified $V_\infty$.

The choice of the parameter $t_d$ determines the position on the Earth's orbit $E_{d'}$ $S_{d'}$. This point remains fixed while $v_{ad'}$ and then, if necessary, $t_s$ is varied in order to establish a transfer orbit.

A selection of $t_d$ and $t_s$ determines the positions $E_{d'}$, $S_{d'}$, and $S_{al'}$, which is a position on the destination planet's orbit. Assume that orbit 1 is computed using $a_{sl}(mln)$ and that $t_{s'}$ is too large. In this case the planet may by at $P_{al'}$ when the vehicle is at $S_{al'}$. Orbit 2 is computed using the
Fig. 8 — Geometry of method D.1
a_s which is based on the coordinates of S_d and S_{al} and the v_{sd}'. The t_s may be such that the planet is at P_{a2} when the vehicle is at S_{al}. At this point in the computations the v_{sd} is changed and new orbits are computed until the orbit n causes P_{an} and S_{al} to coincide, i.e., the vehicle intercepts the planet.

For orbit n, which connects S_d and S_{al}, the required V_{\infty} is determined and compared to the specified V_{\infty}. If the V_{\infty}'s disagree, then the arrival point on the destination planet orbit is changed by taking a different value of t_s. As before, orbits are computed until the vehicle intercepts the planet at S_{a2}. The V_{\infty}'s are again compared, and if they are different a new value of t_s is determined and the process repeats until an orbit is found which will cause the required V_{\infty} to match the specified V_{\infty}. If no such orbit can be found then the parameter t_d must be changed.

D.2 FIXED V_{\infty} PARAMETER t_a

1. Choose t_a and obtain \theta_{pa}', \lambda_{pa}', and r_{pa}'.
2. Choose t_{sa}.
3. Compute t_d = t_a - t_{sa} and obtain \theta_{ed} and r_{ed}.

4-18. These steps are the same as in D.1.

The geometry of motion is different from that of D.1 and is shown in Fig. 9.

In this method the time of arrival is selected, consequently the position of the destination planet at the arrival time. This method is similar to D.1 in that v_{sd} is used as the variable in computing the different orbits which connect the fixed arrival point and a particular departure point on
the Earth's orbit; it differs from D.1 in that a change in $t_s$ causes the departure point to move along the Earth's orbit while the arrival point remains fixed.

Fig. 9 — Geometry of method D.2
VI. DISCUSSION

The methods presented and discussed here are only a few of many which could be employed to establish interplanetary transfer orbits. These methods are not necessarily better than others, but they represent some of the more obvious approaches to the problem and at the same time involve some of the parameters of greatest interest.

The detailed computation procedure given with each method will be of value in preparing a computer program. The computation procedure plus the discussion and figures will give the reader a "feel" for the relative geometry of motion and spatial orientation of the vehicle and planets as a function of time for each of the various parameters.

Several equations are included in the computation procedures, namely Eqs. (8) - (16), for the purpose of reducing the number of iterations required. These equations are obtained by taking the derivatives of various equations of two-body motion. Since the evaluation of these equations is straightforward, though perhaps tedious, their use may in some cases reduce the computation time that would be required using a purely iterative procedure. The advantage of using these equations rather than a simple trial-and-error approach depends somewhat on the type of computer used.
Appendix
DERIVATION OF EQUATIONS

DERIVATION OF EQUATION (8) OF METHOD A.1

According to Eq. (5), $t_s$ is a function of $n_s$, $\eta$, and $\eta_l$. Thus

$$\frac{dt_s}{da_s} = \frac{\partial t_s}{\partial n_s} \frac{dn_s}{da_s} + \frac{\partial t_s}{\partial \eta} \frac{d\eta}{da_s} + \frac{\partial t_s}{\partial \eta_l} \frac{d\eta_l}{da_s}$$

(17)

where

$$n_s = \frac{k' \sqrt{\mu}}{a_s^{3/2}} = \frac{\sqrt{\mu_s}}{a_s^{3/2}}$$

(18)

$$\eta = 2 \sin^{-1} \left[ \frac{1}{2} \left( \frac{r_{sd} + r_{pa} + c}{a_s} \right)^{1/2} \right]$$

(19)

$$\eta_l = 2 \sin^{-1} \left[ \frac{1}{2} \left( \frac{r_{sd} + r_{pa} - c}{a_s} \right)^{1/2} \right]$$

(20)

and where

$$c = \left[ r_{sd}^2 + r_{pa}^2 - 2r_{sd}r_{pa} \cos (v_s - v_{sd}) \right]^{1/2}$$

(21)

For departures from Earth and heliocentric transfer orbits, $r_{sd} = r_{ed}$ and $\mu_s$ is the heliocentric gravitational constant. For convenience the units may be chosen so that $k' = 1$. 
The total derivatives are evaluated as follows:

\[
\frac{dn_s}{da_s} = d \left( \frac{\sqrt{\mu_s}}{a_s^{3/2}} \right) = -\frac{3\sqrt{\mu_s}}{2a_s^{5/2}} \tag{22}
\]

\[
\frac{\eta_1}{da_s} = \frac{\partial \eta_1}{\partial s} + \frac{\partial \eta_1}{\partial \eta} \frac{dr}{da_s} + \frac{\partial \eta_1}{\partial \eta} \frac{dr}{da_s} + \frac{\partial \eta_1}{\partial \eta} \frac{dc}{da_s} + \frac{\partial \eta_1}{\partial \eta} \frac{dc}{da_s}
\]

Since \( t_s \) is the parameter, \( r_{pa} \) is constant. We assume that \( r_{sd} \) is a constant.

Also, because \( (v_s - v_{sd}) \) is fixed, \( c \) is constant. Thus

\[
\frac{dn_s}{da_s} = \frac{\partial \eta_1}{\partial c} = -\frac{1}{a_s} \tan \eta/2 \tag{23}
\]

In a similar manner we find that

\[
\frac{\eta_1}{da_s} = \frac{\partial \eta_1}{\partial s} = -\frac{1}{n_s} \tan \eta_1/2 \tag{24}
\]

The partial derivatives of \( t_s \) depend on the type of transfer orbit and are obtained using Eq. (5). The expressions for the partials are

\[
\frac{\partial t_s}{dn_s} = -\frac{t_s}{n_s} \tag{25}
\]

for all types of transfer orbits.

\[
\frac{\partial t_s}{\partial \eta} = \frac{1}{n_s} (1 - \cos \eta) \tag{26}
\]
where the plus sign is used for direct and perihelion transfer and the minus sign is used for aphelion and indirect transfer.

\[
\frac{\partial t_s}{\partial \eta_1} = \pm \frac{1}{n_s} (1 - \cos \eta_1) \tag{27}
\]

where the plus sign is used for perihelion and indirect transfer and the minus sign is used for direct and aphelion transfer.

Substitution of the derivatives into Eq. (17) gives

\[
\frac{\partial t_s}{\partial a_s} = \sqrt{\frac{a_s}{u_s}} \left[ \frac{3}{2} t_s n_s + (1 - \cos \eta) \tan \eta/2 + (1 - \cos \eta_1) \tan \eta_1/2 \right] \tag{28}
\]

where the multiple signs in the brackets depend on the type of transfer as follows:

- **Direct** = -, +
- **Perihelion** = -, -
- **Aphelion** = +, +
- **Indirect** = +, -

**Derivation of Eq. (9) of Method A.2**

\[
\frac{d(\Delta \nu)}{d a_s} = \frac{d[(v_s - v_{sd}) - (v_s - v_{sd}) \text{ (fixed)}]}{d a_s} = \frac{d(v_s - v_{sd})}{d a_s} \tag{29}
\]

\( (v_s - v_{sd}) \) is related to the coordinates of the departure and destination planets as follows:

\[
v_s - v_{sd} = \cos^{-1} \left[ \cos \lambda_{pa} \cos (\lambda_{pa} - \lambda_{sd}) \right] \tag{30}
\]
where $\Lambda_{sd} = \Lambda_{ed}$. Thus

$$\frac{d(v_s - v_{sd})}{da_s} = \frac{\partial(v_s - v_{sd})}{\partial \Lambda_{pa}} \frac{d\Lambda_{pa}}{da_s} + \frac{\partial(v_s - v_{sd})}{\partial \Lambda_{pa}} \frac{d\Lambda_{pa}}{da_s}$$  \hspace{1cm} (31)$$

Since $t_d$ is the parameter in this method, $\Lambda_{sd}$ is constant and

$$\frac{d\Lambda_{sd}}{da_s} = 0$$

Now, the total derivatives of Eq. (31) are

$$\frac{d\Lambda_{pa}}{da_s} = \frac{\partial \Lambda_{pa}}{\partial t_d} \frac{dt_d}{da_s} + \frac{\partial \Lambda_{pa}}{\partial t_s} \frac{dt_s}{da_s}$$  \hspace{1cm} (32)$$

Since $t_d$ is fixed

$$\frac{d\Lambda_{pa}}{da_s} = \frac{\partial \Lambda_{pa}}{\partial t_s} \frac{dt_s}{da_s}$$

Similarly

$$\frac{d\Lambda_{pa}}{da_s} = \frac{\partial \Lambda_{pa}}{\partial t_s} \frac{dt_s}{da_s}$$

According to Eq. (5), $t_s$ is a function of $n_s$, $\eta$, and $\eta_1$, which are given by Eqs. (18) - (20).

A choice of $t_d$ fixes $r_{sd}$. We assume that $r_{pa}$ is constant; since $(v_s - v_{sd})$ is fixed, $c$ is constant and $dt_s/da_s$ is given by Eq. (26),
which is
\[
\frac{dt_s}{ds} = \sqrt{a_s} \left[ \frac{3}{2} t_s n_s + (1 - \cos \eta) \tan \eta/2 + (1 - \cos \eta_1) \tan \eta_1/2 \right]
\] (28)

The signs of the terms are the same as in the derivation of Eq. (8).

Now, from the geometry
\[
\lambda_{pa} = \tan^{-1} \left[ \tan i_p \cos (\Lambda_{pa} - \Omega_p) \right]
\] (33)

and
\[
\frac{\partial \lambda_{pa}}{\partial t_a} = \frac{\partial \lambda_{pa}}{\partial t_a} = \frac{\tan i_p \cos^2 \lambda_{pa} \cos (\Lambda_{pa} - \Omega_p)}{r_{pa}}
\] (34)

The partial derivative
\[
\frac{\partial \lambda_{pa}}{\partial t_a} = \dot{\lambda}_{pa} \approx \frac{\sqrt{\mu_{sp}}}{r_{pa}}
\] (35)

and Eq. (34) becomes
\[
\frac{\partial \lambda_{pa}}{\partial t_a} = \dot{\lambda}_{pa} \approx \frac{\sqrt{\mu_{sp}}}{r_{pa}} \tan i_p \cos^2 \lambda_{pa} \cos (\Lambda_{pa} - \Omega_p)
\] (36)

The partial derivatives of \((v_s - v_{sd})\) are found using Eq. (30). They are
\[
\frac{\partial (v_s - v_{sd})}{\partial \lambda_{pa}} = \frac{\sin \lambda_{pa} \cos (\Lambda_{pa} - \Lambda_{sd})}{\sin (v_s - v_{sd})}
\] (37)
and

\[
\frac{\partial(v_s - v_{sd})}{\partial \lambda_{pa}} = \frac{\cos \lambda_{pa} \sin (\lambda_{pa} - \lambda_{sd})}{\sin (v_s - v_{sd})}
\]  

(38)

Substitution of Eqs. (34), (35), (37), and (38) into Eq. (31), after using small-angle approximations, i.e., \( \sin \lambda_{pa} = \lambda_{pa} \), \( \cos \lambda_{pa} = 1 \), and \( \tan i_p = i_p \), gives

\[
\frac{d(v_s - v_{sd})}{da_s} = \frac{\sqrt{\mu_s a_p}}{r_{pa}^2 \sin (v_s - v_{sd})} \left[ \lambda_{pa} i_p \cos (\lambda_{pa} - \lambda_{sd}) \right] \frac{dt_s}{da_s}
\]

Since \( \lambda_{pa} \) and \( i_p \) are small for the planets of interest, their product is neglected. In accord with this and the assumptions above

\[
\sin (v_s - v_{sd}) \approx \sin (\lambda_{pa} - \lambda_{sd})
\]

and we get

\[
\frac{d(v_s - v_{sd})}{da_s} = \frac{\sqrt{\mu_s a_p}}{r_{pa}^2} \frac{dt_s}{da_s}
\]

Substituting for \( \frac{dt_s}{da_s} \) gives

\[
\frac{d(v_s - v_{sd})}{da_s} = \frac{\sqrt{a_p}}{r_{pa}^2} \left[ \frac{3}{2} t_s n_a + (1 - \cos \eta) \tan \eta/2 + (1 - \cos \eta_1) \tan \eta_1/2 \right]
\]

(39)
where the multiple signs in the brackets depend on the type of transfer path as follows:

- Direct = -, +
- Perihelion = -, -
- Aphelion = +, +
- Indirect = +, -

**DERIVATION OF Eq. (10) OF METHOD A.3**

\[
\frac{d(\Delta)}{dt_a} = \frac{d(\varphi_{sd} - \varphi_{ed})}{dt_a} = \frac{d\varphi_{sd}}{dt_a} - \frac{d\varphi_{ed}}{dt_a} \tag{40}
\]

Since \((v_s - v_{sd})\) is fixed and \(v_{sd}\) is specified, \(v_s\) is determined. If, in addition, we assume that \(r_{sd}\) and \(r_{pa}\) are constant, \(t_s\) will also be constant, and from \(t_a = t_d + t_s\)

\[
\frac{dt_a}{dt} = \frac{dt_d}{dt}
\]

Eq. (40) becomes

\[
\frac{d\Delta}{dt_a} = \frac{d\varphi_{sd}}{dt_a} - \frac{d\varphi_{ed}}{dt_d} \tag{41}
\]

From the geometry we get

\[
\varphi_{sd} = \varphi_{pa} - \cos^{-1}\left[\cos (v_s - v_{sd})/\cos \varphi_{pa}\right] \tag{42}
\]

and

\[
\frac{d\varphi_{sd}}{dt_a} = \varphi_{pa} - \frac{\cos (v_s - v_{sd}) \sin \varphi_{pa}}{\sin (\varphi_{pa} - \varphi_{sd}) \cos^2 \varphi_{pa}} \tag{43}
\]

Again using small-angle approximations, as in the derivation of Eq. (9),
and Eqs. (35) and (36), we substitute into Eq. (43) and get

\[
\frac{d\lambda}{dt_a} = \frac{\sqrt{\mu_s P}}{r_{pa}^2} \left[ 1 - \frac{\lambda_{pa} \cos (v_s - v_{sd})\cos (\lambda_{pa} - \Omega_p)}{\sin (\lambda_{pa} - \lambda_{sd})} \right]
\]

(44)

The second term on the right side of Eq. (44) is small compared to one except for values of \((\lambda_{pa} - \lambda_{sd})\) near 180°, since the product \(\lambda_{pa} \cos v_{sd}\) is small for all planets of interest. For \((\lambda_{pa} - \lambda_{sd}) \approx 180^\circ\) the inclination of the transfer orbit plane is approximately 90°. Transfer orbits of this inclination would require extremely high velocities; therefore this term is neglected and

\[
\frac{d\lambda_{sd}}{dt_a} = \frac{\sqrt{\mu_s P}}{r_{pa}^2}
\]

(45)

The time rate of change of the Earth's heliocentric longitude is given by

\[
\frac{d\lambda_{ed}}{dt_a} = \frac{\sqrt{\mu_s P}}{r_{ed}^2}
\]

(46)

Substitution of Eqs. (45) and (46) into Eq. (41) gives

\[
\frac{d(\Delta)}{dt_a} = \sqrt{\mu_s} \left[ \frac{\sqrt{P}}{r_{pa}^2} - \frac{\sqrt{P_e}}{r_{ed}^2} \right]
\]

(47)
DERIVATION OF EQUATION (11) OF METHOD B.3

\[
\frac{d(\Delta t_s)}{dt_d} = \frac{d}{dt_d} \left[ t_s - t_s \text{ (fixed) } \right] = \frac{dt_s}{dt_d}
\]

Since \( t_s = f(n_s', \eta, \eta_1) \)

\[
\frac{dt_s}{dt_d} = \frac{\partial t_s}{\partial n_s} \frac{dn_s}{dt_d} + \frac{\partial t_s}{\partial \eta} \frac{d\eta}{dt_d} + \frac{\partial t_s}{\partial \eta_1} \frac{d\eta_1}{dt_d}
\] (48)

The total derivatives are as follows:

\[
\frac{dn_s}{dt_d} = -\frac{3\sqrt{\mu_s}}{2a_s^{5/2}} \frac{ds_s}{dt_d}
\] (49)

If we assume that \( r_{sd} \) and \( r_{pa} \) are constants and use the fact that \( v_{sd} \) is fixed, then using Eq. (1)

\[
\frac{ds_s}{dt_d} = \frac{\partial s_s}{\partial e_s} \frac{de_s}{dt_d} + \frac{\partial s_s}{\partial v_s} \frac{dv_s}{dt_d}
\] (50)

since \( e_s = f(r_{sd}, r_{pa}, v_{sd}, v_s) \). Using Eq. (30)

\[
\frac{dv_s}{dt_d} = \frac{\partial v_s}{\partial \lambda_{pa}} \frac{d\lambda_{pa}}{dt_d} + \frac{\partial v_s}{\partial \lambda_{pa}} \frac{d\lambda_{pa}}{dt_d}
\]

\[
= \frac{\partial v_s}{\partial \lambda_{pa}} \frac{d\lambda_{pa}}{dt_a} + \frac{\partial v_s}{\partial \lambda_{pa}} \frac{d\lambda_{pa}}{dt_a}
\] (51)
since \( t_a = t_d + t_s \) and \( t_s \) is constant.

The partial derivative \( \partial v / \partial p_a \) contains the \( \sin \lambda_{pa} \) and the total derivative \( \partial p_a / \partial t_a \) contains \( \tan i_p \). Since both \( \lambda_{pa} \) and \( i_p \) are small, the product of these derivatives is neglected in Eq. (51) and subsequent steps of the derivation. Thus, Eq. (51) becomes

\[
\frac{dv_s}{dt_d} = \frac{\partial v_s}{\partial p_a} \frac{dp_a}{dt_a}
\]

and Eq. (49) becomes

\[
\frac{dn_s}{dt_d} = \frac{-3 \sqrt{\mu_s}}{2a_s^2} \frac{\partial a_s}{\partial e_s} \frac{\partial e_s}{\partial v_s} \frac{\partial v_s}{\partial p_a} \frac{\partial p_a}{\partial e_s} \frac{\partial e_s}{\partial v_s}
\]

According to the assumptions above and Eq. (19)

\[
\frac{dn}{dt_d} = \frac{\partial n}{\partial a_s} \frac{da_s}{dt_a} + \frac{\partial n}{\partial c} \frac{dc}{dt_a}
\]

Substitution of Eqs. (50) and (52) into Eq. (54) gives

\[
\frac{dn}{dt_d} = \left( \frac{\partial n}{\partial a_s} \frac{\partial a_s}{\partial e_s} \frac{\partial e_s}{\partial v_s} + \frac{\partial n}{\partial c} \frac{\partial c}{\partial v_s} \right) \frac{\partial v_s}{\partial p_a} \frac{\partial p_a}{\partial e_s} \frac{\partial e_s}{\partial v_s}
\]

By using Eq. (20) in a similar manner

\[
\frac{dn_a}{dt_d} = \left( \frac{\partial n_a}{\partial a_s} \frac{\partial a_s}{\partial e_s} \frac{\partial e_s}{\partial v_s} + \frac{\partial n_a}{\partial c} \frac{\partial c}{\partial v_s} \right) \frac{\partial v_s}{\partial p_a} \frac{\partial p_a}{\partial e_s} \frac{\partial e_s}{\partial v_s}
\]
By using Eqs. (19), (20), and (21) we find that

\[ \frac{\partial \eta}{\partial a_s} = -\frac{1}{a_s} \tan \eta/2 \quad \frac{\partial \eta}{\partial c} = \frac{1}{2a_s \sin \eta} \]

\[ \frac{\partial \eta_1}{\partial a_s} = -\frac{1}{a_s} \tan \eta_1/2 \quad \frac{\partial \eta_1}{\partial c} = -\frac{1}{2a_s \sin \eta_1} \]

\[ \frac{\partial \zeta}{\partial v_s} = \frac{1}{c} r_s d_p a \sin (v_s - v_{sd}) \]

By solving Eq. (1) for \( e_s \) and then for \( a_s \) we get

\[ e_s = \frac{r_{sd} - r_{pa}}{r_{pa} \cos v_s - r_{sd} \cos v_{sd}} \]

and

\[ a_s = \frac{r_{sd} (1 + e_s \cos v_{sd})}{1 - e_s^2} \]

The partial derivatives are

\[ \frac{\partial a_s}{\partial e_s} = \frac{a_s (1 + e_s^2) - r_{sd}}{e_s (1 - e_s^2)} \]

and

\[ \frac{\partial e_s}{\partial v_s} = \frac{e_s^2 r_{pa} \sin v_s}{r_{sd} - r_{pa}} \]
Using Eq. (30) we get

\[
\frac{\partial v_s}{\partial \lambda_{pa}} = \frac{\cos \lambda_{pa} \sin (\lambda_{pa} - \lambda_{sd})}{\sin (v_s - v_{sd})}
\]

By substituting for the partial derivatives of Eqs. (53), (55), and (56) and then substituting these equations into Eq. (48) and simplifying, we get

\[
\frac{dt_s}{dt_d} = \left( \frac{\lambda_{pa} \sin (v_s - v_{sd})}{(1 - e_s^2) (r_{sd} - r_{pa})} \right) \left[ -3 \sqrt{\frac{v_s}{a_s}} \frac{\partial t_s}{\partial \eta_{s}} \right]
\]

\[
- \frac{1}{a_s} \tan \eta/2 \frac{\partial t_s}{\partial \eta} - \frac{1}{a_s} \tan \eta/2 \frac{\partial t_s}{\partial \eta_{s}} \right] \frac{\lambda_{pa}}{\lambda_{pa}}
\]

\[
+ \frac{r_{sd} r_{pa}}{c} \sin (v_s - v_{sd}) \left[ \frac{1}{2 a_s \sin \eta} \frac{\partial t_s}{\partial \eta} - \frac{1}{2 a_s \sin \eta_{s}} \frac{\partial t_s}{\partial \eta_{s}} \right] \frac{\lambda_{pa}}{\lambda_{pa}}
\]

If we eliminate the remaining partial derivatives using Eqs. (25), (26), and (27) and simplify, we get

\[
\frac{dt_s}{dt_d} = \sqrt{\frac{a_s}{p}} \frac{r_{pa}}{r_{sa}} \left( \frac{e_s a_s (1 + e_s^2) - r_{sd} \sin v_s}{(1 - e_s^2) (r_{sd} - r_{pa})} \right) \left[ \frac{3}{2} \eta_{ts} \frac{\partial t_s}{\partial \eta_{s}} + (1 - \cos \eta) \tan \eta/2 \right.
\]

\[
+ (1 - \cos \eta_{s}) \tan \eta_{s}/2 \left. \right] + \frac{r_{sd}}{2c} \sin (v_s - v_{sd}) \left[ \frac{\tan \eta/2}{\tan \eta_{s}} + \frac{\tan \eta_{s}/2}{\tan \eta_{s}/2} \right]
\]

(57)
The multiple signs in brackets depend on the transfer orbit as follows:

- **Direct**: \(-, +\) and \(+, +\)
- **Perihelion**: \(-, -\) and \(+, -\)
- **Aphelion**: \(+, +\) and \(-, -\)
- **Indirect**: \(+, -\) and \(-, -\)

**DERIVATION OF Eq. (12) of METHOD C.1**

\[
\frac{d\left(\Delta a_s\right)}{dt_a} = \frac{d\left[a_s\left(\text{min}\right) - a_s\right]}{dt_a} = \frac{d a_s\left(\text{min}\right)}{dt_a} \quad (58)
\]

since for fixed \(V_{sd}\) a choice of \(t_d\) determines \(r_{ed}\) and consequently \(a_s\).

Thus

\[
\frac{da_s}{dt_a} = 0
\]

The equation for \(a_s\left(\text{min}\right)\) is

\[
a_s\left(\text{min}\right) = \frac{1}{4} \left( r_{sd} + r_{pa} + c \right) \quad (59)
\]

where \(r_{sd} = r_{ed}\). We assume that \(r_{pa}\) is constant. As a result of this

\[
\frac{da_s\left(\text{min}\right)}{dt_a} = \frac{\partial a_s\left(\text{min}\right)}{\partial c} \frac{dc}{dt_a}
\]
Using Eq. (2) for $c$ and Eq. (30) we get

$$\frac{dc}{dt_a} = \frac{dc}{\partial(v_s - v_{sd})} \cdot \frac{d(v_s - v_{sd})}{dt_a}$$

$$= \frac{dc}{\partial(v_s - v_{sd})} \left[ \frac{\partial(v_s - v_{sd})}{\partial^\lambda pa} \cdot \lambda_{pa} + \frac{\partial(v_s - v_{sd})}{\partial^\lambda pa} \cdot \lambda_{pa} \right]$$

$$= \frac{dc}{\partial(v_s - v_{sd})} \left[ \frac{\partial(v_s - v_{sd})}{\partial^\lambda pa} \cdot \lambda_{pa} + \frac{\partial(v_s - v_{sd})}{\partial^\lambda pa} \cdot \lambda_{pa} \right]$$

(60)

Since the coefficient of $\lambda_{pa}$ contains the sin $\lambda_{pa}$ and $\lambda_{pa}$ contains tan $i_f$, the second term in the bracket is neglected as in the derivation of Eq. (9).

Equation (58) may now be written as

$$\frac{da_s \text{ (min)}}{dt_a} = \frac{da_s \text{ (min)}}{dc} \cdot \frac{dc}{\partial(v_s - v_{sd})} \cdot \frac{\partial(v_s - v_{sd})}{\partial^\lambda pa} \cdot \lambda_{pa}$$

(61)

By using Eqs. (21), (30), and (59), the partial derivatives are evaluated and then substituted into Eq. (61). After simplification we have

$$\frac{da_s \text{ (min)}}{dt_a} = \sqrt{\mu_s \mu_p} \frac{r_{sd}}{r_{pa}} \frac{4}{4c} \sin (\lambda_{pa} - \lambda_{sd})$$

(61a)

**DERIVATION OF EQ. (13) OF METHOD C.1**

The assumptions made in the derivation of Eq. (12) are valid in this derivation

$$\frac{d(\Delta t_s)}{dt_a} = \frac{d[t_s \text{ (step 9)} - t_s \text{ (step 4)}]}{dt_a} = \frac{dt_s \text{ (step 9)}}{dt_a} - 1$$

(62)
The derivative $\frac{dt_s}{dt_a}$ (step 4) $= 1$ since for $t_d$ (fixed) a change in the
chosen value of $t_a$ causes an equal change in $t_s$.

$$\frac{dt_s}{dt_a} = \frac{\partial t_s}{\partial n} \frac{dn_s}{dt_a} + \frac{\partial t_s}{\partial \eta} \frac{d\eta}{dt_a} + \frac{\partial t_s}{\partial \eta_1} \frac{d\eta_1}{dt_a}$$  \hspace{1cm} (63)

For fixed $V_{sd}$ and $t_d$, $a_s$ is determined and

$$\frac{dn_s}{dt_a} = \frac{d\left( \sqrt{\mu_s/a_s} \right)}{dt_a} = 0$$

Using Eqs. (19), (20), (21), and (60) we get

$$\frac{d\eta}{dt_a} = \frac{\partial \eta}{\partial c} \frac{dc}{dt_a} = \frac{\partial \eta}{\partial c} \left( \frac{dc}{\partial (v_s - v_{sd})} \right) \frac{\partial (v_s - v_{sd})}{\partial \eta_p} \lambda_{pa}$$  \hspace{1cm} (64)

and

$$\frac{d\eta_1}{dt_a} = \frac{\partial \eta_1}{\partial c} \frac{dc}{dt_a} = \frac{\partial \eta_1}{\partial c} \left( \frac{dc}{\partial (v_s - v_{sd})} \right) \frac{\partial (v_s - v_{sd})}{\partial \eta_{pa}} \lambda_{pa}$$  \hspace{1cm} (65)

By replacing the total derivatives and simplifying, Eq. (62) becomes

$$\frac{d(\Delta t_g)}{dt_a} = \frac{\partial (v_s - v_{sd})}{\partial \eta_p} \lambda_{pa} \left[ \frac{\partial n_s}{\partial c} \frac{dt_s}{\partial \eta} + \frac{\partial \eta_1}{\partial c} \frac{dt_s}{\partial \eta_1} \right] - 1$$

$$= \frac{r_{sd} \sqrt{\mu_s \rho}}{c r_{pa}} \sin \left( \lambda_p - \lambda_{sd} \right) \left( \frac{\partial n_s}{\partial \eta} \frac{dt_s}{\partial \eta} + \frac{\partial \eta_1}{\partial c} \frac{dt_s}{\partial \eta_1} \right) - 1$$  \hspace{1cm} (66)
The partial derivatives of Eq. (65), which are given by Eqs. (26), (27), and (56a), are replaced, and we get

\[
\frac{d(\Delta t_s)}{dt_a} = \frac{r_{sd} \sqrt{a_s \rho}}{2c \ r_{pa}} \sin (\Lambda_{pa} - \Lambda_{sd}) \left( \tan \frac{\eta}{2} + \tan \frac{\eta_1}{2} \right) - 1 \ (67)
\]

The signs of the terms in parentheses depend on the type of transfer orbit as follows:

Direct = +, + \hspace{1cm} Perihelion = +, -

Aphelion = -, + \hspace{1cm} Indirect = -, -

**DERIVATION OF Eq. (16) OF METHOD D.1**

By choosing \( t_s \) in addition to the parameter \( t_d' \), the \( t_a \) is determined, and consequently \( (v_s - v_{sd}) \) is a constant. Again we assume that \( r_{pa} \) is constant.

\[
\frac{d\Delta t_s}{dv_{sd}} = \frac{dt_s (\text{step 9})}{dv_{sd}} = \frac{\partial t_s}{\partial n_s} \frac{dn_s}{dv_{sd}} + \frac{\partial t_s}{\partial \eta} \frac{d\eta}{dv_{sd}} + \frac{\partial t_s}{\partial \eta_1} \frac{d\eta_1}{dv_{sd}} \ (68)
\]

The total derivatives are written as follows:

\[
\frac{dn_s}{dv_{sd}} = \frac{d}{dv_{sd}} \left( \frac{s^{3/2}}{a_s^{5/2}} \right) = \frac{3s^{1/2}}{2a_s^{5/2}} \quad \frac{da_s}{dv_{sd}}
\]
Substitution into Eq. (68) gives

\[
\frac{dt}{dv_{sd}} = \left( -\frac{3\sqrt{\mu_s}}{a_s^{\frac{3}{2}}} \frac{dt}{v_s} + \frac{\partial t}{\partial \eta_1} \frac{\partial \eta_1}{\partial a_s} + \frac{\partial t}{\partial \eta_1} \frac{\partial \eta_1}{\partial a_s} \right) \frac{da_s}{dv_{sd}} \tag{69}
\]

\[
\frac{da_s}{dv_{sd}} = \frac{\partial a_s}{\partial e_s} \frac{de_s}{dv_{sd}} + \frac{\partial a_s}{\partial v_{sd}}
\]

\[
= \frac{\partial a_s}{\partial e_s} \left( \frac{de_s}{dv_s} \frac{dv_s}{dv_{sd}} + \frac{de_s}{dv_{sd}} \right) + \frac{\partial a_s}{\partial v_{sd}}
\]

Since \((v_s - v_{sd})\) is constant, \(dv_s = dv_{sd}\) and

\[
\frac{da_s}{dv_{sd}} = \frac{\partial a_s}{\partial e_s} \left( \frac{de_s}{dv_s} + \frac{de_s}{dv_{sd}} \right) + \frac{\partial a_s}{\partial v_{sd}} \tag{70}
\]

where

\[
\frac{\partial a_s}{\partial e_s} = \frac{a(1 + e_s^2) - r_{sd}}{e_s(1 - e_s)}
\]
\[ \frac{\partial c_s}{\partial v_s} = \frac{e_s^2 r_{pa} \sin v_s}{r_{sd} - r_{pa}} \]

\[ \frac{\partial e_s}{\partial v_{sd}} = -\frac{e_s^2 r_{sd} \sin v_{sd}}{r_{sd} - r_{pa}} \]

and

\[ \frac{\partial a_s}{\partial v_{sd}} = -\frac{e_s r_{sd} \sin v_{sd}}{1 - e_s^2} \]

Eq. (70) becomes

\[ \frac{da_s}{dv_{sd}} = \frac{e_s}{1 - e_s^2} \left[ \frac{a_s (1 + e_s^2) - r_{sd}}{r_{sd} - r_{pa}} (r_{pa} \sin v_s - r_{sd} \sin v_{sd}) - r_{sd} \sin v_{sd} \right] \]

(71)

The partial derivatives of Eq. (69) are given by Eqs. (23) - (27).

Using these equations and Eq. (71), Eq. (69) can be written as

\[ \frac{dt_s}{dv_{sd}} = \sqrt{\frac{a_s}{\mu_s}} \frac{e_s}{1 - e_s^2} \left[ \frac{a_s (1 + e_s^2) - r_{sd}}{r_{sd} - r_{pa}} (r_{pa} \sin v_s - v_{sd} \sin v_{sd}) - v_{sd} \sin v_{sd} \right] \]

\[ x \left[ + \frac{3}{2} a_s t_s \tan \eta/2 - (l - \cos \eta) \tan \eta/2 \right] \]
The multiple signs in the second brackets depend on the type of transfer orbit as follows:

Direct = -, +  
Aphelion = +, +  
Perihelion = -, -  
Indirect = +, -
REFERENCES

