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WADD TECHNICAL REPORT 61-187

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ASPECTS OF THE RESPONSE OF STRUCTURES SUBJECT TO SONIC FATIGUE

HASSEL C. SCHJELDERUP
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PASADENA, CALIFORNIA

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AERONAUTICAL SYSTEMS DIVISION

NOTICES

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FLIGHT DYNAMICS LABORATORY
CONTRACT Nr AF 33(616)-7341
PROJECT Nr 13456
TASK Nr 1370

AERONAUTICAL SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

300 - December 1961 - 12-461

FOREWORD

This report was prepared by National Engineering Science Company, Pasadena, California, on Air Force Contract AF33(616)-7341 under Project Nr 1370, "Dynamic Problems in Flight Vehicles", Task Nr 13456, "Resonant Fatigue of Aircraft Structures". The work was administered under the direction of Flight Dynamics Laboratory, Aeronautical Systems Division. Mr. Maurice J. Cote was task engineer for the laboratory.

The studies presented began in May 1960, and were concluded in March 1961. The nominal authors performed the bulk of the work, and other members of the NESCO Technical Staff contributed in their specialties as required. H. C. Schjelderup, NESCO Associate Director of Engineering, served as Project Manager in addition to being one of the principal researchers.

Necessary oscillogram reading and data digitizing was performed by a subcontractor, Associated Aero-Science Corporation, Hawthorne, California.

The cooperation of Boeing Airplane Company, Douglas Aircraft Company, General Dynamics Corporation, and Lockheed Aircraft Company is gratefully acknowledged. Divisions of those corporations provided the data which was used in the experimental portions of the studies.

This report concludes the work on Contract Nr AF 33(616)-7341.

ABSTRACT

The stress in aircraft structure resulting from multi-mode response to sonic excitation is resolved into an alternating stress component superimposed upon a slowly varying mean stress component. It is then found that the probability distribution of those components is nearly independent of the number of modes participating in the response. This finding could have considerable application in simplifying fatigue analysis and testing if it may be shown that the mean stress component has only low order effects upon fatigue life. Some of the possible applications are presented.

A test program for establishing the significance of the mean stress component is outlined.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

William C. Nielsen

WILLIAM C. NIELSEN
Colonel, USAF
Chief, Flight Dynamics Laboratory

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SYMBOLS

A_n	Instantaneous value of the envelope of response in the n_{th} mode
a, b, i, j, k, n	Subscripts denoting mode
N	Number of stress crests or troughs
P(Q)	Probability integral
p(q)	Probability density $= -\frac{d}{dQ} P(Q)$
S	Stress
S_a	Instantaneous peak alternating stress
S_c	Stress crest
S_m	Instantaneous mean stress
S_t	Stress trough
δ	1/2 frequency difference between two adjacent modes
σ	Root mean square stress
ϕ_n	Random phase angle, in n_{th} mode
ψ	Random phase angle
Ω	Average frequency of two adjacent modes
ω_n	Natural frequency in n_{th} mode
Q, R, Z, ζ, τ	Variables of integration

1.0 INTRODUCTION

This report presents the results of a series of analytical and experimental investigations performed to determine the probability distribution of the stress response of structures exposed to high intensity acoustic noise. The object of the investigations was to reduce the number of parameters required to describe the response of structures which may be subject to fatigue damage caused by acoustic noise. Any such reduction could lead to simpler and more accurate predictions of fatigue life.

Accurate prediction of fatigue life of structures loaded by random phonomena is presently conceded to be beyond the state of the art. The reasons for the difficulty are two-fold:

1. Techniques for accurate prediction of the local stress-time history have not been developed sufficiently.

2. Even when the stress-time history is known with sufficient accuracy it is seldom possible to make a direct comparison between the known stress-time history and one whose effects are known through laboratory testing.

Neither of those difficulties is unique to fatigue. However, the latter difficulty in particular is more marked in long life fatigue than it is in high stress loading. This is true in spite of the great amount of laboratory testing which has been performed in the past for determining the fatigue characteristics of materials because the great majority of that testing has been conducted using ideal loading conditions which are rarely duplicated in real structure.

The more widely recognized difference between typical laboratory loading procedures and typical conditions of actual loading has been the existence in many practical structures of a wide range of stress levels applied in random sequence, whereas laboratory tests have principally been restricted to repeated applications to failure of a single stress level. Many theories have been proposed (References 1, 2, 3, 4) for correlating the results of these laboratory fatigue tests with random level loading, and success may be attained within the forseeable future.

However, success in that correlation may not provide a sufficient tool for predicting life, because those efforts to derive theories of cumulative damage neglect an additional difference between laboratory loading procedures and the common conditions of life. That difference is illustrated in Figure 1, where it is indicated that laboratory tests almost invariably consist of repeated cycles of alternating stress about a constant mean stress. This is true whether the more common, single level test (Figure 1-a) or the two or more level tests performed to gain knowledge about the stress interaction (Figure 1-b) are used. Figure 1-c, illustrating a typical aircraft stress record, differs from the laboratory tests in that it does not have a horizontal axis of stress symmetry. The significance of the lack of symmetry has not been established.

In the extreme, the lack of symmetry could be, as Weibull (Reference 5) has suggested, sufficiently influential that it will be impossible to correlate laboratory

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Manuscript released by the authors 14 March 1961 for publication as a WADD Technical Report.

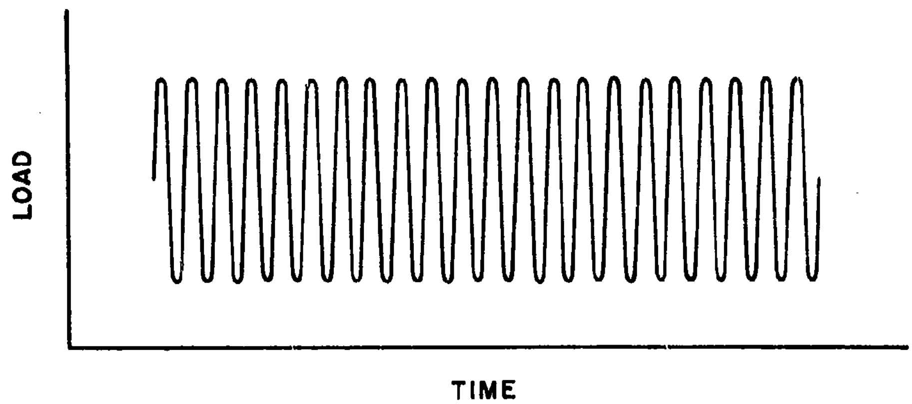

FIG. 1a SINGLE LEVEL LOADING IN FATIGUE TESTING

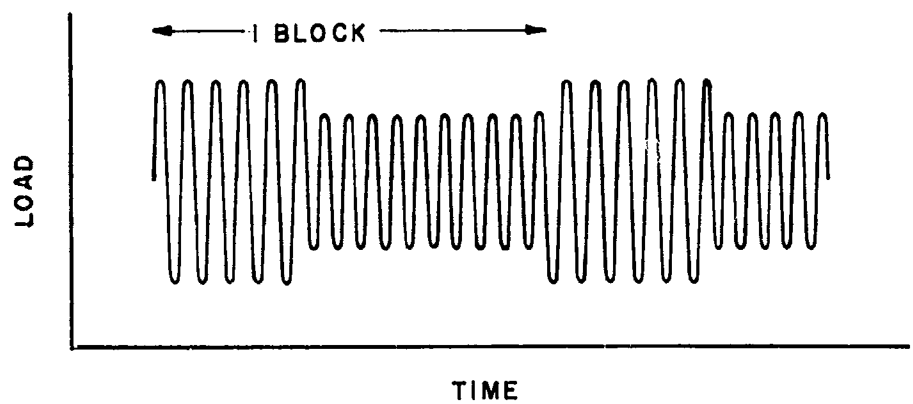

FIG. 1b TWO LEVEL FATIGUE TESTING

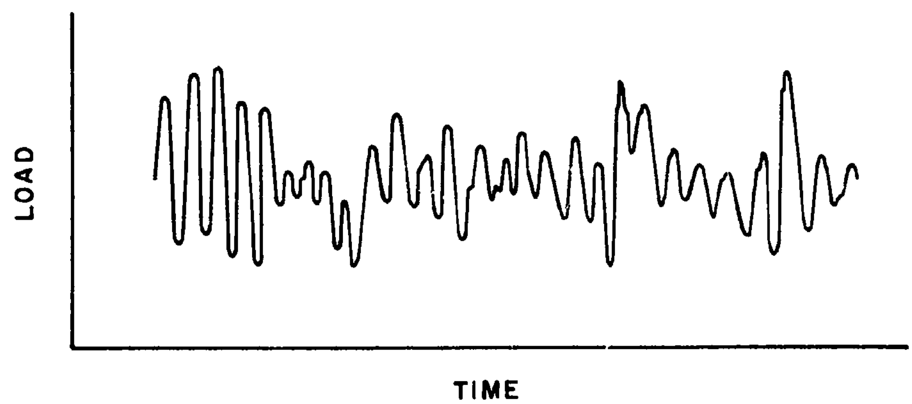

FIG. 1c TYPICAL MULTI-MODE RANDOM STRESS

tests with experience unless the asymmetry of the aircraft stress is essentially reproduced in the laboratory. If that position is found to be correct, then the efforts being expended by students of stress interaction and cumulative damage are of little value, since every real structure will have different asymmetries because of different modal participation in the total stress.

Another approach to the topic of dealing with asymmetric stress patterns, suggested by Schjelderup (Reference 6), involves recognizing that any stress, no matter how complex, can be transformed into a superposition of alternating stresses* and mean stresses.** With that transformation, the alternating stresses become symmetrical by definition. At the same time the variance of the mean stress frequently becomes so small relative to the variance of the alternating stress that it may be approximately correct to replace the mean stress by its long-time average. If this approach may be shown to be valid, then a theory of cumulative damage which has been demonstrated to be valid for the symmetrical stress patterns of the laboratory may be applied to the asymmetric stress patterns of experience.

Schjelderup (Reference 6) observed further that, for the particular stress records he examined, there was considerable consistency in the normalized probability distribution of alternating stresses. If such consistency could be shown to be general, then it would be appropriate to concentrate the effort presently being devoted to multi-level fatigue testing into tests using the distribution of nature. The results of that type of testing could be random S-N curves which could be used directly for predicting life, without recourse to theories of cumulative damage.

If Schjelderup's suggestions and tentative conclusions could be shown to be valid, the number of parameters which are required for describing the response of structures and which must be simulated in the laboratory would be significantly reduced. These suggestions and conclusions are therefore examined herein with the objective of providing guidance for future fatigue analysis and testing.

2.0 RESPONSE OF STRUCTURE TO ACOUSTIC LOADING

2.1 Analytic Determination of Probability Distributions

Oscillatory stresses which can cause fatigue may be excited by a variety of sources. The sonic fatigue which is the subject of this report is most frequently excited by turbulence in the wake of a jet engine, or by boundary layer turbulence. Both of these sources are random and broad band, and the stress response to those sources may therefore be described most generally by its power spectrum.

Inspection of typical spectral densities of measured or calculated stress usually shows the effects of a structural filtering process which transforms the broad band excitation into stress response contained primarily in a small number of rather

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* Instantaneous peak alternating stress is defined as 1/2 the difference between a stress crest and the succeeding stress trough.

** Instantaneous mean stress is defined as the stress midway between a stress crest and the succeeding stress trough.

narrow frequency bands. The instantaneous stress may therefore be described approximately by Equation (1)

$$S(t) = \Sigma\, A_n(t) \cos(\omega_n t + \phi_n) \qquad (1)$$

In Equation (1), ω_n is the natural frequency in the n_{th} structural mode. $A_n(t)$ is the instantaneous value of a random function with root mean square (rms) σ_n, and ϕ_n is the instantaneous value of a random phase angle. If the excitation is Gaussian (normal) and stationary, and the structure responds linearly, then it has been shown (Reference 7) that A_n and ϕ_n vary slowly as compared to $\cos \omega_n t$, and the probability density of A_n is Rayleigh (Equation 2).

$$p(A_n) = \frac{A_n}{\sigma_n^2} \exp\left[-\frac{A_n^2}{2\sigma_n^2}\right] \qquad (2)$$

Occasionally a single mode, the i_{th}, is excited to the effective exclusion of other natural modes of the structure. In that event, Equation (1) is reduced to Equation (3)

$$S = A_i \cos(\omega_i t + \phi_i) \qquad (3)$$

The slow variation of A_i and ϕ_i as compared to the variation of $\cos \omega_i t$, allows Equation (3) to be construed as a quasi-sinusoid, with negligible* instantaneous mean stress and with peak values of alternating stress having the same Rayleigh probability distribution as does A_i. This limiting case of structural response to acoustic noise is of particular interest because most quantitative analyses of acoustic fatigue in the past have assumed symmetrical stress with Rayleigh distribution of peaks, and have, therefore, tacitly assumed response in a single mode.

More general and more realistic response spectral densities, describing stress in several modes, may now be considered. Two limiting, but common, cases of multi-modal response will be studied separately; one where the modal frequencies are widely separated and the other where the frequencies are closely spaced.

Considering first the case where a number of well separated modes, up to the K_{th}, participate in the response to the same order of magnitude and to the approximate exclusion of other modes, Equation (1) becomes Equation (4),

---

* A steady force may exist in addition to the random noise excitation. In that event, the instantaneous mean stress may not be small, but its variation from the average mean stress will be small.

$$S = A_1 \cos(\omega_1 t + \phi_1) + \text{----------} + A_K \cos(\omega_K t + \phi_K) \tag{4}$$

Because the A_n's and ϕ_n's vary slowly, the time derivatives of $S(t)$ may be approximated by Equations (5) and (6),

$$\frac{dS}{dt} = -\omega_1 A_1 \sin(\omega_1 t + \phi_1) + \text{-------} - \omega_K A_K \sin(\omega_K t + \phi_K) \tag{5}$$

$$\frac{d^2 S}{dt^2} = -\omega_1^2 A_1 \cos(\omega_1 t + \phi_1) + \text{------} - \omega_K^2 A_K \cos(\omega_K t + \phi_K) \tag{6}$$

Since ω_K/ω_J has been assumed (well separated modes) to be much greater than one, and each mode has been assumed to be participating to the same order of magnitude, the rate of change of slope (Equation 6) is dominated by the K_{th} term. Therefore, d^2S/dt^2 is large when the K_{th} term in Equation (5) is zero. Consequently dS/dt (Equation 5) will be zero (and define an extrema in Equation 4) within a short time range about the time when the K_{th} term of Equation (5) is zero. Thus, extrema in the composite stress occur at time intervals of, on the average, π/ω_K. With that knowledge, the alternating and mean stress may be defined by Equations (7) and (8) respectively:

$$S_A = A_K + \frac{1}{2} \sum_{n=1}^{J} An \left\{ \cos(\omega_n \tau + \phi_n) - \cos(\omega_n \tau + \frac{\omega_n}{\omega_K} \pi + \phi_n) \right\} \tag{7}$$

$$S_m = \frac{1}{2} \sum_{n=1}^{J} An \left\{ \cos(\omega_n \tau + \phi_n) + \cos(\omega_n \tau + \frac{\omega_n}{\omega_K} \pi + \phi_n) \right\} \tag{8}$$

where τ is any time when,

$$\omega_K \tau + \phi_n = m\pi \qquad m = 0, 1, 2 \cdots\cdots \tag{9}$$

By trigonometric identities, Equations (7) and (8) may be transformed to Equations (10) and (11) respectively:

$$S_A = A_K + \sum_{n=1}^{J} An \sin\left(\frac{\omega_n}{\omega_K} \frac{\pi}{2}\right) \sin(\omega_n \tau + \phi_n') \tag{10}$$

$$S_m = \sum_{n=1}^{j} An \operatorname{Cos}\left(\frac{\omega_n}{\omega_K}\frac{\pi}{2}\right) \operatorname{Sin}(\omega_n \tau + \phi_n'') \tag{11}$$

The alternating stress given by Equation (10) consists of a term, A_k , with Rayleigh probability distribution, plus the sum of (k - 1) terms whose probability distribution is determined by the Rayleigh modulation of a sinusoid. In a similar manner, Equation (11) defines the mean stress as the sum of (k - 1) terms, each term having distribution determined by the Rayleigh modulation of a sinusoid.

A proof is developed in Appendix I of this report which shows that a Rayleigh modulated sinusoid has normal probability. Because the sum of any number of terms, each having normal distribution, has normal distribution, (Reference 8) the mean stress has normal probability. The rms of the instantaneous mean stress, σ_m , is the root of the sum of the squares of its components, as given by Equation (12),

$$\sigma_m = \sqrt{\sum_{n=1}^{j}\left(\sigma_n \operatorname{Cos}\frac{\omega_n}{\omega_K}\frac{\pi}{2}\right)^2} \tag{12}$$

As in the previous, single mode case, the average value of the mean stress is zero unless a steady-state force exists in addition to the random excitation.

Similarly, the alternating stress is composed of a term with Rayleigh distribution, rms = σ_K , and a term with normal distribution rms = σ_Q given by Equation (13),

$$\sigma_Q = \sqrt{\sum_{n=1}^{j}\left(\sigma_n \operatorname{Sin}\frac{\omega_n}{\omega_K}\frac{\pi}{2}\right)^2} \tag{13}$$

The probability density resulting from the sum of a term with Rayleigh distribution and a term with normal distribution is developed in Appendix I, and is given by Equation (14),

$$p(S_A) = \frac{\sigma_Q \exp\left[-\frac{S_A^2}{2\sigma_Q^2}\right]}{\sqrt{2\pi}(\sigma_K^2+\sigma_Q^2)} + \frac{S_A\sigma_K \exp\left[-\frac{S_A^2}{2(\sigma_K^2+\sigma_Q^2)}\right]}{2(\sigma_K^2+\sigma_Q^2)^{3/2}}\left[1 + \operatorname{erf}\frac{S_A\sigma_K}{\sigma_Q\sqrt{2(\sigma_K^2+\sigma_Q^2)}}\right] \tag{14}$$

Equation (14) is plotted as Figure 2, for several values of σ_K/σ_Q . The compound distribution is seen to approach a Rayleigh density rapidly for the larger values σ_K/σ_Q , as would result from the assumed large values of ω_K/ω_j .

On the other hand, if ω_K/ω_j is not significantly greater than one, then an essential assumption in the preceding development is violated and new equations must

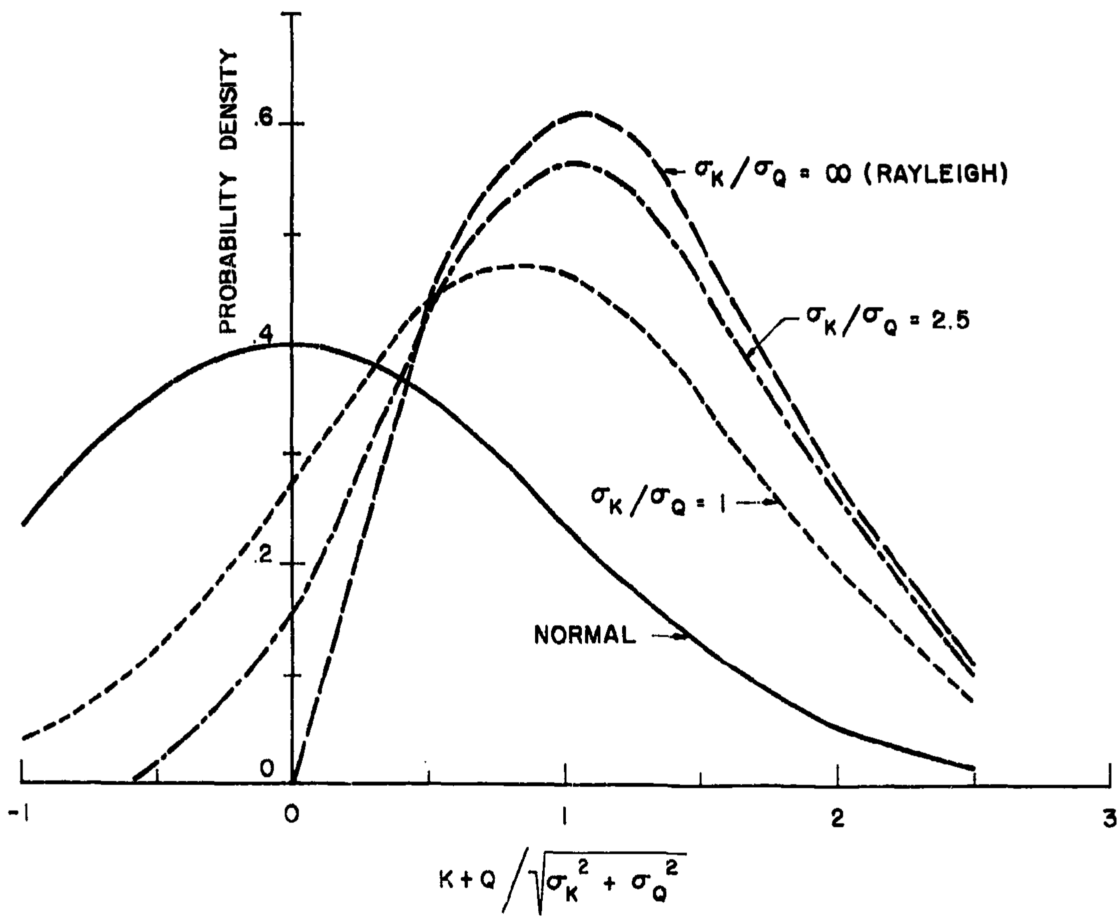


FIGURE 2 COMPOUND PROBABILITY OF K + Q
WHEN
K HAS RAYLEIGH PROBABILITY, rms = σ_K Q HAS NORMAL PROBABILITY, rms = σ_Q

be derived. The probability distribution of mean and alternating stresses for two closely spaced modes of response, the j_{th} and k_{th}, is now determined by a separate development. It will be convenient in this development to use the functions $(\Omega-\delta)$, $(\Omega+\delta)$, instead of ω_j, ω_K. Equation (1) particularized to two modes, may be written in the new nomenclature as Equation (15).

$$S = A_j \operatorname{Cos}\left[(\Omega-\delta)t + \phi_j\right] + A_k \operatorname{Cos}\left[(\Omega+\delta)t + \phi_k\right] \tag{15}$$

The use of trigometric identities allows Equation (15) to be rewritten as Equation (16),

$$S = \sqrt{A_j^2 + A_k^2 + 2A_jA_k \operatorname{Cos}(2\delta t + \phi_k - \phi_j)}\ \operatorname{Sin}(\Omega t + \psi) \tag{16}$$

with the phase angle, ψ, defined:

$$\psi = \operatorname{Tan}^{-1} \frac{A_k \operatorname{Sin}(\delta t + \phi_j) - A_j \operatorname{Sin}(\delta t - \phi_k)}{A_k \operatorname{Cos}(\delta t + \phi_j) + A_j \operatorname{Cos}(\delta t - \phi_k)} \tag{17}$$

The phase angle may be differentiated to show that $\frac{d\psi}{dt}/\Omega$ is of the same order of magnitude as δ/Ω, and is therefore small if the modes are closely spaced. Consequently, Equation (15) may be construed to describe a quasi-sinusoid with frequency,

$$\Omega^* = \Omega + \frac{d\psi}{dt} \simeq \Omega = \frac{\omega_j + \omega_K}{2} \tag{18}$$

modulated by the radical of Equation (16). The probability distribution of the radical is the probability distribution of alternating stress of the composite wave; the mean stress, as in the case of single mode response, has slight variation from its average value.

The probability distribution of the radical is examined in the Appendix where it is shown to be Rayleigh for the case when the root mean square of the j_{th} mode and the k_{th} mode are equal. The general case, $\sigma_j \neq \sigma_K$, presents difficulties in obtaining a closed form, or rigorous proof that the same distribution holds, but it appears likely that the probability distribution is Rayleigh for any ratio of σ_j/σ_K.

The preceeding analytic developments have determined the probability distributions of mean and alternating stress for several idealized cases of structural response to random noise. If these results are extrapolated to the general case of structural response, it may be concluded that the probability distribution of instantaneous mean stress will, in general, be normal, and the distribution of alternating stresses will be between normal and Rayleigh, most frequently approaching more closely to the Rayleigh distribution.

2.2 Experimental Probability Distribution Studies

Data Selection

In order to verify the theoretical results presented above and to determine the effects of the non-linearities which exist in real structure, oscillograms of strain recorded on typical structure subjected to high intensity acoustic loading were obtained. Oscillograms were obtained from:

1. Douglas Aircraft Company, Santa Monica Division
2. Boeing Airplane Company, Wichita Division
3. Convair Division of General Dynamics, Fort Worth
4. Lockheed Aircraft Company, California Division

The oscillograms obtained from Douglas were recorded during ground runs of an RB-66. Boeing provided records made during runs of a B-52, and Convair forwarded results of some B-58 engine runs. The Lockheed records were taken from some specially prepared test panels made to simulate typical aircraft structure. The collected records represent strain recorded on Rib-Skin, Stiffened Skin, Honeycomb and Corrugated Web types of structure, and therefore include the major types of structure which have been damaged by acoustic noise.

Examinations of the power spectral density of strain for each of the collected records showed that the records encompassed a broad range of modal participation, from response in a single lightly damped mode to wide band multi-mode response. Therefore, it was possible to select a total of 24* records and cover, with some duplication, all of the typical structural types plus a wide range of typical modal response. Table I lists the selected traces and presents brief descriptions of the structural type and the modal response represented by each record.

Data Processing and Analysis

The selected records were prepared for data processing and analysis by marking all of the significant crests and troughs. A significant crest (or trough) was differentiated from a minor perturbation, for purposes of data analysis, by application of the requirement that an extremum is included only if it defined an alternating stress peak equal or greater than one-fourth (1/4) the maximum alternating stress determined from a cursory inspection of an entire record. Justification of this truncation procedure is contained in Appendix II.

The data contained on the annotated records, consisting of approximately 2,000 valid crests and troughs for each record, were then translated into the

---

* 23 of the selected records represented raw data and one represented filtered data which was analyzed for a check on procedures of data analysis.

SUMMARY OF PROCESSED STRAIN RECORDS

TABLE I

Record Number	Recorded At	Type Construction	Description of Modal Response of Strain
1	Douglas	Rib Skin, Aluminum	Principally response in a single narrow mode, frequency 490 cps.
2	Douglas	Rib Skin, Aluminum	Response at 450 cps. and 490 cps. Root mean square of higher frequency mode approximately 1.5 times greater than root mean square stress of lower mode.
3	Douglas	Rib Skin, Aluminum	Response at 465 cps. and 515 cps. Root mean square of higher frequency mode approximately 1.8 times greater than that of low frequency mode.
4	Douglas	Rib Skin, Aluminum	Response at 485 cps., 528 cps, and 575 cps. Root mean square of lower frequency mode approximately 1.3 times that of each of the two higher frequency modes.
5	Douglas	Rib Skin, Aluminum	Response at 425 cps., 480 cps. and 575 cps. Root mean square of center mode approximately 2 times greater than that of each of the side-bands.
6	Douglas	Rib Skin, Aluminum	Multi-modal response over the frequency range 350 cps. - 600 cps.
7	Convair	Honey comb, stainless steel	Multi-modal response between 200-500 cps., fairly prominent 50 cps. response.
8	Convair	Corrugated Web-Cap, Aluminum	Broad-band response between 275-260 cps. and fairly prominent 50 cps. response.
9	Convair	Corrugated Web-Cap, Aluminum	Response in a moderately narrow band centered at 110 cps. and nearly white noise below 100 cps.
10	Convair	Corrugated Web-Cap, Aluminum	Response in a broad-band centered at 175 cps. and in a narrower band centered at 450 cps. White noise below 100 cps.
11	Convair	Same as 12	Same as 12, but passed through a 10% band width filter, center frequency 290 cps.
12	Convair	Bonded-beaded panel, Aluminum	Response primarily in a narrow band centered at 290 cps. and a broad-band centered at 50 cps.
13	Boeing	Honeycomb, Aluminum	Response primarily in a broad-band between 180 cps. and 400 cps. Lower level multi-modal (nearly white) response below 100 cps.
14	Boeing	Honeycomb, Aluminum	Multi-modal response from 20 cps. to 400 cps. Fairly prominent mode at 10 cps.
15	Boeing	Honeycomb, Aluminum	Significant response at 10 cps. and 275 cps.
16	Boeing	Honeycomb, Aluminum	Response primarily in moderately wide band centered at 200 cps.
17	Boeing	Honeycomb, Aluminum	Response in moderately wide bands, at 140 cps. and 280 cps. Lower multi-mode response below 100 cps.
18	Boeing	Honeycomb, Aluminum	Broad-band response at 10 cps. and narrow-band response at 500 cps.
19	Lockheed	Rib-Skin Test Panel, Aluminum	Narrow modes at 125 cps. and 175 cps.
20	Lockheed	Rib-Skin Test Panel, Aluminum	Narrow bands at 130 cps. and 190 cps.
21	Lockheed	Fixed Edge Panel, Aluminum	Moderately broad-band response centered at 120 cps.
22	Lockheed	Fixed Edge Panel, Aluminum	Similar to (21) but higher magnitude.
23	Lockheed	Fixed Edge Panel, Aluminum	Narrow band response at 175 cps. Additional lower level response at 275 cps. and 410 cps.
24	Lockheed	Fixed Edge Panel, Aluminum	Narrow modes at 150 cps. and 270 cps.

punched card form suitable for digital computer analyses. A Telemeter Magnetics Inc. "Telereader" and an IBM 523 Summary Key Punch were used for that operation.

A Burroughs 220 digital computer was programmed to receive the cards containing the data and to determine for each record the average mean, and the mean square of the crests, troughs, alternating stresses, and mean stresses. Probability histograms (using intervals of 1% of the respective ranges) were computed for the mean and alternating stresses and the stress crests and troughs.

Results and Discussion

The pertinent results of the data analysis are presented in Table II and Figures 3-15.

Table II, "Some Statistics of the Strain Records," contains the measured and calculated values, (in arbitrary units) of the root mean square total stress, σ', alternating stress σ_a and mean stress σ_m. The rms total stress is calculated from the sum of the squares of the stress crests and troughs, divided by $\sqrt{2}$ in order to approximate the rms of the instantaneous total stress, as is common practice, instead of the rms of the peak values of total stress. The directly computed rms alternating stress is similarly divided by $\sqrt{2}$ to yield σ_a.

When σ_a is normalized with respect to σ', excellent consistency among the records is observed. Therefore, Table II may be accepted as providing empirical evidence that the rms alternating stress may be estimated from the rms total stress with good accuracy and without the necessity for performing the formal procedure of resolving total stress into alternating and mean components.

The normalization does not reveal a similar quantitative consistency in the mean stress component. It does, however, confirm that for the records examined, the variance of the mean stress is substantially smaller than the variance of the total stress.

Table II also contains the range of stress crests and stress troughs and the ratio of the ranges. These values are included because the ratio of the ranges is a convenient measure of the degree of one type of non-linearity which occasionally exists; ratios substantially different from one imply considerable difference in stiffness for positive stress and for negative stress. It is observed that the non-linearities have no apparent effect upon the normalized alternating stress.

Figures 3 to 9 present the measured probability integrals of (normalized) alternating stresses plotted on truncated Rayleigh probability paper, so that a measured distribution which conforms with the theoretical probability distribution will plot as a straight line (see Appendix II). The conformity of the test data to straight lines is, in general, very good. The greatest deviations from straight lines occur near the truncation level of strain and the high value (low probability) levels of strain. Deviations from linearity near the truncation can easily be attributed to human errors in data processing, which result from the annotator of the record attempting to be conservative and including some levels which were slightly below the nominal truncation level. Data points representing high stress levels have an apparent tendency to fall below the line. If that tendency is real, evidence of peak clipping is presented; on the other hand, the high stress points are derived

TABLE II

Some Statistics of the Strain Records

Record Number	σ' RMS Total Stress	σ_a RMS Alt. Stress	$\frac{\sigma_a}{\sigma'}$	σ_m RMS Mean Stress	$\frac{\sigma_m}{\sigma'}$	R_c Range of Crests	R_t Range of Troughs	$\frac{R_c}{R_t}$
1	63.0	61.5	.98	22.5	.36	241	208	1.15
2	65.5	62.5	.96	32.2	.49	286	280	1.02
3	58.0	56.0	.97	21.2	.37	281	232	1.21
4	190.0	186.0	.98	54.0	.29	619	669	.92
5	164.0	159.0	.97	48.4	.30	585	571	1.02
6	87.5	82.5	.95	41.8	.48	387	390	.99
7	176.0	174.0	.99	53.0	.30	632	617	1.03
8	144.0	138.0	.96	59.0	.41	439	813	.54
9	168.0	162.0	.96	65.0	.39	484	836	.58
10	283.0	272.0	.96	114.0	.40	1044	1103	.95
12	324.0	319.0	.98	89.0	.27	855	934	.92
13	469.0	459.0	.98	144.0	.31	1387	1486	.90
14	113.0	107.0	.95	65.0	.57	500	598	.84
15	314.0	304.0	.97	108.0	.34	1102	879	1.25
16	330.0	326.0	.99	82.0	.25	233	1112	1.20
17	506.0	490.0	.97	179.0	.37	1965	1245	1.57
18	198.0	194.0	.98	61.5	.31	673	611	1.10
19	268.0	264.0	.98	64.5	.24	748	939	.80
20	213.0	210.0	.99	53.0	.25	564	646	.86
21	159.0	154.0	.97	55.5	.35	486	562	.86
22	229.0	218.0	.95	97.0	.42	796	955	.83
23	182.0	178.0	.98	52.5	.29	590	561	1.05
24	183.0	179.0	.98	54.5	.30	645	556	1.16
Average			.972					
Standard Deviation			.013					

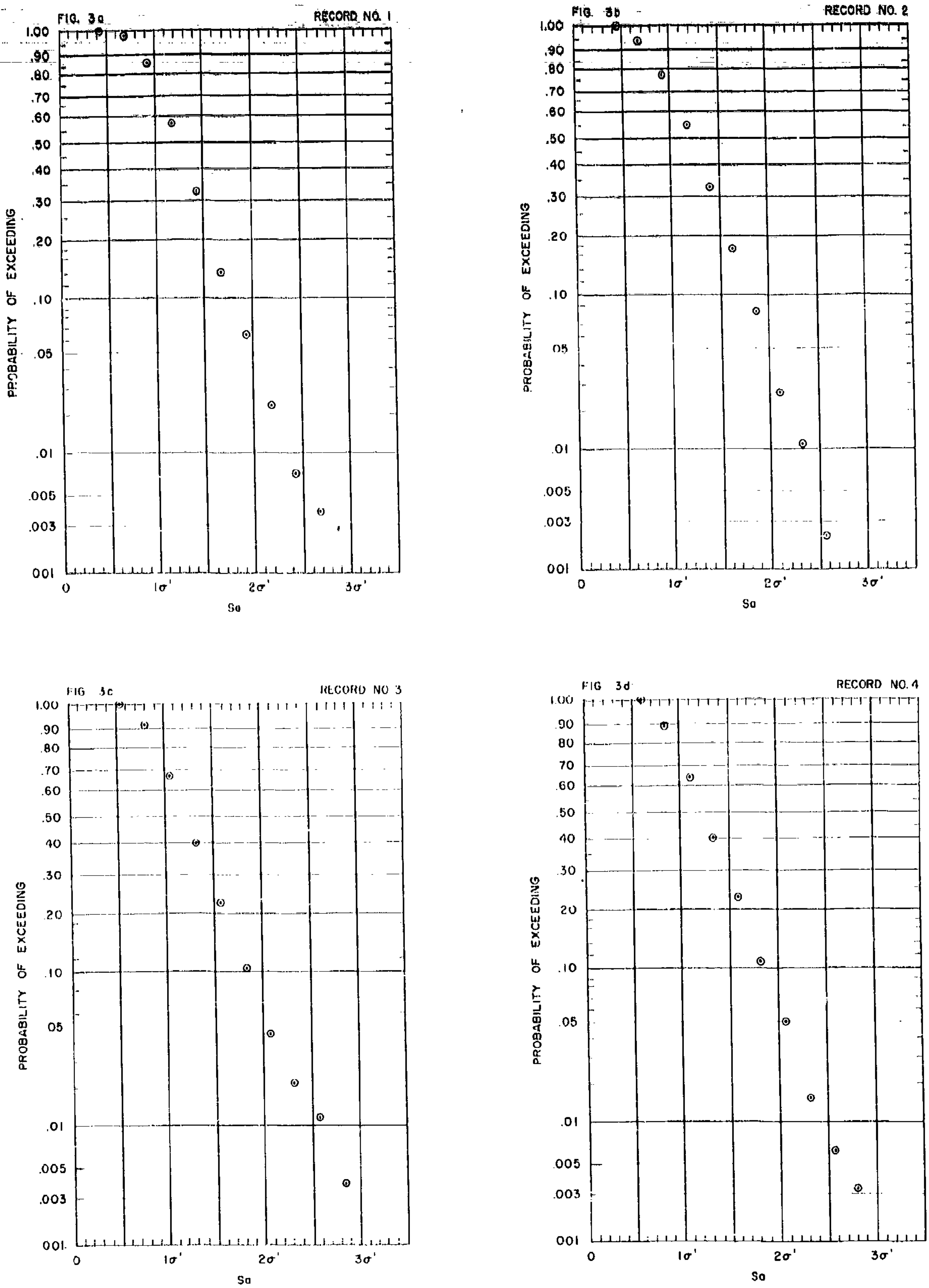


FIGURE 3 PROBABILITY INTEGRAL OF ALTERNATING STRESS PEAKS RECORD NOS. 1, 2, 3, & 4

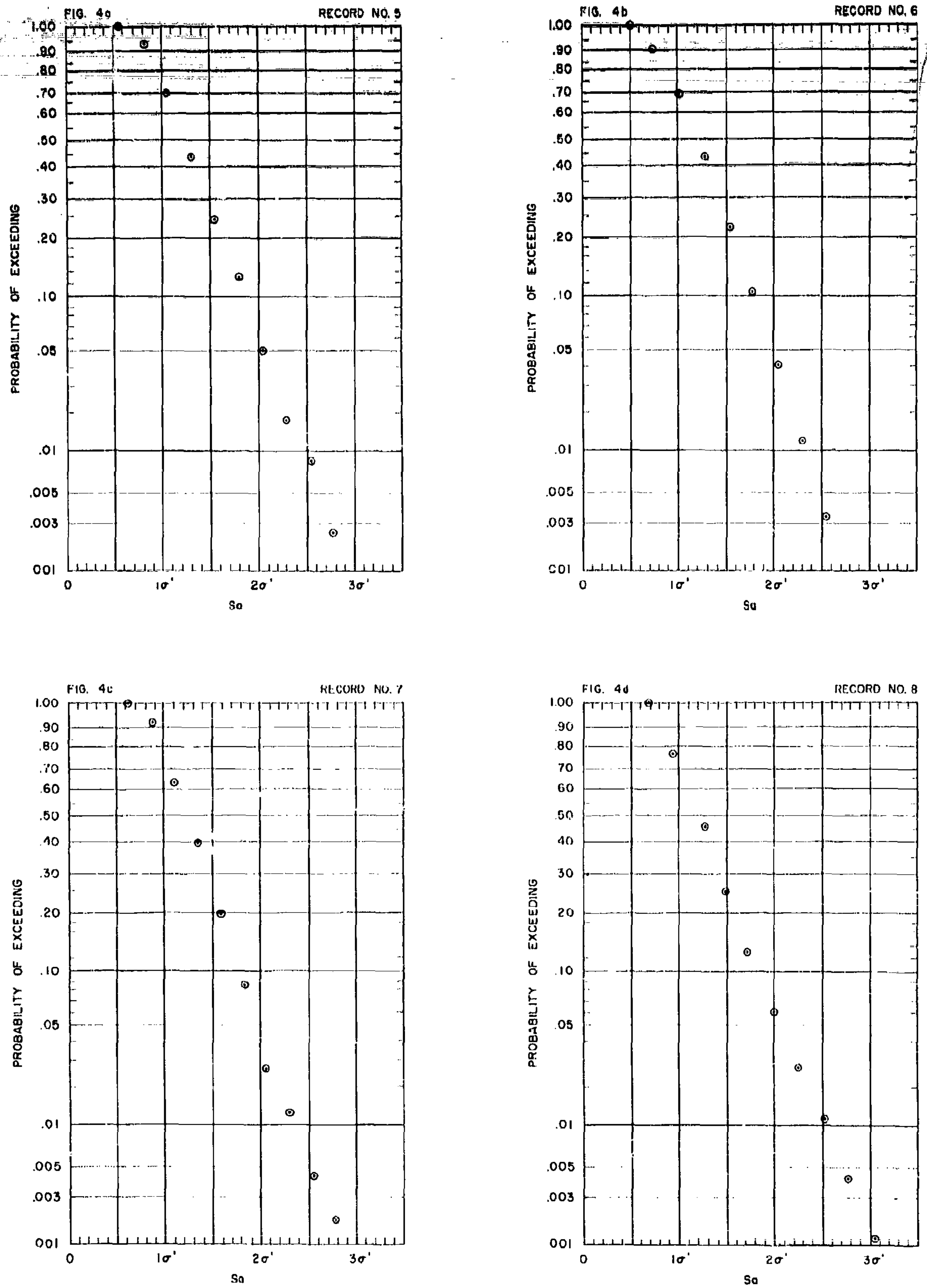


FIGURE 4 PROBABILITY INTEGRAL OF ALTERNATING STRESS PEAKS RECORD NOS. 5, 6, 7, & 8

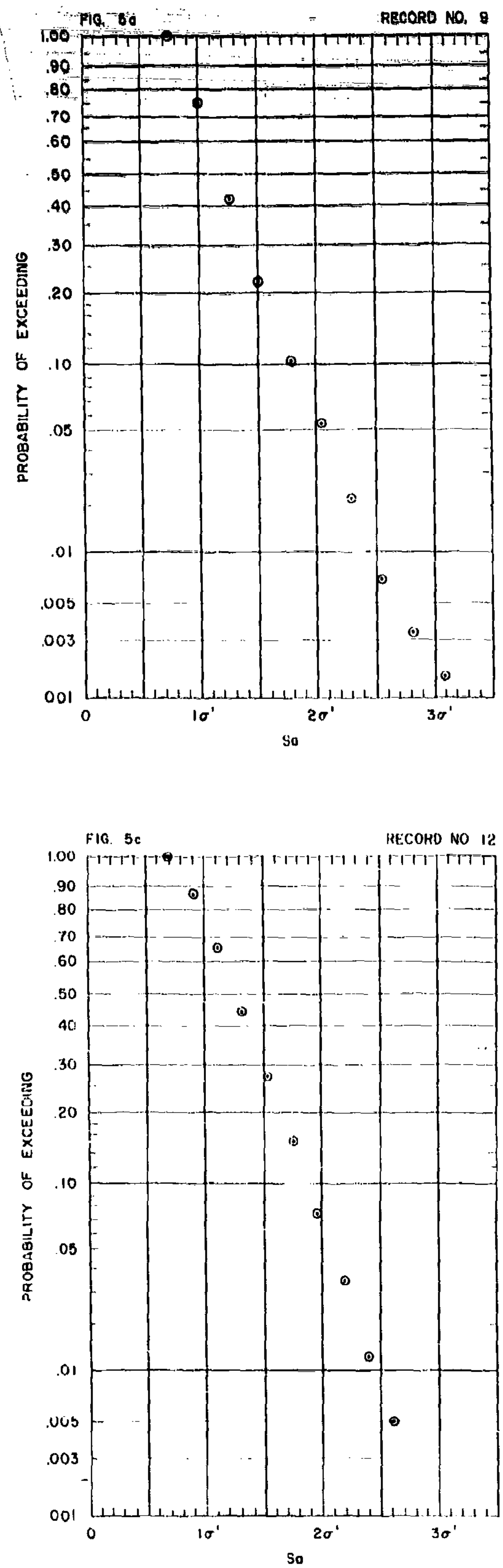


FIGURE 5 PROBABILITY INTEGRAL OF ALTERNATING STRESS PEAKS RECORD NOS. 9, 10, 12, & 13

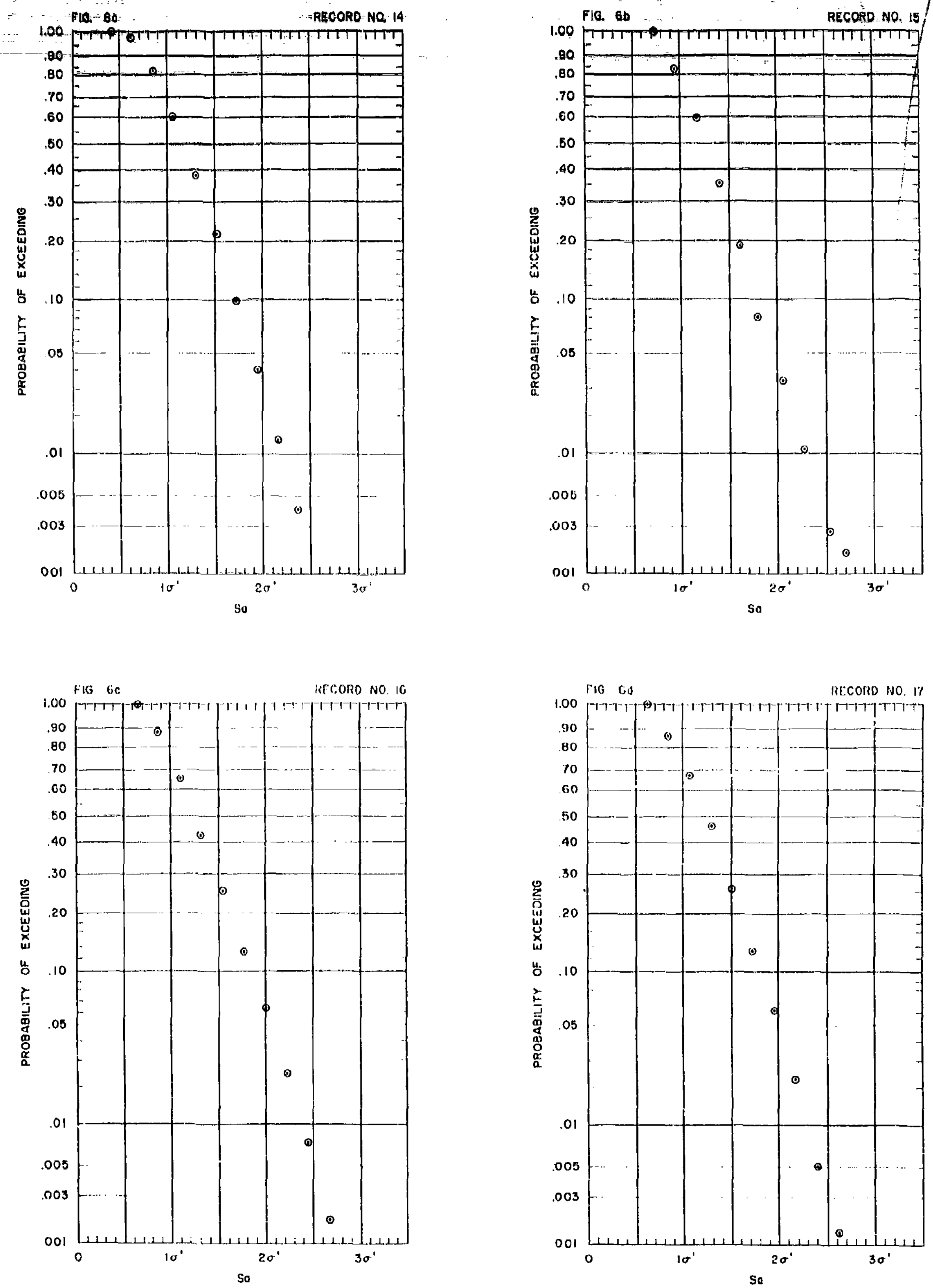


FIGURE 6 PROBABILITY INTEGRAL OF ALTERNATING STRESS PEAKS RECORD NOS. 14, 15, 16, & 17

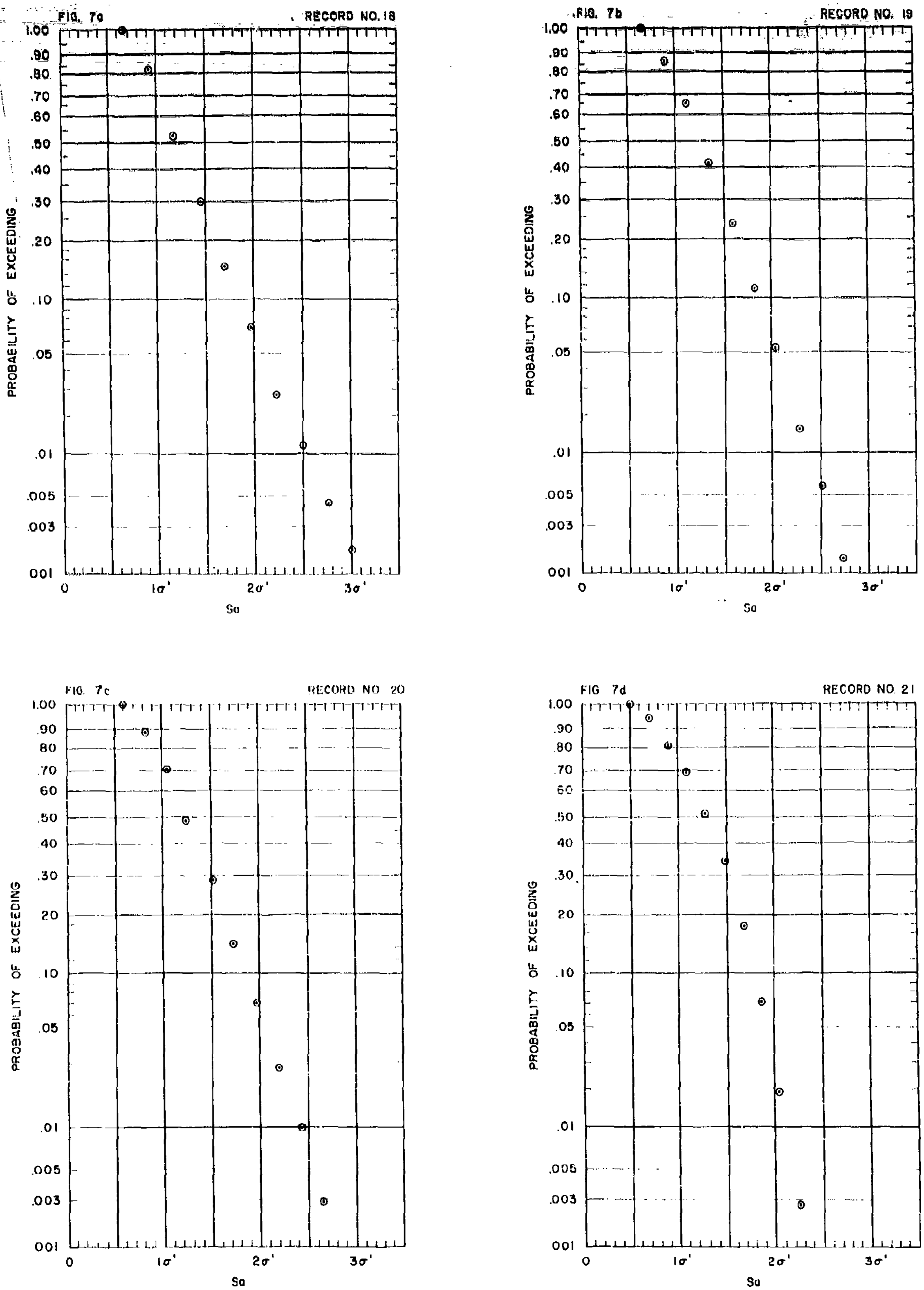
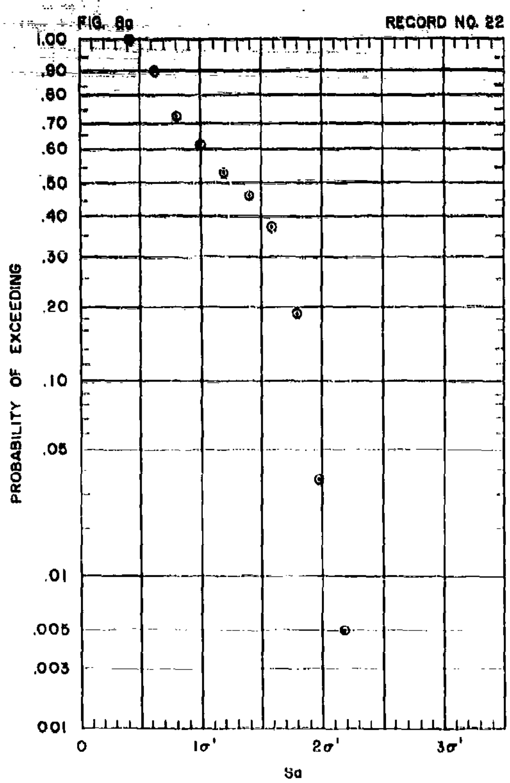
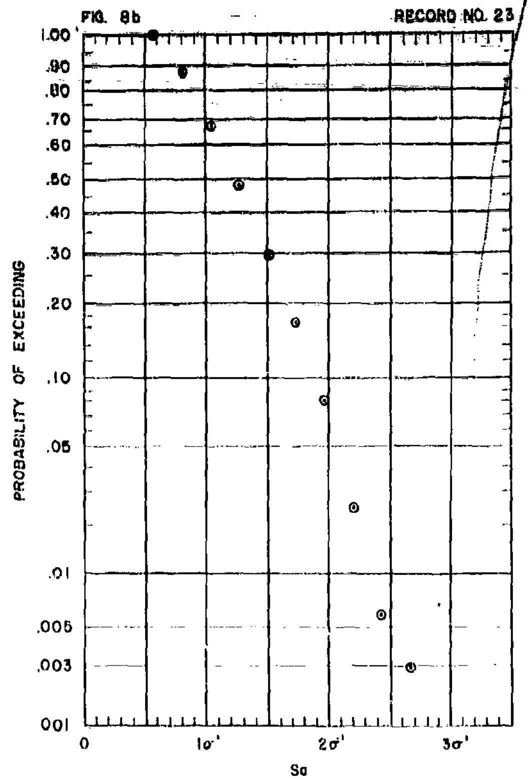


FIGURE 7 PROBABILITY INTEGRAL OF ALTERNATING STRESS PEAKS RECORD NOS. 18, 19, 20, & 21





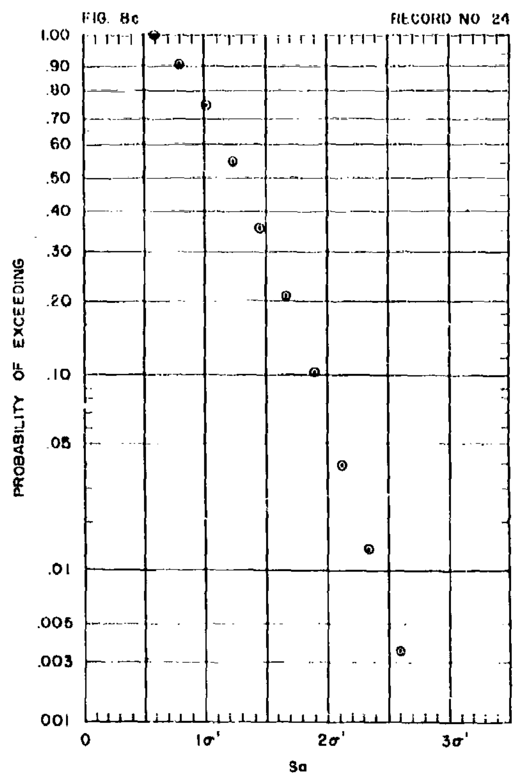


FIGURE 8 PROBABILITY INTEGRAL OF ALTERNATING STRESS PEAKS RECORD NOS. 22, 23, & 24

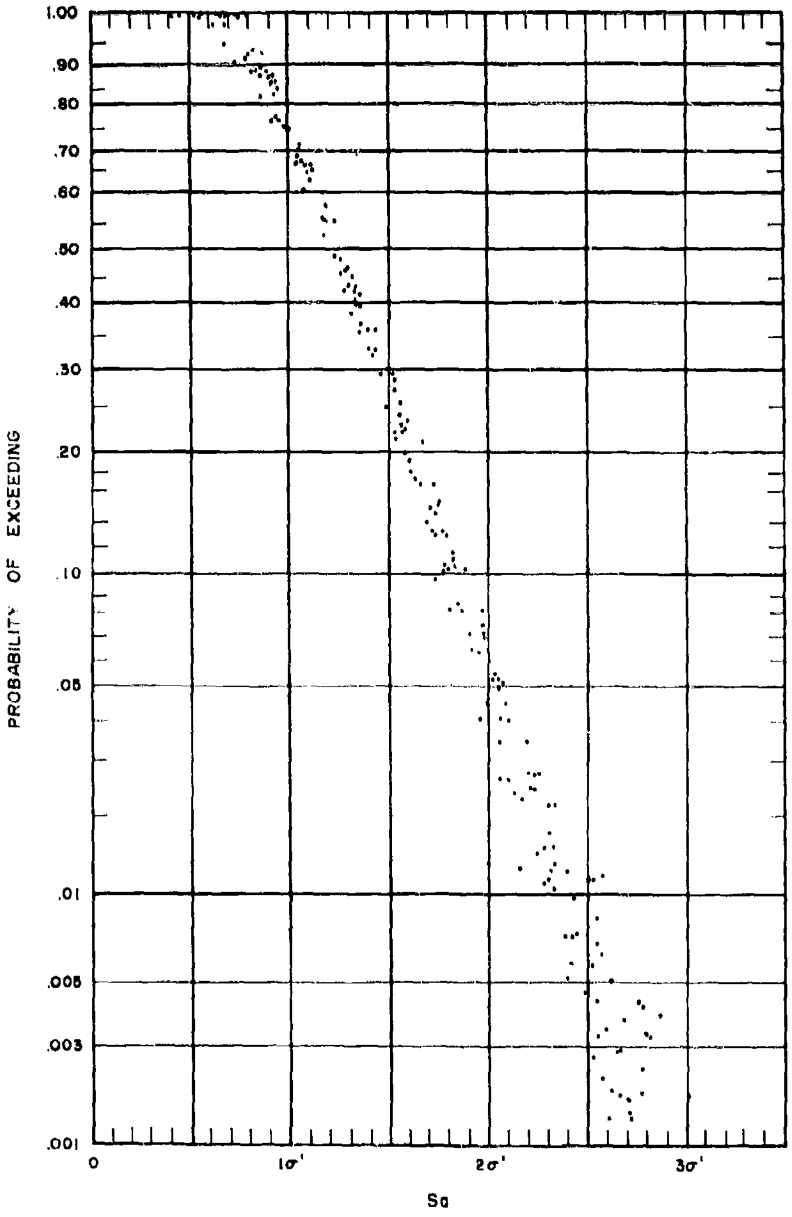


FIGURE 9 COMPOSITE PROBABILITY INTEGRAL OF ALTERNATING STRESS PEAKS

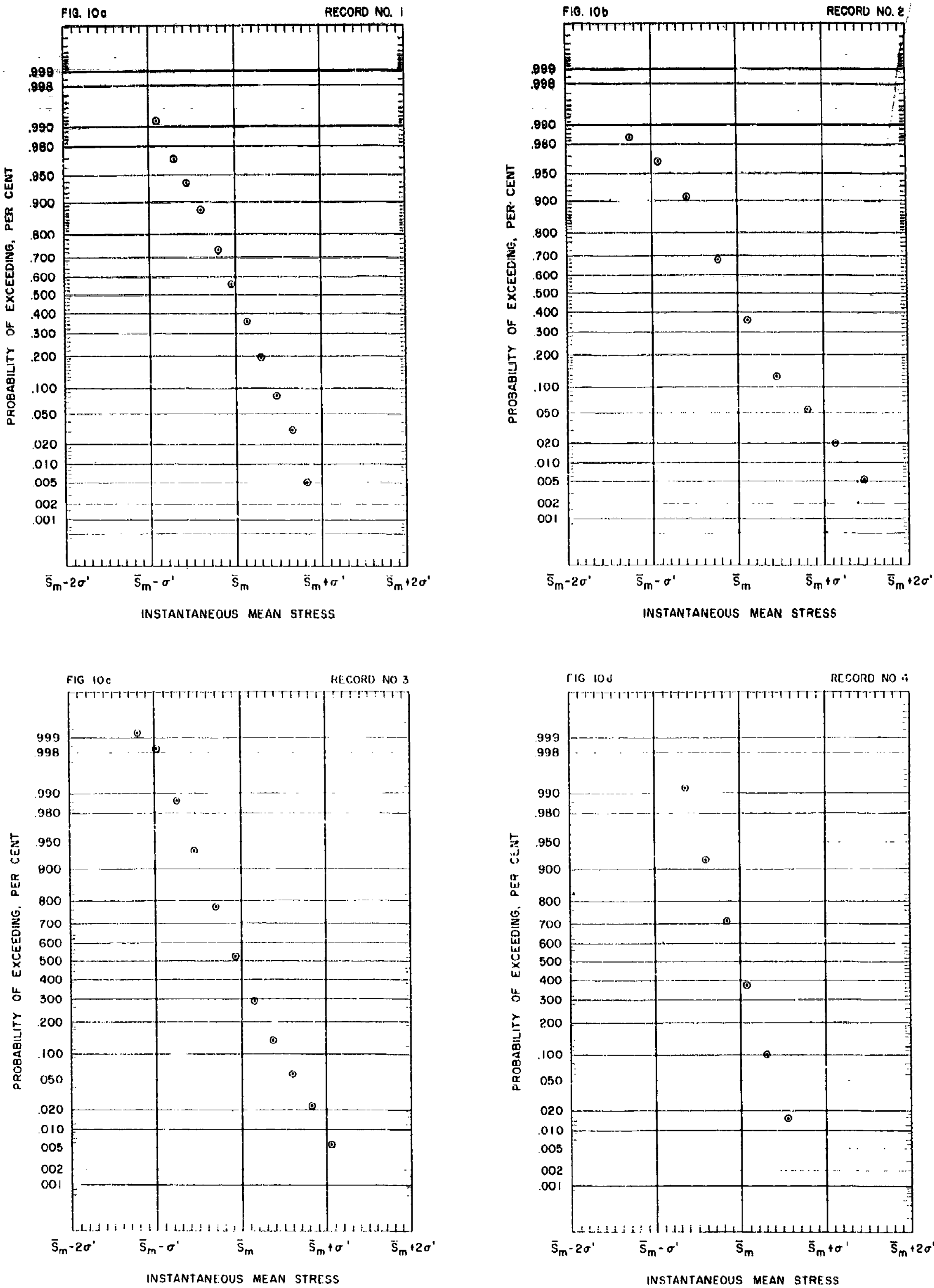


FIGURE 10 PROBABILITY INTEGRAL OF MEAN STRESS RECORD NOS. 1, 2, 3, & 4

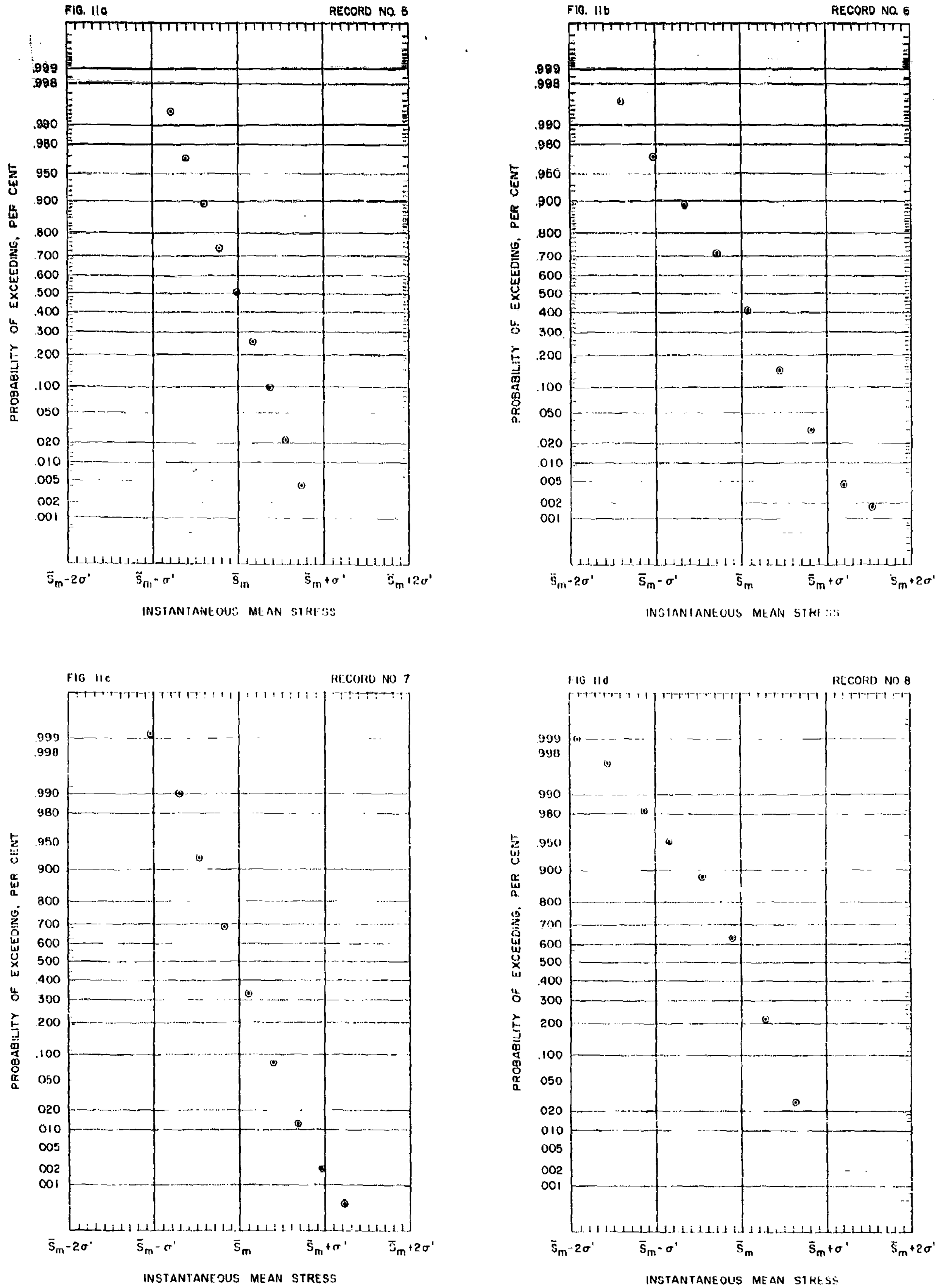


FIGURE 11 PROBABILITY INTEGRAL OF MEAN STRESS RECORD NOS 5, 6, 7, & 8

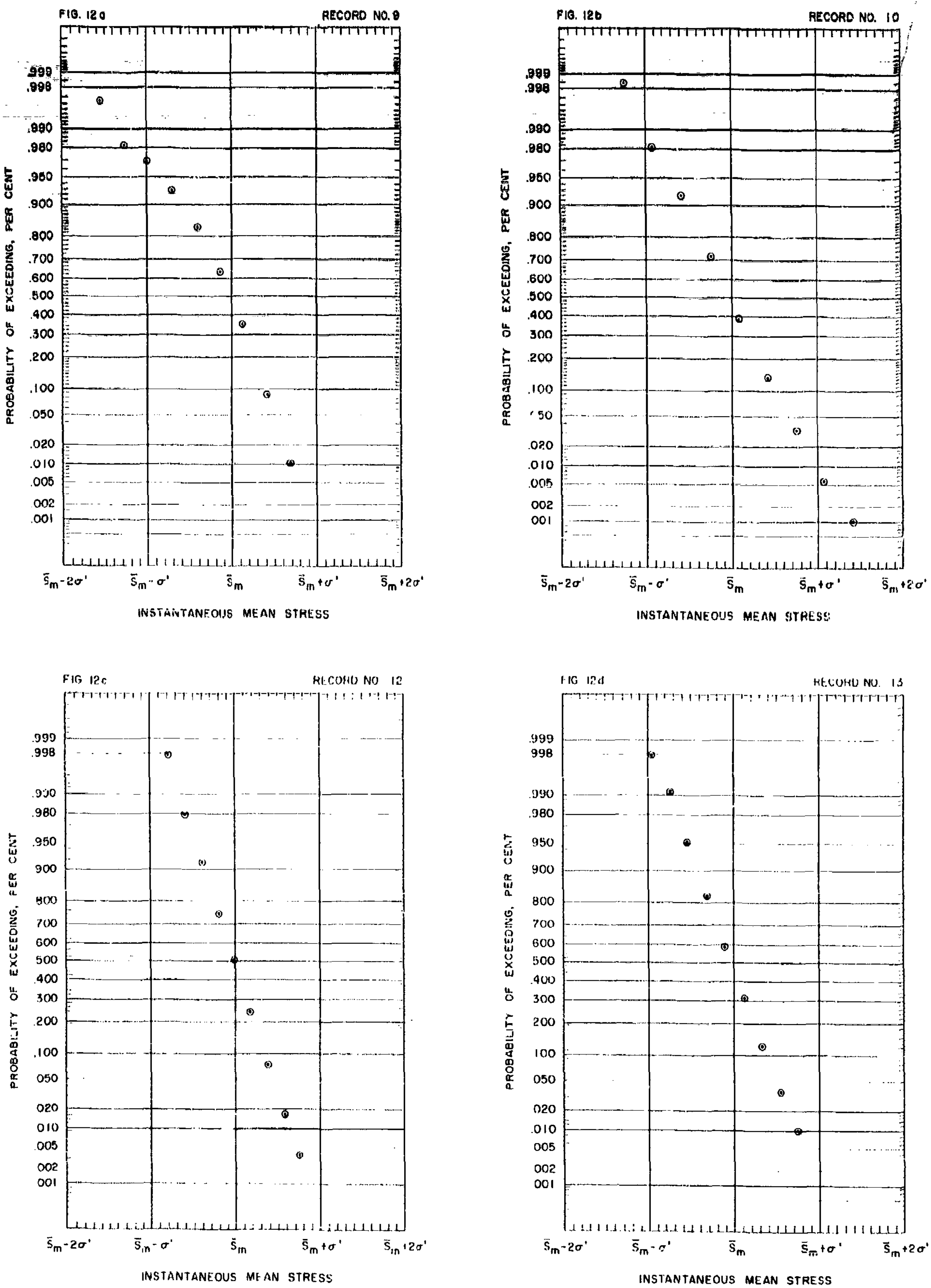


FIGURE 12 PROBABILITY INTEGRAL OF MEAN STRESS RECORD NOS. 9, 10, 12, & 13

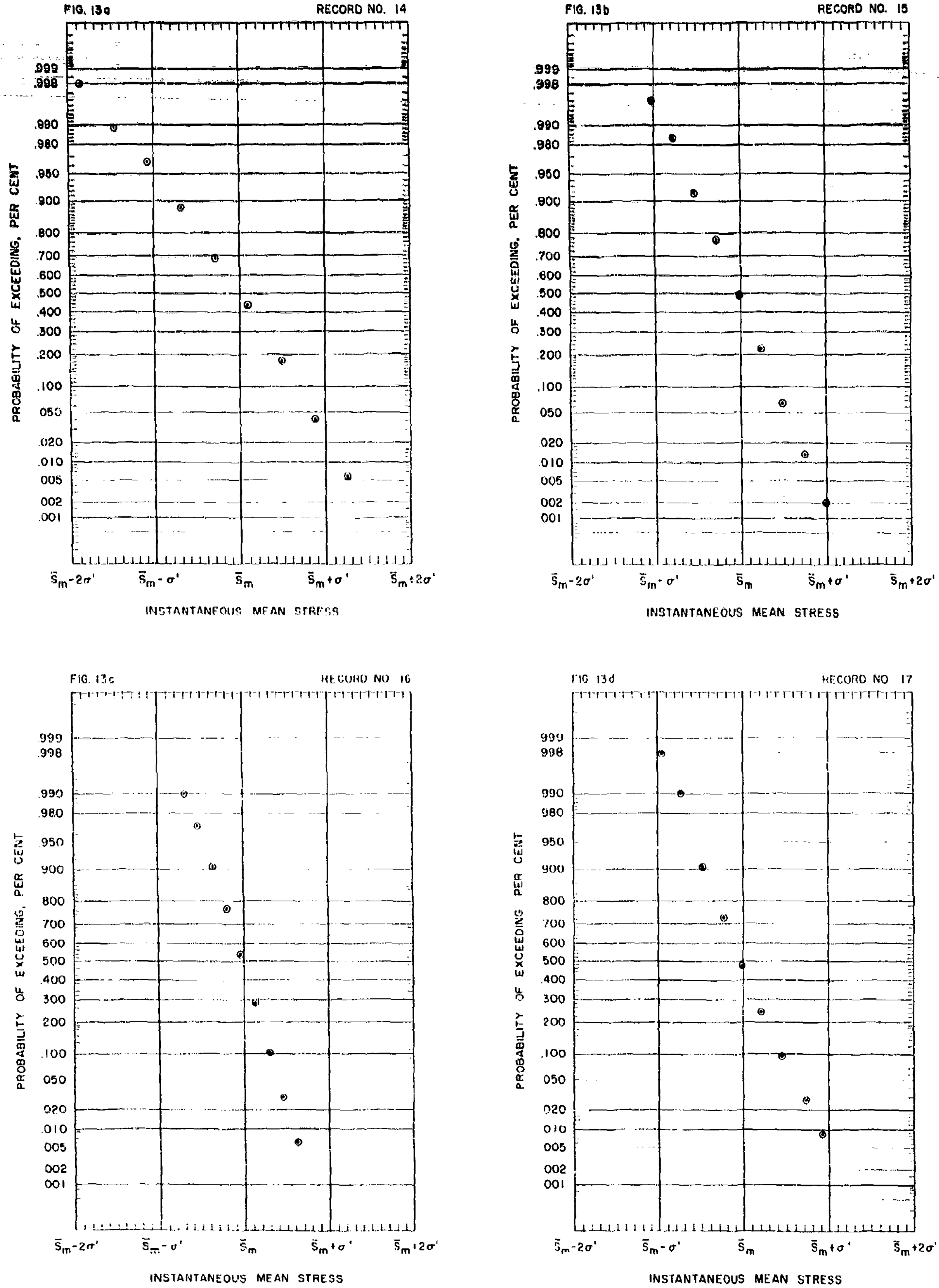


FIGURE 13 PROBABILITY INTEGRAL OF MEAN STRESS RECORD NOS 14, 15, 16, & 17

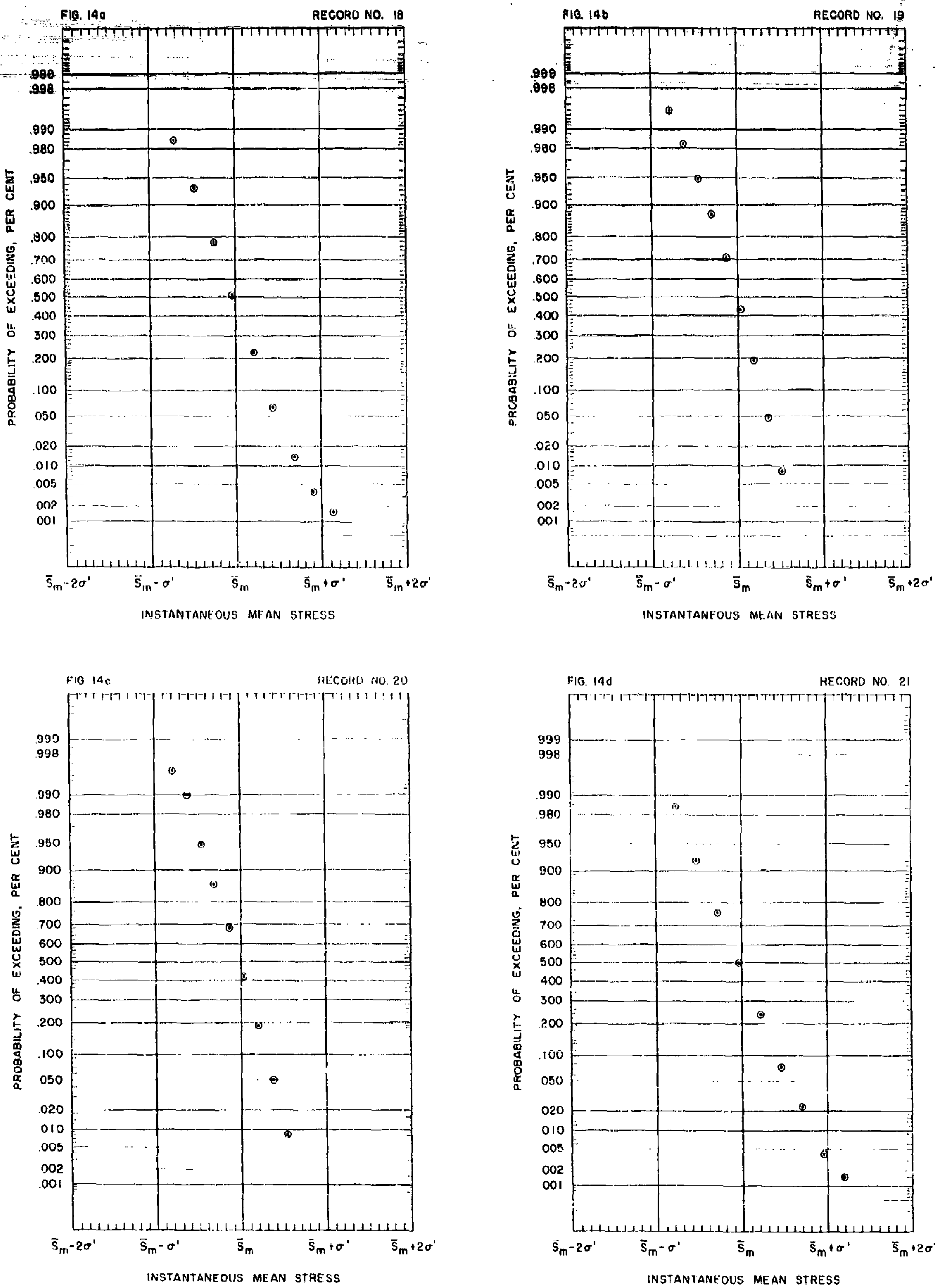
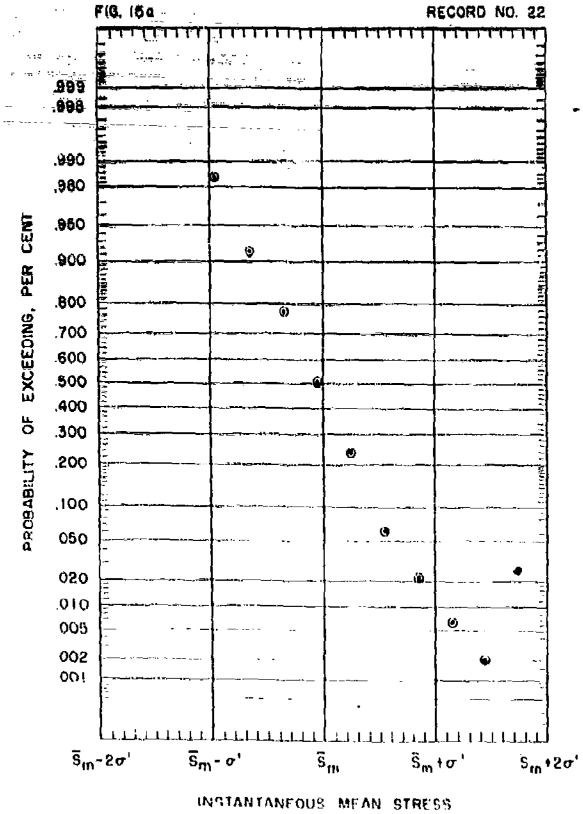
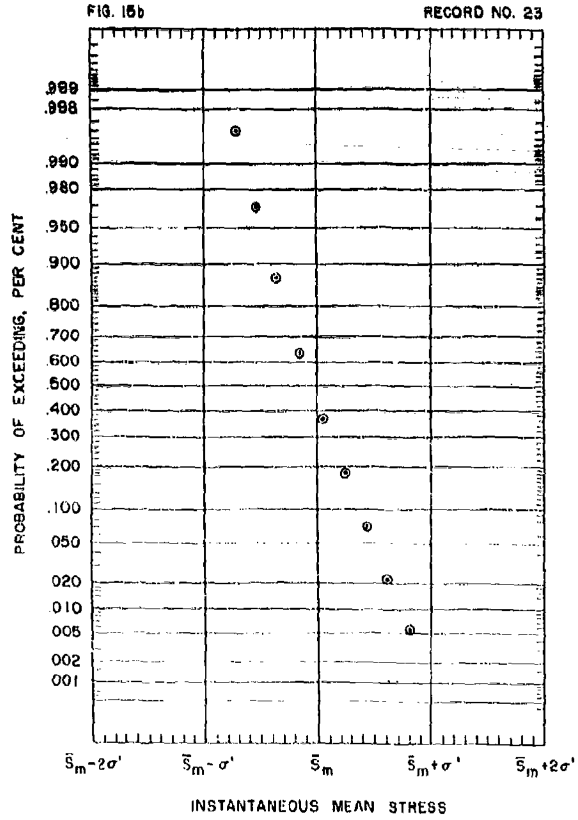


FIGURE 14 PROBABILITY INTEGRAL OF MEAN STRESS RECORD NOS 18, 19, 20, & 21





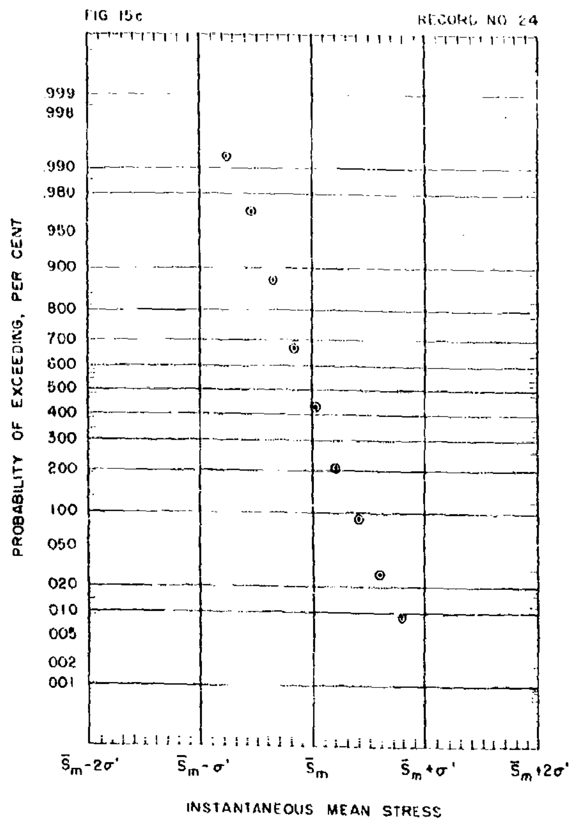


FIGURE 15 PROBABILITY INTEGRAL OF MEAN STRESS RECORD NOS 22, 23, & 24

on the basis of a small number (2-10) of measurements and the confidence which may be attached to their location is doubtful.

It is especially encouraging to observe the probability distribution for records 8 and 9 in Figures 4d and 5a. Those records represent stress response which is quite non-linear, as shown by ratio of range of crests to range of troughs in Table II. However, the probability integrals for those records conform to straight lines at least as well as does the average, demonstrating that the effects of that particular type of non-linearity on the probability distribution of alternating stresses is small.

Figures 7d and 8a for records 21 and 22 show considerable deviation from the straight line. Those records were obtained from Lockheed and pertain to measurements made on specially constructed panels rather than on actual aircraft structure. No further explanation for the deviation of the measured results from the theoretical results is at hand.

Figure 9 is the superposition of Figures 3-8 (with the exception of Figure 7d and 8a where some unknown factors were apparently operating). This figure summarizes the great consistency of the probability distributions and the normalized standard deviation of alternating stress, as presented in the preceding figures and in Table II. The lack of scatter contained in Figure 9 should be considered in light of the variety of structural configurations and modal response represented by the records it summarizes. The consistency in spite of those differences is presented as sufficient evidence that the normalized level and probability distribution of alternating stresses are consistent factors in aircraft subjected to high intensity acoustic noise.

Figures 10-16 present the measured probability integral of mean stress on normal distribution paper. Again, the correspondence to straight lines is good. Figures 11d and 12a show significant deviation from straight lines. Those figures pertain, as did Figures 4d and 5a to the distinctly non-linear records, 8 and 9. It may be useful to observe that this type of non-linearity has more apparent effect upon the mean stress than on the alternating stress.

No composite plot of mean stress probability distribution is presented. Such a plot, if made, would reflect the lack of uniformity of the normalized rms mean stress tabulated in Table II and would show considerable scatter.

3.0 APPLICATIONS

It appears to be established by theory and experiment that it is possible to describe the probability distribution of alternating stresses of structures subjected to high intensity acoustic loading by a simple, consistent, and conveniently reproduced function. This could have great influence in optimizing the techniques of future fatigue testing.

3.0.1 Qualification

Before examining the ramifications of that conclusion, it is appropriate to emphasize that a procedure which resolves the total stress into its mean stress component and alternating stress component is justified only if it can be established that but one of the components is significant in fatigue. This has not yet been established. Consequently, the usefulness of the results presented in Section II above and the validity of the applications which will be presented below remain subject to a test program which should be initiated as soon as possible. Appendix III of this report outlines test procedures which will be sufficient for examining the assumption that the instantaneous value of the varying mean stress has only second order effects upon the life of a structure subjected to random stresses. In the remainder of this section it is assumed that the required test program has been performed with favorable results, and the presence as well as the effects of the varying mean stress may therefore be neglected.

3.1 Life Prediction

Theories of cumulative damage which are more sophisticated than the Palmgren-Miner Hypothesis (Reference 1), have been proposed by several independent researchers (References 2, 3, 4). Although those theories make use of apparently markedly different assumptions concerning the mechanism of interaction between stresses at different levels, they (and others) yield remarkably similar results. Essentially, modern theories of cumulative damage agree that the existence of a range of stresses has the qualitative effect of causing "rotation" of the basic S-N curve of a material. If the amount of "rotation" were known quantitively, then the elementary theory could be applied, using the rotated S-N curve, for suitably accurate random fatigue life predictions. Much of the effort presently being expended on the development of cumulative damage theories is focussed upon deriving techniques for predicting the degree of rotation caused by stress interaction. The experimental methods, in all cases, make use of tests wherein two or more levels of stress are applied in random sequence to test specimens. The probability distributions of the applied stresses are often arbitrary, or are intended to simulate the probability distribution of gust loading.

It may be postulated that the empirical expression for rotation of the basic S-N curve which is derived from results of tests using a particular probability distribution of stresses, will provide the best possible accuracy for cases where the stress probability distribution of usage is identical to the probability distribution of the test. It follows from that postulate, and because the probability distribution of (alternating) stresses in life can be described with engineering accuracy as a Rayleigh distribution, that testing performed in the future with the intent of deriving a theory of cumulative damage for use in acoustic fatigue should incorporate a Rayleigh distribution of stresses.

The results (life versus rms stress) of tests performed with the Rayleigh probability distribution of stresses may, in addition to being incorporated into a theory of cumulative damage, be used directly. Specifically, if a random S-N curve for a particular probability distribution is available and there is reason to believe that a structure will be subjected to stresses with a similar probability distribution, then there is no need for a theory of cumulative damage; it is only necessary to determine the rms alternating stress and the effective frequency and the life may be read directly from the random S-N curve. That procedure is obviously simpler for designers to use than a theory of cumulative damage can reasonably be.

3.2 Production of a Rayleigh Distribution

3.2.1 Production of a Continuous Distribution

A benefit accruing from the fact that the probability distribution of (alternating) stresses has been found to be Rayleigh is the ease, in many cases, of reproducing that distribution. It may be produced in the laboratory by taking advantage of the same natural process which produces it in actual structure; that is, mechanical filtering of a broad band excitation. The natural frequencies of a test specimen may be determined, and the specimen may then be subjected to random vibration with a frequency content centered approximately at the specimen natural frequency which it is desired to excite, and with a band width substantially greater than the anticipated specimen bandwidth. The response stress of the specimen will then approach a perfect Rayleigh distribution.

3.2.2 Simulation by a Discrete Distribution

In many laboratories, test equipment limitations will preclude the use of resonant response to random excitation for production of a Rayleigh distribution. In that event, it is desirable to define a test performed at a finite number of discrete stress levels which will produce the same effects as a Rayleigh distribution. That problem would become trivial if an adequate theory of cumulative damage were in existence, since a straightforward application of a law of cumulative damage could be the calculation of an equivalent stress, Se such that application of N cycles of random stress, rms = σ, or N cycles of constant stress, Se, would cause failure. In reviewing existing theories of cumulative damage, it is natural, because of the ease of application, to investigate first the results of using the elementary theory. When Miner's Law (Reference 1) is used, the accuracy of prediction of Se ranges from fairly good to poor; the goodness of fit is found to be, as would be expected, best when the range of simulated random stresses is low and the S-N curve is well known and poor when the range of random stresses is broad and/or stress concentrations in the structure prevent an adequate knowledge of the S-N curve.

A detailed review of the more elaborate theories of cumulative damage in fatigue does not contain promise of a better method for determining a single level **Se** from a loading probability and an S-N curve. Instead, study of the reasoning behind the more elaborate cumulative damage theories has suggested that discrete amplitude testing which will enable the life of a random loaded structure to be simulated must involve more than one level of stress, with the sequence of levels mixed in a random manner. The highest level of discrete stress must be near the highest level of random stress which occurs significantly often and the lowest level of discrete stress must be approximately as low as the lowest level of random or approximately 33% of the highest stress, whichever is higher. More than two levels of discrete stress will be required if the S-N curve is not well known, either because of stress concentrations or because of the effects of stress interaction.

If a very large number of discrete levels of stress are used, the test would approach exact simulation and no knowledge of the effects of interaction or stress raisers would be required. However, the difficulties of performing multi-level tests would soon approach and surpass difficulties of performing true random tests and the desired economic advantages of discrete testing would be lost. The following is proposed as a compromise:

1. It is assumed that the effects of interaction between stresses at level S_{max} and stresses at level S_{min} are an order of magnitude greater than the effects of interaction between stresses with a range of 25% to 33% as great. That assumption is in accordance with phenomenological theories of fatigue as proposed by Freudenthal (Reference 2), Fuller (Reference 3) and others and is in accordance with considerable experimental data.

2. The assumption in paragraph 1 allows the statement that an adequate overall simulation may be achieved provided the interaction between extreme stress levels is simulated; interaction within fractions of the extreme cause second order effects which may be ignored. That assumption further allows the statement that a form of Miner's Law is usefully accurate for predicting the stress, S', which is equivalent to stresses within the narrow range where interaction is not significant. In the form used,

$$S'_{i-j} = \left[\frac{\int_{S_i}^{S_j} S^b p(S)\, dS}{\int_{S_i}^{S_j} p(S)\, dS} \right]^{\frac{1}{b}}$$

where p (S) is the probability density of stresses and b is the inverse slope of the S-N diagram in log-log representation. (If $S_i - S_j$ is reasonably narrow, a straight line representation will be completely adequate.)

3. If a range, $S_j - S_i$ may be found so that S'_{i-j} is essentially independent of b over a realistically wide range of b, then S' may be specified without quantitative knowledge of b. Material variability, stress concentrations, and stress interactions, etc. which affect b thus become second order effects in evaluating the accuracy of the life which is determined by the testing procedure.

To that end, the values of S' which are intended to simulate random stresses between S_i and S_j, for the Rayleigh probability density and several realistic values of b, have been computed. The results follow:

TABLE III

Single Stress Level Equivalent to a Segment of a Rayleigh Density of Stress

Si	Sj	S'(b=4)	S'(b=6)	S'(b=8)
0	σ	.735	.773	.80 *
σ	2σ	1.51	1.56	1.60
2σ	3σ	2.39	2.42	2.45
3σ	∞	3.34	3.38	3.41

*(need not be included in test procedure)

The differences in stress levels within the rows of Table III are seen to be small, indicating that an accurate knowledge of the shape of the S-N curve is of second order importance for a realistically wide range of b. Consequently, it is possible to prescribe a three-level test which is nearly equivalent to a random test. Typically, if it is desired to simulate a Rayleigh distribution, rms = σ , it could be done by means of blocks of stress, each block consisting of 1,110 repetitions of stress equal to 3.4σ , 12,440 cycles of stress equal to 2.4σ , and 47,100 cycles of 1.6σ . Each block would be equivalent to 100,000 cycles of random stress (39,350 cycles of stress at levels less than σ may be ignored) and the sequence of high, low and medium stresses would be varied within each succeeding block in a random manner. N blocks would be equivalent in life to 100,000N cycles of random stress.

3.3 Proof Testing

It is reasonable to assume that, even after a reliable theory of cumulative damage in fatigue is developed it will continue to be appropriate to perform fatigue tests on subassemblies, major assemblies, and complete structures, in order to evaluate specific designs and assembly techniques. Simulation of the probability distribution of stresses is more important, if anything, for this type of testing than it is for the laboratory type experimentation discussed in Section 3.2 above. Simulation is both automatic and perfect when the proof testing is accomplished by simply tying down the test airplane and letting the engine run; however, when economic considerations dictate testing of local areas using siren excitation, then the considerations of paragraph 3.2.2 apply and a simulated random test should be performed.

3.4 Accelerated and Aggravated Testing

Fatigue proof testing of aircraft structure is accelerated testing whenever it is assumed that the stress history during flight is so trivial compared to the stress history during the take off run, that simulating the take off stress and the total time at the stress level is equivalent to simulating the entire life of the airplane. Granting that assumption, proof testing at maximum ground levels, for perhaps one hundred hours, is considered to qualify an airplane design for its entire life of up to 10,000 flight hours.

Although an important amount of acceleration of airplane life is contained in testing at normal ground stress levels, there is a natural desire to shorten test times still further. This may be accomplished with reasonable accuracy provided the random S-N curve of the material or joint being tested is known and it is possible to apply an exaggerated stress level. This technique is discussed at some length by Schjelderup (Reference 9) and will not be considered further here, except to remark that both normal test techniques which assume negligible effects upon life from the stress levels of cruise and the aggravated tests described by Schjelderup (Reference 9) assume better knowledge of random S-N curves than is usually at hand. Substantial improvement in confidence that may be attached to accelerated testing would result from a greatly enlarged library of random S-N curves.

4.0 CONCLUSIONS AND RECOMMENDATIONS

Any stress history, no matter how complex, may be resolved into an alternating stress superimposed upon a varying mean stress. The resolving procedure is essentially a high-pass filtering process for the alternating stress and a low-pass process for the mean stress.

The filtering tends to make the probability distribution of the alternating stress component conform approximately to the probability distribution characteristic of the highest frequency mode which is present to a significant degree in the total stress. Therefore, the probability distribution of peaks of the alternating stress component approaches a Rayleigh distribution when the stress is excited by acoustic noise.

Similarly, the probability distribution of instantaneous values of the stress in lower frequency modes tends to dominate the distribution of instantaneous mean stresses. Consequently, a normal distribution of mean stresses tends to occur.

Examination of oscillograms containing strain measured upon a wide variety of typical aircraft structures exposed to high intensity acoustic noise has confirmed that the expected probability distribution of mean and alternating stresses are observed, within limits of engineering accuracy.

Resolving total stress into mean and alternating components will be extremely useful provided it may be demonstrated by test that the instantaneous value of mean stress has only slight effect upon the fatigue damage caused by the alternating stress component. A test program to determine the effect of variable mean stress with variance of 1% to 30% the variance of the alternating stress should therefore be initiated in the immediate future. An outline of a sufficient test program for determining those effects is included in this report as Appendix III.

If the results of the recommended test program are favorable, then future fatigue testing of laboratory specimens and real structures should reflect the probability distribution of which has been found to be nearly universal for structures excited by high intensity acoustic noise. The results of such fatigue testing will be random S-N curves which may be used directly in the design and testing of structures and which may be used as data for establishing a general theory of cumulative damage in fatigue.

APPENDIX I

COMPOUND PROBABILITIES

In paragraph 2, preceding, the form of the mean stress and alternating stress was found to be the product and/or sum of two or more random and mutually independent functions. The probability functions resulting from those compound results are derived herein.

1. Consider the function, $S = A \cos\theta$ where A has Rayleigh probability with variance of σ_a^2, and all values of θ are equally probable.

Since all values of θ are equally likely, the probability that $\cos\theta > \frac{Q}{\tau}$ is,

$$P_{\cos\theta}\left(\frac{Q}{\tau}\right) = 1 \qquad \frac{Q}{\tau} < -1$$

$$= \frac{1}{\pi}\cos^{-1}\frac{Q}{\tau} \qquad -1 < \frac{Q}{\tau} < 1$$

$$= 0 \qquad \frac{Q}{\tau} > 1$$

Since A has a Rayleigh density, the probability that A is bounded by $\tau, \tau + d\tau$ is

$$p_A(\tau) = 0 \qquad \tau < 0$$

$$= \frac{\tau}{\sigma_a^2}\exp\left[-\frac{\tau^2}{2\sigma_a^2}\right] \qquad \tau > 0$$

If A is bounded by τ, $\tau + d\tau$, and $\cos\theta > \frac{Q}{\tau}$, then $A\cos\theta > Q$. The probability, then, that $A\cos\theta > Q$, is the product of the above mutually independent probabilities, summed over all values of τ,

$$P(Q) = \frac{1}{\pi}\int_{|Q|}^{\infty}\frac{\tau}{\sigma_a^2}\cos^{-1}\frac{Q}{\tau}\exp\left[-\frac{\tau^2}{2\sigma_a^2}\right] d\tau$$

The above expression for the Probability Integral of $A\cos\theta$, is not integrated easily. However, if its derivative is taken,

$$\frac{d}{dQ} P(Q) = \frac{1}{\pi} \int_{|Q|}^{\infty} \frac{\tau \exp\left[-\frac{\tau^2}{2\sigma_a^2}\right]}{\sigma_a^2 \sqrt{\tau^2 - Q^2}} d\tau$$

the integration may then be performed and multiplied by (-1) to yield the probability density of A Cos θ ,

$$p(Q) = \frac{1}{\sqrt{2\pi}\,\sigma_a} \exp\left[-\frac{Q^2}{2\sigma_a^2}\right]$$

which is recognized as the normal probability density.

2. Consider the function, A + B, where A has Rayleigh probability, variance of σ_a^2 , and B has normal probability, variance of σ_b^2 . Since A has Rayleigh distribution the probability that $A > (Q - \tau)$ is,

$$\begin{aligned} P_A(Q-\tau) &= \exp\left[-\frac{(Q-\tau)^2}{2\sigma_a^2}\right] & \tau < 0 \\ &= 1 & \tau > 0 \end{aligned}$$

B has normal density. Therefore, the probability that B is bounded by $\tau, \tau + d\tau$ is,

$$p_B(\tau) = \frac{1}{\sqrt{2\pi}\,\sigma_b} \exp\left[-\frac{\tau^2}{2\sigma_b^2}\right] d\tau$$

If $A > Q - \tau$, and B is bounded by $\tau, \tau + d\tau$, then $A + B > Q$. Since values of A and B are assumed mutually independent, the joint probability is the product of the probabilities. Summed over all τ , the resultant probability is,

$$P_{A+B}(Q) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{Q} \frac{\exp\left[-\frac{\tau^2}{2\sigma_b^2} - \frac{(Q-\tau)^2}{2\sigma_a^2}\right]}{\sigma_b} d\tau + \int_{Q}^{\infty} \frac{\exp\left[-\frac{\tau^2}{2\sigma_b^2}\right]}{\sigma_b} d\tau \right]$$

The integration is straightforward and the result is,

$$P_{A+B}(Q) = \frac{\sigma_a \exp\left[-\frac{Q^2}{2(\sigma_a^2 + \sigma_b^2)}\right]}{2(\sigma_a^2 + \sigma_b^2)^{1/2}} \left[1 + \operatorname{erf} \frac{Q\sigma_a}{\sigma_b\sqrt{2(\sigma_a^2 + \sigma_b^2)}}\right] + \frac{1}{2}\left[1 - \operatorname{erf} \frac{Q}{\sqrt{2}\sigma_b}\right]$$

or, taking the negative derivative to yield the probability density,

$$P_{A+B}(Q) = \frac{Q\sigma_a \exp\left[-\frac{Q^2}{2(\sigma_a^2+\sigma_b^2)}\right]}{2(\sigma_a^2+\sigma_b^2)^{3/2}}\left[1 + \text{erf}\,\frac{Q\sigma_a}{\sigma_b\sqrt{2(\sigma_a^2+\sigma_b^2)}}\right] + \frac{\sigma_b \exp\left[-\frac{Q^2}{2\sigma_a^2}\right]}{\sqrt{2\pi}(\sigma_a^2+\sigma_b^2)}$$

3. Consider the function $A^2 + B^2 + 2AB \cos\theta$, where A has Rayleigh distribution, variance σ_a^2, B has Rayleigh distribution, variance σ_b^2. All values of θ are equally probable.

The probability integral of $(A^2 + B^2)$ may be found readily, as may the probability density of $2AB \cos\theta$. However, reflection shows that it is not correct to perform the convolution of $P(A^2 + B^2)$ with $p(2AB \cos\theta)$ to determine $P(A^2 + B^2 + 2AB \cos\theta)$ because the instantaneous value of (AB) is not independent of the instantaneous value of $(A^2 + B^2)$. That is, 2AB is equal to or less than $A^2 + B^2$.

It is necessary to make use of trigonometric identities to write,

$$A^2 + B^2 + 2AB \cos\theta = \left[A^2 + B^2\right]\left[1 + \sin(2\,\text{Tan}^{-1}\,\tfrac{A}{B})\,\cos\theta\right]$$

In that form, the instantaneous value of $(A^2 + B^2)$ does not, in any way, govern or limit the instantaneous value of A/B, and a convolution integral may therefore be performed on the (now independent) bracketed terms. That is, if $A^2+B^2 \geq Q/\tau$, and $1 + \sin 2\,\text{Tan}^{-1} A/B \cos\theta$ is bounded by τ, $\tau + d\tau$, then,

$$\left[A^2 + B^2\right]\left[1 + \sin 2\,\text{Tan}^{-1} A/B \cos\theta\right] > Q.$$

The probability that $A^2 + B^2 > Q/\tau$ will first be determined; $A^2 + B^2$ will be greater than Q/τ, provided

$$A^2 > \frac{Q}{\tau} - V$$

B^2 bounded by V, V + dV.

Since A and B each have Rayleigh distribution, the above probabilities may be written immediately:

$$P_{A^2}\left(\frac{Q}{\tau}-V\right) = \exp\left[-\frac{(Q/\tau - V)}{2\sigma_a^2}\right] \qquad V < \frac{Q}{\tau}$$

$$= 1 \qquad V > \frac{Q}{\tau}$$

$$= 0 \qquad V < 0$$

$$p_{B^2}(V) = \frac{1}{2\sigma_b^2}\exp\left[-\frac{V}{2\sigma_b^2}\right] dV \qquad V > 0$$

Therefore,

$$P_{A^2+B^2}\left(\frac{Q}{\tau}\right) = \frac{1}{2\sigma_b^2}\left[\int_0^{Q/\tau}\exp\left[\frac{V}{2}\left(\frac{1}{\sigma_a^2}-\frac{1}{\sigma_b^2}\right)-\frac{Q}{2\tau\sigma_a^2}\right] dV + \int_{Q/\tau}^{\infty}\exp\left[-\frac{V}{2\sigma_h^2}\right] dV\right]$$

which may be integrated directly to yield

$$P_{A^2+B^2}\left(\frac{Q}{\tau}\right) = \frac{\sigma_b^2\exp\left[-\frac{Q}{2\tau\sigma_b^2}\right]-\sigma_a^2\exp\left[-\frac{Q}{2\tau\sigma_a^2}\right]}{\sigma_b^2-\sigma_a^2}.$$

The probability that $1 + \mathrm{Sin}2\mathrm{Tan}^{-1}A/B\ \mathrm{Cos}\,\theta$ is bounded by $\tau, \tau + d\tau$ is now found. This is done by first finding the probability integral of (A/B), determining from it the probability integral of $\mathrm{Sin}2\mathrm{Tan}^{-1}A/B$, and then performing a convolution integral of that probability with the probability density of $\mathrm{Cos}\,\theta$.

Proceeding as in the previous cases, it is seen that $A/B > Z$ provided $A > Z\phi$, and B bounded by $\phi - d\phi, \phi$. A and B have Rayleigh probabilities; therefore

$$P_A(Z\phi) = \exp\left[-\frac{Z^2\phi^2}{2\sigma_a^2}\right] \qquad Z\phi > 0$$

$$p_B(Z) = \frac{\phi}{\sigma_b^2}\exp\left[-\frac{\phi^2}{2\sigma_b^2}\right] d\phi \qquad \phi > 0$$

$$P_{\frac{A}{B}}(Z) = \int_0^{\infty}\frac{\phi}{\sigma_b^2}\exp\left[-\frac{\phi^2}{2\sigma_b^2}-\frac{Z^2\phi^2}{2\sigma_a^2}\right] d\phi$$

The integration may be performed to yield,

$$P_{\frac{A}{B}}(Z) = \frac{\sigma_a^2}{\sigma_a^2 + Z^2\sigma_b^2}$$

or, the probability that A/B is bounded by Z, 1/Z (Z < 1) is,

$$P(Z) - P\left(\frac{1}{Z}\right) = \frac{\sigma_a^2\sigma_b^2(1-Z^4)}{(\sigma_a^4+\sigma_b^4)Z^2 + \sigma_a^2\sigma_b^2(1+Z^4)}$$

If A/B is between Z, 1/Z then $\mathrm{Sin}2\mathrm{Tan}^{-1}$ A/B is between $2A/1+Z^2$, and 1 .

Defining $2Z/1 + Z^2 \equiv R/S$, we may substitute in the preceding result to state that the probability that $\mathrm{Sin}2\mathrm{Tan}^{-1}$ A/B > R/S, is,

$$P\left(\frac{R}{S}\right) = \frac{4\sigma_a^2\sigma_b^2\, S\sqrt{S^2 - R^2}}{4\sigma_a^2\sigma_b^2 S^2 + (\sigma_a^2 - \sigma_b^2)R^2} \qquad S > R$$

Now, if $\mathrm{Sin}2\mathrm{Tan}^{-1}$A/B > R/S and Cos θ is bounded by S, S + dS, then $\mathrm{Sin}2\mathrm{Tan}^{-1}$A/B Cos θ>R. The probability density of Cos θ is well known,

$$p(S) = \frac{1}{\pi}\frac{dS}{\sqrt{1-S^2}} \qquad -1 < S < 1$$

$$= 0 \qquad S^2 > 1$$

therefore,

$$p(R) = \frac{4\sigma_a^2\sigma_b^2}{\pi}\int_R^1 \frac{S\sqrt{S^2 - R^2}}{\sqrt{1-S^2}\left[R^2(\sigma_a^2-\sigma_b^2)+4\sigma_a^2\sigma_b^2 S^2\right]}\,dS$$

The change of variable, $\zeta^2 = S^2 - R^2/1-S^2$ and separation into partial fractions transforms the above integral into the more convenient form,

$$P(R) = \frac{1}{\pi}\left\{\int_0^\infty \frac{d\zeta}{1+\zeta^2} - \int_0^\infty \frac{d\zeta}{1 + \frac{\zeta^2}{R^2}\left[\frac{1+\left(\frac{R(\sigma_a^2-\sigma_b^2)}{2\sigma_a\sigma_b}\right)^2}{1+\left(\frac{\sigma_a^2-\sigma_b^2}{2\sigma_a\sigma_b}\right)^2}\right]}\right\}$$

which may be integrated immediately to yield,

$$P_{\mathrm{Sin}\,2\,\mathrm{Tan}^{-1}\frac{A}{B}\mathrm{Cos}\,\theta\,(R)} = \frac{1}{2}\left(1 - \frac{R(\sigma_a^2 + \sigma_b^2)}{\sqrt{R^2(\sigma_a^4 + \sigma_b^4) + \sigma_a^2\sigma_b^2(4-2R^2)}}\right)$$

Now, if $\mathrm{Sin}2\mathrm{Tan}^{-1}A/B > R$, then $1 + \mathrm{Sin}2\mathrm{Tan}^{-1}A/B\mathrm{Cos}\,\theta > 1 + R$.

Define $1 + R \equiv \tau$, and it is seen that the probability that $1 + \mathrm{Sin}2\mathrm{Tan}^{-1}A/B$ $\mathrm{Cos}\,\theta > R + 1$ is,

$$P(\tau) = \frac{1}{2}\left(1 - \frac{(\tau-1)(\sigma_a^2 + \sigma_b^2)}{\sqrt{(\sigma_a^2 + \sigma_b^2)^2 + (\sigma_a^2 - \sigma_b^2)^2(\tau^2 - 2\tau)}}\right)$$

The negative differential of the above function may be taken to yield the probability that $(1 + \mathrm{Sin}2\mathrm{Tan}^{-1}A/B\mathrm{Cos}\,\theta)$ is between τ, $\tau + d\tau$.

$$p(\tau) = \frac{2\sigma_a^2\sigma_b^2(\sigma_a^2 + \sigma_b^2)\,d\tau}{\left[(1-\tau)^2(\sigma_a^2 - \sigma_b^2)^2 + 4\sigma_a^2\sigma_b^2\right]^{3/2}}$$

and the convolution integral of that, together with the previously determined probability that $A^2 + B^2 > Q/\tau$, yields the desired result; that is the probability that $A^2 + B^2 + 2AB\mathrm{Cos}\,\theta > Q$

$$P(Q) = \frac{2\sigma_a^2\sigma_b^2(\sigma_a^2 + \sigma_b^2)}{\sigma_b^2 - \sigma_a^2}\int_0^2 \frac{\sigma_b^2\exp\left[-\frac{Q}{2\tau\sigma_b^2}\right] - \sigma_a^2\exp\left[-\frac{Q}{2\tau\sigma_a^2}\right]}{\left[(1-\tau)^2(\sigma_a^2 - \sigma_b^2)^2 + 4\sigma_a^2\sigma_b^2\right]^{3/2}}\,d\tau$$

The indicated quadrature may be performed readily in closed form, for the special case $\sigma_a = \sigma_b$, and yields a Rayleigh probability integral, $P(Q) = \exp\left[-Q/2(\sigma_a^2 + \sigma_b^2)\right]$ Efforts to perform the integration in closed form for the general case, $\sigma_a \neq \sigma_b$, have not been successful. However, numerical integration yielded results for the points calculated which were, again, in conformance with a Rayleigh distribution. It is extremely likely that the solution to the general case, if it could be found, has a Rayleigh distribution.

APPENDIX II

TRUNCATION AND RECTIFICATION OF DATA

Data Processing

When a strain record contains the results of response in several modes, there may be a large number of points of zero slope separated by small changes of strain. At present no universally accepted criterion exists for deciding which of those extrema are significant in fatigue analysis.

It is certainly sufficient to make no distinction between minor perturbations and significant fluctuations and to include all points of zero slope in a data analysis of stress peaks. The question of whether it is necessary, as well as sufficient, arises when a high frequency low amplitude mode is excited along with lower frequency high amplitude modes. That condition of modal response, together with the assumptions that stress peaks are the damaging mechanism in fatigue and all points of zero slope in a stress record are significant peaks, led Weibull (Reference 5) to conclude that every stress record is an entity which cannot be compared to anything except a nearly identical stress record.

A different difficulty arises when considering alternating stress, instead of stress peaks, as the damaging mechanism. In that case, inclusion of all extrema can transform a normal stress-time history into a superposition of a small alternating stress component upon a large mean stress component. That condition is the reverse of what was considered when the Alternating Stress Hypothesis (Reference 6) was proposed and it may be concluded that either we may not include all minor extrema or the alternating stress hypothesis must be abandoned.

A common technique for discarding minor perturbations involves assuming that fluctuations which do not cause zero crossings are trivial, and a valid crest is the largest excursion from zero between two adjacent zero crossings. This assumption, however, is frequently self-defeating. For example, consider the extreme case when a significant steady-state stress exists. In that event, many (or all)of the stress fluctuations will not cross true zero and will be ignored. That case may perhaps be handled by (mentally) filtering out the steady component and sketching in a line which apparently represents the average stress. Using that line as a first approximation to zero, instantaneous mean stress (point midway between a "line defined" crest and a "line defined" trough) may be tabulated and a second approximation to zero may be determined by,

$$\bar{S}_{m_2} = \frac{\sum_{i=1}^{N} S_{m_i}}{N}$$

However, the process may converge slowly if a significant high frequency component exists. Furthermore, when a significant very low frequency component (approaching zero frequency) exists, the calculated zero-line may still ignore an excessive number of important stress fluctuations.

Another approach, which has gained favor in other activities including ocean wave analysis and gust analysis, involves defining a "minimum included" strain change. This concept may be applied by examining each crest, level S_i, together with the trough preceding it, S_{i-1}, and the one following, S_{i+1}. If $(S_i - S_{i-1})$ and $S_i - S_{i+1})$ are both greater than the minimum to be included, then S_i is defined to be a valid crest. If that condition is not met, then S_i is considered to be a minor perturbation and is ignored in further data processing. Figure 16 illustrates the procedure. This definition of significant crests and troughs appears to be the best when analyzing records in terms of alternating stress and mean stress, provided a physically meaningful truncating level may be determined.

In deriving a truncation level, it becomes necessary to examine a number of records in the light of some knowledge of fatigue. A cursory examination of a strain record may be performed and the strain fluctuation which appears to be the largest S_{max}, may be selected. Considerable fatigue testing (Reference 10) has shown that fluctuations smaller than $S_{max}/3$ will have slight effect upon the fatigue life unless they constitute the great majority of all stress cycles; examinations of typical strain response caused by acoustic noise shows that fluctuations smaller than $S_{max}/3$ constitute approximately one-half of the total number of fluctuations. By those criteria, then, a truncation level of $S_{max}/3$ may be appropriate.

However, more detailed study of the typical strain records reveals that discarding the lower 50% of the strain fluctuations invariably discards the mode of the distribution. This is undesirable because it increases the potential error incurred in matching an empirical probability distribution to the measured one.

Use of a truncation level of $S_{max}/4$ reduces the number of discarded fluctuations to 33% of the total, and includes the mode of the distribution with a small margin.

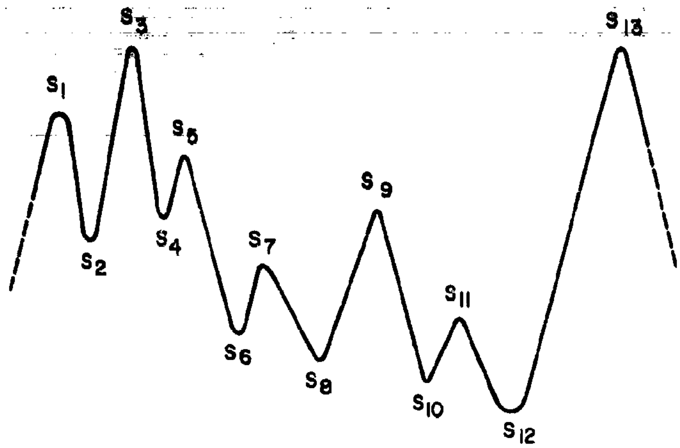
A larger margin may be achieved by use of still smaller truncation levels. However, if a level as small as $S_{max}/5$ is used, then it is found that a significant number of fluctuations are dissected into a trivial alternating stress superimposed upon a large mean stress, thereby jeopardizing the alternating stress hypothesis.

The truncation level of $S_{max}/4$ appears to be the reasonable compromise, and that level has been used throughout this report.

Data Presentation

Special graphic displays of data are often developed to facilitate comparison between theoretical and experimental results. One of the most convenient displays of results is one which represents the results as straight lines if the theory is confirmed. Deviations from the straight lines are good qualitative measures of the deviation of experimental results from theoretical results.

Theory suggests a probability distribution of alternating stresses which approaches a Rayleigh distribution. However, the truncation procedure used in the data processing distorts the experimental results so that they may not be compared directly to a Rayleigh distribution. Therefore, special graph paper was prepared, upon which a truncated Rayleigh distribution plots as a straight line. The truncation level used in preparing the paper, 0.9σ, was chosen on the basis that truncation occurs at a level of $S_{max}/4$, with S_{max} determined from a cursory examination of a strain record. Experience has shown that the maximum determined from that type



$$S_{13} - S_{12} = S_{max}$$

$S_1 - S_2 > S_{max}/4$	
	S_2 Valid
$S_3 - S_2 >$ "	
	S_3 Valid
$S_3 - S_4 >$ "	
	S_4 Minor
$S_5 - S_4 <$ "	
	S_5 Minor
$S_5 - S_6 >$ "	
	S_6 Minor
$S_7 - S_6 <$ "	
	S_7 Minor
$S_7 - S_8 >$ "	
	S_8 Valid
$S_9 - S_8 >$ "	
	S_9 Valid
$S_9 - S_{10} >$ "	
	S_{10} Minor
$S_{11} - S_{10} <$ "	
	S_{11} Minor
$S_{11} - S_{12} >$ "	
	S_{12} Valid
$S_{13} - S_{12} >$ "	

FIGURE 16 PROCEDURE FOR DISCRIMINATING BETWEEN VALID CRESTS AND TROUGHS, AND MINOR EXTREMA

of inspection is generally a level which is exceeded by approximately 1 to 2 peaks per thousand; therefore, S_{max} is approximately 3.6σ for a Rayleigh distribution, and $S_{max}/4$ is approximately 0.9σ

For the instantaneous mean stress, theory implies a Gaussian distribution. The truncation procedure will exert some bias upon the mean stress, since most of the stress fluctuations which are ignored in the data processing will have small mean stress. However, efforts to derive a procedure which can correct for this bias have not been successful. Because the mean stress has been assumed to cause only second order effect upon fatigue life in any event, it was elected not to pursue those efforts and to tentatively neglect the bias. Therefore, the probability distribution of mean stress is plotted upon normal probability paper.

APPENDIX III

TEST PLAN FOR DETERMINING

EFFECTS OF MEAN STRESS

Much of the value of the results of the studies reported here depends upon the verification of the hypothesis that a random varying mean stress, with variance smaller than the variance of the alternating stress, will have slight effect upon the fatigue life of a structure. Therefore, it is considered appropriate to include herein a possible test plan for examining that hypothesis.

It appears that the most convenient test program for evaluating the relative effects of mean stress and alternating stress will make use of the same natural processes which tend to cause typical aircraft structure to be subjected to variable mean stress and alternating stress; that is, response of a several degree of freedom elastic system to broad-band random excitation. A test specimen can be designed to have two well separated resonant frequencies in the frequency band of interest. When both resonant modes are excited, the response in the higher frequency mode will be the alternating stress and the response in the lower frequency mode will be the mean stress. The exciter may most conveniently be a random vibration system with equipment for shaping the power spectrum of the vibratory input so that the relative response of the two modes may be adjusted.

Test procedure should consist of first exciting the higher resonant mode alone. A "run" would provide excitation of the higher frequency mode at a particular mean square stress level until failure is observed, thereby establishing a point on a random S-N curve. Sufficient runs should be made to determine random fatigue curves for each material at negligible mean stress. Succeeding tests of nominally identical specimens should then be made with the lower frequency mode excited simultaneously with the higher frequency. The result would be a family of curves of root mean square alternating stress versus number of cycles of alternating stress to failure, for several realistic ratios of the root mean square mean stress to the root mean square alternating stress. In order to confirm the hypothesis under trial, the differences between the several curves of the family should be small; significant differences would be cause for rejecting the hypothesis and showing instead, that the detailed form of the stress power spectral density is important and must be taken into account by some method beyond the scope of this report.

The total test program should include tests of several materials and should include tests of both smooth and notched specimens. All specimens should be instrumented sufficiently to determine stress at the point of failure.

It is urged that the above test program be initiated as soon as possible, so that if the results should be favorable it will be possible to take advantage of the other results of this report.

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