REGULARITY OF FUNDAMENTAL SOLUTIONS OF HYPERBOLIC EQUATIONS

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Introduction.

Fundamental solutions play a decisive role in the method of Hadamard [11] for solving the Cauchy problem for hyperbolic equations with variable coefficients, of the second order. In the case of analytic coefficients, he constructed the fundamental solution as a series of functions, each term being determined by the previous ones by solving fairly simple differential systems. Convergence of the series is proved by employing the method of majorants of Cauchy.

For higher order hyperbolic equations with constant coefficients, which are homogeneous in the highest derivatives, the fundamental solution was given by Herglotz [12] in a closed form for m even, $m \geq n$ (m is the order of the equation and n is the number of space-dimensions). A closed form was later given by Petrowski [17] for $m \geq n$, and by F. John [13] and Gelfand-Shapiro [9] (see also [10; Chapter 1]) for all m, n . More recently that form was derived by Borovikov [2] as a consequence of a general formula for fundamental solutions of partial differential equations with constant coefficients.

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Recently, Babitch[w] [1], extending the scheme of Gelfand-Shapiro, has constructed fundamental solutions for hyperbolic equations with analytic coefficients of any order, by representing them as series $G = \Sigma \int u_k f_k d\sigma$ (integration on a parameter σ). The method depends on the construction of some special solutions ("quasi" plane-waves) which are employed in the successive construction of the sequences u_k, f_k . Convergence is proved by generalizing the proof of Hadamard [11]. Using this construction, Babitch proved that the fundamental solution G with pole at a point $(0,x^0)$ is analytic at all points (t,x) $(0 < t < \varepsilon$, ε sufficiently small) which do not lie on the bicharacteristics through $(0,x^0)$ (i.e., on the characteristic conoid with vertex $(0,x^0)$). This result may also be formulated in the following way: (The strict) Huygen's principle is valid for the property of analyticity of solutions.

Babitch also proved that for sufficiently smooth coefficients (and not necessarily analytic), the fundamental solution is differentiable up to any given order at the points (t,x) as above. Finally, he extended all the above results to hyperbolic systems of any order.

The result about the differentiability of the fundamental solution was previously proved by Courant-Lax [3] and by Lax [14], by different methods, for first order hyperbolic systems.

The purpose of the present paper is to extend the results of Babitch in the following way: We consider classes $C\{M_q\}$ consisting of all the C^∞ functions (in some set) whose q-th derivatives are bounded by $H^q M_q$ for all $q \geq 1$ (H is a constant depending on f), where M_q is a given sequence of positive numbers satisfying

$$\binom{k}{h} M_h M_{k-h} \leq A M_k \quad \text{for all} \quad 0 \leq h \leq k < \infty \text{ (A constant).}$$

(*) After the manuscript was completed, it came to our attention that most of the results of Babitch [1] were also obtained, independently, by D. Ludwig, "Exact and asymptotic solution of the Cauchy problem", Comm. Pure Appl. Math., vol. 13 (1960), pp. 473-508.

We then prove that if the coefficients of the hyperbolic equation belong to $C\{M_q\}$, then the fundamental solution belongs to $C\{\hat{M}_q\}$ in any set lying in $0 \leq t \leq \varepsilon$, which excludes the bicharacteristics through $(0, x^0)$, where (with $\tilde{M}_q$ defined by (3.28))

$$\hat{M}_q \leq \frac{M_{3q}}{q!} + \frac{M_{2q}\tilde{M}_q}{q!}, \text{ and if } M_q = q! \text{ then } \hat{M}_q = q! \ .$$

For $M_q = q!$ we thus get a new proof for the analytic case considered by Babitch.

Our procedure starts (as that of Hadamard and Babitch) by constructing "quasi" plane-waves and then sequences u_k, f_k. However, we stop at a certain $k = p$ and proceed to evaluate derivatives D^r or $\check{u} = \sum_{k=0}^{p} u_k f_k$ (or $\int \check{u}\, d\sigma$) and of $\hat{u} = G_\sigma - \check{u}$ (where $G = \int G_\sigma d\sigma$) separately. The derivation of the estimates for the u_k is technically the most lengthy step in our proof. It employs techniques which we used in earlier works [4] - [7]. As for $\hat{u}$, it satisfies a certain hyperbolic equation and, to evaluate $\hat{u}$ we employ well known energy inequalities. In estimating $D^r G$, we take p to be dependent on r (in fact, $p = r + d_0$; d_0 depending on m, n).

We briefly describe the structure of the paper:

In §1 we prove auxiliary lemmas to the effect that various nonlinear operations are closed in classes $C\{M_q\}$. In §2 we solve the Cauchy problem for general first order nonlinear equations within the class $C\{M_q\}$, i.e., we prove (Theorem 1) that if all the data and the equation belong to classes $C\{M_q\}$, then the same is true of the solution.

In §3 we write down the formal procedure of constructing a fundamental solution in the analytic case and then state the main theorem (Theorem 2) of the paper. A theorem (Theorem 3) on interior estimates for hyperbolic equations (analogous to the main theorems in [4][5] for elliptic and parabolic equations) is proved in §4. The proof of Theorem 2 is given in §5. It uses the results of §§1, 4. In §6 we prove (Theorem 4) Huygen's principle for the property of smoothness in the $C\{M_q\}$ – sense, and also mention briefly the case of hyperbolic systems of any order.

1. Auxiliary Lemmas

Let D be an open set, or the closure of an open set, in the n-dimensional euclidean space with coordinates $x = (x_1, \ldots, x_n)$. Let $\{M_q\}$ be a monotone increasing sequence of (positive) numbers with $M_1 \geq 1$, which satisfy for some constant A and all $0 \leq p \leq q < \infty$,

(1.1) $$\binom{q}{p} M_p M_{q-p} \leq AM_q \quad .$$

Taking, in particular, $p = 0$ and $p = 1$ we conclude:

(1.2) $$M_q \geq A_1 q M_{q-1} \quad , \quad M_q \geq (A_1)^q q! \quad (A_1 \text{ constant}) .$$

By $C\{M_q ; D\}$ we mean the class of C^∞ (infinitely differentiable) functions $f(x)$ on D which satisfy for some constants H_0, H (depending on f)

(1.3) $$|D_x^q f(x)| \leq H_0 H^q M_q \qquad (0 \leq q < \infty) .$$

Here D_x^q denotes any partial derivative $\partial^q/\partial x_1^{q_1} \dots \partial x_n^{q_n}$. The class $C\{q! \; ; D\}$ consists of all the functions which are analytic in $\bar{D}$, the closure of D.

If f depends on a parameter λ, we say that $f(x,\lambda)$ belongs to $C\{M_q \; ; D\}$ <u>uniformly</u> with respect to λ, if (1.3) holds with H_0, H independent of λ.

If (1.2) is replaced by

$$
(1.4) \qquad \begin{aligned} |D_x^q f(x)| &\le H_0 && (0 \le q < a) \\ |D_x^q f(x)| &\le H_0 H^{q-a} M_{q-a} && (a \le q < \infty) \end{aligned}
$$

for some integer $a > 0$, then the class is denoted by $C\{M_{q-a} \; ; D\}$. For $a \le 0$, the class $C\{M_{q-a} \; ; D\}$ is defined by (1.3) with M_q replaced by M_{q-a}.

For conveniency we set $M_q = 1$ if $q < 0$. We then can express (1.4) in the equivalent form:

$$
|D_x^q f(x)| \le H_1 H_2^q M_{q-a} \qquad (0 \le q < \infty) .
$$

Lemma 1. Let $u_1, \dots, u_h$ be functions of $x \in D$ which satisy the inequalities $(1 \leq i \leq h)$

$$\left|D_x^r u_i(x)\right| \leq H_0 \qquad (0 \leq r \leq a)$$

(1.5)

$$\left|D_x^r u_i(x)\right| \leq H_0 H^{r-a} M_{r-a} \qquad (a < r \leq p)$$

where $a \geq 2$, and let V be an h-dimensional open set which contains the set $\{u(x) = (u_1(x), \dots, u_h(x)) ; x \in D\}$. Finally, let $F(u_1, \dots, u_h)$ be a function defined in V and satisfying

$$\left|D_u^r F(u)\right| \leq K_0 \qquad (0 \leq r \leq a)$$

(1.6)

$$\left|D_u^r F(u)\right| \leq K_0 K^{r-a} M_{r-a} \qquad (a < r \leq p) .$$

Then, if H is sufficiently large depending on K_0, K, H_0, the following inequalities hold:

$$\left|D_x^r F(u(x))\right| \leq BH_0 \qquad (0 \leq r \leq a)$$

(1.7)

$$\left|D_x^r F(u(x))\right| \leq BH_0 H^{r-a} M_{r-a} \qquad (a < r \leq p) ,$$

where B is a constant depending only on K_0, K, H_0.

Remark. The lemma is not true if $a = 0$ or $a = 1$.

Proof. For $a = 2$ the proof, in a slightly different form, is given in [6; pp. 47-50]. The proof for $a > 2$ is obtained by some obvious modifications of the proof for $a = 2$.

Corollary 1. *If the* u_i *belong to* $C\{M_{q-a}; D\}$, $a \geq 2$, *and if* $F(u)$ *belongs to* $C\{M_{q-a}; V\}$, *then* $F(u(x))$ *belongs to* $C\{M_{q-a}; D\}$.

In particular (with $M_q = q!$), an analytic function of an analytic function is analytic.

We shall also need a more detailed result in the special case $h = 1$, $F(u) = u^i$:

Lemma 2. *Let* $F(u) = u^i$, $u = u(x_1, \dots, x_n)$ *and assume that*

$$|D_x^q\, u(x)| \leq H_0 \qquad (0 \leq q \leq a)$$

$$|D_x^q\, u(x)| \leq H_0 H^{q-a} M_{q-a} \qquad (a < q \leq i),$$

where $a \geq 2$. *Then*

$$|D_x^q\, F(u(x))| \leq K_0 \qquad (0 \leq q \leq a)$$

$$|D_x^q\, F(u(x))| \leq \frac{H_0^i K_0^{i-1}}{H^{i-1}} H^{q-a} q(q-1)\dots(q-i+2)\, M_{q-i-a+1} \quad (a < q \leq i)$$

where K_0 *depends only on* H_0.

Proof. The proof for $a = 2$ follows from (16), (18) of [6]. The proof for $a \geq 2$ is very similar to the case $a = 2$.

Later on we shall deal with polynomials of the form

(1.8) $\sum_{i=0}^{m} b_i(x)\lambda^i$ ($x \in \bar{D}$, D open bounded set)

having the property: for all $x \in \bar{D}$ the roots $\lambda_1(x), \ldots, \lambda_m(x)$ of (1.8) are real and distinct. We shall then need:

Lemma 3. If a polynomial (1.8) has the above property and if the coefficients $b_i(x)$ belong to $C\{M_{q-a} ; D\}$ for some $a \geq 2$, then the $\lambda_j(x)$ also belongs to $C\{M_{q-a} ; D\}$.

Proof. Consider the polynomials $\sum_{i=0}^{m} b_i \lambda^i$ where the b_i vary in a complex neighborhood N_ε of the $b_i(x)$, that is, $\Sigma|b_i - b_i(x)| < \varepsilon$. Since the roots $\lambda_k = \lambda_k(b)$ are continuous functions of $b = (b_0, \ldots, b_m)$, all the $\lambda_k(b)$ are distinct if ε is sufficiently small. Hence, by a well known theorem, the $\lambda_k(b)$ are analytic functions of $b \in N_\varepsilon$. Therefore, by (1.2), they belong to $C\{M_{q-a} ; N_\varepsilon\}$, for any a. Since the $b_i(x)$ belong to $C\{M_{q-a} ; D\}$, the assertion of the lemma follows by Corollary 1.

We next need an extension of the Implicit Function Theorem. We consider a system

(1.9) $F_i(x_1, \ldots, x_n ; y_1, \ldots, y_h) = 0$ $(1 \leq i \leq h)$

and assume that at some point (x^0, y^0)

(1.10) $F_i(x_1^0, \ldots x_n^0; y_1^0, \ldots, y_h^0) = 0$ $(1 \leq i \leq h)$

$$(1.11) \qquad \left.\frac{\partial(F_1, \ldots, F_h)}{\partial(y_1, \ldots, y_h)}\right|_{(x^0,y^0)} \neq 0$$

If the F_i are functions of differentiability class $C^p (p \geq 1)$, then in some neighborhood N of (x^0, y^0) the only solution of (1.9) is given by some functions

$$y_i = f_i(x) \qquad (1 \leq i \leq h)$$

defined in a certain neighborhood D of x^0 and f_i are of class C^p.

We shall prove:

Lemma 4. *Let the F_i satisfy, in addition to the foregoing assumptions, the inequalities* $(1 \leq i \leq h)$

$$(1.12) \qquad \begin{aligned} &|D^r F_i(x,y)| \leq K_0 && (0 \leq r \leq a + 1) \\ &|D^r F_i(x,y)| \leq K_0 K^{r-a-1} M_{r-a-1} && (a + 1 < r \leq p), \end{aligned}$$

where D^r is any r-th *partial derivative with respect to* (x,y), and $a \geq 2$. *Then the solution* $y_i = f_i(x)$ of (1.9) *satisfies*

$$(1.13) \qquad \begin{aligned} &|D_x^r f_i(x)| \leq H_0 && (0 \leq r \leq a) \\ &|D_x^r f_i(x)| \leq H_0 H^{r-a} M_{r-a} && (a < r \leq p) \end{aligned}$$

where H_0 is determined so that (1.13) *holds for* $0 \leq r \leq a$ *and* H *then depends only on* K_0, K, H_0 *and on a lower bound on the absolute value of* $\partial(F_1, \ldots, F_h) / \partial(y_1, \ldots, y_h)$.

<u>Corollary 2</u>. If the F_i belong to $C\{M_{q-a-1}; N\}$ then the f_i belong to $C\{M_{q-a}; D\}$.

The analytic case $(M_q = q!)$ is of course well known, but the standard proofs are different from the present one.

<u>Proof</u>. The proof is by induction on r. The assertion (1.13) for $r < a + 1$ is valid by the choice of H_0. We now assume that (1.13) holds for all $0 \leq r < q$ $(q \leq p)$ and proceed to prove it for q. Differentiating (1.9) with respect to x_j we get

$$\frac{\partial F_i}{\partial x_j} + \sum_{k=1}^{h} \frac{\partial F_i}{\partial y_k} \frac{\partial f_k}{\partial x_j} = 0 \qquad (1 \leq i \leq h) . \tag{1.14}$$

We next apply D_x^{q-1} (where D_x now means total x-differentiation) to both sides of (1.14) and obtain

$$\sum_{k=1}^{h} \frac{\partial F_i}{\partial y_k} D_x^q f_k = \frac{\partial^q}{\partial x^q} F_i + \sum_{k=1}^{h} \sum_{j=1}^{q-1} \binom{q-1}{j} \left[D_x^j \frac{\partial F_i}{\partial y_k} \right] D_x^{q-j} f_k . \tag{1.15}$$

Here we used Leibnitz' rule

$$D^q(fg) = \sum_{i=0}^{q} \binom{q}{i} D^i f D^{q-i} g$$

where $\binom{q}{i} D^i f D^{q-i} y$ means that there are $\binom{q}{i}$ terms of the form $D^i f D^{q-i} g$, D being any partial derivative.

The functions $y_k = f_k(x)$ satisfy (1.13) for all $r \leq q - 1$. Hence, applying Lemma 1 we get

$$\left| D_x^j \left(\frac{\partial F_i(x,y)}{\partial y_k} \right) \right| \leq B\, H_0 H^{j-a} M_{j-a} \qquad (a \leq j \leq q-1) ,$$

provided H is sufficiently large (depending on K_0, K, H_0) .

Substituting this into (1.15) we find that the right side is bounded by

$$(1.16) \qquad K_0 K^{q-a-1} M_{q-a-1} + \sum_{j=1}^{q-1} \binom{q-1}{j} B H_0 \hat{H}^{j-a} M_{j-a} H_0 \hat{H}^{q-j-a} M_{q-j-a} \quad ,$$

where we use the convention:

$$\hat{H}^i = H^i \text{ if } i \geq 0 \, , \; \hat{H}^i = 1 \text{ if } i < 0 \, .$$

Now,

$$(1.17) \qquad \sum_{j=1}^{q-1} \binom{q-1}{j} M_{j-a} M_{q-j-a} \leq B_1 M_{q-a}$$

as follows from calculations similar to [6; p. 49]; B_i are used to denote constants depending only on K_0, K, H_0 and on a lower bound on $|\partial(F_1, \ldots, F_h)/\partial(y_1, \ldots, y_h)|$.

Substituting (1.17) into (1.16) we find that the right side of (1.15) is bounded by

$$B_2 H_0 H^{q-a-1} M_{q-a} \quad ,$$

provided H is also $> K$. We finally solve the linear system (1.15) for $D_x^q f_k$ and get

$$\left| D_x^q f_k(x) \right| \leq B_3 H_0 H^{q-a-1} M_{q-a} \quad .$$

Taking $H \geq B_3$ the proof of (1.13) for $r = q$ is completed.

Remark. If some of the F_i are linear functions in the y_k , then the assertion of Lemma 4 remains true assuming that these F_i satisfy (1.12) with $a + 1$ replaced by a . Corollary 2 also remains true assuming that these F_i belong to $C\{M_{q-a} ; N\}$.

a Manifold S is said to belong to class $C\{M_{q-a}\}$ if it can be covered by a finite number of patches, each having a local representation in terms of a function, say, $f(y)$ of class $C\{M_{q-a}\}$. A family of manifolds $S(x)$ is said to belong to class $C\{M_{q-a}\}$ if the f's are of class $C\{M_{q-a}\}$ in the variables (y,x). A family $S(x)$ is said to belong <u>strongly</u> to class $C\{M_{q-a}\}$ if (i) there exists one-to-one correspondence $y(x) \to y(x')$ between the points of $S(x)$ and $S(x')$ whenever $|x - x'| < \delta$ (for some $\delta > 0$), and (ii) in the local representation of $S(x)$, say $y_i = g(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k, x)$, where $y = y(x)$, g is of class $C\{M_{q-a}\}$ in all the variables. The local representation of any $S(x)$ is assumed to be valid also for all $S(x')$ with $|x - x'| < \delta$.

<u>Lemma 5</u>. <u>Let</u> $S(x)$ <u>be a family of</u> n-<u>dimensional manifolds with boundaries</u> $\Sigma(x)$ <u>which are</u> $(n-1)$ - <u>dimensional manifolds with no boundary</u>. <u>Assume that</u> $S(x)$ <u>and</u> $\Sigma(x)$ <u>belong strongly to</u> $C\{M_{q-a}\}$ <u>for</u> $x \in D$ (D <u>open bounded set</u>), <u>where</u> $a \geqslant 2$. <u>Let</u> $u(x,y)$ <u>belong to</u> $C\{M_{q-a}\}$ <u>for</u> (x,y) <u>in an open set</u> V <u>which contains the closure of</u> $\{(x,y)\,;\, y \in S(x),\ x \in D\}$. <u>Then the integral</u>

$$I(x) = \int_{S(x)} u(x,y)\,dS_y(x)$$

<u>belongs to</u> $C\{M_{q-a}; D\}$.

<u>Proof</u>. Differentiating I twice we obtain

$$D_x I(x) = \int_{S(x)} [Du(x,y) + u(x,y)\Delta(x,y)]\, dS_y(x)$$

$$+ \int_{\Sigma(x)} u(x,y)\,\Gamma(x,y)\, d\Sigma_y(x)\,, \tag{1.18}$$

and

$$D_x^2 I(x) = \int_{S(x)} \{D[Du + u\Delta] + [Du + u\Delta]\,\Delta\}\, dS_y(x)$$

$$+ \int_{\Sigma(x)} \{[Du + u\Delta]\Gamma + D(u\Gamma)W\}\, d\Sigma_y(x) \tag{1.19}$$

where $Du = D_x u + D_y u D_x y$ and where $\Delta = \Delta_i$, $\Gamma = \Gamma_i$, $W = W_i$ if $D = D_i$. Δ and Γ, W can be represented in terms of the functions of the local representations of $S(x)$ and $\Sigma(x)$ respectively and of their first derivatives. We then i nd that

$$|D^r L| \leq A_0 A^r M_{r-a+1} \qquad (0 \leq r < \infty) \quad \text{for} \quad L = \Delta, \Gamma, W , \tag{1.20}$$

where D^r now means any r-th partial derivatives with respect to (x,y) .

In deriving (1.19) we made us of the fact that $\Sigma(x)$ has no boundary.

We can now proceed to differentiate $I(x)$ any number of times. Introducing the notation

$$f_{r+1} = Df_r \cdot W , \quad f_0 = f$$

we have:

$$D_x^q I(x) = \int_{S(x)} (Du + u\Delta)^q \, dS_y(x)$$

$$+ \int_{\Sigma(x)} \left\{ \sum_{i=1}^{q-1} D\left\{[(Du + u\Delta)^{q-i}\Gamma]_{i-1}\right\} + (u\Gamma)_{q-1} \right\} d\Sigma_y(x) \quad . \tag{1.21}$$

We shall prove that for any $q \geq 0$

$$|D_x^q I(x)| \leq H_0 H^q M_{q-a} \quad . \tag{1.22}$$

In proving it, we shall use the inequalities

$$|D^r u(x,y)| \leq K_0 K^r M_{r-a} \tag{1.23}$$

which follow from the assumptions of the lemma.

Using Lemma 2 and (1.20), (1.23) we find that

$$|D^r[(Du + u\Delta)^{q-i}]| \leq \frac{K_1}{2^{q-i}} K_2^r M_{r-a+1} \tag{1.24}$$

for all $r \geq 0$, where K_i are appropriate constants.

Next, it can be proved by induction on s that if a function g satisfies, for all $r \geq 0$,

(1.25) $$|D^r g| \leq N_0 N^r M_{r-b}$$

where $b \geq 0$, then

(1.26) $$|D^r g_s| \leq N_0 N^{r+s} M_{r+s-b}$$

provided N is sufficiently large (depending only on A_0, A of (1.20)).

with $b=a-1$,

Applying this fact to $g = (Du + u\Delta)^{q-1}\Gamma$ (which satisfies (1.25) by (1.20), (1.24) combined) and to $g = u\Gamma$, we obtain from (1.21) the inequality (1.22) with appropriate constants H_0, H (independent of q).

From the proof of Lemma 5 one can easily establish:

<u>Lemma 5'.</u> <u>Let</u> $S(x,\gamma)$, $\Sigma(x,\gamma)$ <u>satisfy, for each</u> γ $(\alpha \leq \gamma \leq \beta)$, <u>the assumptions of Lemma</u> 5 <u>and let</u> $S(x,\gamma)$, $\Sigma(x,\gamma)$ <u>belong strongly to</u> $C\{M_{q-a}\}$ $(a \geq 2)$. <u>Finally, let</u> $u(x,y,\gamma)$ <u>belong to</u> $C\{M_{q-a}\}$ <u>in an open set</u> V <u>containing the closure of</u> $\{(x,y,\gamma)\ ;\ x \in D,\ \alpha \leq \gamma \leq \beta,\ y \in S(x,\gamma)\}$. <u>Then the integral</u>

$$I(x) = \int_a^b \int_{S(x,\gamma)} u(x,y,\gamma)\, dS_y(x,\gamma)\, d\gamma$$

<u>belongs to</u> $C\{M_{q-a}\ ;\ D\}$.

2. Cauchy Problem for Nonlinear First Order Equations

Consider the differential equation

(2.1) $$F(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0$$

where F is a C^∞ function in all its arguments and $p_j = \partial z/\partial x_j$. The Cauchy problem consists in finding a unique solution $z = z(x_1, \ldots, x_n)$ of (2.1) (which is an n-dimensional manifold) passing through a given (n-1)-dimensional manifold

$$x_i^0 = \psi_i(t_1, \ldots, t_{n-1}) \qquad (1 \leq i \leq n)$$

(2.2)

$$z^0 = \psi_{n+1}(t_1, \ldots, t_{n-1}) \quad .$$

In order to solve the problem, values p_i^0 of p_i corresponding to (x_i^0, z^0) must first be found or be given. These values must necessarily satisfy the equations

$$F(x_1^0, \ldots, x_n^0, z^0, p_1, \ldots, p_n) = 0$$

(2.3)

$$\sum_{i=1}^{n} p_i \frac{\partial \psi_i}{\partial t_h} = \frac{\partial \psi_{n+1}}{\partial t_h} \qquad (1 \leq h \leq n-1) \ .$$

One is thus led to assume that for the initial manifold (2.2) the following conditions hold: (i) There exists a solution $p_i = p_i^0$ of (2.3), and (ii) The $n \times n$ matrix

$$\left(\frac{\partial F}{\partial p_i}, \quad \frac{\partial(x_1^0, \ldots, x_n^0)}{\partial(t_1, \ldots, t_{n-1})}\right)$$

is non-singular on the initial manifold, when $p_i = p_i^0$.

We remark that if (ii) is violated, there may exist more than one solution or no solution at all to the Cauchy problem.

The solution of (2.1), (2.2) (with $p_i = p_i^0$ for $x_i = x_i^0$, $z = z^0$) is constructed with the aid of an auxiliary system of ordinary differential equations (the characteristic equations)

$$\frac{dx_i}{ds} = \frac{\partial F}{\partial p_i} , \quad \frac{dz}{ds} = \sum_{h=1}^{n} p_h \frac{\partial F}{\partial p_h} , \tag{2.4}$$

$$\frac{dp_i}{ds} = -\left(\frac{\partial F}{\partial x_i} + p_i \frac{\partial F}{\partial z}\right) \qquad (1 \leq i \leq n) .$$

We solve this system with the initial values x_i^0 , z^0 , p_i^0 . The solution

$$\begin{aligned} x_i &= x_i(s,t_1, \ldots, t_{n-1}) \\ z &= z(s,t_1, \ldots, t_{n-1}) \\ p_i &= p_i(s,t_1, \ldots, t_{n-1}) \end{aligned} \tag{2.5}$$

is C^∞ in $(s,t_1, \ldots, t_{n-1})$. Using the assumption (ii) one finds that if s is sufficiently small, then

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(s,t_1, \ldots, t_{n-1})} \neq 0 . \tag{2.6}$$

Hence, we can solve $s,t_1, \ldots, t_{n-1}$ in terms of $x_1, \ldots, x_n$. Substituting this into z in (2.5), we obtain a C^∞ function $z = g(x_1, \ldots, x_n)$ which can be proved to be a solution of the Cauchy problem (the p_i are proved to be $\partial z/\partial x_i$) .

We shall now prove:

Theorem 1. *If F belongs to $C\{M_{q-b}\}$ (in all its variables) for some $b \geq 3$, and if the ψ_i belong to $C\{M_{q-b}\}$, then the solution $z = z(x_1, \ldots, x_n)$ belongs to $C\{M_{q-b}\}$.*

For simplicity the domains where the classes $C\{M_{q-b}\}$ are defined has not been mentioned. The domain where $z(x_1, \ldots, x_n)$ is of class $C\{M_{q-b}\}$ is some neighborhood of $(x_1^0, \ldots, x_n^0)$.

Proof. Set $a = b - 1$; then $a \geq 2$. We can write the system (2.4) and the initial conditions $(s = 0)$ in the form

$$(2.7) \qquad \frac{dv_i(s,t)}{ds} = \Phi_i(v_1, \ldots, v_N)$$

$$(2.8) \qquad v_i(0,t) = v_i^0(t) \qquad (1 \leq i \leq N)$$

where $t = (t_1, \ldots, t_{n-1})$, $N = 2n + 1$, $v_i = x_i$ if $1 \leq i \leq n$, $v_{n+1} = z$, $v_{n+1+i} = p_i$ if $1 \leq i \leq n$. Φ_i are functions of class $C\{M_{q-a}\}$ in $(v_1, \ldots, v_n)$ and v_i^0 are functions of class $C\{M_{q-a}\}$ in t. Indeed, in view of our assumptions, all that remains to show is that the p_i^0 belong to $C\{M_{q-a}\}$ and that follows by Corollary 2 of §1 and the remark following the proof of Lemma 4.

We next perform the transformation

$$w_i(s,t) = v_i(s,t) - v_i(0,t)$$

and obtain

$$(2.9) \qquad \frac{dw_i(s,t)}{ds} = \Psi_i(s,t,w_1, \ldots, w_N)$$

$$(2.10) \qquad w_i(0,t) = 0 \qquad (1 \leq i \leq N)$$

where, by Lemma 1 (or its corollary), the Ψ_i are of class $C\{M_{q-a}\}$ in all the variables.

We shall need the following lemma:

Lemma 6. If the ψ_i belong to $C\{M_{q-a}\}$ for some $a \geq 2$, then the solution w_j of (2.9), (2.10) also belongs to $C\{M_{q-a}\}$.

From the lemma it follows that x_i, z, p are of class $C\{M_{q-a}\}$ in $(s, t_1, \ldots, t_{n-1})$. Applying the remark following the proof of Lemma 4 (concerning Corollary 2) we conclude that z and p_i are of class $C\{M_{q-a}\}$ in $(x_1, \ldots, x_n)$ (Indeed, we take F_i to be $x_i - x_i(s, t_1, \ldots, t_{n-1})$, i.e., linear in the x_j.). Since $p_i = \partial z/\partial x_i$, z is then of class $C\{M_{q-a+1}\} = C\{M_{q-b}\}$, and the proof of Theorem 1 is completed.

It remains to prove Lemma 6.

Proof of Lemma 6. We first prove that it is enough to establish the inequalities

$$|D_t^r\, w_i(s,t)| \leq H_0 \qquad (0 \leq r \leq a) \tag{2.11}$$

$$|D_t^r\, w_i(s,t)| \leq H_0 H^{r-a} M_{r-a} \qquad (a < r < \infty) \tag{2.12}$$

for $i = 1, \ldots N$. Indeed we shall prove that if (2.11), (2.12) hold, then

$$|D_s^p\, D_t^r\, w_i(s,t)| \leq \overline{H}_0 \qquad (0 \leq p + r \leq a) \tag{2.13}$$

$$|D_s^p\, D_t^r\, w_i(s,t)| \leq \overline{H}_0 \overline{H}^{p+r-a} M_{p+r-a} \qquad (a < p + r < \infty) \tag{2.14}$$

for some $\overline{H}$ depending on $H_0, H, \overline{H}_0$; $\overline{H}_0$ is chosen so that (2.13) is satisfied.

We proceed to establish (2.14) by induction on p ; assuming it to hold for all $p \leq q$ we shall prove it for $p = q + 1$. The case $p = 0$ follows by (2.11), (2.12).

Applying $D_s^p D_t^r$ to both sides of (2.9) we obtain

$$D_s^{p+1} D_t^r w_i = D_s^p D_t^r \psi_i(s,t,w_1, \ldots, w_N) . \tag{2.15}$$

Using Lemma 1, the assertion readily follows if $\bar{\bar{H}}$ is appropriately large (independently of p, r) .

We remark, in passing, that the ψ_i are of class $C\{M_q'\}$ in s where $M_q' \equiv 1$, and we can therefore establish (using (2.11), (2.12)) the inequalities

$$|D_s^p D_t^r w_i| \leq \bar{\bar{H}}_0 \bar{\bar{H}}^{p+r-a} M'_{r-a} \qquad (a \leq p + r < \infty) . \tag{2.16}$$

These stronger inequalities, however, are not needed in proving Theorem 1.

It remains to prove (2.11), (2.12). We first choose H_0 so that (2.11) hold and then proceed to prove (2.12) by induction on r : assuming it to hold for all $r \leq q - 1$ we shall prove it for $r = q$.

Applying D_t^q to both sides of (2.9) and integrating with respect to s we get, using (2.10),

$$D_t^q w_i(s,t) = \int_0^s D_t^q \psi_i(\sigma, t, w_1, \ldots, w_N) d\sigma . \tag{2.17}$$

Expanding $D_t^q \psi_i$ by the formula of total differentiation, we see that all the derivatives $D_t^j w_k$ appearing in the expansion are of order less than q with the exception of the terms

$$\left(\frac{\partial}{\partial w_k} \psi_i\right) D_t^q w_k \quad .$$

Hence, applying Lemma 1, we obtain

$$|D_t^q \psi_i| \leq A_1 \sum_{k=1}^{N} |D_t^q w_k| + A_2 H_0 H^{q-a-1} M_{q-a} \quad ,$$

where A_i are used to denote constants depending only on the ψ_j and on H_0. Substituting this inequality into (2.17), taking absolute values on both sides of the resulting inequality and finally summing over $i = 1, \ldots, N$ we obtain

$$(2.18) \qquad \varphi(s) \leq A_3 \int_0^s \varphi(\sigma)d\sigma + A_3 sH_0 H^{q-a-1} M_{q-a}$$

where

$$(2.19) \qquad \varphi(s) = \sum_{k=1}^{N} |D_t^q w_k(s,t)| \quad .$$

Integrating both sides of (2.18) with respect to s and taking s sufficiently small $(2sA_3 \leq 1)$ we obtain an estimate for $\int \varphi(\sigma)d\sigma$. Substituting back into (2.18), we conclude that

$$\varphi(s) \leq A_4 \; sH_0 H^{q-a-1} M_{q-a}.$$

Hence, if $A_4 s \leq H$ then (2.12) follows for $r = q$.

Remark. Lemma 6 is valid without any restriction on the smallness of s. To prove it one modifies the last argument in the proof and uses Lemma 8 of §4.

From the proof of Theorem 1 we get:

Corollary. If the initial values ψ_i depend on a parameter λ and are of class $C\{M_{q-b}\}$ $(b \geq 3)$ in (t,λ), then the solution $z = z(x,\lambda)$ is of class $C\{M_{q-b+1}\}$ in (x,λ) (in fact, in class $C\{M_{q-b}\}$ in x and $C\{M_{q-b+1}\}$ in λ).

Indeed, the initial values of p_i^0 are then of class $C\{M_{q-b+1}\}$ in (t,λ) (by using Corollary 2 and the remark following the proof of Lemma 4) and we then can proceed as in the proof of Theorem 1, slightly modifying Lemma 6.

3. Formulas for Fundamental Solutions

The formulas of 3.2, 3.3 are taken from Babitch [1].

3.1. Definitions

Consider the differential operator

$$Pu \equiv P(t,x,\frac{\partial}{\partial t},\frac{\partial}{\partial x})u \equiv \frac{\partial^m u}{\partial t^m} + \sum_{k_0+k_1+\dots+k_n=m} a_{k_0 k_1 \dots k_m}(t,x)\,\frac{\partial^{k_0+k_1+\dots+k_m} u}{\partial t^{k_0}\partial x_1^{t_1}\dots\partial x_n^{t_n}}$$

and denote its principal part by $P_0(t,x,\frac{\partial}{\partial t},\frac{\partial}{\partial x})$. P is said to be <u>hyperbolic</u> (with respect to the t-direction) if for every real vector $\xi = (\xi_1, \dots, \xi_n) \neq 0$ the algebraic equation $P_0(t,x,\lambda,\xi) = 0$ has m real and distinct roots λ_i, for any value of (t,x).

We shall consider in this paper only <u>uniformly</u> hyperbolic operators, in the following sense:

(i) If we denote by $\lambda_i(\xi,t,x)$ the roots of $P_0 = 0$, then

$$\inf_{(\xi,t,x)} |\lambda_i(\xi,t,x) - \lambda_j(\xi,t,x)| > 0$$

for all $i,j = 1, \dots, m$, where $i \neq j$, ξ varies on $|\xi| = 1$, $0 \leq t \leq \mathcal{E}_0$ for some $\mathcal{E}_0 > 0$, and x varies in the euclidean space R^n.

(ii) The coefficients $a_\alpha(t,x)$ of P and the first derivatives of a_α, $|\alpha| = m$, are uniformly bounded in the strip: $|t| \leq \mathcal{E}_0$, $x \in R^n$.

Under these assumptions and the assumptions that the a_α are sufficiently smooth, the following Cauchy problem has been solved by Petrowski [16], Leray [15] and Gårding [8]:

$$(3.1)\qquad Pu = f(t,x)$$

$$(3.2)\qquad \left.\frac{\partial^h u}{\partial t^h}\right|_{t=0} = \varphi_h(x) \qquad (0 \le h \le m-1)\ .$$

The degree of smoothness of the solution depends on the degree of smoothness of the a_α, f and the φ_h. Petrowski's work contains a gap; Leray's work is complete, whereas Gårding's work is a slight improvement of Leray's results and mostly a simplification of the methods. He considers also non-smooth data, assuming that P satisfies only (i), (ii).

A <u>fundamental solution</u> G_j $(0 \le j \le m-1)$ of the Cauchy problem with pole (t^0,x^0) is a distribution G_j in x, with t as a parameter, which satisfies the equation $PG_j = 0$ and the initial conditions

$$(3.3')\qquad \left.\frac{\partial^h G_j}{\partial t^h}\right|_{t=t^0} = \begin{cases} 0 & \text{if} \quad h \ne j \qquad (0 \le h \le m-1) \\ \delta(x-x^0) & \text{if} \quad h = j \end{cases}$$

where $\delta(x)$ is the Dirac measure with support at the origin.

Babitch [1] considered only the case $j = m-1$. We consider first this case but in §6 we discuss the general case. We set $G = G_{m-1}$ so that

$$(3.3)\qquad \left.\frac{\partial^h G}{\partial t^h}\right|_{t=t^0} = 0 \ \text{if}\ 0 \le h \le m-2\ , \quad \left.\frac{\partial^{m-1} G}{\partial t^{m-1}}\right|_{t=t^0} = \delta(x-x^0)\ .$$

3.2. "Quasi" Plane-Waves

If the coefficients of P_0 are constants, then for any real vector $\sigma = (\sigma_1, \ldots, \sigma_n) \neq 0$ and for any function f there are m solutions $f(\gamma)$ of $P_0 u = 0$, where $\gamma = vt + x\cdot\sigma$ $(x\cdot\sigma = \Sigma\, x_i\sigma_i)$ and v is any one of the m real and distinct roots of $P_0(v,\sigma) = 0$. The solutions $f(\gamma)$ may be considered as plane-waves.

To construct an analogue of γ in the general case (γ may then be viewed as a "quasi" plane-wave) we solve the problem

(3.4) $$P_0(t, x, \frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial x}) = 0$$

(3.5) $$\gamma\Big|_{t=0} = \sigma\cdot x \ .$$

Equation (3.4) is of the form (2.1) and the initial conditions analogous to (2.2) can be given by setting $s = t$ and

(3.6) $$x_i\Big|_{s=0} = y_i \quad (1 \leq i \leq n)\ , \quad x_0\Big|_{s=0} = t\Big|_{s=0} = 0$$

(3.7) $$\gamma\Big|_{s=0} = \sigma\cdot y\ ,$$

where we (sometimes) set $x_0 \equiv t$.

Conditions on the p_i at $s = 0$ (which satisfy the analogue of (2.3)) are given by

(3.8) $$p_i\Big|_{s=0} = \frac{\partial\gamma}{\partial x_i}\Big|_{s=0} = \sigma_i \qquad (1 \leq i \leq n)\ ,\ p_0 = v\ .$$

If $x_i = y_i$ on $s = 0$ $(1 \leq i \leq n)$ and if $t = 0$ on $s = 0$ then (3.5) implies (3.7), (3.8). Conversely, (3.7) and (3.8) imply (3.5). The values $p_0 = v$ are calculated from

(3.9) $$P_0(0,y,v,\sigma) = 0$$

and there are m distinct solutions $v = v_i(y,\sigma)$ $(1 \leq i \leq m)$, thus giving rise to m distinct solutions $\gamma = \gamma^{(i)}(x,t)$.

For later purposes we write down the characteristic system corresponding to the Cauchy problem (3.4), (3.6)–(3.8):

(3.10) $$\frac{dx_i}{ds} = \frac{\partial P_0(t,x,p_0,p)}{\partial p_i}, \quad \frac{d\gamma}{ds} = \sum_{j=1}^{n} p_j \frac{\partial P_0(t,x,p_0,p)}{\partial p_j},$$
$$\frac{dp_i}{ds} = -\frac{\partial P_0(t,x,p_0,p)}{\partial x_i} \qquad (0 \leq i \leq n \ ; \ p = (p_1,\ldots,p_n)).$$

Solutions of (3.10) are called bicharacteristics.

3.3. Formal Construction of Fundamental Solutions

For simplicity we take $(t^0, x^0) = (0,0)$.

We shall use the formula of Gelfand-Shapiro [9] (see also [10; Chapter 1]):

(3.11) $$\delta(x) = \int_{|\sigma|=1} \varphi_n(x\cdot\sigma)d\sigma, \quad \varphi_n(r) = \begin{cases} c_n \delta^{(n-1)}(r) & (n \text{ odd}) \\ c_n r^{-n} & (n \text{ even}) \end{cases}$$

where c_n is a constant. It should be stresses that by r^{-2k} we understand the distribution defined by (see [10; p.73])

$$(r^{-2k},\varphi) = \int_0^\infty r^{-2k} \{\varphi(r) + \varphi(-r) - 2[\varphi(0) + \frac{r^2}{2!}\varphi''(0) + \dots + \frac{r^{2k-2}}{(2k-2)!}\varphi^{(2k-2)}(0)]\}dr$$

and by r^{-2k-1} we understand the distribution defined by

$$(r^{-2k-1},\varphi) = \int_0^\infty r^{-2k-1}\{\varphi(r) - \varphi(-r) - 2[r\varphi'(0) + \frac{r^3}{3!}\varphi^{(3)}(0)+\dots+\frac{r^{2k-1}}{(2k-1)!}\varphi^{(2k-1)}(0)]\}dr.$$

We have: $\frac{d}{dr} r^{-h} = -hr^{-h-1}$, and the distribution r^h for $h > 0$ is the ordinary function r^h.

We intend to find a solution G_σ of $PG_\sigma = 0$, satisfying

(3.12) $$\left.\frac{\partial^h G_\sigma}{\partial t^h}\right|_{t=0} = 0 \quad (0 \le h \le m-2), \quad \left.\frac{\partial^{m-1} G_\sigma}{\partial t^{m-1}}\right|_{t=0} = \varphi_n(x\cdot\sigma).$$

Then, the fundamental solution G would be

(3.13) $$G = \int_{|\sigma|=1} G_\sigma d\sigma .$$

We shall find G_σ in the form

(3.14) $$G_\sigma = \sum_{j=1}^m G_{j\sigma}, \quad G_{j\sigma}(t,x) = \sum_{k=0}^\infty u_{kj\sigma}(t,x)\, f_k(\gamma_\sigma^{(j)})$$

and, for simplicity, we write

(3.15) G_j for $G_{j\sigma}$, u_{kj} for $u_{kj\sigma}$, $\gamma^{(j)}$ for $\gamma_\sigma^{(j)}$.

The series $G_j = \sum_k u_{kj} f_k(\gamma^{(j)})$ has to satisfy $PG_j = 0$ and ΣG_j has to satisfy (3.12), formally. We do not consider here the question of convergence.

We introduce the operators P_s by the identity

$$P(uf(\gamma)) = \sum_{s=0}^{m} P_s(u) \frac{d^{m-s} f}{d\gamma^{m-s}} . \tag{3.16}$$

P_s are linear differential operators of order s and their coefficients are linear combinations of products of derivatives $D^j \gamma$ $(j \leq s + 1)$ with coefficients which are coefficients a_α of P. In particular,

$$P_0(u) = P_0(t, x, \frac{\gamma}{\gamma}, \frac{\gamma}{\gamma})u \equiv \sum_{k_0+\ldots+k_n=m} a_{k_0 \ldots k_n}(t,x) \gamma_0^{k_0} \ldots \gamma_n^{k_n} u \equiv Au \tag{3.17}$$

$$P_1(u) = \Sigma \frac{\partial A}{\partial \gamma_i} \frac{\partial u}{\partial x_i} + \Big(\frac{1}{2} \sum_{i,j} \frac{\partial^2 A}{\partial \gamma_i \partial \gamma_j} \gamma_{ij} + B\Big)u , \tag{3.17}$$

$$B = \sum_{k_0+\ldots+k_n=m-1} a_{k_0 \ldots k_n}(t,x) \gamma_0^{k_0} \ldots \gamma_n^{k_n}$$

where $\gamma_i = \partial\gamma/\partial x_i$, $\gamma_{ij} = \partial^2\gamma/\partial x_i \partial x_j$ $\quad (0 \leq i, j \leq n)$.

Taking f_k such that $f_{k+1}(r) = \int f_k(r)dr$ we conclude that $u = \Sigma u_k(t,x) f_k(\gamma)$ is a solution of $Pu = 0$ if

$$P_0(u_0) = 0 , \; P_1(u_0) + P_0(u_1) = 0 , \ldots, \; P_m(u_k) + P_{m-1}(u_{k+1}) + \ldots + P_0(u_{k+m}) = 0 , \; \ldots \; . \tag{3.18}$$

We now make a special choice of γ , namely, we take γ to be one of the m "quasi" plane-waves of 3.2. Then $A = 0$ and the first equation $P_0(u_0) = 0$ in (3.18) is satisfied, in view of (3.17). Also, $P_0(u_k) = 0$ for any k .

If we use (3.10), then the second equation becomes

$$\frac{du_0}{ds} + \left(\frac{1}{2} \Sigma \frac{\partial^2 A}{\partial\gamma_i \partial\gamma_j} \gamma_{ij} + B\right)u_0 = 0 \qquad (s = t) ,$$

and the general equation of (3.18) becomes

$$\frac{du_k}{dt} + \left(\frac{1}{2} \Sigma \frac{\partial^2 A}{\partial\gamma_i \partial\gamma_j} \gamma_{ij} + B\right)u_k = -\sum_{i=2}^{m} P_i(u_{k-i+1}) \tag{3.19}$$

provided we agree to set $u_{-1} = u_{-2} = \dots = u_{-m+1} = 0$.

Since we have to deal with m distinct $\gamma = \gamma^{(h)}$, we get the equations

$$\frac{du_{kh}}{dt} + \left(\frac{1}{2} \Sigma \frac{\partial^2 A_h}{\partial\gamma_i^{(h)} \partial\gamma_j^{(h)}} \gamma_{ij}^{(h)} + B_h\right)u_{kh} = -\sum_{i=2}^{m} P_{ih}\,(u_{k-i+1,\ h}) \tag{3.20}$$

where A_h, B_h, P_{ih} are obtained from A , B, P_i by taking $\gamma = \gamma^{(h)}$.

In order to satisfy (3.12), formally, we first apply $\partial^h/\partial t^h$ to $u = \sum_{k=0}^{\infty} u_k\, f_k(\gamma)$ and obtain:

$$\frac{\partial^h u}{\partial t^h} = \sum_{k=0}^{\infty} \left[u_k(\gamma_0)^h + Q_{h1}(u_{k-1}) + \dots + Q_{hi}(u_{k-i}) + \dots + Q_{hh}(u_{k-h})\right] f_{k-h}(\gamma) \tag{3.21}$$

where $u_{-j} = 0$ if $j \geq 1$. Here

$$Q_{hi} = q_{hi} \frac{\partial^i}{\partial t^i}$$

and q_{hi} is a polynomial in γ and its derivatives up to order h .

If we take

$$f_{-m+1}\,(r) = \begin{cases} c_n \delta^{(n-1)}(r) & (n \text{ odd}) \\ c_n\, r^{-n} & (n \text{ even}) , \end{cases} \tag{3.22}$$

then it follows that the initial conditions (3.12) are equivalent to

$$\left[\sum_{j=1}^{m} u_{kj}(\gamma_0^{(j)})^h + \dots + Q_{hij}(u_{k-i,j}) + \dots + Q_{hhj}(u_{k-h,j})\right]_{t=0}$$

$$(3.23) \qquad = \begin{cases} 1 & \text{if } h = m-1 \text{ and } k = 0 \\ & \text{otherwise,} \end{cases}$$

where Q_{hij} is Q_{hi} with $\gamma = \gamma^{(j)}$.

Since $\gamma_0^{(j)}$ are all distinct, (3.23) defines the u_{kj} uniquely in terms of the preceding u_{sj}, $s < k$ (recall our convention: $u_{sj} = 0$ if $s < 0$).

We finally write down the sequence f_k (as follows by (3.22) and the rule $f_{k+1} = \int f_k$).

For n odd,

$$(3.24) \qquad f_k(r) = \begin{cases} c_n \delta^{(n-m-k)}(r) & \text{if } n-m-k \geq 0 \\ c_n \dfrac{r^{k+m-n-1}\mathcal{E}(r)}{(k+m-n-1)!} & \text{if } n-m-k < 0 \end{cases}$$

where $\mathcal{E}(r) = 0$ if $r < 0$, $\mathcal{E}(r) = 1$ if $r \geq 0$.

For n even

$$(3.25) \qquad f_k(r) = \begin{cases} \dfrac{c_n}{(k+m-1-n)\dots(1-n)\, r^{n+1-k-m}} & \text{if } n+1-k-m > 0 \\ \dfrac{(-1)^n c_n}{(n-1)!} \log \dfrac{1}{|r|} & \text{if } n+1-k-m = 0 \\ \dfrac{(-1)^n c_n\, r^{k+m-n-1}}{(n-1)!\,(k+m-n-1)!}\left(\log \dfrac{1}{|r|} + 1 + \dots + \dfrac{1}{k+m-n-1}\right) & \\ & \text{if } n+1-k-m < 0 . \end{cases}$$

3.4. Statement of the Main Result

The characteristic conoid of P with the origin as vertex is the set of all points (t,x) satisfying the system of equations:

$$(3.26) \qquad \gamma_\sigma^{(j)}(t,x) = 0\,, \quad \frac{\partial}{\partial \psi_i}\gamma_\sigma^{(j)}(t,x) = 0 \qquad (1 \le i \le n-1)$$

for some (σ, j), where $|\sigma| = 1$, $1 \le j \le m$. Here ψ_i are local coordinates on the unit sphere.

Let V be an open bounded set of points (t,x), or a closure of such a set, which does not intersect the characteristic conoid, and let V^ε be the set $V \cap \{0 \le t \le \varepsilon\}$, for any $\varepsilon > 0$. Let W_ρ^ε be the set

$$\{(t,x)\ ;\ 0 \le t \le \varepsilon\,,\ \ |x| \le \rho\}$$

for any $0 < \rho \le \infty$. Finally let R, R_0 be any positive numbers such that $R < R_0$ and $V^\varepsilon \subset W_R^\varepsilon$ for sufficiently small ε, say (for simplicity) for $\varepsilon \le \varepsilon_0$ where ε_0 appears in 3.1.

As is well known [15] [16] there exists $\varepsilon_1 > 0$ such that for any $\varepsilon \le \varepsilon_1$ the following holds: For every f and φ_h, the solution u of

$$Pu = f(t,x)$$

$$(3.27) \qquad \left.\frac{\partial^h u}{\partial t^h}\right|_{t=0} = \varphi_h(x) \qquad (0 \le h \le m-1)$$

in W_R^ε depends on f, φ_h and the coefficients of P only in $W_{R_0}^\varepsilon$, that is to say: if one changes f, φ_h outside $W_{R_0}^\varepsilon$ in any manner and if one changes the coefficients of P outside $W_{R_0}^\varepsilon$ in such a way that (i), (ii) and the bounds con-concerned are preserved (ε_1 depends on these bounds) then the solution Υ of the modified problem (3.27) coincides with the solution u of (3.27) in W_R^ε. In other words, $W_{R_0}^\varepsilon$ contains the domain of dependence of W_R^ε.

In the sequel we shall assume (see Theorem 2) that the coefficients a_α of P are C^∞ in $W_{R_0}^{\varepsilon_0}$, and we shall consider the behavior of the fundamental solution G in $V^\varepsilon \subset W_R^\varepsilon$. Hence, if $\varepsilon \leq \varepsilon_1$, we may modify the a_α outside $W_{R_0}^\varepsilon$ without affecting G in V^ε. We can use mollifiers for this purpose and thus achieve C^∞ coefficients for P also outside $W_{R_0}^\varepsilon$. Hence, in proving Theorem 2 we may assume that the coefficients of P are C^∞ in W_∞^ε for all $\varepsilon \leq \varepsilon_1$.

Let $\{\widetilde{M}_q\}$ be a sequence of numbers which satisfy the same properties that $\{M_q\}$ was assumed to satisfy, and in addition,

$$(3.28) \qquad M_{p+q} \leq (A')^{p+q+1}\, \widetilde{M}_p M_q\,, \quad M_p \leq \widetilde{M}_p \qquad \text{for all } p,q \leq 0$$

where A' is a constant. For instance, if $M_p = (\delta p)!$ $(\delta \geq 1)$, then we can take $\widetilde{M}_p = M_p$.

We can now state the main result of the paper.

Theorem 2. If the coefficients of P belong to $C(M_{q-a}; W_{R_0}^{\varepsilon_0})$ for some $a \geq \varphi(m,n)$, then the fundamental solution $G(t,x)$ belongs to $C(\hat{M}_{q-a+d}; V^\varepsilon)$ for some ε sufficiently small, where φ and $d = d(m,n)$ depend only on m,n; $\varphi(m,n) \geq d(m,n)$, and where

$$(3.29) \qquad \hat{M}_{p-b} \leq \frac{M_{3p-b}}{p!} + \frac{M_{2p-b}\,\widetilde{M}_p}{p!} \qquad \text{for } b \geq 0$$

and $\hat{M}_p = M_p$ if $M_p = p!$.

Corollary 1. $M_q = q!$ is the analytic case and our result for this case coincides with that of Babitch [1].

Corollary 2. If the coefficients of P belong to $C((\delta q)! \; ; W_{R_0}^{\varepsilon_0})$ then G belongs to $C([(3\delta - 1)q]! \; ; V^{\varepsilon})$, for any $\delta > 1$.

The proof of Theorem 2 is given in §5. An auxiliary result on interior estimates for hyperbolic equations, which is of intrinsic interest, is proved in §4.

4. A Theorem on Interior Estimates

In this section we prove that for the Cauchy problem

$$(4.1) \qquad Pu = f$$

$$(4.2) \qquad \left.\frac{\partial^h u}{\partial t^h}\right|_{t=0} = 0 \qquad (0 \leq h \leq m-1)$$

the following is true: If the coefficients of P belong to $C(M_{q-b}; W_{R_0}^{\alpha})$ for some $\alpha > 0$, $R_0 > 0$, then the succesive derivatives of u in W_R^{α} can be estimated in terms of the successive derivatives of f in $W_{R_0}^{\alpha}$, provided the latters are bounded by $A^q M_{q-b}$. Here R_0 is sufficiently large, depending on R, α and P. The result is formulated in Theorem 3 below, and this theorem will be needed in §5, for the proof of Theorem 2.

Theorem 3 is analogous to results derived by the author in [4] [5] for elliptic and parabolic equations. It would be strictly analogous if R were to be $R_0 - \varepsilon$ for any $\varepsilon > 0$, but such an assertion cannot be expected to hold for hyperbolic equations. The proof of Theorem 3 is based on different tools than those used in [4] [5], although a part of the technique is similar to [4] [5].

To formulate the theorem we introduce the norm

$$||v(t,.)||_\rho = \left(\int_{|x|<\rho} |v(t,x)|^2 \, dx \right)^{\frac{1}{2}} \quad .$$

We assume that P is a uniformly hyperbolic operator with C^∞ coefficients in W_∞^α and that the coefficients belong to $C(M_{q-b}; W_{R_0}^\alpha)$, where $b \geq 0$. We further assume that R_1, R_2 can be found such that $R < R_1 < R_2 < R_0$ and (a) the domain of dependence of W_R^α is contained in $W_{R_1}^\alpha$, and (b) the domain of dependence of $W_\infty^\alpha - W_{R_0}^\alpha$ is contained in $W_\infty^\alpha - W_{R_2}^\alpha$.

We can now state:

__Theorem 3__. __Let the foregoing assumptions be satisfied and let__ u __be a solution of__ (4.1), (4.2) __in__ W_R^α. __If__ f __satisfies, for__ $0 \leq t \leq \alpha$,

(4.3) $$\|D^q f(t,.)\|_{R_0} \leq E_0 E^q M_{q-b} \qquad (0 \leq q \leq p)$$

__then__ u __satisfies, for__ $0 \leq t \leq \alpha$,

(4.4) $$\|D^{q+m-1} u(t,.)\|_R \leq K_0 K^q M_{q-b} \qquad (0 \leq q \leq p)$$

__where__ K_0, K __depend only on__ E_0, E __and on__ P, R, R_0, R_1, R_2 .

__Proof.__ Note, first, that all the derivatives of u in (4.4) exist by [8] [15] [16]. We shall prove (4.3) for any given q, $0 \leq q \leq p$ (without using induction on q).

We modify f into a function $\tilde{f}$ defined as follows:

$$(4.5)\qquad \tilde{f}(t,x) = \begin{cases} f(t,x) & \text{in } W^{\alpha}_{R_1} \\ \varphi(x)\, f(t,x) & \text{in } W^{\alpha}_{R_2} - W^{\alpha}_{R_1} \\ 0 & \text{in } W^{\alpha}_{\infty} - W^{\alpha}_{R_2} \end{cases},$$

where $\varphi(x) = \varphi(|x|)$, and

$$(4.6)\qquad \varphi(\,(R_2 - R_1)r + R_1) = \zeta(r^2)\ .$$

The function $\zeta(r)$ is required to satisfy the conditions:

$$\zeta(0) = 1\ ,\quad \zeta^{(k)}(0) = 0\quad (1 \le k \le q)\ ,\quad \zeta^{(j)}(1) = 0\quad (0 \le j \le q)\ .$$

Then, the same conditions are satisfied by $\zeta(r^2)$, and by (4.5) (4.6) it follows that $\tilde{f}$ is of differentiability class C^q in W^{α}_{∞}.

We want to find $\zeta(r)$ as a polynomial of degree $2q + 2$.

Then,

$$\zeta'(r) = r^q(1 - r)^q(\beta r + \gamma)$$

for some β, γ Integrating and using the conditions $\zeta(0) = 1$, $\zeta(1) = 0$, we get

$$\zeta(r) = \beta \int_0^r s^{q+1}(1 - s)^q\, ds + \gamma \int_0^r s^q(1 - s)^q\, ds + 1$$

where β, γ satisfy

$$\beta B(q + 2,\ q + 1) + \gamma B(q + 1,\ q + 1) + 1 = 0\ .$$

Here

$$B(x,k) = \frac{k!}{x(x + 1)\ \dots\ (x + k - 1)!} \qquad (k\ \text{ integer } > 0)$$

is the Beta function.

If we take $\beta = 1$ then

$$\gamma = -\frac{1}{2} - \frac{1}{B(q+1,\ q+1)} \to -\frac{1}{2} \quad \text{as} \quad q \to \infty \ .$$

Let $\eta(r)$ be the polynomial

$$\eta(r) = \int_0^r s^q\,(1+s)^{q+1}\,ds + 1 \ .$$

Then it is clear that

$$|D_r^k\,\zeta(r^2)| \le D_r^k\,\eta(r^2) \ .$$

By expanding $D_x^k\,\zeta(r^2)$ and comparing each term in the expansion with that of $D_r^k\,\zeta(r^2)$ we find that

$$(4.7) \qquad |D_x^k\,\zeta(r^2)| \le D_r^k\,\eta(r^2) \ .$$

The right side of (4.7) is easily seen to be bounded by $B_1^{k+1}\,q^k$, where B_i are used in this section to denote appropriate constants depending only on P, R, R_0, R_1, R_2 .

Using (4.6) we get

$$(4.8) \qquad |D_x^k\,\varphi(x)| \le B_2^{k+1} q^k \qquad (0 \le k \le q) \ .$$

Hence,

$$(4.9) \qquad \|D^k\,f(t,.)\|_\infty \le E_0 E^k\,M_{k-b} + \sum_{j=0}^{k} \binom{k}{j} |D^j\varphi|\ \|D^{k-j}\,f(t,.)\|_{R_0} \ .$$

At this point we introduce a new sequence $\{\overline{M}_{k-b}\}$ defined by

$$(4.10) \qquad \overline{M}_{k-b} = M_{k-b}\,\frac{q^k}{k!} \qquad (0 \le k \le q)$$

and notice that $\overline{M}_k \ge M_k$. Also, by (1.2) and (1.1),

$$\binom{k}{j} q^j\,M_{k-j-b} \le B_3^j\,\frac{q^j}{j!}\binom{k}{j} M_{j-b}\,M_{k-j-b} \le B_3^j\,M_{k-b}\,\frac{q^k}{k!}$$

that is

$$(4.11) \qquad \binom{k}{j} q^j M_{k-j-b} \leq B_3^j \bar{M}_{k-b} \qquad (0 \leq j \leq k \leq q) \quad .$$

Using (4.11), we obtain from (4.9)

$$\|D^k f(t,.)\|_\infty \leq E_0 E^k M_{k-b} + B_4 E_0 \bar{E}^k \bar{M}_{k-b}$$

where $\bar{E} = \max\{E, 2B_2 B_3\}$. Hence,

$$(4.12) \qquad \|D^k f(t,.)\|_\infty \leq \bar{E}_0 \bar{E}^k \bar{M}_{k-b} \qquad (0 \leq k \leq q)$$

where $\bar{E}_0 = E_0(1 + B_4)$.

Let u be the solution in W_∞^α of

(4.13) $$P\tilde{u} = \tilde{f}$$

(4.14) $$\left.\frac{\partial^j \tilde{u}}{\partial t^j}\right|_{t=0} = 0 \qquad (0 \le j \le m-1) .$$

By the choice of R_1 , R_2 we conclude:

(4.15) $$\tilde{u} = 0 \text{ in } W_\infty^\alpha - W_{R_0}^\alpha , \quad \tilde{u} = u \text{ in } W_R^\alpha .$$

Hence, if we prove by induction on h that

(4.16) $$\|D^{h+m-1}\tilde{u}(t,\cdot)\| \le K_1 K_2^h \bar{M}_{h-b} \qquad (0 \le h \le q)$$

then, taking $h = q$ we obtain (4.4) with any $K > eK_2$. K_1 and K_2 will be proved to depend on $\bar{E}_0$, $\bar{E}$ and on P, R, R_0, R_1, R_2 .

Before starting with the proof of (4.16), we need the following fact which we state as a lemma.

Lemma 7. *If*

$$|D^k g| \le B_5^{k+1} M_{k-b} ,$$

$$\|D^k h\| \le N_0 N^k \bar{M}_{k-b}$$

for $0 \le k \le q$, *then*

$$\|D^k(fg)\| \le B_6 N_0 N^k \bar{M}_{k-b} \qquad (0 \le k \le q)$$

provided $N > 2B_5$. B_6 *is independent* of N_0 , N .

<u>Proof</u>. We first observe that if $0 \leq j \leq k \leq q$,

$$\binom{k}{j} M_{k-j-b} \overline{M}_{j-b} \leq A\, M_{k-b} \frac{q^j}{j!} \leq A\, M_{k-b} \frac{q^k}{q!} ,$$

that is,

$$\binom{k}{j} M_{k-j-b} \overline{M}_{j-b} \leq A \overline{M}_{k-b} , \tag{4.17}$$

where we made use of (1.1).

Using (4.17) we get

$$\|D^k(gh)\| \leq \sum_{j=0}^{k} \binom{k}{j} |D^{k-j}h| \; \|D^j g\| \leq B_6 N_o N^k \overline{M}_{k-b} ,$$

where we made use of the fact that $N > 2B_5$.

In proving (4.16) we shall make use of the energy inequality [8 ; Theorem 7.1]:

$$\sum_{|\alpha|=m-1} \|D^\alpha v(t,\cdot)\|_\infty \leq B_7 \sum_{|\alpha|=m-1} \|D^\alpha v(0,\cdot)\|_\infty + \tag{4.18}$$

$$+ B_7 \int_0^t \|Pv(\tau,\cdot)\|_\infty \, d\tau ,$$

where B_7 depends only on P. We shall apply it to $v = D^h \tilde{u}$. Hence, we first have to estimate the left side of (4.16) for $t = 0$.

We shall prove, by induction on h, that

$$\|D_t^{h+m} D_x^r \tilde{u}(0,\cdot)\|_\infty \leq K_3 K_4^{2h+r} \overline{M}_{h-b} \qquad (0 \leq h+r \leq q). \tag{4.19}$$

Here K_3, K_4 depend on the same quantities as K_1, K_2.

For $h = -1$ the left side of (4.19) is zero. We now assume that (4.19) holds for all $h < k$ and proceed to prove it for $h = k$.

Applying $D_t^k D_x^r$ $(k+r \leq q)$ to both sides of (4.13) and taking $t = 0$ we get

$$(4.20) \qquad D_t^{k+m} D_x^r \tilde{u} = - D_t^k D_x^r \Big(\sum_{|\alpha| \leq m} a_\alpha D^\alpha \tilde{u} \Big) + D_t^k D_x^r \tilde{f} \equiv I_1 + I_2 ,$$

where $D^\alpha = D_t^{\alpha_0} \ldots D_{x_n}^{\alpha_n}$ and $\alpha_0 < m$.

$\|I_1\|$ can be estimated by using the inductive assumption, making use of the assumption that $a_\alpha \in C\,(M_{q-b}; W_{R_0}^\alpha)$ (and recalling that $u = 0$ in $W_\infty^\alpha - W_{R_0}^\alpha$, by (4.15)), and employing Lemma 7. $\|I_2\|$ was already estimated in (4.12). Combining these estimates we get

$$\|D_t^{k+m} D_x^r \tilde{u}(0,\cdot)\|_\infty \leq B_8 K_3 K_4^{2k+r-1} \bar{M}_{k-b} ,$$

provided K_4 is sufficiently large, depending only on $\bar{E}_0$, $\bar{E}$, P, R, R_0, R_1, R_2. Taking $K_4 \geq B_8$ the proof of (4.19), by induction, is completed.

We can now proceed to establish (4.16).

Proof of (4.16). We assume (4.16) to hold for all $h < k$ and shall prove it for $h = k$. The case $h = 0$ follows from the energy inequality (4.18) applied to $v = \tilde{u}$.

Applying D^k to both sides of (4.13) we get

$$(4.21) \qquad P(D^k \tilde{u}) = D^k \tilde{f} - \sum_{|\alpha| \leq m} \sum_{s=0}^{k-1} \binom{k}{s} (D^{\alpha+s} \tilde{u}) D^{k-s} a_\alpha \equiv J_1 + J_2 .$$

J_2 consists of two sums. The first sum, J_{21}, contains all the terms involving derivatives of $\tilde{u}$ of order $k + m - 1$ and its norm is bounded by

$$B_9 \sum_{|\alpha|=m} \|D^{\alpha+k-1} u(t,\cdot)\|_\infty \quad .$$

The other sum, $J_2 - J_{21}$, can be estimated using the inductive assumption and Lemma 7. We obtain

$$\|J_2 - J_{21}\| \le B_{10} K_1 K_2^{k-1} \bar{M}_{k-b} \quad .$$

Finally, $\|J_1\|$ is estimated by (4.12).

Taking $K_2 > \bar{E}$, $K_1 > \bar{E}_0$ we get:

(4.22) $$\|P(D^k \tilde{u})(t,\cdot)\|_\infty \le B_9 \sum_{|\alpha|=m} \|D^{\alpha+k-1} \tilde{u}(t,.)\|_\infty + B_{11} K_1 K_2^{k-1} \bar{M}_{k-b} .$$

We now apply (4.18) with $v = D^k \tilde{u}$. Using (4.19), (4.22) we obtain, if $K_2 > K_4$ and $K_1 > K_3$,

(4.23) $$\sum_{|\alpha|=m} \|D^\alpha D^{k-1} \tilde{u}(t,\cdot)\|_\infty \le B_{12} K_1 K_2^{k-1} \bar{M}_{k-b} + B_{12} \int_0^t \sum_{|\alpha|=m} \|D^\alpha D^{k-1} \tilde{u}(\tau,\cdot)\|_\infty d\tau \ .$$

In this inequality $D^{k-1} \tilde{u}$ is one specific $(k-1)$-th derivative. (Strictly speaking, we only obtain (4.23) with

$$\sum_{|\alpha|=m-1} \|D^\alpha D^k \tilde{u}\|$$

on the left, but then (4.23) follows very easily.)

We now need:

__Lemma 8.__ __Let__ $y(\tau)$, $Q(\tau)$ __be continuous non-negative functions for__ $\tau \ge 0$ __and suppose that__

$$y(\tau) \le H \int_0^\tau y(\xi)d\xi + Q(\tau) \quad , \quad \tau \ge 0 \ .$$

__Then__

$$\int_0^\tau y(\xi)d\xi \le e^{H\tau} \int_0^\tau e^{-H\xi} Q(\xi)\, d\xi \ .$$

Applying the lemma to (4.23) and taking K_2 sufficiently large (depending on B_{12} and α $(t \leq \alpha)$), the proof is completed.

Corollary 1. Using Sobolev's lemma we conclude from (4.4) that for all (t,x) in W_R^α,

$$|D^{q+m-1-\nu}\, u(t,x)| \leq \bar{K}_0\, K^q\, M_{q-b} \qquad (0 \leq q \leq p) \tag{4.29}$$

where $\bar{K}_0 = K_0\, B_{13}$ and $\nu = [\frac{n+2}{2}]$.

Corollary 2. From (4.24) it follows that if f belongs to $C\{M_{q-b}; W_{R_0}^\alpha\}$ then u belongs to $C\{M_{q-b+m-1-\nu}; W_R^\alpha\}$.

Corollary 3. If $f = f(t,x,\lambda)$ depends on a parameter λ and it satisfies

$$\|D_0^q\, f(t,\cdot,\lambda)\|_R \leq K_0\, K^q M_{q-b} \qquad (0 \leq q \leq p)\,.$$

where D_0^q means any q-th partial derivatives with respect to (t,x,λ), and $0 \leq t \leq \alpha$, $\lambda \in \Lambda$, then

$$\|D_0^{q+m-1}\, u(t,\cdot,\lambda)\|_R \leq K_0\, K^q\, M_{q-b} \qquad (0 \leq q \leq p)\,.$$

The proof is similar to that of Theorem 3. Indeed, $\tilde{f}$ is defined in the same way as before. In (4.19) we replace D_x^r by $D_{x\lambda}^r$ which means: any r-th partial derivative with respect to (x,λ), and then proceed to prove it by induction on h.

Finally, in proving (4.16) we apply D_0^k to both sides of (4.13).

Remark. Given any $R < R_0$ we can find R_1, R_2 as in the assumptions of Theorem 3, provided α is sufficiently small. This fact will be used in §5.

4. Proof of the Theorem 2.

The proof is divided into seven steps. In 4.1 - 4.4 we only use the restriction $\varphi(m,n) \geq m + 4$.

4.1. Estimates for $\gamma^{(j)}$

Consider the Cauchy problem (3.4),(3.6)-(3.8) (with $v = v_j$) for $\gamma = \gamma^{(j)}$. By Lemma 3 and (1.2), the initial values $p_0 = v_j$ which are determined as solutions of the polynomial equation (3.9), whose coefficients are of class $C\{M_{q-a}\}$ in (y,σ), are of class $C\{M_{q-a}\}$ in (y,σ).

We can now apply Theorem 1 and its corollary (as $a \geq 3$) and conclude that $\gamma^{(j)} = \gamma_\sigma^{(j)}(t,x)$ is of class $C\{M_{q-a+1}\}$ in (t,x,σ). Thus,

$$(5.1) \qquad |D_*^r \gamma_\sigma^{(j)}(t,x)| \leq A_0 A^r M_{r-a+1} \qquad (0 \leq r < \infty),$$

where D_* is used in this section to denote any partial derivative with respect to (t,x,σ).

4.2. Estimates for u_{kh}

We set $b = a - 3$.

The coefficient

$$(5.2) \qquad C_h(t,x) \equiv \frac{1}{2} \sum_{i,j} \frac{\partial^2 A_h}{\partial \gamma_i^{(h)} \partial \gamma_j^{(h)}} \gamma_{ij}^{(h)} + B_h$$

in (3.20) is of class $C\{M_{q-b}\}$ (as follows by (5.1), using Lemma 1). Hence,

$$(5.3) \qquad |D_*^r C_h(t,x)| \leq A_1 A_2^r M_{r-b} \qquad (0 \leq r < \infty),$$

where A_1, A_2 are constants.

We next derive estimates for the coefficients of the operators P_{ij} , Q_{hij} appearing in (3.20) and (3.23), respectively.

By the paragraph containing (3.16) and the fact that $\gamma^{(j)}$ belongs to $C\{M_{q-a+1}\}$ (and hence $D^{s+1}\gamma^{(j)}$ belongs to $C\{M_{q-a+s+2}\}$) it follows, using Lemma 1, that the coefficients of P_{sj} belong to $C\{M_{q-a+s+2}\}$ provided $a-(s+2) \geq 2$. A closer look at P_{sj} shows that if

$$P_{sj}(v) = \sum_{|\alpha| \leq s} P_{s\alpha j}(t,x) D^{\alpha} v$$

where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, $|\alpha| = \Sigma\alpha_i$, $D^{\alpha} = D_t^{\alpha_0} D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}$, then

$$P_{s\alpha j} \text{ belongs to } C\{M_{q-a+s-|\alpha|+2}\} , \tag{5.4}$$

provided $a-(s-|\alpha|+2) \geq 2$. Here it is where our assumption $a \geq m+4$ enters, as $1 \leq s \leq m$.

We turn to Q_{hij}. By the sentence following (3.21) it follows, upon using Lemma 1, that

$$Q_{hij} = q_{hij}(t,x) \frac{\partial^i}{\partial t^i}$$

and

$$q_{hij} \text{ belongs to } C\{M_{q-a+m}\}. \tag{5.5}$$

We set

$$c = \min (a-3, a-m)$$

so that $b \geq c \geq 2$, and express (5.4), (5.5) in more detail, namely,

$$|D_*^r P_{s\alpha j}| \leq A_3 A_4^r M_{r-a+s-|\alpha|+2} \qquad (0 \leq r < \infty) \tag{5.6}$$

$$|D_*^r q_{hij}| \leq A_3 A_4^r M_{r-c} \qquad (0 \leq r < \infty) \tag{5.7}$$

where A_3 , A_4 are constants.

We shall now establish the desired inequalities for $u_{kh}(t,x)$ (recall (3.15)) in five steps. Throughout the rest of this section, we denote by D_o^r any r-th partial derivative with respect to the variables (x,σ). D^r has the same meaning with respect to the variables (t,x).

The first step consists in proving that

$$|D_t^q \, D_o^r \, u_{oh} \, (0,x)| \leq H_1 \, H_2^{\,q+r-b} \, M_{q+r-b} \tag{5.8}$$

for all $1 \leq h \leq m$, and all q, r such that $q + r > b$. Here H_1 is chosen in such a way that

$$|D_t^q \, D_o^r \, u_{oh}(0,x)| \leq H_1 \qquad \text{if } \; q + r \leq b.$$

The proof of (5.8) is by induction on q. To prove it for $q = 0$ we use equations (2.23) for $k = 0$, namely,

$$\sum_{j=1}^{m} u_{oj} \, (\gamma_o^{(j)})^h = \begin{cases} 0 & \text{if } \; 0 \leq h \leq m-2 \\ 1 & \text{if } \; h = m-1 \, . \end{cases}$$

The u_{oj} can be uniquely solved by using Cramer's rule, as the coefficients matrix is non-singular. Since by Lemma 1, the product of functions in $C\{M_{q-b}\}$ is again in $C\{M_{q-b}\}$ and since $1/f$ belongs to $C\{M_{q-b}\}$ if $f \neq 0$ and f belongs to $C\{M_{q-b}\}$, we conclude that the u_{oj} belong to $C\{M_{q-b}\}$ (as $\gamma_o^{(j)} = \partial\gamma^{(j)}/\partial t$ belongs $C\{M_{q-b}\}$, by (5.1)). This establishes (5.8) for $q = 0$.

We next assume that (5.8) holds for all $q < p$ and proceed to establish it for $q = p$. We shall make use of the differential equation (3.20) for $k = 0$, namely,

$$\frac{du_{oh}}{dt} + C_h \, u_{oh} = 0 \, . \tag{5.9}$$

Applying $D_t^{p-1} D_o^r$ to both sides of (5.9) and then taking $t = 0$ we obtain (at points $(0,x,\sigma)$)

$$D_t^p D_o^r u_{oh} = - \sum_{\ell=0}^{r} \sum_{s=0}^{p-1} \binom{p-1}{s} \binom{r}{\ell} D_t^s D_o^\ell u_{oh} D_t^{p-1-s} D_o^{r-\ell} C_h .$$

The right side is bounded by

$$A_5 H_1 \sum_{\ell=0}^{r} \sum_{s=0}^{p-1} \binom{p-1}{s} \binom{r}{\ell} \hat{H}_2^{s+\ell-b} A_2^{p-1+r-s-\ell} M_{s+\ell-b} M_{p-1+r-s-\ell-b}$$

where A_i are constants independent of r,p and where we use the notation:

$$\hat{H}_j^e = \begin{cases} H_j^e & \text{if } e \geq 0 \\ 1 & \text{if } e < 0 . \end{cases}$$

Taking $H_2 > 2A_2$ and using (1.1) we get

$$|D_t^p D_o^r u_{oh}(0,x)| \leq A_6 H_1 H_2^{p-1+r-b} M_{p-1+r-b} .$$

Taking $H_2 > A_6$, the proof of (5.8) for $q = p$ follows. (In fact, we have proved (5.8) with M_{q+r-b} replaced by $M_{q+r-b-1}$, but this will not be used in the sequel, as it does not yield any improvement for the estimates of the $u_{kh}(t,x)$.)

<u>The second step</u> consists in proving that

(5.10) $$|D_t^q D_o^r u_{kh} (0,x)| \leq H_3 H_4^{2k+q+r-b} M_{k+q+r-c} ,$$

for all $1 \leq h \leq m$, and all q,r,k such that $2k + q + r > b$. H_3 is chosen in such a way that

$$|D_t^q D_o^r u_{kh} (0,x)| \leq H_3 \qquad \text{if} \quad 2k + q + r \leq b.$$

From the proof given below it follows that $M_{k+q+r-c}$ can be replaced in (5.10) by

$M_{k+q+r-c-1}$, but this will not be used in the sequel.

Since $c \leq b$, (5.10) for $k = 0$ follows from (5.8) if we take $H_3 \geq H_1$, $H_4 \geq H_2$. We now proceed by induction on k: We assume that (5.10) holds for all u_{ph} with $p < k$ and proceed to prove it for $p = k$. H_3 is fixed and H_4 is still to be determined (independently of k,q,r).

We employ another induction on q. We then first have to establish (5.10) for $q = 0$. We make use of (3.23) and we write these equations in the form

$$(5.11) \qquad \sum_{j=1}^{m} u_{kj}(\gamma_o^{(j)})^h = -\tilde{Q}_{kh} .$$

We first need to estimate $D_o^r \tilde{Q}_{kh}$ at points $(0,x,\sigma)$. This expression consists of a finite sum (the number of terms is bounded independently of k,r), the general term of which is

$$D_o^r (q_{hij} D_t^i u_{k-i,j}) = \sum_{s=0}^{r} \binom{r}{s} D_o^{r-s} q_{hij} D_o^s D_t^i u_{k-i,s} \qquad (1 \leq i \leq h,k).$$

Using (5.7) and the inductive assumptions we get the bound

$$A_7 \sum_{s=0}^{r} \binom{r}{s} A_4^{r-s} H_3 \hat{H}_4^{2(k-i)+i+s-b} M_{r-s-c} M_{k+s-c} .$$

Taking $H_4 > 2A_4$ we thus obtain

$$(5.12) \qquad |D_o^r \tilde{Q}_{kh}| \leq A_8 H_3 H_4^{k+r-b-1} M_{k+r-c} .$$

We are now ready to establish (5.10) for $9 = 0$ by induction on r. For $r = 0$ we simply solve (5.11) and use (5.12) with $r = 0$, taking $H_4 > A_8$. Assuming the validity of (5.10) for $D_o^s u_{kr}$ with $s < r$, we shall prove it for $s = r$. We apply D_o^r to both sides of (5.11) and obtain

$$(5.13)\qquad \sum_{j=1}^{m} (D_o^r u_{kj})(\gamma_o^{(j)})^h = - \sum_{j=1}^{m} \sum_{s=0}^{r-1} \binom{r}{s} D_o^s u_{kj} D_o^{r-s} (\gamma_o^{(j)})^h - D_o^r \tilde{Q}_{kh} .$$

Using the inductive assumption and the inequalities (5.1) which hold not only for the $\gamma^{(j)}$ but also for any power $(\gamma^{(j)})^h$ (by Lemma 1) with different A_o, A, we find that the first sum on the right side of (5.13) is bounded by

$$A_9 H_3 H_4^{2k+r-1} M_{k+r-c} .$$

The second term is estimated by (5.12). Hence, taking $H_4 > A_8 + A_9$, the proof of (5.10) for $q = 0$ is completed.

We now proceed by induction on q. Assuming (5.10) to hold for all $q < p$, we shall establish it for $q = p$. We shall make use of the differential equations (3.20) which we write in the abbreviated form

$$(5.14)\qquad \frac{du_{kh}}{dt} + C_h u_{kh} = - \tilde{P}_{kh} .$$

We first estimate $D_t^{p-1} D_o^r \tilde{P}_{kh}$ at points $(0,x,\sigma)$. It suffices to estimate the general term

$$J \equiv D_t^{p-1} D_o^r (P_{i\alpha h} D^\alpha u_{k-i+1,h}) \qquad (2 \le i \le m,\ |\alpha| \le i, k \ge 1).$$

Using the inductive assumptions and (5.6) we get

$$|J| \le \sum_{s=0}^{p-1+r} \binom{p-1+r}{s} | D_*^{p-1+r-s} P_{i\alpha h}| \, |D^\alpha D_*^s u_{k-i+1,h}|$$

$$\le A_{10} H_3 H_4^{2k+p-1+r-b} \max_s \binom{p-1+r}{s} M_{p-1+r-s-a+i-|\alpha|+2} M_{k-i+1+|\alpha|+s-c}$$

$$\le A_{11} H_3 H_4^{2k+p-1+r-b} M_{k+p+r-c} .$$

Hence,

$$(5.15) \qquad |D_t^{p-1}\ D_o^r\ \tilde{P}_{kh}| \le A_{12}\ H_3\ H_4^{2k+p-1+r-b}\ M_{k+p+r-c}\ .$$

We now proceed to estimate $D_t^p\ D_o^r\ u_{kh}(0,x)$. Applying $D_t^{p-1}\ D_o^r$ to both sides of (5.14) and taking $t = 0$, we obtain

$$D_t^p\ D_o^r\ u_{kh} = - \sum_{\ell=0}^{p-1} \sum_{s=0}^{r} \binom{p-1}{\ell}\binom{r}{s} (D_t^{p-1-\ell}\ D_o^{r-s}\ C_h)(D_t^{\ell}\ D_o^{s}\ u_{kh})$$
$$- D_t^{p-1}\ D_o^r\ \tilde{P}_{kh}\ .$$

The first sum is estimated by using (5.3) and the inductive assumptions, and we obtain the bound

$$A_{13}\ H_3\ H_4^{2k+p-1+r-b}\ M_{k+p-1+r-c}\ .$$

Combining this with the inequality (5.15) and taking $H_4 > A_{12} + A_{13}$, the proof of (5.10) is completed.

The third step consists in proving that, for $1 \le h \le m$,

$$(5.16) \qquad |D_t^q\ D_o^r\ u_{oh}(t,x)| \le H_5\ H_6^{q+r-b}\ M_{q+r-c} \qquad (q + r > b)$$

(note that the argument is (t,x) and not $(0,x)$ as in the first two steps). H_5 is chosen in such a way that

$$|D_t^q\ D_o^r\ u_{oh}(t,x)| \le H_5 \quad \text{if} \quad q + r \le b.$$

The proof is by induction on q. To prove it for $q = 0$ we employ induction on r. Assuming (5.16) with $q = 0$ to hold for all D_o^{ℓ}, $\ell < r$, we proceed to prove it for $\ell = r$.

Applying D_o^r to both sides of (5.9) we get

(5.17) $$\frac{d}{dt}\,(D_o^r\, u_{oh}) + C_h(D_o^r\, u_{oh}) = F_{oh}$$

and

(5.18) $$|F_{oh}| \le \sum_{s=0}^{r-1} \binom{r}{s} |\, D_o^s\, u_{oh}|\,|D_o^{r-s}\, C_h| \le A_{14}\, H_5\, H_6^{\,r-1-b}\, M_{r-c}\ ,$$

as follows by using (5.3) and the inductive assumption.

Integrating (5.17) with respect to t and using (5.8) with $q = 0$ and (5.18), we get

(5.19) $$|D_o^r\, u_{oh}(t,x)| \le H_1\, H_2^{\,r-b}\, M_{r-c} + A_{14}\, H_5\, H_6^{\,r-1-b}\, M_{r-c} + A_{15} \int_0^t |D_o^r\, u_{oh}(\tau,x)|\, d\tau .$$

Integrating both sides of (5.19) with respect to t, we can then eliminate the integral on the right side of (5.19) and thus obtain (5.16) with $q = 0$, provided we take $H_6 > H_2$, $H_6 > A_{16}$, for appropriate A_{16} :

We proceed to prove that if (5.16) holds for all $q < p$ then it holds for $q = p$. Applying $D_t^{p-1}\, D_o^r$ to both sides of (5.9) and using (5.3) and the inductive assumption (in a similar way to the calculations in step 1), the desired inequality easily follows if $H_6 > A_{17}$, for appropriate A_{17}.

<u>The fourth step</u> consists in proving that, for $1 \le h \le m$,

(5.20) $$|D_t^q\, D_o^r\, u_{kh}(t,x)| \le H_7\, H_8^{\,2k+q+r-b}\, M_{2k+q+r-c} \qquad (2k + q + r > b)$$

where H_7 is chosen in such a way that

$$|D_t^q\, D_o^r\, u_{kh}(t,x)| \le H_7 \qquad \text{if} \quad 2k + q + r \le b.$$

The proof is by induction on k. The case $k = 0$ is step 3. In order to establish the inductive passage from $k-1$ to k, we employ another induction on q. Thus, we first have to prove the case $q = 0$, that is,

$$|D_o^r u_{kh}(t,x)| \leq H_7 H_8^{2k+r-b} M_{2k+r-c} \qquad (2k + r > b). \tag{5.21}$$

To prove (5.21) we employ induction on r. The case $r = 0$ will not be described here since it follows by a part of the argument given below for the inductive passage from $r-1$ to r.

In order to perform this passage, we apply D_o^r to both sides of (3.20) and obtain

$$\frac{d}{dt}(D_o^r u_{kh}) + C_h(D_o^r u_{kh}) = - \sum_{s=0}^{r-1} \binom{r}{s} D_o^s u_{kh} D_o^{r-s} C_h$$

$$- \sum_{i=2}^{m} D_o^r P_{ih}(u_{k-i+1,h}) \equiv F_{rkh} \quad . \tag{5.22}$$

The first sum is estimated by

$$A_{18} H_7 H_8^{2k+r-1-b} M_{2k+r-c} , \tag{5.23}$$

where use is being made of (5.3) and the inductive assumption.

In the second sum, each term is a sum of terms of the form

$$D_o^r(P_{i\alpha h} D^\alpha u_{k-i+1,h}). \tag{5.24}$$

Using (5.6) and the inductive assumption get (by calculation similar to step 2) a bound (5.23) but with a different A_{18}.

Hence,

$$|F_{rkh}| \leq A_{19} H_7 H_8^{2k+r-1-b} M_{2k+r-c} . \tag{5.25}$$

We now integrate (5.22) with respect to t and proceed by an argument of step 3.

The proof of (5.21) is thus established.

Having proved (5.20) for $q = 0$ to proceed to establish the inductive passage from $q-1$ to q. This is done simply by applying $D_t^{q-1} D_o^r$ to both sides of (5.14). Since a similar argument appears in step 2, we omit further details.

Remark. The inequalities

$$(5.26) \qquad |D_t^q D_o^r u_{kh}(t,x)| \le H_7 H_8^{2k+q+r-b} M_{k+q+r-c}$$

may seem more natural than (5.20). If they are true then the next step is superfluous and Theorem 2 can also be improved by having

$$\hat{M}_{p-b} \le \frac{M_{2p-b}}{p!} + \frac{M_{p-b}\tilde{M}_p}{p!} .$$

However, it seems to us that (5.26) is not true. The difficulty in trying to establish it is that the sum of the orders of differentiation and the subindices of the u's of (5.24) is $k + r + 1$, if $|\alpha| = i$, and not $k + r$. Therefore, in order to carry out the inductive passage from $k-1$ to k, more weight should be given to the index k. We are thus led to establishing (5.20) with $M_{\lambda k+q+r-c}$ for some $\lambda \ge 1$. The previous proof works well only if $\lambda \ge 2$.

The fifth step (and the final one) consists in combining the results of the second and fourth steps in order to improve the results of the fourth step.

By Taylor's formula we have:

$$D_t^q D_o^r u_{kh}(t,x) = \sum_{\nu=0}^{N-1} \frac{t^\nu}{\nu!} D_t^{\nu+q} D_o^r u_{kh}(0,x) + R_{kh}^N$$

where

$$|R_{kh}^N| \le \frac{t^N}{N!} | D_t^{N+q} D_o^r u_{kh}(\tilde{t},x)|$$

for some $\tilde{t}$, $0 < \tilde{t} < t$.

Using (5.10),(5.20) we get

$$(5.27)\qquad \begin{aligned} |D_t^q D_o^r u_{kh}(t,x)| \le \sum_{\nu=0}^{N-1} \frac{t^\nu}{\nu!} H_3 H_4^{2k+q+r+\nu} M_{k+q+r+\nu-c} \\ + \frac{t^N}{N!} H_7 H_8^{2k+q+r+N} M_{2k+q+r+N-c} \end{aligned}$$

If $M_q = q!$ then the last term on the right side of (5.27) tends to zero as $N \to \infty$, provided $t \le \varepsilon$ and ε is sufficiently small (i.e., $3 \varepsilon H_8 < 1$). On the other hand, the sum on the right side of (5.27) is bounded, independently of ν, by

$$A_{20} H_3 H_4^{2k+q+r} (k + q + r - c)! ,$$

provided ε is sufficiently small (say $3 \varepsilon H_4 < 1$).

Introducing the notation

$$(5.28)\qquad M_{k,s} = \begin{cases} M_s & \text{if } M_q = q! \\ M_{k+s} & \text{otherwise} \end{cases}$$

we conclude from (5.27) and from step 4, that

$$(5.29)\qquad |D_*^r u_{kh}(t,x)| \le H_o H^{k+r} M_{k,k+r-c}$$

for some constants H_o, H. This is the final form of our estimates for the u_{kh}. Note that (t,x) varies in $W_{R_o}^{\varepsilon}$, and σ varies on $|\sigma| = 1$.

4.3. Estimates for $f_k(\gamma^{(j)})$

From (3.24),(3.25) we see that $f_k(x)$, for $k \ge n - m + 2$ is of differentiability class $C^{k-n+m-2}$ and we can write

$$f_k(r) \approx \frac{r^{k-n+m-1}}{(k-n+m-1)!} \tag{5.30}$$

in the following sense: Each derivative D_r^q $(q \leq k-n+m-2)$ of any side of (5.30) is bounded by a constant (independent of q,k) times the derivative D_r^q of the other side.

From the proof of Lemma 2 it is seen (using (5.30)) that the lemma remains true if $F(u) = u^i$ is replaced by

$$F(u) = i!\, f_{i+n-m+1}(u) .$$

Making use of (5.1) we thus get

$$|D_*^q f_k(\gamma^{(j)})| \leq K_1 \frac{A_o^{\,i-1} K_o^{\,i-1}}{A^{i-1}} A_o A^{q-a-1} \frac{q(q-1) \ldots (q-i+2)}{(k-n+m-1)!} M_{q-i-a+2} \tag{5.31}$$

where $i = k - n + m - 1$ and $a + 1 < q \leq k - n + m - 2 = i - 1$, and

$$|D_*^q f_k(\gamma^{(j)})| \leq K_1 \frac{1}{(k-n+m-1)!} \qquad \text{if } 0 \leq q \leq a + 1. \tag{5.32}$$

K_i are constant independent of q,k.

Taking $A > A_o K_o$ we conclude, from (5.31), that

$$|D_*^q f_k(\gamma^{(j)})| \leq K_2 \frac{A_o A^{q-a}}{(k-n+m-1)!} M_{q-a+1} \qquad (a+1 < q < k-n+m-2). \tag{5.33}$$

The inequalities (5.32), (5.33) are true also for all $q \geq 0$ at points (t,x,σ) where $\gamma_\sigma^{(j)}(t,x) \neq 0$. Indeed, for n odd, $f_k(r) = 0$ if $k \leq n-m$, and $r \neq 0$ whereas $f_k(r) = C_{k,n} r^{k+m-n-1}$ if $k > n-m$, where $C_{k,n}$ is constant, positive for $r > 0$ and zero for $r < 0$. Hence we only have to consider the case $k > n-m$, $\gamma^{(j)}(t,x) > 0$ and we then apply Lemma 2 and derive (5.32), (5.33) for all $q \geq 0$.

For n even proof of (5.32), (5.33) for all $q \geq 0$ (at points where $\gamma^{(j)} \neq 0$

is somewhat different. Let $|\gamma^{(j)}(t,x)| > r_o$ for some $r_o > 0$. We then observe that $(k+m-n-1)!\, f_k(r)$ belongs to $C\{(q-\alpha)!\}$, uniformly in k, on any set $\{r\ ;\ r_o \leq |r| \leq r_1\}$. Combining this remark with (5.1) and using Lemma 1, the proof of (5.32), (5.33) follows.

4.4. Estimates for $u_{kh}(t,x)f_k(\gamma^{(h)})$

Let $q \leq k - n + m + 1$. Then, using (5.29) and (5.32),(5.33) we get

$$|D_*^q[u_{kh}(t,x)f_k(\gamma^{(h)})]| \leq \frac{K_3}{(k-n+m-1)!} \sum_{r=0}^{q} \binom{q}{r} H_o \hat{H}^{k+r} A_o A^{q-r} M_{k,k+r-c}$$

(5.34)

$$M_{q-r-a+1} \leq \frac{K_4\, H^{k+q}}{(k-n+m-1)!} M_{k,k+q-c} \quad .$$

This inequality remains true for all $q > 0$ at points (t,x,σ) where $\gamma_\sigma^{(h)}(t,x) \neq 0$, and then K_4 depends on r_o where r_o is any constant $< |\gamma_\sigma^{(h)}(t,x)|$. Hence, for such points (t,x,σ),

$$(5.35) \qquad |D_*^q [\sum_{k=0}^{N} u_{kh}(t,x)f_k(\gamma^{(h)})]| \leq \frac{K_5\, H^{N+q}}{(N-n+m-2)!} M_{N,N+q-c}$$

4.5. Estimates for the "remainder" $G_h - \sum_{k=0}^{N} u_{kh}(t,x)f_k(\gamma^{(h)})$

We set

$$(5.36) \qquad U_{pj}(t,x) = \sum_{k=0}^{p} u_{kj}(t,x)f_k(\gamma^{(j)})$$

$$(5.37) \qquad R_{ph}(t,x) = G_h(t,x) - U_{ph}(t,x).$$

We continue to use the abbreviations (3.15). R_{ph} may be considered as the "remainder" of the (generally) divergent series $\sum_{k=0}^{\infty} u_{kh} f_k(\gamma^{(h)})$. We shall estimate in this subsection $D_*^q R_{ph}$ for $q \leq p - n + m - 1 - \nu$ where $\nu = [\frac{n+2}{2}]$. Setting $P = P(t,x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x})$ it is easily seen that R_{ph} satisfies formally, the equation

$$(5.38) \qquad P R_{ph} = - L_{ph} ,$$

where
$$L_{ph} = P_{mh} (u_{p,h}) f_p(\gamma^{(h)}) + [P_{mh}(u_{p-1,h}) + P_{m-1,h}(u_{p,h})] f_{p-1} (\gamma^{(h)})$$
$$+ \dots + [P_{mh}(u_{p-m+2}) + P_{m-1,h}(u_{p-m+3,h}) + \dots + P_{2h}(u_{ph})] f_{p-m+2}(\gamma^{(h)}),$$

and the initial conditions

$$(5.39) \qquad \frac{\partial^j}{\partial t^j} R_{ph} \Big|_{t = 0} = 0 \qquad\qquad (0 \leq j \leq m - 1).$$

(5.38) is an hyperbolic equation and the nonhomogeneous part is of differentiability class C^{p-n} in $W_{R_0}^{\ell}$ (if $m = 1$, $L_{ph} \equiv 0$). By Gårding [8] and the fact that the domain of dependence of W_R is contained in W_{R_0}, it follows R_{ph} exists in W_R^{ℓ} and is of class C^{p+m-n}. Using the definitions (5.36),(5.37) and (3.15), it follows that G is the fundamental solution. Since p can be made arbitrarily large, G is differentiable up to any order s at points (t,x) where

$$(5.40) \qquad U_{ph}(t,x) \equiv \int_{|\sigma| = 1} [\sum_{h=1}^{m} U_{ph\sigma}(t,x)] d\sigma$$

is differentiable up to the order s, for some $p \geq s + n - m$.

We proceed to estimate $D_*^q R_{ph}$ in two steps.

The first step consists in deriving estimates for L_{ph}.

In order to find a bound on $D_*^q L_{ph}$, it suffices to find a bound on

$$(5.41) \qquad D_*^q \{P_{m-j,h}(u_{p-i+j,h})\, f_{p-i}(\gamma^{(h)})\} \qquad (0 \le i\ ,\ j \le m-2)$$

We first estimate

$$(5.42) \qquad D_*^r \{P_{m-j,h}(u_{p-i+j,h})\}\ .$$

It suffices to treat the general term of (5.42):

$$I \equiv D_*^r \{P_{m-j,\alpha,h}\ D^\alpha\, u_{p-i+j,h}\} \qquad (|\alpha| \le m-j).$$

Using (5.6),(5.29) we get, after some (by now) standard calculation,

$$(5.43) \qquad |I| \le B_1\, B_2^{p+r}\, M_{p,p+r-e}$$

where B_i are used to denote appropriate constants, and where the symbol e will be used in what follows to denote various constants, all of which are of the form

$$e = a - e_1\ ,\quad e_1 \text{ depending only on } m,n.$$

The $\varphi(m,n)$ appearing in the statement of Theorem 2 is taken to be larger than the maximum of the various $e_1(m,n)$, so that $e \ge 0$.

An inequality similar to (5.43) holds also for (5.42). Using this and (5.32), (5.33) we find that each term of (5.41) is bounded by

$$B_3\, B_4^{p+q}\, \frac{M_{p,p+q-e}}{p!}$$

provided $q \le p - i - n + m - 2$. Since $i \le m - 2$ we conclude that

$$(5.44) \qquad |D_*^q L_{ph}(t,x)| \le B_5 B_6^{p+q} \frac{M_{p,p+q-e}}{p!} \quad \text{for } (t,x) \text{ in } W_{R_o}^\varepsilon ,$$

for all $0 \le q \le p - n$.

Using (3.28) we get

$$(5.45) \qquad |D_*^q L_{ph}(t,x)| \le B_7 B_8^{p+q} \frac{M_{p,p-e}}{p!} \tilde{M}_q \quad \text{for } (t,x) \text{ in } W_{R_o}^\varepsilon .$$

<u>The second step</u> consists of applying Corollaries 1,3 of Theorem 3 to the solution

$$R_{ph}/(B_8^p \frac{M_{p,p-e}}{p!})$$

of the system (5.38),(5.39) with L_{ph} replaced by

$$L_{ph}/(B_8^p \frac{M_{p,p-e}}{p!}) .$$

Recalling the remark at the end of §4 we conclude that if ε is sufficiently small, depending only on R, R_o and P, then

$$(5.46) \qquad |D_*^{q+m-1-\nu} R_{ph}(t,x)| \le B_9 B_{10}^{p+q} \frac{M_{p,p-e}\tilde{M}_q}{p!} \quad \text{in } W_R^\varepsilon ,$$

for $0 \le q \le p - n$.

4.6. Division into Cases: The First Case

We are now going to divide the points (σ,h) into two classes, and complete the proof of the theorem by treating each class separately and then combining the two results.

For every point (t^o,x^o) in V, the system of equations (3.26) is not satisfied. Hence, for any given σ and h ($|\sigma| = 1$ and $1 \le h \le m$), either

$$\gamma_\sigma^h (t^\circ , x^\circ) \neq 0 \quad , \quad \text{or}$$

$$\gamma_\sigma^h (t^\circ,x^\circ) = 0 \quad \text{but} \quad \mathrm{grad}_\psi \, \gamma_\sigma^h(t^\circ,x^\circ) \neq 0 \quad (\psi = (\psi_1, \ldots ,\psi_{n-1})).$$

It is clear that, for any h, we can divide the unit sphere $|\sigma| = 1$ into a finite number of smooth regions $\Sigma_{\mu h}$ such that for all σ in any $\Sigma_{\mu h}$ either

$$\text{(i)} \quad \gamma_\sigma^{(h)} (t^\circ,x^\circ) \neq 0$$

or, for all σ in that $\Sigma_{\mu h}$

$$\text{(ii)} \quad \mathrm{grad}_\psi \, \gamma_\sigma^h (t^\circ,x^\circ) \neq 0.$$

Both cases may occur simultaneously.

It is clearly enough to derive the estimates on G for (t,x) in a small neighborhood V_o of (t°,x°), $V_o \in V^\varepsilon$, since then we can apply the Heine-Borel principle and complete the proof of the theorem.

We can take V_o and the $\Sigma_{\mu h}$ in such a way that if (ii) holds, but (i) does not hold, then

(5.47) for some $\eta > 0$, $\gamma^h (t,x) = \gamma$ $\quad (|\gamma| < \eta, \sigma \in \Sigma_{\mu h}, (t,x) \in V_o)$ can be uniquely solved in terms of one of the ψ_i, and $\mathrm{grand}_\psi \, \gamma_\sigma^{(h)}(t,x) \neq 0$.

We shall now estimate derivatives of $G_{h\sigma} \equiv R_{ph\sigma} + U_{ph\sigma}$ for $\sigma \in \Sigma_{\mu h}$, in case (i)

By (5.35),(5.46) we get, for $q = p - 2n - 2$,

$$|D_*^q \, G_{h\sigma}(t,x)| \leq B_{11}(B_{12})^q \left[\frac{M_{q, 2q-a+\alpha}}{q!} + \frac{M_{q, q-a+\alpha} \tilde{M}_q}{q!} \right] \tag{5.48}$$

for some $\alpha = \alpha(m,n)$.

4.7. The Second Case; Completion of the Proof

We next consider the case (ii) and estimate derivatives of

$$G_{h,\mu}(t,x) = \int_{\sigma \in \Sigma_{\mu h}} G_{h\sigma}(t,x)dS_{\psi} \quad . \tag{5.49}$$

Since $G_{h\sigma} = R_{ph\sigma} + U_{ph\sigma}$ and the derivatives of $R_{ph\sigma}$ have already been estimated in 4.5 (see (5.46)), it remains to estimate derivatives of

$$U_{ph,\mu}(t,x) = \int_{\sigma \in \Sigma_{\mu h}} U_{ph\sigma}(t,x)dS_{\psi} \quad . \tag{5.50}$$

We may clearly assume that (i) is not satisfied; hence (5.47) holds. Let σ^0 be the unique $\sigma \in \Sigma_{\mu h}$ for which $\gamma_{\sigma}^{(h)}(t^0,x^0) = 0$. Because of (5.47), there exists $\eta_0 > 0$ such that

$$\text{if } |\sigma - \sigma^0| \geq \eta_0 \, , \; \sigma \in \overline{\Sigma}_{\mu h} \, , \; (t,x) \in \overline{V}_0 \, , \text{ then } \gamma_{\sigma}^{h}(t,x) \neq 0 \, , \tag{5.51}$$

provided the diameter of V_0 is sufficiently small, which we may assume.

We now split the integral (5.50) into two integrals:

$$U_{ph,\mu} = \int_{\sigma \in \Sigma_{\mu h}} = \int_{\sigma \in \Sigma'} + \int_{\sigma \in \Sigma''} \equiv J_p' + J_p'' \quad . \tag{5.52}$$

To define Σ' consider the family of surfaces

$$\gamma_{\sigma}^{h}(t,x) = \gamma \tag{5.53}$$

for $-\eta < \gamma < \eta$. Because of (5.47) this is a family of $(n-2)$ - dimensional surfaces in the local parameters $\psi_1, \ldots, \psi_{n-1}$, the parameters of the family are (t,x,γ) . Denote this family by $T(t,x,\gamma)$. For each γ', γ'' in the interval

$-\frac{\eta}{N} < \gamma < \frac{\eta}{N}$ there is a one-to-one correspondence ξ between the points $\psi = \psi(t',x',\gamma')$ on $T(t',x',\gamma')$ and $\psi = \psi(t'',x'',\gamma'')$ on $T(t'',x'',\gamma'')$ provided (t',x') and (t'',x'') belong to V_0, and N is sufficiently large. We determine on $T(t,x,\gamma)$ a set $\pi(t,x,\gamma)$ in the following way:

$\pi(t^0,x^0,0)$ is the intersection of $T(t^0,x^0,0)$ with an $(n-1)$ - dimensional sphere in the ψ - space. If the radius of the sphere is taken to be sufficiently small, then $\pi(t^0,x^0,0)$ is a manifold. $\pi(t,x,\gamma)$ is defined to be the set corresponding to $\pi(t^0,x^0,0)$ by the mapping ξ. Let $T_0(t,x,\gamma)$ be the interior of $\pi(t,x,\gamma)$ in $T(t,x,\gamma)$. Then it is clear that if the diameter of V_0 is sufficiently small and if N is sufficiently large, then $\pi(t,x,\gamma)$ and $T_0(t,x,\gamma)$ belong strongly to $C\{M_{q-a-1}\}$ (see §1). Also, $\pi(t,x,\gamma)$ has no boundary.

Note now that if we decrease V_0, η and N remain unchanged but η_0 in (5.51) can be decreased. Hence, without loss of generality we may assume that the family $\{T_0(t,x,\gamma)\ ;\ -\frac{\eta}{N} < \gamma < \frac{\eta}{N}\}$ contains the $(n-1)$ - dimensional set in the ψ - space which corresponds to the set $|\sigma - \sigma^0| < \eta_0$.

We define

$$\Sigma' = \Sigma'(t,x) = \{T_0(t,x,\gamma)\ ;\ -\frac{\eta}{N} < \gamma < \frac{\eta}{N}\} .$$

Then, by (5.51), $|\gamma_\sigma^h(t,x)| \geq r_0 > 0$ in $\Sigma_{\mu h} - \Sigma'$.

Hence, in Σ'' (5.35) is valid, and using (3.28) we obtain

$$|D_*^q\, U_{ph\sigma}(t,x)| \leq B_{13}\,(B_{14})^{p+q}\,\frac{M_{p,p-e}}{p!}\,\tilde{M}_{q-2} .$$

Applying Lemma 5' , we obtain, for the particular choice $q = p - n$,

$$|D^q J_p''(t,x)| \le B_{15}(B_{16})^{p+q} \frac{M_{p,p-e}}{p!} M_p \quad . \tag{5.54}$$

We turn to J_p' and introduce on $\Sigma_{\mu h}$ the form (see [10; pp. 272-3])

$$d\sigma = d\gamma \, dO \qquad \text{where} \quad dO = dS_\gamma / \left|\frac{\partial\gamma_\sigma(h)}{\partial\nu}\right| ;$$

dS_γ is the element of area on the manifold $T_o(t,x,\gamma)$ and $\partial/\partial\nu$ is the normal derivative to $T_o(t,x,\gamma)$. We obtain

$$J_p'(t,x) = \int_{\frac{\eta}{N}}^{\frac{\eta}{N}} \sum_{k=0}^{p} f_k(\gamma) \left\{ \int_{T_o(t,x,\gamma)} u_{kh\sigma}(t,x) \frac{dS_\gamma}{\left|\frac{\partial\gamma_\sigma(h)}{\partial\nu}\right|} \right\} d\gamma \ . \tag{5.55}$$

Denoting the inner integral by $\Phi_{kh}(t,x,\gamma)$, we claim that for any $q \ge 0$,

$$|D_+^q \Phi_{kh}(t,x,\gamma)| \le B_{17}(B_{18})^{k+q} M_{k,k-e} \tilde{M}_q \tag{5.56}$$

where D_+^q is any q-th partial derivative with respect to (t,x,γ) .

Indeed, this follows by employing Lemma 5 while making use of the estimates (5.29), (5.1).

It is now easy to complete the derivation of the estimates for J_p' . In fact,

$$D^q J_p'(t,x) = \sum_{k=0}^{p} \int_{-\frac{\eta}{N}}^{\frac{\eta}{N}} f_k(\gamma) \, D^q \Phi_k(t,x,\gamma) d\gamma \equiv \sum_{k=0}^{p} J_k \ .$$

Consider first the case where n is odd. If $k \le n - m$ then, by (3.24),

$$J_k(t,x) = c_n D_\gamma^{n-m-k} D^q \Phi_k(t,x,\gamma)\Big|_{\gamma = 0} , \tag{5.57}$$

and if $k > n - m$ then

$$(5.58) \qquad |J_k(t,x)| \le \frac{B_{19}}{(k+m-n-1)!} \sup_{\gamma} |D^q \bar{\Phi}_k(t,x,\gamma)| \ .$$

Combining both cases for k, and using (5.56), we get

$$(5.59) \qquad |D^q J_p'(t,x)| \le B_{20}(B_{21})^{p+q} \frac{M_{p,p-e}}{p!} \tilde{M}_q \ .$$

For n even, if $n+1-k-m < 0$ then (see (3.25)) we obtain an estimate similar to (5.58). If $n + 1 - k - m \ge 0$ then we obtain a result similar to (5.57); this follows by using the definition of the distribution r^{-h} ($h > 0$) and Taylor's formula.

Combining the estimates (5.59), (5.54) and using (5.52) we get, for $q = p - 2n - 2$,

$$(5.60) \qquad |D^q U_{ph,\mu}(t,x)| \le B_{22}(B_{23})^p \frac{M_{p,p-e}}{p!} \tilde{M}_p \ .$$

Combining this inequality with (5.46) we obtain for $G_{h,\mu}$ the inequality (5.48) with different B's and α.

Since in the first case we have, by (5.48) (see the definition (5.49),

$$(5.61) \qquad |D^q G_{h,\mu}(t,x)| \le B_{24}(B_{12})^q \left[\frac{M_{q,2q-a+\alpha}}{q!} + \frac{M_{q,q-a+\alpha}\tilde{M}_q}{q!}\right] ,$$

we find that (5.61) holds in each of the cases.

Summing over h,μ we obtain the same inequality (with different B_{24}) for $G(t,x)$. This completes the proof of the theorem.

6. Concluding Remarks

6.1. Other Fundamental Solutions

Theorem 2 remains true also for the other fundamental solutions $G_j(t,x)$ defined by the initial conditions (3.3'), with $t^0 = 0$, $x^0 = 0$. The only difference is in the definition of the f_k. f_{k+1} is defined as $\int f_k$, but instead of (3.22) we now take

$$f_{-j}(r) = \begin{cases} c_n \delta^{(n-1)}(r) & (n \text{ odd}) \\ c_n r^{-n} & (n \text{ even}) . \end{cases} \tag{6.1}$$

Formulas (3.24), (3.25) have to be modified accordingly.

Obviously, all the results remain true for the fundamental solutions $G_j(t,x;t^0,x^0)$ with pole at (t^0,x^0).

6.2. Smoothness of Solutions: Huygen's Principle

Let u be a solution of the Cauchy problem

$$\begin{aligned} &Pu = 0 \\ &\left.\frac{\partial^j u}{\partial t^j}\right|_{t=0} = \varphi_j(x) \qquad (0 \leq j \leq m-1) . \end{aligned} \tag{6.2}$$

For any point (t^*,x^*) where $0 < t^* < \varepsilon$ (the same ε which appears in the statement of Theorem 2), denote by $C(t^*,x^*)$ the intersection of $t = 0$ with the characteristic conoid with center (t^*,x^*). Let C^* be an open neighborhood of $C(t^*,x^*)$, and assume that

$$\varphi_j(x) = \varphi_{1j}(x) + \varphi_{2j}(x) \qquad (0 \leq j \leq m-1) \tag{6.3}$$

where φ_{1j} is of the class $C\{M_{q-a}\}$ in some ball $|x| \leq R'$ and where φ_{2j} is any function (say, bounded and measurable) which vanishes on C^*.

Let u_1 and u_2 be the solutions of the Cauchy problem (6.2) with the initial conditions φ_{1j} and φ_{2j} respectively.

If $R' \geq R_0$ where R_0 depends only on ε and P, then, by Theorem 3, u_1 is of class $C\{M_{q-a}\}$ in some neighborhood N of (t^*,x^*).

As for u_2, we can represent it in terms of the fundamental solutions, namely,

$$u_2(t,x) = \sum_{j=0}^{m-1} G_j(t,x;0,x^0) * \varphi_j(x^0) \quad . \tag{6.4}$$

Since $\varphi_j(x^0) = 0$ if $x^0 \in C^*$, we can apply Theorem 2 and conclude that u_2 belongs to $C\{\hat{M}_{q-a+d};N_0\}$ if N_0 is sufficiently small neighborhood of (t^*,x^*). Hence:

Theorem 4. Under the assumptions of Theorem 2 and the forgoing assumptions concerning the φ_j, the solution u of the Cauchy problem (6.2) belongs to $C\{\hat{M}_{q-a+d}\}$ in some sufficiently small neighborhood of (t^*,x^*).

This theorem may be viewed as an Huygen's principle for the smoothness of solutions, namely, if the initial values belong to $C\{M_{q-a}\}$ in some neighborhood of $C(t^*,x^*)$ and are arbitrary elsewhere, and if the decomposition (6.3) holds in some ball $|x| \leq R'$, then the solution u is in a corresponding class $C\{\hat{M}_{q-a+d}\}$ in some neighborhood of (t^*,x^*).

6.3. Hyperbolic Systems of equations

The most general hyperbolic systems for whom the Cauchy problem has been solved by Petrowski [16] and Leray [15] are

$$\frac{\partial^{n_\rho} u_\rho}{\partial t^{n_\rho}} = \sum_{j=1}^{N} \sum_{\substack{k_0+k_1+\dots+k_n \leq n_j \\ k_0 < n_j}} A_{k_0k_1\dots k_n,j\rho} \frac{\partial^{k_0+k_1+\dots+k_n} u_j}{\partial t^{k_0}\partial x_1^{k_1}\dots\partial x_n^{k_n}} \tag{6.5}$$

$$(1 \leq \rho \leq N) \quad ,$$

where the A's are matrices of order $N \times N$ with coefficients depending on (t,x). The condition of hyperbolicity is the following:

For any real vector $\sigma \neq 0$, the matrix

$$(e_{j\rho}) = \Big(\sum_{\Sigma k_i = n_j} A_{k_0 k_1 \ldots k_n, j\rho} \, v^{k_0} \sigma_1^{k_1} \ldots \sigma_n^{k_n} - v^{n_j} \delta_{j\rho} \Big)$$

can be transformed into a matrix

$$\begin{pmatrix} N_1 & 0 & \ldots & 0 \\ 0 & N_2 & \ldots & 0 \\ 0 & 0 & \ddots & N_k \end{pmatrix}$$

where the roots of each polynomial det. (N_h) are real and distinct.

Consider now the special case $k = 1$. If all the $n_j = 1$ then the system is the one considered by Courant-Lax [3] and Lax [14].

The formalism of §3 was extended to hyperbolic systems (with $k = 1$) by Babitch [1]. An energy inequality analogous to (4.18) is valid also for hyperbolic systems (Leray [15]). Using these tools, Theorems 2-4 can be extended without difficulty to hyperbolic systems. Since the proofs are quite analogous and the methods are the same, we omit all the details. We only mention here the definition of a fundamental matrix:

<u>Fundamental matrix</u> with pole at the origin is a matrix $G = (G^h_{jk})$ of order $N \times N$ having the following properties:

(a) Each column is a distribution in x, with t as a parameter, which satisfies (6.5).

(b) For $0 \leq i \leq n_j - 1$

$$\frac{\partial^i}{\partial t^i} G^h_{jk}\Big|_{t=0} = 0 \text{ if } i \neq h \; ; \quad \frac{\partial^h}{\partial t^h} G^h_{jk}\Big|_{t=0} = \delta_{jk}\, \delta(x) \, .$$

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