NAVWEPS REPORT 7837

AN EMPIRICAL STUDY OF THREE STOCHASTIC APPROXIMATION TECHNIQUES APPLICABLE TO SENSITIVITY TESTING (U)

U.S. NAVAL WEAPONS EVALUATION FACILITY

ALBUQUERQUE, NEW MEXICO

U N C L A S S I F I E D

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By

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FOREWORD

In order that more meaningful data can be obtained in certain proposed sensitivity tests associated with nuclear weapon vulnerability studies, a study of old methods in comparison with new ones was deemed desirable.

The results presented here will aid an experimenter in determining the feasibility of using stochastic approximation techniques. Such techniques have wide application in industry and their use is not confined to the evaluation of weapon systems.

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ABSTRACT

The rates of convergence of three stochastic approximation estimators are studied empirically using a Monte Carlo sampling procedure. The results are presented in tabular form and various conclusions are made as to the utility of each estimator in the light of these results.

CONTENTS

INTRODUCTION

Sensitivity testing deals with a continuous variable which cannot be determined in practice. For example, suppose it is desirable to know the amount of mass of a high explosive such that the probability that an explosive response will occur when the mass is subjected to a jet-fuel fire is less than some specified level, say α. There are levels of mass at which less than 100α percent will respond and levels where more than 100α percent will respond. Clearly, the critical value of mass at which exactly 100α percent will respond cannot be measured. All one can do is select a sample arbitrarily and determine whether the critical value for a sample is less than or greater than the mass of each element of the sample.

This situation arises in many fields of research. In selecting insecticides, a critical dose is associated with each insect but cannot be measured. One can only try some dose and observe whether or not the preassigned percentage of insects are killed, i.e., observe whether or not the desired dose for the insect is less than the chosen dose. The same difficulty arises in pharmaceutical research dealing with germicides, anaesthetics, and other drugs, in testing strengths of materials, and in several areas of engineering and developmental research.

In true sensitivity experiments, it is not possible to make more than one observation on a given specimen. Once a test has been made, the specimen is altered (e.g., the explosive is destroyed, the insect weakened) so that a bona fide result cannot be obtained from a second test on the sa. : specimen. The common procedure in experiments of this kind is to divide the sample of specimens into several groups (usually, but not necessarily, of the same size) and to test one group at a chosen level, and a second group at a second level, etc. The data consist of the numbers affected and not affected at each level. Several methods of analyzing such data (variously called sensitivity data, all-or-none data, or quantal responses) are available (Ref. 1 and 2).

Most of the methods commonly used are applicable only in special cases, most of which are based on various assumptions concerning the distributions of the estimators, especially if confidence limits are desired. A method, devised relatively recently (and seldom used for various reasons), is available to the experimenter in which he may estimate any critical value in its range with some assurance that after a large number of trials the estimator will approximate closely the desired critical value. The method, called a stochastic approximation method, was formulated by Robbins and Monro and published in 1951 in the _Annals of Mathematical Statistics_ (Ref. 3).

Briefly stated, stochastic approximation is concerned with the regression of a variable y on a variable x, and seeks the value $x = \theta$ for which the regression value of y is some preassigned number, $y = \alpha$. The estimation procedure for θ is sequential and _distribution-free_. Despite its extreme simplicity in application and the wide variety of the situations in which it may be useful, the technique has not been taken advantage of bv empirical research workers. One reason for this may be that the existing literature is addressed primarily to the professional mathematician. Another reason may be that the mathematical theory itself is not yet complete for relatively small samples.

A desirable feature of stochastic approximation is the lack of assumptions required. In many problems, the researcher has no clear picture of the structure of the relationship he wishes to study and would prefer, if possible, not to commit himself to hypothesize the precise shapes of the regression or other distribution features. In such cases, he needs a procedure which is distribution-free.

Theoretically, the problem reduces to solving the regression equation

$$M(x) = \alpha \tag{1}$$

This problem has been studied by Robbins and Monro (Ref. 3), Blum (Ref. 4), Keston (Ref. 5), and others (Ref. 6, 7, and 8). Using the notation of Robbins and Monro, M(x) denotes the expected value at level x of the response, say Y, of a certain experiment. M(x) is assumed to be a continuous monotone function of x, but is unknown to the experimenter, and it is desired to find the solution $X = \theta$ of the equation $M(x) = \alpha$ where α is a given constant. The Robbins and Monro method is one in which successive experiments are performed at levels X_1, X_2, ... in such a way that X_j will tend to θ in probability.

Except for an unpublished study by Teichrow[1] and an application of the Robbins and Monro technique described by Louis and Ruth Guttman (Ref. 9), little is available to the experimenter to guide him in the use of stochastic approximation methods. The purpose of this report is to give the experimenter information

[1] Teichrow, D., "An Empirical Investigation of the Stochastic Approximation Method of Robbins and Monro."

which will aid him in determining the feasibility of using stochastic approximation methods; and also, if he decides to use the techniques, in determining which of the three available estimators he should use. The proofs that two of the three estimators converge with probability one to the desired value are available in statistical literature and will not be discussed here.

The report is divided into two parts. The first is a discussion and description of the estimators. The second part is an empirical comparison of the convergence properties of the three estimators.

Since the form of $M(x)$ is not known to the experimenter, the means used here to study the convergence properties is to employ a Monte Carlo sampling scheme to simulate a test in which stochastic approximation methods will be used. Upon repeated simulations of trials for various forms of $M(x)$, various convergence properties of each of the three estimators can be observed.

The primary interest here lies in sensitivity testing, sometimes called quantal response testing; therefore, the empirical study made is a simulation of this type of testing. A similar study could be made by assigning a continuous distribution function to the observed random variable $Y(x)$.

THREE STOCHASTIC APPROXIMATION ESTIMATORS

For each real number x, let $Y(x)$ be a random variable such that $E[Y(x)] = M(x)$ exists. Assume that the regression equation $M(x) = \alpha$ has a single root at $x = \theta$, which is to be estimated, and that $(x - \theta)[M(x) - \alpha] > 0$ for all $x \neq \theta$. An initial value

x_1 and a sequence $[c_j]$ of positive numbers are selected. The $(j + 1)$st approximation to Θ is defined inductively by the recursive formula

$$x_{j+1} = x_j + c_j(\alpha - y_j) \tag{2}$$

where y_j is the observed value of the random variable at $x = x_j$. The letter j denotes the trial number.

Each of the three estimators can be written in the form of Eq. 2. However, the difference lies in the way the sequence $[c_j]$ is defined.

The sequence $[c_j]$ which defines estimator I (the Robbins-Monro estimator) is a fixed sequence of positive elements with the following properties:

(a) $$\sum_{j=1}^{\infty} c_j = \infty$$

(b) $$\sum_{j=1}^{\infty} c_j^2 < \infty$$

The sequence $[1/j]$ has these properties.

The second estimator (estimator II proposed by Keston) is defined by Eq. 1, where the sequence $[c_j]$ is defined in the following way from the sequence

$$c_1 = a_1$$

$$c_2 = a_2$$

$$c_j = a_{t(j)}$$

where $t(j) = 2 + \sum_{i=3}^{j} \delta[(x_i - x_{i-1})(x_{i-1} - x_{i-2})]$

and $\delta(x) = 1$ if $x \leq 0$
$= 0$ if $x > 0$

Thus every time $(x_j - x_{j-1})$ differs in sign from $(x_{j-1} - x_{j-2})$, another a_k is taken. A further restriction on the sequence $[a_k]$ other than the properties (a) and (b) is

$$\text{(c)} \qquad a_{k+1} \leq a_k$$

It is important to note that the elements of $[c_j]$ for $j > 2$ are random variables.

Keston's rule for selecting the members of $[c_j]$ is based on the conjecture that in the neighborhood of $x = \theta$, θ being the solution of Eq. 1, it seemed likely that frequent fluctuations in the sign of $(x_j - \theta) - (x_{j+1} - \theta) = x_j - x_{j+1}$ indicate that $|x_j - \theta|$ is small where a few fluctuations in the sign of $x_j - x_{j+1}$ indicate that x_j is far away from θ.

It can be shown that there exists a θ', not necessarily identical with θ, where fluctuations in the sign occur more frequently in a finite number of trials. The value $x = \theta'$ is defined by the intersection of the line $Y(x) = \alpha$ and the locus of the medians of the densities $dH(y \mid x)/dy$ for any x. It should be noted that if the density $dH(y \mid x)/dy$ is symmetric, then Keston's conjecture is obviously correct. Even though the fluctuation would be expected to occur at θ' instead of θ, this does not affect the convergence in probability of

$$x_{j+1} = x_j + c_j(\alpha - y_j) \tag{3}$$

to θ, as Keston has proved.

Let x_j be the value such that the variation in the algebraic sign of $x_j - x_{j+1}$ is maximum. Suppose that $x_{j-1} < x_j$. In order for a variation in the sign to occur, $x_{j+1} < x_j$; where x_{j+1} is defined by Eq. 3.

Let U denote a random variable whose density is the point binomial. The variable U takes on the value unity with the probability P_x where

(4) $$P_x = Pr[X_{j+1} < x_j \mid x_{j-1} < x_j]$$

From Eq. 3, it follows that

(5) $$P_x = Pr[Y(x_j) > \alpha]$$

Clearly, U has maximum variance at $P_x = 1/2$. Therefore, that value of x such that

(6) $$Pr[Y(x_j) > \alpha] = 1/2$$

is the desired value of θ'.

If $x_{j-1} > x_j$, a similar argument leads to the conclusion that the value of x such that

(7) $$Pr[Y(x_j) < \alpha] = 1/2$$

is the desired θ'. Hence, θ' is the value of x defined by the intersection of the line $M(x) = \alpha$ and the locus of the medians of $dH(Y \mid x)/dy$.

Since the sequence $[x_j]$ converges to θ with probability one, there exists a J such that for all $j > J$

$$Pr[\sup_{x_j} |X_j - \theta| < |\theta - \theta'|] = 1 - e \qquad \theta' \neq \theta \text{ and } e > 0$$

That is, there exists a neighborhood of θ which does not contain θ' such that after some trial number N almost surely all x_j will lie inside the neighborhood. Hence, there will exist almost surely only a finite number of sign changes in a neighborhood of θ' if θ is not in the neighborhood of θ'. But, for a finite number of trials, the experimenter cannot be assured that the sign changes are occurring in the neighborhood of θ or θ'.

In order to obtain an indication of how this fact would affect the sequence $[c_j]$, consider the difference between the median and means of two rather common skewed densities: the triangular and the gamma.

Consider first the following form of the triangular distribution:

$$f(x) = \begin{cases} \frac{2}{cb} x & 0 \leq x \leq b \\ \frac{2}{c(c-b)} (c - x) & b \leq x \leq c \end{cases}$$

Table 1 presents values of the ratio of the median to c, the ratio of the mean to c, and their difference for various values of b/c. Note that for small values of c, the difference between the median and the mean can be slight.

Table 2 presents the ratio of the median to β, the ratio of the mean to β, and their difference for various values of α, when the gamma density is of the following form:

$$f(x) = \frac{1}{\beta^{\alpha+1}\Gamma(\alpha + 1)} x^{\alpha} e^{-x/\beta} \qquad x \geq 0$$

TABLE 1. Comparison of the Mean and Median for the Triangular Density Function

b/c	Median/c	Mean/c	Difference/c
.5	.500	.500	.000
.6	.548	.533	.015
.7	.592	.567	.025
.8	.632	.600	.032
.9	.671	.633	.038
1.0	.707	.667	.040

TABLE 2. Comparison of Mean and Median for the Gamma Density Function

α	Median/β	Mean/β	Difference/β
0	.693	1.000	.307
1	1.678	2.000	.322
2	2.674	3.000	.326
3	3.672	4.000	.328
4	4.671	5.000	.329
5	5.670	6.000	.330
6	6.670	7.000	.330
7	7.669	8.000	.331
8	8.669	9.000	.331
9	9.669	10.000	.331
10	10.669	11.000	.331

From the data in Table 2, it appears that even for small values of β, the difference between the median and the mean can be relatively large.

It should be noted that the mean and the median are identical in the binomial distribution if, and only if, p = 1, 0, or 1/2 where p + q = 1. The importance of the binomial distribution is that it is the basic distribution for quantal response problems.

It is hard to justify the use of an estimator computed from a small number of trials simply because it is known to converge to the desired value as the number of trials increases without bound. The fact that no other estimators have been proposed and found better, in some sense could be a just reason for using the stochastic approximation estimator. Therefore, it seems desirable to compare the two stochastic approximation estimators previously described with an estimator (estimator III) which seems to be the one which would be most naturally proposed by an experimenter who had no knowledge of the Robbins—Monro or the Keston estimators.

An experimenter who wishes to determine an x such that $M(x) = \alpha$ would most logically select an x_1 which he would consider as being close to the desired value and then compare the random variable $Y(x_1)$ with α.

If $Y(x_1)$ exceeded α, then $x_2 < x_1$ would be selected according to the magnitude of $\alpha - Y(x_1)$. Similarly, if $Y(x_1)$ was less than α, $x_2 > x_1$ would be selected. Clearly, if $Y(x_1) = \alpha$, the experimenter would continue testing at x_1. If after j tests $Y(x_{j-1}) < \alpha$ and $Y(x_j) > \alpha$ or $Y(x_{j-1}) > \alpha$ and $Y(x_j) < \alpha$, then it seems logical that the experimenter would interpolate in order to obtain x_{j+1}. Also, it seems a desirable procedure to shorten the steps that one takes after each trial in a small neighborhood of the desired value of x. A modification of Keston's procedure for shortening the step length seems intuitively adequate.

Mathematically, this procedure can be described by the recursive formula, Eq. 1, where c_j is an element of a sequence $[c_j]$ defined by the following rule:

$$c_1 = a_1$$

$$c_2 = a_2$$

If $c_{j-1} = a_k$ for $k \geq 2$, then

$$c_j = \begin{cases} a_k & \text{when } \alpha \notin (y_j, y_{j-1}) \\ (x_j - x_{j-1})/(y_j - y_{j-1}) & \text{when } \alpha \in (y_j, y_{j-1}) \end{cases}$$

$$c_{j+1} = \begin{cases} a_k & \text{when } c_j = a_k \text{ and } \alpha \notin (y_{j+1}, y_j) \\ (x_{j+1} - x_j)/(y_{j+1} - y_j) & \text{when } \alpha \in (y_{j+1}, y_j) \\ a_{k+1} & \text{when } \alpha \notin (y_{j+1}, y_j) \\ & \text{and } c_j = (x_j - x_{j-1})/(y_j - y_{j-1}) \end{cases}$$

When a_k is an element of a sequence $[a_k]$ having the following properties:

(a) $a_k > 0$ for $k = 1, 2, \ldots$

(b) $a_k > a_{k+1}$ for $k = 1, 2, \ldots$

(c) $\sum_1^{\infty} a_k = \infty$

(d) $\sum_1^{\infty} a_k^2 < \infty$

That is, if $\alpha \in (y_j, y_{j-1})$, then x_{j+1} is obtained by linear interpolation. A new a_k is selected after each period of linear interpolation. An end of a period occurs if $\alpha \in (y_j, y_{j-1})$ but $\alpha \notin (y_{j+1}, y_j)$; hence, c_{j+1} is the next unused element of the sequence $[a_k]$.

APPLICATION OF STOCHASTIC APPROXIMATION METHODS TO QUANTAL RESPONSE PROBLEMS

Let the random variable Y take on only two values, unity with the probability M(x) and zero with the probability 1 - M(x). This type of a response has been called quantal response. Let there be two real numbers, a and b (a < b), such that

$$Y(x) = 0 \qquad \text{for all } x \leq a$$

and

$$Y(x) = 1 \qquad \text{for all } x \geq b$$

Assume that a = 0 and b = 1. Then the regression function M(x) will have the following properties:

$$\begin{aligned} M(x) &= 0 && \text{for } x \leq 0 \\ &= f(x) && \text{for } 0 \leq x \leq 1 \\ &= 1 && \text{for } x \geq 1 \end{aligned}$$

In a neighborhood of $x = \theta$, the root of the regression equation $M(x) = \alpha$, we know that there exists a small neighborhood of θ in which

(8) $\Pr[|x_{j+1} - \theta| \geq |x_j - \theta| \text{ and } (x_{j+1} - \theta)(x_j - \theta) \geq 0]$

the probability of making an incorrect decision at x_j is an increasing function of x as x tends toward θ.

Since $Y(x_j)$ can take on only the values of zero and unity, and assuming $\alpha \neq 1$ or $\alpha \neq 0$, then $Pr[Y(x_j) = \alpha \mid x_j = \theta] = 0$, or the value of the probability statement 8 is unity.

Suppose, however, that at each level x_j a sample of $k > 1$ Y's are taken. Since the sample mean

$$\overline{Y}(x_j) = \frac{1}{k}\sum_{1}^{k} Y_i(x_j) \qquad \text{where } Y_i = \begin{cases} 0 & \text{if no response occurs} \\ 1 & \text{if a response occurs} \end{cases}$$

has the same expected value as the random variable $Y(x)$, the recursive formula $x_{j+1} = x_j + c_j[\alpha - \overline{y}(x_j)]$ will converge with probability one to the same limit as $x_{j+1} = x_j + c_j[\alpha - y(x_j)]$ for estimators I and II.

Let us consider a special application of the general stochastic approximation technique, that is, the problem to which stochastic approximations would be most applicable: the quantal response problem or sensitivity testing.[2] This is a test in which the experimenter wants to determine a level of x such that the probability of a response as defined by the problem will be some preassigned value, say α. Let $M(x)$ be defined by Eq. 1 where $f(x)$ is monotonically increasing in its range. Let us now consider the upper and lower tolerance equations, L_1 and L_2, respectively, such that $1 - 2\gamma$ percent of the observed $\overline{Y}(x)$ will be expected to fall between them. Let us represent these by

[2] A good example would be to determine that dosage of radiation to which a specified laboratory animal can be subjected such that the probability of his death after subjection to the dosage would be less than 10 percent.

$$L_1 = f(x) + m \frac{f(x) - f^2(x)}{k}$$

and
$$L_2 = f(x) - m \frac{f(x) - f^2(x)}{k}$$

Differentiating

$$\frac{dL_1}{dx} = f'(x)[1 + \frac{m}{k} - \frac{2m}{k} f(x)] \geq 0$$

when k, the sample size, and m are selected so that

$$\Pr[f(x) - m\sigma_{\overline{y}} < \overline{Y}(x) < f(x) + m\sigma_{\overline{y}}] = 1 - 2\gamma$$

and k is sufficiently large so that $m/k \leq 1$. Similarly,

$$\frac{dL_2}{dx} = f'(x)[1 - \frac{m}{k} + \frac{2m}{k} f(x)] \geq 0$$

That is, both tolerance equations are monotone and increasing with x as long as $m/k \leq 1$.

In order to gain further insight, consider Fig. 1 A desirable quality of a test would be conditions such that the length of the interval $I(\theta) = [x(L_1), x(L_2)]$ be minimized. The length of $I(\theta)$ depends upon slope and curvature of $f(x)$ in the neighborhood of θ and the distribution function of $\overline{Y}$, say $G(\overline{y} \mid x)$. Since $k\overline{Y}$ is distributed as

$$\binom{k}{k\overline{y}} M(x)^{k\overline{y}}[1 - M(x)]^{k-k\overline{y}}$$

increasing the sample size k decreases the variance

$$\mathrm{Var}(\overline{Y} \mid x) = \frac{M(x)[1 - M(x)]}{k}$$

We note that $\lim_{k \to \infty} I(\theta) = 0$ and that the density $g(\bar{y} \mid x)$ becomes symmetric as k increases. Hence, for large samples, we are assured that as the trials proceed we will move toward θ with a probability of at least $1 - \gamma$ at each trial when $x \notin I(\theta)$. It is only in those trials at levels of x which are contained in $I(\theta)$ that the probability the next step will be toward θ is less than $1 - \gamma$.

Figure 1 illustrates that each sample size fixes the tolerance equations $L_1(x)$ and $L_2(x)$. Note that the probability of moving toward θ at each x_j exceeds or is equal to $1 - \gamma$ if $x \notin I(\theta)$. Since cost and sample size are usually directly related, it would be desirable to minimize k, the sample size. If $|x_j - \theta|$ is relatively large, a small sample size seems to be desirable. When $|x_j - \theta|$ is relatively small, a larger sample size requires the length of $I(\theta)$ to decrease and the likelihood that $x \notin I(\theta)$ to increase.

The effect of increasing sample size with number of trials has been studied empirically. (See Tables 4–8, pp. 22–26.)

THE MONTE CARLO SAMPLING PLAN TO STUDY THE RATES OF CONVERGENCE OF THE ESTIMATORS

Due to the number of uncontrollable parameters involved, perhaps the most practical means available at this time to study convergence properties of the three estimators is a Monte Carlo procedure. The procedure used is as follows:

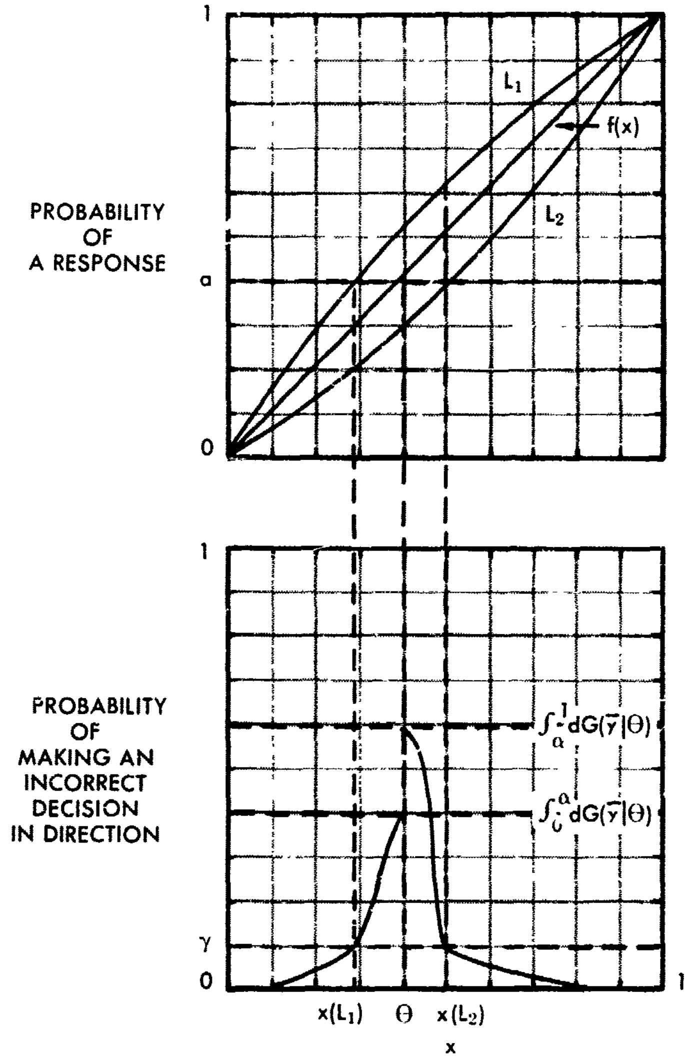

FIG. 1. The Regression Function and the Associated Curve Illustrating the Probability of Making an Incorrect Decision in Direction.

1. Define $M(x)$, α, Δk, and k, where k is the size of the sample taken at each level of x, and Δk is an increment which will be added to k with increasing trials.

2. Letting $x_1 = \alpha$, compute $M(x_1)$.

3. Generate k random numbers (r_i, $i = 1, 2 \ldots k$) from a uniform density.

4. Compare each random number r_i with $M(x_1)$. If $r_i > M(x_1)$, assign the value of zero to Y_i. If $r_i \leq M(x_1)$, assign the value of unity to Y_i.

5. Compute $\overline{y}_1 = \frac{1}{k} \sum_{i=1}^{k} y_i$

6. Substitute $\overline{y}_1$ into the recursive formula to determine x_2.

7. If $(x_j - x_{j-1})(x_{j-1} - x_{j-2}) < 0$, an increment of Δk is added to the sample size.

This procedure was programmed for the IBM 704 and continued for a desired number of trials. By repeating the process several times, various conclusions can be made.

In the study, each test was composed of a simulation of forty-nine trials. Each test was repeated one hundred times. Average values for x_7, x_{14}, x_{21}, x_{28}, x_{35}, x_{42}, and x_{49} were tabulated (Tables 4–8) for various values of α, k, and Δk.

In practice, the form of $M(x)$ is unknown to the experimenter, but it was necessary to define the form of $M(x)$ to perform the sampling plan. Five forms of $M(x)$ were selected in order for a relatively complete grid to be placed over the unit square (Fig. 2).

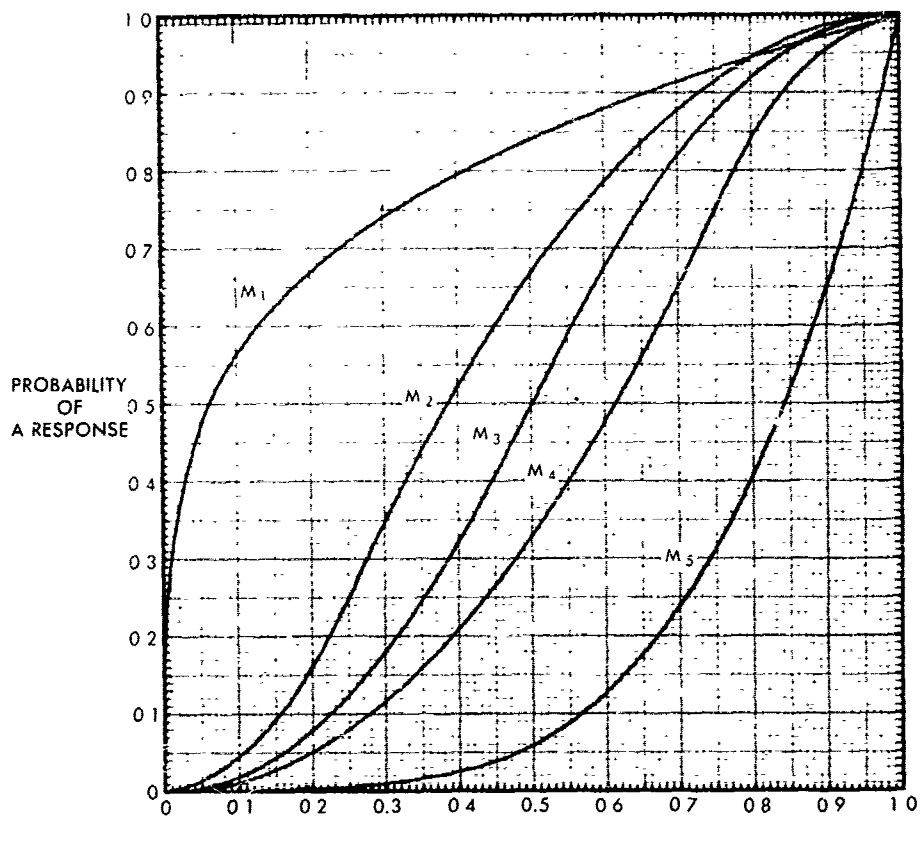

FIG. 2. The Five Forms of M(x).

These were

$$M_1(x) = x^{1/4} \qquad 0 \leq x \leq 1$$

$$M_2(x) = \begin{cases} 4x^2 & 0 \leq x \leq 1/4 \\ 1 - 4(1 - x)^2 /3 & 1/4 \leq x \leq 1 \end{cases}$$

$$M_3(x) = \begin{cases} 2x^2 & 0 \leq x \leq 1/2 \\ 1 - 2(1 - x)^2 & 1/2 \leq x \leq 1 \end{cases}$$

$$M_4(x) = \begin{cases} 4x^2 /3 & 0 \leq x \leq 3/4 \\ 1 - 4(1 - x)^2 & 3/4 \leq x \leq 1 \end{cases}$$

$$M_5(x) = x^4 \qquad 0 \leq x \leq 1$$

The form of $dH(y \mid x)/dy$ is defined by the quantal response property as the point binomial.

The values of α considered here with their associated θ_i for $i = 1, 2, 3, 4, 5$, where θ_i is the x value of the intersection of $M_i(x) = \alpha$ and $M_i(x)$, are tabulated in Table 3.

TABLE 3. Data for Sampling Procedure

α	θ_1	θ_2	θ_3	θ_4	θ_5
.05	.00006	.11180	.15811	.19365	.47287
.10	.00010	.15811	.22361	.27386	.56234
.30	.00810	.27543	.38730	.47434	.74008
.50	.06250	.38763	.50000	.61237	.84090

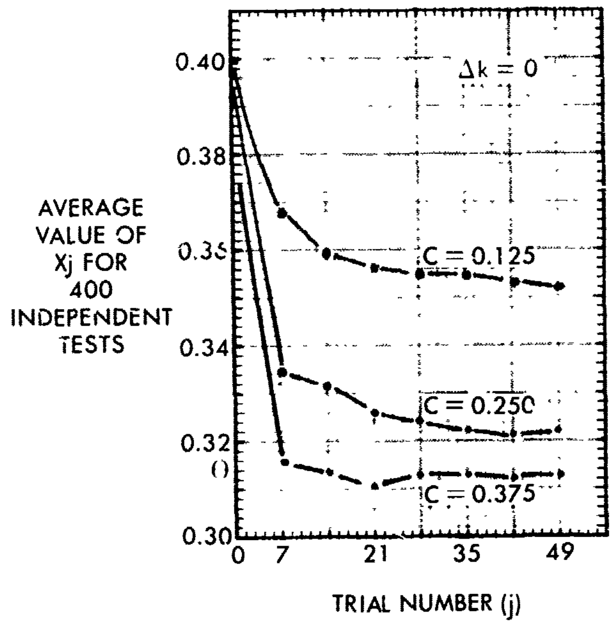

FIG. 3. The Effect of the Selection of [c/j] on the Rate of Convergence.

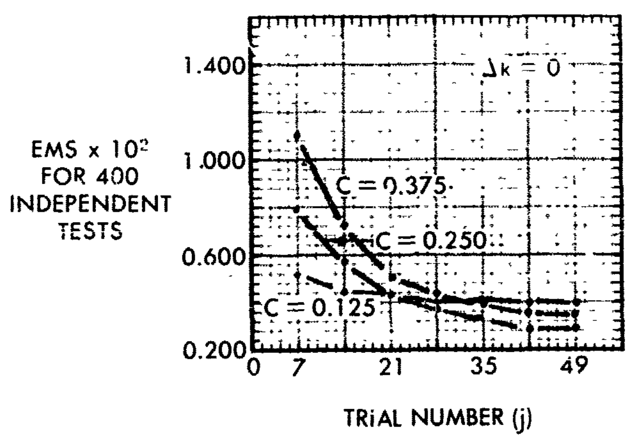

FIG. 4. The Variation of the Estimator for Various Values of c.

Various sample sizes, ranging from one to twenty, were used in simulating the test. Also, a scheme in which the sample size increases by an increment of five as the number of trials increased was considered. When $(x_j - x_{j-1})(x_{j-1} - x_{j-2}) < 0$, the sample size was increased.

The sequence $[c_j]$ for the empirical study was $[c/j]$ where $c = 0.250$ and j the trial number. The choice of 0.250 is arbitrary and is not optimum for all forms of $M(x)$.

The selection of $c = 0.250$ was based on the data summarized in Fig. 3 and 4. Three choices of $c(c = 0.125, 0.250, 0.375)$ were studied empirically using estimator I. From Fig. 3, a "good" value of c in terms of minimum error in accuracy, in a sequence of form $[c/j]$, would be in the range of from 0.250 to 0.375. Figure 4 shows that the greater variability of the estimator for a small sample size for $c = 0.375$ may offset its value as an estimator even though it is associated with the minimum bias of the three cases studied here.

The results of the Monte Carlo simulation are tabulated in Tables 4–8.

TABLE 4. Values of x_j for Estimators I, II, and III When α = .05

Sample size (k)	Trial number (j)	θ_1 = .00006			θ_2 = .112			θ_3 = .158			θ_4 = .194			θ_5 = .473		
		I	II	III	I	II	III	I	II	III	I	II	III	I	II	III
1	7	.084	.099	.092	.067	.087	.086	.077	.085	.091	.076	.091	.093	.081	.094	.094
	14	.080	.075	.064	.073	.086	.121	.084	.099	.130	.083	.122	.134	.090	.137	.137
	21	.075	.058	.037	.075	.073	.148	.088	.101	.164	.087	.141	.171	.095	.181	.181
	28	.072	.058	.016	.077	.069	.167	.090	.113	.188	.090	.146	.200	.099	.225	.224
	35	.070	.037	.005	.078	.074	.175	.092	.110	.208	.092	.148	.226	.101	.261	.268
	42	.068	.015	.007	.079	.081	.183	.093	.112	.224	.094	.152	.246	.104	.298	.310
	49	.066	.020	.009	.080	.087	.187	.095	.120	.231	.095	.153	.260	.106	.331	.351
5	7	.052	.050	.041	.073	.079	.085	.076	.086	.088	.078	.087	.091	.081	.094	.094
	14	.043	.034	.019	.078	.091	.105	.083	.106	.118	.085	.111	.123	.090	.137	.137
	21	.039	.016	.010	.081	.094	.114	.087	.117	.135	.089	.127	.147	.095	.180	.180
	28	.035	.010	.004	.082	.097	.120	.090	.123	.147	.092	.136	.164	.099	.220	.222
	35	.033	.004	.003	.083	.099	.125	.091	.127	.152	.094	.142	.174	.101	.258	.263
	42	.032	.005	.002	.085	.101	.127	.093	.130	.158	.096	.148	.182	.104	.291	.301
	49	.031	.004	.001	.085	.103	.128	.094	.132	.162	.097	.151	.187	.106	.318	.335
10	7	.041	.035	.032	.072	.076	.078	.075	.085	.085	.078	.086	.086	.081	.094	.094
	14	.034	.018	.009	.077	.087	.091	.082	.102	.106	.085	.109	.112	.090	.137	.137
	21	.030	.008	.003	.080	.091	.098	.085	.111	.118	.089	.120	.128	.095	.179	.179
	28	.027	.004	.002	.082	.094	.103	.088	.117	.125	.092	.128	.138	.099	.219	.219
	35	.025	.002	.001	.083	.096	.105	.090	.121	.130	.094	.132	.145	.101	.255	.256
	42	.023	.002	.001	.084	.097	.107	.091	.123	.134	.096	.136	.151	.104	.284	.288
	49	.022	.001	.000	.085	.098	.108	.093	.125	.137	.097	.140	.155	.106	.306	.314
20	7	.040	.036	.024	.072	.075	.078	.076	.082	.086	.078	.086	.089	.081	.094	.094
	14	.032	.016	.005	.077	.082	.093	.083	.097	.111	.085	.104	.120	.090	.137	.137
	21	.027	.006	.002	.080	.086	.100	.087	.103	.127	.089	.114	.142	.095	.180	.180
	28	.024	.002	.001	.081	.088	.105	.089	.108	.134	.092	.119	.155	.099	.221	.223
	35	.022	.002	.000	.083	.090	.107	.091	.111	.141	.094	.123	.163	.101	.256	.264
	42	.020	.001	.000	.084	.091	.108	.093	.113	.145	.095	.126	.169	.104	.280	.302
	49	.019	.001	.000	.085	.092	.110	.094	.115	.148	.093	.128	.175	.106	.296	.335

TABLE 5. Values of x_j for Estimators I, II, and III When $\alpha = .10$

Sample size (k)	Trial number (j)	$\theta_1 = .0001$			$\theta_2 = .158$			$\theta_3 = .224$			$\theta_4 = .274$			$\theta_5 = .562$		
		I	II	III	I	II	III	I	II	III	I	II	III	I	II	III
1	7	.042	.048	.018	.129	.128	.165	.149	.151	.168	.150	.174	.189	.161	.187	.187
	14	.036	.014	.012	.134	.117	.207	.157	.164	.227	.161	.212	.248	.180	.272	.275
	21	.031	.023	.001	.135	.121	.215	.161	.175	.267	.168	.212	.297	.190	.348	.359
	28	.027	.010	.007	.136	.130	.228	.164	.180	.286	.172	.220	.327	.197	.416	.440
	35	.024	.010	.009	.137	.138	.234	.166	.186	.295	.176	.224	.348	.203	.465	.508
	42	.021	.006	.008	.137	.139	.239	.168	.189	.306	.178	.228	.359	.207	.482	.556
	49	.020	.006	.006	.138	.140	.238	.170	.193	.311	.181	.239	.370	.211	.495	.588
5	7	.017	.015	.013	.129	.132	.134	.144	.151	.159	.151	.164	.163	.161	.186	.187
	14	.009	.004	.001	.135	.139	.148	.154	.168	.183	.163	.190	.200	.179	.269	.271
	21	.005	.002	.001	.138	.142	.153	.159	.179	.193	.170	.203	.219	.189	.339	.344
	28	.002	.001	.001	.139	.144	.155	.163	.184	.200	.174	.213	.230	.196	.393	.398
	35	.001	.000	.000	.141	.145	.157	.165	.188	.207	.178	.219	.237	.202	.422	.434
	42	.001	.000	.000	.141	.146	.157	.167	.191	.210	.180	.222	.242	.206	.438	.459
	49	.001	.000	.001	.142	.147	.158	.169	.194	.212	.183	.226	.247	.210	.451	.475
10	7	.007	.002	.002	.127	.132	.138	.143	.151	.159	.149	.161	.167	.161	.186	.187
	14	.002	.001	.001	.132	.137	.147	.154	.168	.189	.161	.182	.210	.179	.268	.272
	21	.001	.000	.000	.135	.139	.154	.159	.176	.202	.168	.191	.235	.189	.331	.350
	28	.000	.000	.000	.137	.141	.157	.163	.180	.209	.173	.197	.247	.197	.367	.417
	35	.000	.000	.000	.138	.143	.157	.166	.183	.215	.176	.201	.254	.202	.389	.468
	42	.000	.000	.000	.139	.144	.158	.168	.185	.218	.179	.205	.257	.207	.404	.500
	49	.010	.003	.000	.140	.145	.158	.170	.187	.219	.181	.207	.260	.210	.415	.520

TABLE 6. Values of x_j for Estimators I, II, and III When $\alpha = .30$

Sample size (k)	Trial number (j)	$\theta_1 = .008$			$\theta_2 = .275$			$\theta_3 = .387$			$\theta_4 = .474$			$\theta_5 = .740$		
		I	II	III	I	II	III	I	II	III	I	II	III	I	II	III
1	7	.100	.084	.080	.287	.284	.311	.367	.348	.371	.396	.397	.435	.472	.520	.540
	14	.074	.050	.058	.277	.272	.336	.369	.354	.402	.409	.423	.476	.514	.635	.681
	21	.060	.035	.045	.279	.274	.336	.371	.361	.415	.415	.432	.481	.538	.666	.721
	28	.049	.028	.044	.278	.275	.326	.374	.364	.420	.421	.439	.500	.553	.683	.746
	35	.045	.024	.041	.279	.276	.328	.375	.365	.422	.424	.441	.513	.564	.695	.753
	42	.040	.019	.035	.279	.275	.329	.375	.370	.423	.425	.443	.522	.573	.703	.756
	49	.036	.019	.033	.279	.275	.328	.376	.374	.429	.426	.444	.526	.580	.712	.759
5	7	.081	.048	.044	.280	.279	.287	.352	.356	.356	.385	.396	.404	.470	.533	.536
	14	.046	.019	.018	.278	.276	.283	.360	.366	.376	.403	.424	.436	.515	.651	.662
	21	.033	.014	.015	.278	.276	.281	.364	.373	.381	.412	.435	.447	.538	.681	.703
	28	.026	.012	.013	.277	.276	.283	.366	.376	.382	.415	.442	.454	.554	.698	.714
	35	.020	.011	.012	.277	.275	.285	.368	.377	.383	.420	.446	.459	.565	.707	.721
	42	.018	.010	.012	.277	.275	.282	.370	.378	.384	.423	.448	.460	.574	.712	.728
	49	.016	.010	.012	.277	.274	.281	.370	.380	.385	.425	.452	.463	.582	.715	.731
10	7	.076	.044	.038	.281	.283	.280	.358	.362	.363	.391	.397	.410	.471	.533	.535
	14	.041	.016	.014	.280	.280	.282	.365	.370	.379	.408	.421	.445	.515	.661	.669
	21	.029	.012	.013	.279	.280	.276	.369	.374	.378	.415	.431	.457	.538	.692	.722
	28	.022	.010	.012	.278	.279	.277	.370	.377	.382	.420	.437	.461	.554	.704	.725
	35	.019	.009	.010	.278	.278	.275	.372	.378	.386	.424	.440	.465	.565	.710	.730
	42	.016	.009	.010	.278	.278	.275	.373	.379	.387	.427	.443	.468	.574	.713	.732
	49	.014	.009	.010	.277	.277	.276	.373	.380	.389	.429	.445	.467	.581	.716	.733

TABLE 7. Values of x_j for Estimators I, II, and III When $\alpha = .50$

Sample size (k)	Trial number (j)	$\theta_1 = .062$ I	II	III	$\theta_2 = .388$ I	II	III	$\theta_3 = .500$ I	II	III	$\theta_4 = .612$ I	II	III	$\theta_5 = .841$ I	II	III
1	7	.300	.281	.279	.436	.435	.428	.511	.504	.510	.569	.569	.596	.721	.739	.728
	14	.261	.211	.187	.429	.422	.402	.505	.501	.497	.580	.576	.599	.753	.787	.790
	21	.241	.182	.153	.426	.413	.412	.504	.500	.494	.584	.586	.602	.766	.810	.811
	28	.226	.168	.126	.424	.404	.403	.501	.500	.493	.585	.589	.606	.776	.820	.818
	35	.216	.155	.106	.421	.402	.400	.500	.500	.498	.588	.592	.607	.782	.822	.820
	42	.207	.144	.099	.420	.400	.394	.500	.503	.499	.590	.593	.608	.787	.824	.827
	49	.202	.138	.098	.416	.398	.394	.501	.502	.497	.591	.593	.609	.791	.826	.829
5	7	.317	.244	.258	.425	.413	.428	.504	.504	.494	.573	.578	.586	.721	.769	.770
	14	.274	.154	.156	.416	.403	.410	.503	.504	.495	.584	.593	.599	.758	.813	.821
	21	.251	.124	.120	.411	.399	.404	.503	.503	.496	.590	.599	.603	.774	.825	.830
	28	.237	.110	.098	.408	.397	.398	.503	.501	.498	.593	.602	.605	.784	.829	.834
	35	.226	.100	.088	.406	.395	.396	.502	.500	.499	.594	.603	.607	.790	.832	.836
	42	.218	.093	.084	.404	.394	.394	.502	.500	.499	.596	.603	.607	.795	.834	.838
	49	.211	.089	.080	.404	.394	.393	.502	.500	.499	.597	.604	.609	.798	.834	.839
10	7	.316	.255	.266	.425	.424	.411	.499	.502	.499	.573	.577	.585	.722	.770	.770
	14	.273	.159	.135	.416	.412	.403	.499	.500	.499	.584	.588	.600	.759	.810	.827
	21	.250	.128	.096	.413	.407	.396	.500	.499	.501	.587	.593	.605	.775	.821	.831
	28	.236	.114	.079	.410	.403	.392	.499	.500	.503	.590	.597	.606	.786	.825	.834
	35	.224	.105	.073	.408	.401	.390	.499	.499	.502	.592	.599	.606	.793	.829	.835
	42	.216	.100	.070	.407	.400	.391	.500	.499	.502	.593	.600	.608	.798	.831	.838
	49	.209	.096	.068	.406	.399	.392	.500	.499	.500	.595	.602	.608	.801	.832	.839

TABLE 8. Values of x_j for Estimators I, II, and III When k = 1 and Δk = 5

α	Trial Number (j)	θ_1 = .00006			θ_2 = .112			θ_3 = .158			θ_4 = .194			θ_5 = .473		
		I	II	III	I	II	III	I	II	III	I	II	III	I	II	III
.05	7	.095	.105	.081	.073	.077	.087	.077	.087	.092	.078	.086	.090	.081	.094	.094
	14	.091	.080	.056	.078	.078	.120	.085	.116	.129	.084	.115	.130	.090	.137	.137
	21	.088	.057	.034	.081	.069	.137	.089	.129	.158	.089	.140	.166	.095	.181	.181
	28	.085	.042	.012	.082	.067	.141	.092	.126	.176	.091	.159	.193	.099	.225	.224
	35	.083	.026	.004	.083	.069	.135	.094	.118	.179	.093	.154	.208	.101	.267	.268
	42	.082	.010	.001	.084	.077	128	.095	.119	.176	.095	.143	.212	104	.307	.311
	49	.080	.007	.001	.085	.083	.125	.096	.122	.172	.096	.143	.213	.106	.324	.352
		θ_1 = .0001			θ_2 = .158			θ_3 = .224			θ_4 = .274			θ_5 = .562		
.10	7	.044	.040	.026	.134	.137	.147	.149	.147	.177	.147	.169	.180	.161	.187	.187
	14	.037	.012	.004	.136	.134	.144	.159	.166	.228	.158	.209	.247	.180	.272	.273
	21	.031	.002	.000	.139	.131	.167	.164	.164	.248	.165	.217	.277	.190	.349	.356
	28	.028	.000	.000	.139	.137	.166	.167	.177	.243	.170	.224	.287	.197	.412	.433
	35	.025	.000	.000	.140	.142	.163	.169	.186	.237	.173	.234	.286	.203	.461	.495
	42	.022	.000	.000	.141	.145	.162	.171	.193	.233	.176	.238	.282	.207	.470	.530
	49	.020	.000	.000	.141	.147	.161	.173	.197	.231	.178	.242	.280	.211	.484	.541
		θ_1 = .008			θ_2 = .275			θ_3 = .387			θ_4 = .474			θ_5 = .740		
.30	7	.082	.082	.063	.272	.280	.290	.363	.351	.391	.389	.380	.425	.475	.522	.544
	14	.051	.025	.018	.273	.274	.286	.366	.362	.391	.403	.412	.448	.519	.619	.672
	21	.038	.012	.010	.273	.274	.282	.369	.369	.388	.411	.430	.455	.542	.666	.703
	28	.031	.009	.009	.273	.274	.279	.370	.373	.387	.416	.440	.460	.558	.696	.717
	35	.027	.009	.009	.273	.275	.278	.371	.375	.386	.420	.447	.463	.569	.711	.725
	42	.024	.008	.008	.274	.275	.277	.372	.378	.387	.422	.452	.465	.578	.719	.729
	49	.021	.008	.008	.274	.275	.276	.373	.379	.387	.425	.455	.466	.585	.724	.731
		θ_1 = .062			θ_2 = .388			θ_3 = .500			θ_4 = .612			θ_5 = .841		
.50	7	.301	.278	.263	.433	.435	.419	.497	.488	.497	.565	.570	.579	.713	.729	.772
	14	.259	.204	.168	.422	.419	.399	.499	.492	.500	.576	.586	.596	.749	.797	.821
	21	.240	.159	.115	.417	.413	.394	.499	.494	.501	.581	.595	.603	.766	.818	.832
	28	.225	.131	.093	.415	.408	.392	.499	.496	.502	.584	.598	.606	.777	.826	.835
	35	.215	.115	.082	.412	.405	.390	.499	.496	.501	.586	.601	.607	.784	.830	.838
	42	.207	.104	.077	.410	.403	.389	.499	.497	.501	.588	.603	.609	.789	.833	.838
	49	.201	.097	.072	.409	.401	.389	.499	.498	.500	.590	.604	.609	.793	.834	.839

CONCLUSIONS

The most significant result of the empirical study is perhaps the apparent slowness with which estimator I converges to θ especially when $|x_j - \theta|$ is relatively large. For a test which involves less than fifty trials, estimator I when compared with II and III appears the least desirable in terms of accuracy. Figures 5 and 6 illustrate and emphasize the slowness of its convergence. A good rule is that <u>unless the experimenter is certain that the initial value, x_1, is close to θ, he should avoid using estimator I</u> (the Robbins-Monro stochastic approximation method).

On comparing estimators II and III, it is apparent that there are cases in which II appears better in terms of average accuracy than III, and vice versa. When $\alpha = 0.50$, the data from Table 7 indicate that III is slightly better for all sample sizes. Also, it should be noted that increasing the sample size had little effect in increasing the rate of convergence for all the estimators, I, II, and III. This is not true for other values of α. However, with sample size 10, estimator III gives a close approximation such that $|x_{49} - \theta| < 0.006$ for all θ_i for $i = 1, 2, 3, 4, 5$ If accuracy of the estimator is of first importance when estimating θ for $\alpha = 0.50$, the experimenter can be assured that estimator III will on the average give results with very good accuracy.

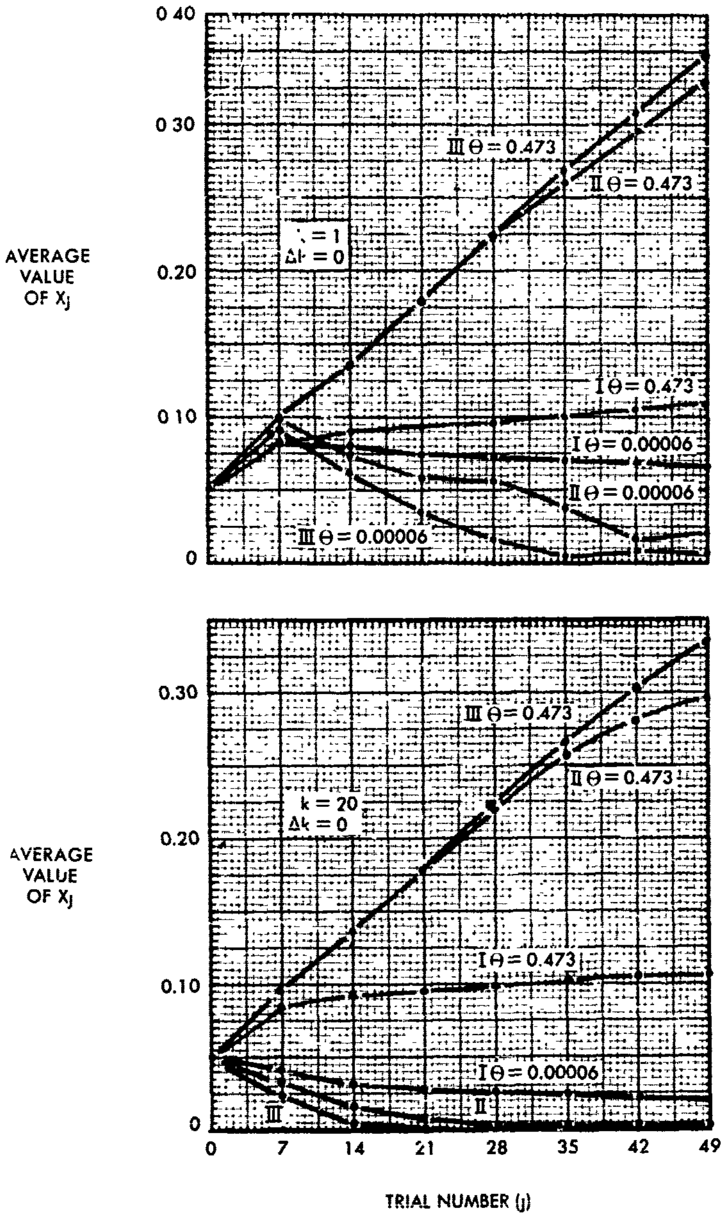

FIG. 5. Comparisons of the Rates of Convergence.

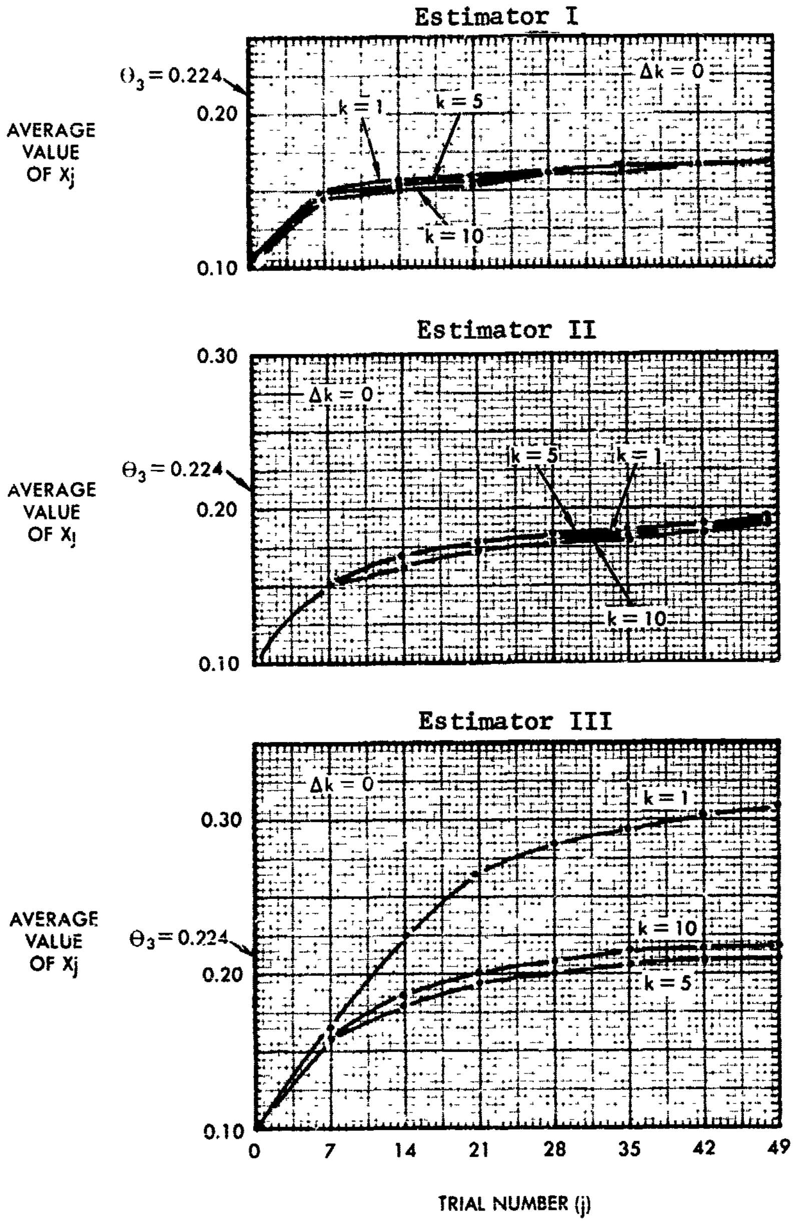

FIG. 6. The Effect of Sample Size on the Rate of Convergence when α = 0.10.

On comparing estimators II and III for values of α other than $\alpha = 0.50$, it is clear that II is better for sample size one, but III becomes better with increasing sample size. The data indicate that, for small sample sizes (1 and 5) and $\alpha = 0.05$, III overestimates (see Fig. 3). In order to explain this, consider the following rationale.

Recalling that for sample size one

$$Y_j = \begin{array}{ll} 0 & \text{if no response occurs} \\ 1 & \text{if a response occurs} \end{array}$$

then if $\alpha \in (y_j, y_{j-1})$, a linear interpolation restricts x_{j+1} such that $x_j < x_{j+1} < x_{j-1}$ or $x_{j-1} < x_{j+1} < x_j$. Suppose that $\alpha = 0.05$, then one would expect in the neighborhood of Θ that only one out of twenty trials would result in a response. Hence, there would occur on the average twenty steps to the right for one to the left. But when the one does occur, $x_{j+1} \in (x_{j-1}, x_j)$ or $x_{j+1} \in (x_j, x_{j-1})$, which offsets the large step back to the right which occurs in using I and II. Hence, one can expect estimator III to overestimate toward the left in the limit for $\alpha < 0.50$ and sample size one. It is assumed that α is always less than or equal to 0.50. But when $\alpha = 0.50$, the linear interpolation is meaningful and apparently there is little or no bias (see Table 7).

As the sample size increases, the error in accuracy for estimator III becomes smaller, indicating that either the symmetry of the density $dH(y \mid x)/dy$ or the decrease in the size of the variance of $\overline{Y}$ affects the convergence properties of III to Θ.

Consider the function $p_k(x)$, which defines the probability that the direction of the next step from x will not be in the direction of θ (see lower part of Fig. 1).

That is

$$p_k(x) = \begin{cases} s(x) & x \neq \theta \\ 1 & x = \theta \end{cases}$$

where

$$0 \leq s(x) \leq \max\left[\int_{\alpha}^{1} dG(\overline{y} \mid x), \int_{0}^{\alpha} dG(\overline{y} \mid x)\right]$$

which in the limit as k increases without bound becomes $p_k(x) = 0$. This is sufficient for the estimator $x_{j+1} = x_j + c_j(\alpha - \overline{y}_j)$ to converge in the limit to θ as k tends to ∞, and j tends to ∞.

The results support the following rules: <u>For small sample sizes and a large number of trials, avoid using estimator III. For sample sizes larger than five and a small number of trials, estimator III gives greater accuracy.</u>

The direct relationship between small error in accuracy and large sample sizes poses a problem of efficiency of estimators, that is, the resolving of the problem of whether larger samples with a small number of trials is more desirable than unit sample sizes with a large number of trials. The solution depends on the nature of the test and must be solved for the specific test, hence, will not be considered here.

Increasing the sample size sequentially by increments of five (see Table 8) does not, in the cases studied, increase significantly the accuracy of the estimators, especially when the comparisons are based on sample sizes larger than one. This method can be used when sample sizes are not restrictive and relatively good accuracy is important. However, it was observed that sample sizes will in some cases exceed one hundred experimental units at the forty-ninth trial. The accuracy of estimator III is increased perhaps the most from such a scheme. It is important to note that increasing the sample size has little or no effect on the rate of convergence of estimators I and II.

The main results of this study are that estimator I, although of historical and theoretical importance, appears impractical for purposes of application, and the choice of using II or III depends upon the conditions surrounding the tests and must be determined for each test.

Appendix

PROPERTIES OF THE SEQUENCE $[c_j]$ ASSOCIATED WITH ESTIMATOR III

Let $H(y \mid x)$ be a family of distribution functions depending on the real parameter x, and let

$$M(x) = \int_{-\infty}^{+\infty} y\,dH(y \mid x) \tag{9}$$

be the corresponding regression function. It is assumed that $M(x)$ is unknown to the experimenter, who is, however, allowed to take observations on $H(y \mid x)$ for any value of x.

The recursive formula

$$x_{j+1} = x_j + c_j(\alpha - y_j) \tag{10}$$

defines a sequence $[x_j]$ which in the limit would be desirable to converge with probability one to θ, which is a root of the equation

$$M(x) = \alpha \tag{11}$$

The value c_j is an element of a sequence defined by the following rule:

$$\begin{aligned} c_1 &= a_1 \\ c_2 &= a_2 \end{aligned} \tag{12}$$

If $c_{j-1} = a_k$ for $k \geq 2$, then

$$c_j = \begin{cases} a_k & \text{when } \alpha \notin (y_j,\ y_{j-1}) \\ (x_{j+1} - x_j)/(y_{j+1} - y_j) & \text{when } \alpha \in (y_j,\ y_{j-1}) \end{cases}$$

$$c_{j+1} = \begin{cases} a_k & \text{when } c_j = a_k \text{ and } \alpha \notin (y_{j+1}, y_j) \\ (x_{j+1} - x_j)/(y_{j+1} - y_j) & \text{when } \alpha \in (y_{j+1}, y_j) \\ a_{k+1} & \text{when } \alpha \notin (y_{j+1}, y_j) \text{ and} \\ & c_j = (x_j - x_{j-1})/(y_j - y_{j-1}) \end{cases}$$

When a_k is an element of a sequence, $[a_k]$, having the following properties:

(a) $a_k > 0$ for $k = 1, 2, 3, \ldots$

(b) $a_k > a_{k+1}$ for $k = 1, 2, 3, \ldots$

(c) $\sum_1^\infty a_j = \infty$

(d) $\sum_1^\infty a_j^2 < \infty$

It is assumed that $M(x)$ is a continuous function and $H(y \mid x)$ is such that

$$\Pr[Y > \alpha \mid x < \theta] < \Pr[Y > \alpha \mid x = \theta]$$

and $$\Pr[Y > \alpha \mid x > \theta] > \Pr[Y > \alpha \mid x = \theta]$$

These conditions and the restrictions listed below are the only restrictions placed on $M(x)$ and $H(y \mid x)$.

(a) $|M(x)| \leq c + |d|x$ c and d are real constants

(b) $\int_{-\infty}^{\infty} |y - M(x)|^2 dH(y \mid x) \leq \sigma^2 < \infty$

(c) $M(x) < \alpha$ for $x < \theta$, $M(x) > \alpha$ for $x > \theta$

(d) $$\inf_{\delta_1 \leq |x-\theta| \leq \delta_2} |M(x) - \alpha| > 0$$

for every pair of numbers (δ_1, δ_2) with $0 < \delta_1 < \delta_2 < \infty$

The properties of the sequence $[c_j]$ will be presented in the form of seven lemmas and a single theorem.

Lemma 1

If the elements c_k and c_{k-1} of the sequence $[c_j]$ are such that $c_{k-1} \epsilon [a_j]$ and $c_k = (x_k - x_{k-1})/(y_k - y_{k-1})$, then $0 < c_k < c_{k-1}$.

Proof. Since $c_{k-1} \epsilon [a_j]$, $c_{k-1} > 0$. If $y_k < \alpha < y_{k-1}$, then $x_k < x_{k-1}$. Similarly, if $y_{k-1} < \alpha < y_k$, then $x_{k-1} < x_k$. It follows immediately that $c_k = (x_k - x_{k-1})/(y_k - y_{k-1}) > 0$.

It remains to be proved that $c_k < c_{k-1}$. Since $x_k = x_{k-1} + c_{k-1}(\alpha - y_{k-1})$, we can write $c_k = c_{k-1}(\alpha - y_{k-1})/(y_k - y_{k-1})$. Noting that both $y_k < \alpha < y_{k-1}$ and $y_{k-1} < \alpha < y_k$ imply that $0 < (\alpha - y_{k-1})/(y_k - y_{k-1}) < 1$, it can be concluded that $c_k < c_{k-1}$. It should be noted that if $x_k < x_{k-1}$, then $y_{k-1} < y_k$ cannot be true. This follows immediately from the recursive formula, Eq. 10.

Lemma 2

For every k such that $c_k = (x_k - x_{k-1})/(y_k - y_{k-1})$ and $c_{k+1} = (x_{k+1} - x_k)/(y_{k+1} - y_k)$, $c_{k+1} < c_k$.

Proof. From the proof of Lemma 1, we know that $c_{k+1} = c_k(\alpha - y_k)/(y_{k+1} - y_k)$, since $c_k > 0$ and $0 < (\alpha - y_k)/(y_{k+1} - y_k) < 1$, it follows that $c_{k+1} < c_k$.

It should be noted that in general c_{j+1} is not less than c_j for all $j = 1, 2, \ldots$.

Lemma 3

For each k and J the probability that $c_j = a_k$ for all $j \geq J$ is zero.

Proof. Let $c_j = a_k$ for all $j \geq J$, where $j = 1, 2, \ldots$. The sequence $[x_j]$ is monotone which converges to a finite value, say A, if the sequence is bounded, and diverges to either $-\infty$ or $+\infty$ if unbounded.

Let $[x_j]$ be non-increasing and bounded below by its limit A. Then for each $j > J$ there exists an $e_j > 0$ such that $x_j = A + e_j a_k$. The sequence $[e_j]$ is a non-increasing sequence of positive elements such that $\lim_{j\to\infty} e_j = 0$.
Clearly then,

(13) $$0 \leq x_{j+1} - A \leq e_j a_k$$

Simplifying,

$$0 \leq x_j - A + a_k(\alpha - y_j) \leq e_j a_k$$

$$0 \leq e_j a_k + a_k(\alpha - y_j) \leq e_j a_k$$

$$0 \leq e_j + (\alpha - y_j) \leq e_j$$

$$\alpha \leq y_j \leq \alpha + e_j$$

Let us now consider the probability of such an event, that is, $Pr[\alpha \leq Y_j \leq \alpha + e_j]$. If $H(y \mid x)$ is continuous, then, as $j \to \infty$ and $e_j \to 0$, $Pr[\alpha \leq Y_j \leq \alpha + e_j] \to 0$. However, if $H(y \mid x)$ is discrete, $Pr[\alpha \leq Y_j \leq \alpha + e_j]$ may not necessarily converge to zero as $e_j \to 0$. But $[\alpha \leq Y_j < \alpha + e_j]$ must hold for all j greater than that one for which the inequality 13 holds. Clearly, as e_j tends to zero, the probability of such an event is

$$\prod_1^\infty \Pr[Y_j = \alpha] \leq \prod_1^\infty \left\{ \max_{A \leq x \leq x_1} (\Pr[Y_j = \alpha]) \right\} = 0$$

A similar argument holds when the sequence $[x_j]$ is non-decreasing and bounded.

Suppose the sequence $[x_j]$ is unbounded, then either $\lim_{j\to\infty} x_j = \infty$ or $\lim_{j\to\infty} x_j = -\infty$. In order for these events to occur, $y_j < \alpha$ or $y_j > \alpha$ for all $j > J$, respectively. Let us investigate the probability of such events, that is, $\Pr[Y_j > \alpha, Y_{j+1} > \alpha, \ldots]$ $= \Pr[\lim_{j\to\infty} x_j = -\infty]$ and $\Pr[Y_j < \alpha, Y_{j+1} < \alpha, \ldots] = \Pr[\lim_{j\to\infty} x_j = +\infty]$ Consider the latter of the two cases.

$$\begin{aligned}\Pr[Y_j < \alpha, Y_{j+1} < \alpha, \ldots] &= \Pr[Y_j < \alpha] \Pr[Y_{j+1} < \alpha \mid Y_j < \alpha] \ldots, \\ &\qquad \Pr[Y_{j+L} < \alpha \mid Y_j < \alpha, \ldots, Y_{j+L-1} < \alpha] \\ &= \prod_L^\infty \Pr[Y_{j+L} < \alpha]\end{aligned}$$

There exists only a finite number of L such that $x < \theta$. It follows then that

$$\begin{aligned}\Pr[Y_j < \alpha, Y_{j+1} < \alpha, \ldots] &\leq \prod_1^\infty \Pr[Y_j < \alpha \mid x_j > \theta] \\ &\leq \prod_1^\infty \Pr[Y < \alpha \mid x = \theta] \\ &= 0\end{aligned}$$

A similar argument holds when $\lim_{j\to\infty} x_j = -\infty$, and the lemma is proved.

Lemma 4

If $c_j = (x_j - x_{j-1})/(y_j - y_{j-1})$ for all $j > J$, then $\lim_{j\to\infty} (x_j - x_{j-1}) = 0$ almost surely is true for all c_j.

<u>Proof</u>. Suppose $x_{2j} > x_{2j-1}$ and $\alpha < y_{2j}$. In order that c_j have the form restricted by the hypothesis of the lemma, $y_{2j-1} < \alpha < y_{2j}$ for all $j \geq J$. The sequences $[x_{2j-1}]$ and $[x_{2j}]$ are monotone; the first is increasing, the second is decreasing. Since x_{j+1} is obtained by a linear interpolation between x_j and x_{j-1}, both sequences are bounded above and below. Let $\lim_{j\to\infty} x_{2j-1} = A$ and $\lim_{j\to\infty} x_{2j} = B$. Let $B - A = \Delta$, where $\Delta \geq 0$. Then for every $j > J$, there exists an $e_{2j-1} > 0$ such that $x_{2j-1} = A - e_{2j-1}$. The sequence $[e_{2j-1}]$ is monotonically decreasing and converges to zero. With each j there exists an e_{2j} such that $x_{2j} = B + e_{2j}$. The sequence $[e_{2j}]$ is monotonically decreasing and converges to zero as j increases without bound. Consider

$$(14) \qquad x_{2j+1} = x_{2j} + [(x_{2j} - x_{2j-1})/(y_{2j} - y_{2j-1})](\alpha - y_{2j})$$

$$= B + e_{2j} + (B + e_{2j} - A + e_{2j-1}) \cdot [(\alpha - y_{2j})/(y_{2j} - y_{2j-1})]$$

$$= B + \Delta[(\alpha - y_{2j})/(y_{2j} - y_{2j-1})] + e_{2j} + (e_{2j} + e_{2j-1})[(\alpha - y_{2j})/(y_{2j} - y_{2j-1})]$$

Taking the limit of both sides,

$$\lim_{j\to\infty} x_{2j+1} = B - \Delta[\lim_{j\to\infty} (y_{2j} - \alpha)/(y_{2j} - y_{2j-1})]$$

it is clear that

$$\Pr[\lim_{j\to\infty} (Y_{2j} - \alpha)/(Y_{2j} - Y_{2j-1}) = D] = 0 \quad \text{for any } D$$

Since the left side of Eq. 14 converges and $[\lim_{j\to\infty} (Y_{2j} - \alpha)/(Y_{2j} - Y_{2j-1})]$ almost surely does not exist, $\Delta = 0$, that is $A = B$. It follows immediately, then, that for almost all c_j $\lim_{j\to\infty} (x_j - x_{j-1}) = 0$, the desired result.

Lemma 5

Let the sequence $[z_k]$ be the union of all subsequences of $[Z_k]$ such that $\lim_{k\to\infty} z_k = \infty$, where Z_k is the number of times that the kth element of $[a_k]$ appears in the sequence $[c_j]$. Then, $\Pr[\lim_{k\to\infty} z_k = \infty] = 0$.

Proof. From Lemma 3, we know that for each k, Z_k is almost always finite. Since the sum of a denumerable number of sets of measure zero is also of measure zero, we can conclude that the probability of at least one element of the sequence of infinite terms in $[Z_k]$ being infinite is also zero. This still does not assure us that the sequence $[z_k]$ is almost always bounded.

Let $\lim_{k\to\infty} z_k = \infty$. Then for each $L > 0$, there must exist a k such that $z_k > L$. Consider the probability of such an event, that is,

$$\Pr[z_k > L] = \Pr[Y_1 > \alpha, \ldots, Y_L > \alpha]$$

or

$$\Pr[z_k > L] = \Pr[Y_1 < \alpha, \ldots, Y_L < \alpha]$$

But, from the proof of Lemma 3, we know

$$\lim_{L\to\infty} \Pr[Y_1 > \alpha, \ldots, Y_L > \alpha] = 0$$

or

$$\lim_{L\to\infty} \Pr[Y_1 < \alpha, \ldots, Y_L < \alpha] = 0$$

Hence, we can conclude

$$\Pr[\lim_{L\to\infty} z_k = \infty] = 0$$

That is, the sequence $[z_k]$ is almost surely a bounded sequence.

Lemma 6

For every J, the probability that $c_j = (x_j - x_{j-1})/(y_j - y_{j-1})$ for all $j \geq J$ is zero.

Proof. Suppose each element of the sequence $[c_j]$ takes on the form defined by the hypothesis of the lemma. Then the sequences $[x_{2j}]$ and $[x_{2j-1}]$ are monotonically decreasing and increasing sequences, respectively, when $y_{2j-1} < \alpha < y_{2j}$ for all $j \geq J$. Similarly, the sequences are monotonically increasing and decreasing, respectively, if $y_{2j} < \alpha < y_{2j-1}$ for all $j \geq J$.

By Lemma 4, we know that both these sequences converge to a common limit, A. Consider a neighborhood of A, say $\nu(A)$, such that at least one of the following probabilities is less than unity for all $x \in \nu(A)$: $\Pr[Y_{2j} > \alpha \mid x \in \nu(A)]$ and $\Pr[x_{2j-1} < \alpha \mid x \in \nu(A)]$. The existence of $\nu(A)$ is assured by the continuity of $M(x)$. Suppose that at least one of the probabilities above is identically equal to unity, or at least in the limit equal to unity as $j \to \infty$ and $x \to A$. It is assumed that the variance of the random variable Y is finite for all values of x and that $M(x)$ is continuous. Then if

$$\lim_{\substack{x\to A \\ j\to\infty}} \Pr[Y_{2j} > \alpha \mid x_{2j} \in \nu(A)] = 1$$

this must imply

$$\lim_{\substack{x\to A \\ j\to\infty}} \Pr[Y_{2j-1} < \alpha \mid x_{2j-1} \in \nu(A)] = 0$$

and vice versa.

Let there be a J such that $c_j = (x_j - x_{j-1})/(y_j - y_{j-1})$ for all $j \geq J$. Consider the probability of such an event, that is,

$$\Pr[Y_1 < \alpha, \ldots, Y_{2j-1} < \alpha, \ldots] \Pr[Y_2 > \alpha, \ldots, Y_{2j} > \alpha, \ldots]$$
$$\leq \lim_{j\to\infty}(\max_{x_j \epsilon \nu} \Pr[Y_{2j-1} < \alpha])^j \lim_{j\to\infty}(\max_{x_j \epsilon \nu} \Pr[Y_{2j} > \alpha])^j$$

This is true since in $\nu(A)$ either max $\Pr[Y_{2j-1} < \alpha]$ or max $\Pr[Y_{2j} > \alpha]$ must be less than unity. Therefore, at least one of the limits will be identically zero.

Lemma 7

Let the sequence $[z_k]$ be the union of all subsequences of $[Z_k]$ such that $\lim_{k\to\infty} z_k = \infty$ where Z_k is the number of elements of the sequence $[c_j]$ having the form $(x_j - x_{j-1})/(y_j - y_{j-1})$ which lie between any two successive members of the sequence $[a_j]$. Then $\Pr[\lim_{k\to\infty} z_k = \infty] = 0$.

Proof. Let $\lim_{j\to\infty} z_j = \infty$, then for each $2L > 0$ there exists a j such that $z_j > 2L$. Let us now consider the probability of such an event, that is, $\Pr[Y_1 < \alpha, Y_2 > \alpha, \ldots, Y_{2k-1} < \alpha, Y_{2k} > \alpha, \ldots, Y_{2L-1} < \alpha, Y_{2L} > \alpha]$. But, from the proof of Lemma 3, we know that $\lim_{L\to\infty} \Pr[Y_1 < \alpha, \ldots, Y_{2L-1}] \Pr[Y_2 > \alpha, \ldots, Y_{2L}] = 0$. It follows then that $\Pr[\lim_{j\to\infty} z_j = \infty] = 0$.

Theorem 1

Any given sequence $[c_j]$ is almost surely a member of the class of sequences $[b_j]$ where $[b_j]$ is defined by the following properties:

(a) $b_j > 0$ for all j

(b) $\sum_1^\infty b_j = \infty$

(c) $\sum_1^\infty b_j^2 < \infty$

Proof. Consider any sequence $[c_j]$ as defined in rule 12. By Lemma 1, each element of the sequence is necessarily positive. Condition (a) is satisfied.

Lemma 3, Lemma 6, and Lemma 7 assure us that every element of the sequence $[a_j]$ is almost surely contained in $[c_j]$. Therefore, since $c_j > 0$ for all j,

$$\sum_1^\infty c_j \geq \sum_1^\infty a_j, \text{ but } \sum_1^\infty a_j = \infty, \text{ then } \sum_1^\infty c_j = \infty$$

Hence, condition (b) is satisfied.

In order to show that the sequence $[c_j]$ satisfies condition (c), consider the following infinite sum:

$$\sum_1^\infty c_j^2 = a_1^2 + a_2^2 + \ldots + a_2^2 + c_{11}^2 + c_{12}^2 + \ldots + c_{1M_1} + a_3^2$$
$$+ \ldots + a_3^2 + c_{21}^2 + \ldots + c_{2M_2}^2, \text{ etc.}$$

Where a_1 occurs once, a_2 occurs k_2 times, a_3 occurs k_3 times, etc. By Lemma 5, the sequence $[k_j]$ is almost surely bounded. By Lemma 7, the sequence $[M_j]$ is almost surely bounded. Let $k = \max_j k_j$ and $M_j = \max_j M_j$.

If the sum $\sum_1^\infty c_j^2$ is convergent, it is absolutely convergent. The rearrangement of terms will not affect the convergence or the sum. Hence,

$$\sum_1^\infty c_j^2 \leq k \sum_1^\infty a_j^2 + M \sum_1^\infty a_j^2 = (k + M) \sum_1^\infty a_j^2 < \infty$$

which is the desired result, condition (c).

What is unusual about the theorem is that the conditions

(a) $\qquad c_j^2 > 0$

(b) $$\sum_{1}^{\infty} c_j^2 < \infty$$

(c) $$\sum_{1}^{\infty} c_j = \infty$$

are identical to those required by Blum (Ref. 4) in his theorem which proves that the limit point of the sequence $[x_j]$ is θ with probability one for estimator I. The theorem can be stated as follows: Let M(x) be the regression function corresponding to the family $H(y \mid x)$. Assume that M(x) is a Lebesque-measurable function satisfying

(a) $$|M(x)| \leq c + d\,|x|$$

(b) $$\int_{-\infty}^{\infty} |y - M(x)|^2 \, dH(y \mid x) \leq \sigma^2 < \infty$$

(c) $$M(x) < \alpha \text{ for } x < \theta, \; M(x) > \alpha \text{ for } x > \theta$$

(d) $$\inf_{\delta_1 \leq |x-\theta| \leq \delta_2} |M(x) - \alpha| > 0$$

for every pair of numbers (δ_1, δ_2) with $0 < \delta_1 < \delta_2 < \infty$

Let $[b_j]$ be a sequence of positive numbers such that

(e) $$\sum_{1}^{\infty} b_j = \infty$$

(f) $$\sum_{1}^{\infty} b_j^2 < \infty$$

Let x_1 be an arbitrary number. Define a sequence of random variables recursively by

(g) $$x_{j+1} = x_j + b_j(\alpha - y_j)$$

where Y_j is a random variable distributed according to $H(y \mid x)$. Then x_j converges to θ with probability one.

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An Empirical Study of Three Stochastic Approximation Techniques Applicable to Sensitivity Testing (U), by Patrick L. Odell. Albuquerque, NWEF, 15 August 1961, 47 pp. (NAVWEPS Report 7837), UNCLASSIFIED.

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