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A NOTE ON SEMIDEFINITE MATRICES

by

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ABSTRACT

It is of general interest to find criteria for a matrix to be positive (or negative)-semidefinite. The usual characterization of semidefinite matrices in terms of their principal minors can be rather laborious to implement practically. We present here an elementary proof of a known alternate characterization of a semidefinite matrix in terms of its null-space and of its largest characteristic value. An iterative procedure is also suggested which may be useful in deciding the semidefiniteness of a matrix.
A NOTE ON SEMIDEFINITE MATRICES

In what follows $A$ will always represent a real, symmetric, $n \times n$ matrix. If, for each $x \in \mathbb{R}^n$ (*) it is true that $\langle xA \rangle^T \geq 0$ (**) then we say that $A$ is positive-semidefinite, denoted: p.s.d.; if $\langle xA \rangle^T(yAy^T) > 0$ for all $x, y \in \mathbb{R}^n$ we say that $A$ is semidefinite, denoted s.d. We first prove the following:

THEOREM 1. The following are equivalent:

(i) $A$ is s.d.
(ii) $\langle xAy^T \rangle^2 \leq \langle xAy^T \rangle(yAy^T)$, all $x, y \in \mathbb{R}^n$
(iii) $x \in \mathbb{R}^n$, $xA = 0 \Rightarrow xA^2 = 0$
(iv) $x \in \mathbb{R}^n$, $xA^2 = 1 \Rightarrow (xA^T)^2 > 0$
(v) $x \in \mathbb{R}^n$, $xA = 0 \Rightarrow xA = 0$

PROOF: We show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii)

Suppose $A$ is s.d., let $x, y \in \mathbb{R}^n$. Consider the real quadratic polynomial $p$ defined by:

$$p(\lambda) = (x + \lambda y)A(x + \lambda y)^T = xA^T + 2\lambda xAy^T + \lambda^2 yAy^T.$$

Since $A$ is s.d., $p$ does not change sign, i.e., its discriminant is non-positive, whence:

$$4(xAy^T)^2 - 4\langle xAy^T \rangle(yAy^T) \leq 0,$$

(*) $\mathbb{R}^n = \{x \mid x = (x_1, \ldots, x_n) and x_i is a real number for i = 1, \ldots, n\}$.

(**) If $x \in \mathbb{R}^n$, $x^T$ denotes the transpose of $x$. 
giving the desired result.

(ii) $\Rightarrow$ (iii).

Suppose $x \in \mathbb{R}^n$ and $xAx^T = 0$, then, from (ii), $(xAy)^T \leq 0$, i.e., $xAy^T = 0$, for all $y \in \mathbb{R}^n$. Thus $xA = 0$, but $xA^2 x^T = (xA)(xA)^T = 0$.

(iii) $\Rightarrow$ (iv).

If $x \in \mathbb{R}^n$ and $(xAx^T)^2 \leq 0$ then $xAx^T = 0$ and, by (iii), $xA^2 x^T = 0$, contradicting $xA^2 x^T = 1$.

(iv) $\Rightarrow$ (v).

If $x \in \mathbb{R}^n$ and $xAx^T = 0$ then, by (iv), $xA^2 x^T \leq 0$ (because if $xA^2 x^T > 0$ then we could normalize $x$ to get $xA^2 x^T = 1$, $xAx^T = 0$). However, $xA^2 x^T = (xA)(xA)^T$, and thus $xA^2 x^T \geq 0$ with equality holding if and only if $xA = 0$.

(v) $\Rightarrow$ (i).

Suppose (i) is false, i.e., there exist $x, y \in \mathbb{R}^n$ such that $xAx^T > 0$, $yAy^T < 0$. By suitable normalization [dividing $x$ by $(xAx)^{1/2}$ and $y$ by $(-yAy)^{1/2}$], we may assume that $xAx^T = 1$, $yAy^T = -1$. Now let:

1. $\lambda = -xAy^T + \left[1 + (xAy)^T\right]^{1/2}$
2. $z = \lambda x + y$

We claim that $zA \neq 0$ and $zAz^T = 0$, thus contradicting (v). First, if $zA = 0$ then multiplying (2) by $Ax^T$ and $Ay^T$ we get:

$0 = \lambda xAx^T + yAx^T = \lambda + xAy^T$

$0 = \lambda xAy^T + yAy^T = \lambda xAy^T - 1$.

Combining the last two equations:

$0 = \lambda xAy^T - 1 = (-xAy^T)(xAy^T) - 1$

$= -1 - (xAy^T)^2$.
a contradiction, thus \( zA \neq 0 \). However,

\[
zA z^T = (\lambda x + y) A (\lambda x + y)^T = \\
= \lambda^2 x A x^T + 2\lambda x A y^T + y A y^T \\
= \lambda^2 + 2\lambda x A y^T - 1,
\]

and \( \lambda \) was chosen to be precisely one of the two (real) roots of the preceding quadratic polynomial in \( \lambda \).

q. e. d.

Several comments are in order. Obviously, \( A \) is s.d. if and only if \( -A \) is p. s. d. Condition (ii) of Theorem 1 is a generalization of the Cauchy-Schwartz inequality, namely:

\[
(3) \quad (uv^T)^2 \leq (uu^T)(vv^T) \quad \text{all } u, v \in \mathbb{R}^n,
\]

for if we take \( A \) to be the \( nxn \) identity matrix which is clearly p. s. d., we obtain (3) from (ii) - Theorem 1. Condition (v) - Theorem 1, or its obvious equivalents (iii) and (iv), states that if we consider \( xA \), the image under the linear transformation \( A \) of a point \( x \) in \( \mathbb{R}^n \), then \( A \) cannot be perpendicular to \( x \) unless \( x \) is in the null-space of \( A \). Alternately, (v) - Theorem 1 states that if \( x \) is not in the null-space of \( A \) then its image under \( A \) cannot be perpendicular to \( x \).

We proceed next to obtain results which are, in a sense, "refinements" of conditions (ii) (see Lemma 1 below) and (iv) (see Theorem 2) of Theorem 1. Lemma 1 is a generalization of the well known fact, associated with the Cauchy-Schwartz inequality, stating that equality holds in (3) if and only if \( u, v \) are linearly dependent. We shall apply Lemma 1 in the proof of Theorem 3.
LEMMA 1
Let $A$ be s.d.. If $x, y \in \mathbb{R}^n$ then $(xAy)^2 = (xAx)(yAy)$ if and only if $x_A, y_A$ are linearly dependent.

PROOF: If, say, $x_A = \lambda y_A$, where $\lambda$ is a real number, then $xAy = \lambda yAy_T$ while $xAx = \lambda yAx = \lambda xAy = \lambda^2 yAy_T$. Whence it follows that $(xAy)^2 = \lambda^2 (yAy_T)^2 = (xAx)(yAy).

On the other hand, suppose $(xAy)^2 = (xAx)(yAy)$. If $xAx = 0$ or $yAy_T = 0$ then, by (v) - Theorem 1, $x_A = 0$ or $y_A = 0$ and we certainly can conclude that $x_A, y_A$ are linearly dependent. Otherwise, say, $xAx > 0$ and $yAy_T > 0$, consequently $xAy \neq 0$. Let $\rho = \text{signum}(xAy)$ and let:

$$a = (yAy_T)^{1/2}$$
$$\beta = -\rho(xAx_T)^{1/2},$$

then $\alpha, \beta \neq 0$ and:

$$(\alpha + \beta y)A(\alpha + \beta y)^T = \alpha^2 xAx_T + \beta^2 yAy_T + 2\alpha\beta xAy_T =$$
$$= 2(xAx_T)(yAy_T) - 2\rho(xAy_T)(xAx_T)^{1/2}(yAy_T)^{1/2} =$$
$$= 2(xAx_T)(yAy_T) - 2\|xAy_T\|^{1/2}(xAx_T)^{1/2}(yAy_T)^{1/2} =$$
$$= 2(xAx_T)(yAy_T) - 2(xAx_T)(yAy_T) = 0.$$

Thus, $(\alpha + \beta y)A(\alpha + \beta y)^T = 0$ and, by (v) - Theorem 1, $0 = (\alpha + \beta y)A = \alpha x_A + \beta y_A$.

The preceding lemma was motivated, in part, by an examination of (ii) - Theorem 1 in case $A$ is the identity matrix, in that case (since the square of the identity is the identity), (iv) - Theorem 1 states: $x \in \mathbb{R}^n$, $xx_T = 1$ implies $(xx_T)^2 > 0$, which is, of course, true. We notice, though, that $(xAx_T)^2$ has then a positive lower bound, namely 1. In general, this
will be the case, i.e., a positive lower bound will exist for \((xAx^T)^2\) in 
(iv) - Theorem 1, whenever \(A\) is s.d. Clearly, when \(A\) is identically 
zero any positive number will serve as a lower bound, because there is no 
\(x \in \mathbb{R}^n\) for which \(xA^2x^T = 1\), thus we will exclude \(A = 0\) in the next theorem:

**Theorem 2**

Suppose \(A\) is p.s.d. and \(A \neq 0\), then there exist a positive real number \(\mu\) and an \(x_0 \in \mathbb{R}^n\) such that:

\[
\begin{align*}
(4) & \quad x \in \mathbb{R}^n, \ xA^2x^T = 1 \Rightarrow xAx^T \geq \mu \\
(5) & \quad x_0A^2x_0^T = 1 \quad \text{and} \quad x_0Ax_0^T = \mu.
\end{align*}
\]

**Proof:** Let

\[
X = \left\{ x \mid x \in \mathbb{R}^n \quad \text{and} \quad xA^2x^T = 1 \right\}
\]

\[
\mu = \inf_{x \in X} xAx^T.
\]

Since \(A\) is p.s.d. and \(A \neq 0\), \(\mu\) is well defined and in fact \(\mu \geq 0\) and satisfies (4). By definition of \(\mu\), there exists a sequence \(x_k\) such that

\[
\begin{align*}
(6) & \quad x_k \in X \quad \text{for} \quad k = 1, 2, \ldots \\
(7) & \quad x_kAx_k^T \quad \text{converges to} \quad \mu.
\end{align*}
\]

We consider two cases:

**Case 1.** The sequence \(x_k\) has a bounded subsequence. In this eventuality the \(x_k\) have a point of accumulation \(x_0\), for which it must be true (by (6) and (7) and because \(X\) is closed) that \(x_0 \in X\) and \(x_0Ax_0^T = \mu\). Thus \(x_0\) satisfies (5). That \(\mu\) is positive then follows from (v) - Theorem 1. The two preceding facts, together with the remark above that \(\mu\) satisfies 4, complete the proof.
Case 2. The sequence \( \{x_k\} \) has no bounded subsequence, i.e., we may assume that \( |x_k| = (x_k^T x_k)^{1/2} \to \infty \) and \( |x_k| > 0, \ k = 1, 2, \ldots \). We define another sequence \( \{y_k\} \) by:

\[
y_k = \frac{x_k}{|x_k|}.
\]

Now, \( y_k A y_k^T \) converge to zero, because \( x_k A x_k^T \) converge to \( \mu \) and also \( y_k A^2 y_k^T \) converge to zero, because \( x_k A^2 x_k = 1 \) all \( k \). However, \( |y_k| = 1 \), thus the \( y_k \)'s have an accumulation point \( y \), for which it must be true that \( y A y^T = 0 \). Thus \( y A = 0 \) by (v) - Theorem 1.

Next we observe that from the definition of \( y \) and the \( y_k \)'s it follows that whenever \( y \) has a non-zero component then infinitely many \( x_k \)'s have the same component non-zero, and in fact of the same sign. We may assume that an appropriate subsequence of \( x_k \) has been selected so that whenever \( y \) has a positive (negative) component then all the \( x_k \)'s have the same component positive (negative). Now, if \( \{\lambda_k\} \) is any sequence of real numbers then:

\[
(x_k + \lambda_k y) A (x_k + \lambda_k y)^T = x_k A x_k^T
\]

and

\[
(x_k + \lambda_k y) A^2 (x_k + \lambda_k y)^T = x_k A^2 x_k^T,
\]

because \( y A = 0 \). We can thus replace \( x_k \) by \( x_k + \lambda_k y \), \( k = 1, 2, \ldots \), and (6) and (7) will still hold. However, by an appropriate choice of \( \lambda_k \) we can reduce the number of non-zero components in each of the \( x_k \)'s, eventually (repeating the above process, if necessary) we obtain a sequence \( \{x_k\} \), satisfying (6)-(7) and which has an accumulation point, thus reducing it to case 1. q.e.d.

-6-
As an immediate consequence of Theorem 2 we can "strengthen" (iv) - Theorem 1.

Corollary
If $A$ is s.d. and $A \neq 0$ then

$$\text{minimum } \left\{ \left( xA^T x \right)^2 \mid x \in \mathbb{R}^n \text{ and } x^T A^2 x = 1 \right\}$$

exists and is positive.

PROOF: As noted before, if $A$ is s.d., then either $A$ is p.s.d. or $-A$ is p.s.d., in either case the square of the $\mu$ in Theorem 2 is the required minimum and the $x_0$ of the same theorem is the required minimizing $x$.

The $\mu$ and $x_0$ of Theorem 2 are, as one might expect, intimately related to the characteristic values of $A$. This is brought forth in the next theorem.

**THEOREM 3**
Let $A$ be p.s.d., $A \neq 0$. Let $\mu$ and $x_0$ be as in Theorem 2 and let $\lambda_n$ be the largest characteristic value of $A$, then $\lambda_n = \mu^{-1}$ and $x_0 A$ is a characteristic vector of $A$ corresponding to $\lambda_n$.

PROOF: Suppose $\lambda$ is any characteristic value of $A$, i.e., there exists an $x \in \mathbb{R}^n$, $x \neq 0$, such that $xA = \lambda x$, whence $x^T A^2 x = \lambda x^T A x$. If $\lambda = 0$ then certainly $\lambda \leq \mu^{-1}$. Assuming $\lambda \neq 0$, it follows that $xA \neq 0$ (because $x \neq 0$) and thus, by (v) - Theorem 1, $x A^T > 0$. Let $y = (x^T A^2 x)^{-1/2} x$, then $y^T A^2 y = 1$ and, by definition of $\mu$, $y A y^T \geq \mu$. However, $y A y^T = (x A x^T)^{-1} (x A x^T) = = \lambda^{-1}$, thus $\lambda \leq \mu^{-1}$. We have just demonstrated that $\lambda \leq \mu^{-1}$ for any characteristic value $\lambda$ of $A$, thus $\lambda_n \leq \mu^{-1}$.
To complete the proof of this theorem it will suffice to show that there is a characteristic value $\lambda$ of $A$ such that $\lambda = \mu^{-1}$, and $(x_0 A) A = \lambda (x_0 A)$, $x_0$ being as in Theorem 2. Let $x = x_0$ be a minimizing $x_0$ in question.

Since $A$ and $A^2$ are p. s. d. (the square of any real symmetric matrix is p. s. d.), and $x A \neq 0$ ($x A x^T = x_0 A x_0^T = \mu > 0$), it follows that $x A^3 x^T = (x A) A (x A)^T > 0$, and $x A^4 x = (x A) A^2 (x A)^T > 0$. Thus, if we define

$$\rho = 2(x A^3 x^T)(x A^4 x^T)^{-1}$$

then $\rho$ is positive. Next let

$$y = x - \rho x A$$

The motivation for the above definition of $y$ is as follows: we know $x$ minimizes a certain function, namely $x A x$, since we wish to derive from this fact some properties of $x$ we examine how $x A x$ will change in the direction of its gradient, namely $2x A$. As defined in (9), $y$ is a translation from $x$ precisely in the direction of that gradient, the particular value of $\rho$ chosen is designed to keep $y$ within the "feasibility" set, i.e., $y A^2 y = 1$.

We check next the last mentioned condition:

$$y A^2 y^T = (x - \rho x A) A^2 (x - \rho x A)^T =$$

$$= x A^2 x^T - 2 \rho x A^3 x^T + \rho^2 x A^4 x^T =$$

$$= 1 - 2 \rho \left[ x A^3 x^T - \frac{\rho}{2} (x A^4 x^T) \right]$$

$$= 1 - 2 \rho \left[ x A^3 x^T - (x A^3 x^T) (x A^4 x^T)^{-1} (x A^4 x^T) \right]$$

$$= 1.$$

One can, incidentally, readily check that the particular value of $\rho$, as given in (8), is the only value of $\rho$ (other than $\rho = 0$) which yields $y A^2 y = 1$. Now,
since \( yA^2 y^T = 1 \), we must have, by definition of \( \mu \),

\[
(10) \quad yAy^T - xAx^T \geq 0.
\]

However,

\[
yAy^T - xAx^T = (x - \rho xA)x(A - \rho xA)^T - xAx^T =
\]

\[
= -2\rho xA^2 x^T + \rho^2 xA^3 x^T =
\]

\[
= 2\rho \left[ \frac{\rho}{2} (xA^3 x^T) - (xA^2 x^T) \right] =
\]

\[
= 2\rho (xA^4 x^T)^{-1} \left[ (xA^3 x^T)^2 - (xA^2 x^T)(xA^4 x^T) \right].
\]

Thus, since \( \rho > 0 \), \((xA^4 x^T)^{-1} > 0 \) and because (10) holds, we have:

\[
(11) \quad (xA^3 x^T)^2 \geq (xA^2 x^T)(xA^4 x^T).
\]

We now refer to inequality (3), which is a special case of (ii) - Theorem 1 with \( A \) being the identity, letting \( u = xA \), \( v = xA^2 \) we get:

\[
(12) \quad (xA^3 x^T)^2 \leq (xA^2 x^T)(xA^4 x^T).
\]

Combining (11) and (12), we get:

\[
(13) \quad (xA^3 x^T)^2 = (xA^2 x^T)(xA^4 x^T).
\]

However, from Lemma 1, again with \( A \) being the identity matrix, we then know that \( xA, xA^2 \) are linearly dependent. Since \( xA \neq 0 \), it follows that there is a real number \( \lambda \) such that \( xA^2 = \lambda xA \), multiplying by \( x^T : 1 = xA^2 x^T = \lambda xAx^T \), and \( \lambda = (xAx^T)^{-1} = \mu^{-1} \). q.e.d.

As a final general result, we specialize (ii) - Theorem 1, and Lemma 1, for the case where \( A \) is non-singular.
THEOREM 4

Let $A$ be p. s. d. and non-singular then,

$$ (14) \quad (uv^T)^2 \preceq (uAu^T)(vA^{-1}v^T) \quad \text{all} \quad u, v \in \mathbb{R}^n $$

and equality holds above if and only if $u, vA^{-1}$ are dependent.

PROOF: We first note that $A^{-1}$ must be symmetric because $AA^{-1} = I$, thus $I^T = I = (AA^{-1})^T = (A^{-1})^T A^T = (A^{-1})^T A$. But the inverse is unique, thus $A^{-1} = (A^{-1})^T$. Next, let $u, v \in \mathbb{R}^n$, we let

$$ (15) \quad x = u, \quad y = vA^{-1}. $$

One readily checks that:

$$ xAy^T = uv^T, \quad xAx = uAu^T, \quad yAy = vA^{-1}v^T. $$

Thus the desired inequality (14) follows from (ii) - Theorem 1. Now if (14) is actually an equation, then from Lemma 1, using $x, y$ as defined in (15), we get $u, vA^{-1}$ are linearly dependent. The converse also follows readily.

q.e.d.

Note: The condition of equality in (14) is directly connected with characteristic vectors of $A$ (and of course, those of $A^{-1}$), for suppose (14) is an equation and $u = v \neq 0$, then one sees immediately that $uA = \lambda u$ for some real number $\lambda$. The corresponding converse also holds in this case.

An iterative scheme, for deciding the definiteness of $A$, based on the proof of Theorem 3 might go as follows:

(a) By examining the diagonal elements of $A$ we have decided that, if at all, $A$ is p. s. d.
(b) We have an \( x \) such that \( x^T A \neq 0 \); if \( x^T A x \leq 0 \) then \( A \) is not p. s. d., if \( x^T A x > 0 \) normalize \( x \) so that \( x^T A^2 x = 1 \) and proceed to (c)

(c) We have an \( x \) such that \( x^T A \neq 0 \), \( x^T A^2 x = 1 \); perform the transformation given by (8) and (9). There are three cases:

\textbf{Case 1.} if \( y^T A y > x^T A x \) then \( A \) is not p. s. d.

\textbf{Case 2.} if \( y^T A y < x^T A x \) return to beginning of (c), using \( y \) as the new "test" vector.

\textbf{Case 3.} if \( y^T A y = x^T A x \) we have isolated a characteristic vector of \( A \), return to (b) using, as \( x \), a vector independent of all characteristic vectors thus far obtained.

The preceding is, of course, "informal" in the sense that the iterative procedure described above has not been shown to converge.
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