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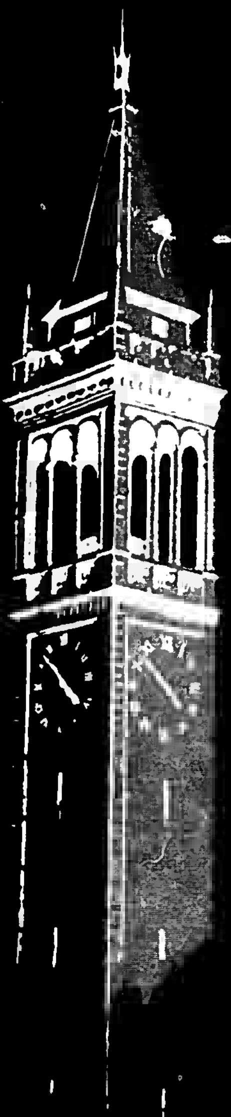

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SUMMARY

In this study a simplified analytic test of stability of linear discrete systems is obtained. This test also yields the necessary and sufficient conditions for a real polynomial in the variable z to have all its roots inside the unit circle in the z-plane. The new stability constraints require the evaluation of only half the number of Schür-Cohn determinants[1, 2]. It is shown that for the test of a fourth-order system only a third order determinant is required and for the fifth-order only two determinants are required. The test is applied directly in the z-plane and yields the minimum number of constraint terms. Stability constraints up to the fifth-order case are obtained and for the n^{th} order case are formulated. The simplicity of this criterion is equivalent to that of the Lienard-Chipart criterion[3] for the continuous case which has a decisive advantage over the Routh-Hurwitz criterion[4, 5].

INTRODUCTION

It is known that linear time-invariant discrete systems can be described by constant coefficient linear difference equations. These equations can be easily transformed into the function of the complex variable z by the z-transform method. One of the problems in the analysis of such systems is the test for stability, i. e., to determine the necessary and sufficient conditions for the roots of the system characteristic equation to lie inside the unit circle in the z-plane. These stability tests involve both graphical procedure such as Nyquist locus, Bode diagrams, and the root-locus, and analytical methods such as Schür-Cohn or Routh-Hurwitz criteria. Because of the higher order determinants to be evaluated using the presented form of the Schür-Cohn criterion, many authors in the past have used either a unit shifting transformation[6*] or bilinear transformation[7]. The latter transformation maps the inside of the unit circle in the $z = e^{Ts}$ plane into the left half of the w-plane** and then applies the Routh-Hurwitz criterion. This transformation involves two

* This transformation uses $p = z-1$

** This transformation uses $z = \dfrac{w+1}{W-1}$

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difficulties: a) algebraic manipulation for higher-order systems becomes complicated, and b) the final constraints on the coefficients in the z-plane become unwieldy and require algebraic red·ctions to yield the minimum number of terms. Because of these limitations this criterion is not usually used for systems higher than second-order.

A recent investigation by this author has shown that the evaluation of the Schür-Cohn determinants can be simplified considerably by making use of the real coefficients of the polynomial in z, so that the manipulation involved in testing for the zeros of a polynomial are comparable to those using the "transformed (or modified) Routh-Hurwitz criterion", thus avoiding the bilinear transformation [8, 9].

The study in this paper represents a major simplification of the earlier work, where it is shown that only half the number of determinants are required for obtaining the stability constraints. This simplification has a decisive advantage over the modified Routh-Hurwitz criterion and, indeed, higher-order systems can easily be tackled using the proposed stability test.

THEORETICAL BACKGROUND

In this section we review the simplifications which had been obtained in an earlier publication [8, 9] and explain in detail the manipulations involved.

Schür-Cohn criterion[1, 2]:

If for the polynomials

$$F(z) = a_0 + a_1 z + a_2 z^2 + \ldots a_n z^n \tag{1}$$

all the determinants of the matrices

$$\Delta_k = \begin{bmatrix} a_0 & 0 & 0 & 0 \ldots & 0 & a_n & a_{n-1} & \ldots & a_{n-k+1} \\ a_1 & a_0 & 0 & 0 \ldots\ldots & 0 & 0 & a_n & \ldots & a_{n-k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{k-1} & a_{k-2} & & a_{k-3} \ldots & a_0 & 0 & 0 & \ldots & a_n \\ \bar{a}_n & 0 & & 0 \ldots & 0 & \bar{a}_0 & \bar{a}_1 & \ldots & \bar{a}_{n-1} \\ \bar{a}_{n-1} & \bar{a}_n & & 0 \ldots & \cdot & \cdot & \cdot & \ldots & \cdot \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \bar{a}_{n-k+1} & \bar{a}_{n-k+2} & & \bar{a}_{n-k+3} & \bar{a}_n & 0 & 0 & \ldots & \bar{a}_0 \end{bmatrix} \tag{2}$$

$k = 1, 2, 3, \ldots n$, $\bar{a}_k$ = complex conjugate of a_k

are different from zero, then F(z) has no zeros on the circle $|z|=1$ and μ zeros in this circle, μ being the number of variations in sign in the sequence 1, $|\Delta_1|$, $|\Delta_2|$ ..., $|\Delta_n|$. The proof of the above theorem is quite involved and is available in the literature[1, 2]. This criterion was first introduced by Cohn in 1922, and since that time neither engineers nor mathematicians have simplified it to a usable form.

For a system of order n to be stable, all the n zeros of its characteristic n^{th} order equation must lie within the unit circle, i.e., the sequences, 1, $|\Delta_1|$, $|\Delta_2|$, ... $|\Delta_n|$ must have n variations in sign. The stability criterion can, therefore, be expressed by the constraints[4]:

$$|\Delta_k| < 0 \,, \quad k \text{ odd}$$

$$|\Delta_k| > 0 \,, \quad k \text{ even}, \quad k = 1, 2, \ldots n \tag{3}$$

For a discrete or a sampled-data system, all the coefficients of the characteristic equation are real. Hence, the conjugate sign in (2) is superfluous. It is the utilzation of this fact that leads to the simplification of (2).

As noticed from (2), the highest-order determinant $|\Delta_n|$ is of order 2n, while the characteristic equation is of order n. Hitherto this constituted one of the discouraging facts in widely using the criterion for higher-order sampled-data systems. A recourse to transformation to other planes was therefore attempted to yield easier stability tests.

SIMPLIFICATION OF THE STABILITY CONSTRAINT EQUATION[8, 10]

Since all a_k in equation (2) are real, the matrix can be written as:*

$$\Delta_k = \begin{bmatrix} P_k & Q_k \\ Q_k^T & P_k^T \end{bmatrix} \tag{4}$$

where the superscript T denotes transpose and

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* The author acknowledges the helpful correspondence with Dr. N. H. Choksy with regard to the material in this section.

$$P_k = \begin{bmatrix} a_0 & 0 & 0 \dots 0 \\ a_1 & a_0 & 0 \dots 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{k-1} & a_{k-2} & a_{k-3} \cdots a_0 \end{bmatrix} \quad (5) \quad , Q_k = \begin{bmatrix} a_n & a_{n-1} \cdots & a_{n-k+1} \\ 0 & a_n \quad \cdots & a_{n-k+2} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & a_n \end{bmatrix} \qquad (6)$$

It is noticed that all the diagonal terms of P_k and Q_k are equal and both are symmetric with its cross diagonal. It is this characteristic of the Schür-Cohn matrix that leads to the following simplification, using a unitary transformation.

Let I_k be the k-order identity matrix, I_k^+ the k-order permutation matrix,

$$I_k^+ = \begin{bmatrix} 0 & 0 \dots & 0 & 1 \\ 0 & 0 \dots & 1 & 0 \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad (7)$$

and U_k the 2k-order unitary matrix

$$U_k = \begin{bmatrix} I_k & 0 \\ 0 & I_k^+ \end{bmatrix}, \text{ (note that } U_k^{-1} = U_k) \qquad (8)$$

Let $\Lambda_k = U_k^{-1} \Delta_k^T U_k$, then by actual substitution for Δ_k^T, it is readily evident that:

$$|\Lambda_k| = |\Delta_k^T| = |\Delta_k| \qquad (9)$$

and

$$\Lambda_k = \begin{bmatrix} X_k & Y_k \\ Y_k & X_k \end{bmatrix} \qquad (10)$$

where

$$X_k = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{k-1} \\ 0 & a_0 & a_1 & \cdots & a_{k-2} \\ \cdot & & & & \\ \cdot & 0 & 0 & & a \\ 0 & 0 & 0 & & 0 \end{bmatrix} = P_k^T \tag{11}$$

and

$$Y_k = Q_k I_k^+ = \begin{bmatrix} a_{n-k+1} & \cdots & a_{n-1} & a_n \\ a_{n-k+2} & \cdots & a_n & 0 \\ \cdot & & & \\ \cdot & & & \\ a_{n-1} & & 0 & 0 \\ a_n & & 0 & 0 \end{bmatrix} \tag{12}$$

Hence

$$|\Delta_k| = |\Lambda_k| = \begin{vmatrix} P_k^T & Q_k I_k^+ \\ I_k^+ Q_k^T & I_k^+ P_k I_k^+ \end{vmatrix} = \begin{vmatrix} X_k & Y_k \\ Y_k & X_k \end{vmatrix} \tag{13$^+$}$$

$$= \begin{vmatrix} X_k + Y_k & Y_k + X_k \\ Y_k & X_k \end{vmatrix} \tag{14}$$

$$= \begin{vmatrix} X_k + Y_k & 0 \\ y_k & X_k - Y_k \end{vmatrix} \tag{15}$$

$$= |X_k + Y_k| \; |X_k - Y_k| \tag{16}$$

$^+$ One can easily verify that $P_k^T = I_k^+ P_k I_k^+$ and $Q_k I_k^+ = I_k^+ Q_k^T$

Thus the Schür=Cohn determinant $|\Delta_k|$ is reduced to the product of two k-order determinants[1, 2] which is considerably easier to evaluate than the direct evaluation of the 2k-order determinant $|\Delta_k|$. If a_k are complex, then this simplification is no longer possible[1, 2].

THE SYMMETRICAL PROPERTIES OF $|X_k + Y_k|$ $|X_k - Y_k|$ [8]

Now $|X_k + Y_k|$ is a homogeneous polynomial of dimension k in the variables $a, \ldots, a_n$. The polynomial $|X_k - Y_k|$ is identical to the polynomial $|X_k + Y_k|$ except for a change of sign of those monomial terms which have an odd number of elements from Y_k, i.e.,

$$|X_k + Y_k| = A_k + B_k \tag{17}^+$$

$$|X_k - Y_k| = A_k - B_k \tag{18}$$

where $A_k(B_k)$ is the sum of all monomial terms which do not change (do change) sign when Y_k is replaced by $-Y_k$ in $|X_k + Y_k|$.

<u>IDENTIFICATION OF A_k AND B_k</u> (which we designate as the stability constants)[8].

1) Let all the a_i's in the matrix Y_k in (12) be denoted by b_i's; then expand the determinant $|X_k + Y_k|$ in terms of a_i and b_i.

2) After expansion, examine every term which is a product of a_i's and b_i's; if it contains an even number (including zero) of b_i's, then it is assigned to A_k; otherwise assign the term to B_k.

3) After collecting the terms of A_k and B_k, replace all the b_i's by the a_i's. Hence

$$|\Delta_k| = (A_k + B_k)(A_k - B_k) = A_k^2 - B_k^2 \tag{19}^+$$

+ From (11) and (12), by replacing all the a's of Y_k by b's, we obtain A_k and B_k by first expanding the following determinant:

$$\underset{k=1,2,\ldots,n-1}{|X_k + Y_k|} = \begin{vmatrix} a_0+b_{n-k+1} & a_1+b_{n-k+2} & a_2+b_{n-k+3} \cdots a_{k-2}+b_{n-1} & a_{k-1}+b_n \\ b_{n-k+2} & a_0+b_{n-k+3} & a_1+b_{n-k+4} \cdots a_{k-3}+b_n & a_{k-2} \\ b_{n-k+3} & b_{n-k+4} & a_0+b_{n-k+5} \cdots \quad a_{k-4} & a_{k-3} \\ \vdots & \vdots & \vdots \qquad\qquad \vdots & \vdots \\ b_{n-2} & b_{n-1} & b_n \;\; 0 \;\; 0 \cdots \;\; a_1 & a_2 \\ b_{n-1} & b_n & 0 \;\; 0 \;\; 0 \;\; 0 \;\; a_0 & a_1 \\ b_n & 0 & 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 & a_0 \end{vmatrix} \tag{20a}$$

and for stability from (3) this reduces to

$$|A_k| > |B_k|, \text{ k even}$$

$$|A_k| < |B_k|, \text{ k odd, } k = 1, 2, \ldots n \tag{20}$$

Therefore, the application of the Schür-Cohn criterion now reduces to the evaluation of determinants up to order n only for the n^{th} order polynomial.

It should be noted that the last simplification is useful only for design procedures where the coefficients of F(z) are given in other than numerical values. However, if a_k's are given in numerical values, the use of equations (17 and 18) is preferred.

EQUIVALENCE OF THE LAST CONSTRAINT $|A_n| \gtrless |B_n|$ TO A SIMPLE AUXILIARY CONSTRAINT[9]:

The constraint which we will introduce involves the exclusion of certain real roots outside of the unit circle. It constitutes a necessary (but not sufficient) condition for the roots of F(z) to lie inside the unit circle. This constraint is given by

$$F(z)\Big|_{z=1} > 0 \quad (1) \quad \text{and } F(z)\Big|_{z=-1} \begin{array}{l} > 0 \text{ for n even (2a)} \\ < 0 \text{ for n odd (2b)} \end{array} \tag{21}$$

The alternate constrain $F(z)\Big|_{z=1} < 0$ and $F(z)\Big|_{z=-1} \begin{array}{l} < 0 \text{ for n even} \\ > 0 \text{ for n odd} \end{array}$ is also possible. However, we may exclude this, without loss of generality, by always letting a_n be positive, in which case to satisfy (1) and (2a, 2b) requires[11] that $F(z)\Big|_{z\to\infty} > 0$ and $F(z)\Big|_{z=-\infty} \begin{array}{l} > 0 \text{ for n even} \\ < 0 \text{ for n odd.} \end{array}$

Lemma 1. If (1) and (2b) are satisfied, then there exists at least one real root of F(z) = 0 between plus and minus one. Also the total number of such roots is <u>odd</u>[12].

Lemma 2. If (1) and (2a) are satisfied, then the total number of real roots that lies between plus and minus one is <u>zero</u> or <u>even</u>[13].

To show the equivalence of the constraint $|A_n| \gtrless |B_n|$ to the above auxiliary constraint it is simple to distinguish between two cases:

(a) n is odd: Suppose we satisfy the constraint constants up to A_{n-1} and B_{n-1}, then a generalization by Marden[2] of the Schür-Cohn criterion indicates that there exist (n-1) roots inside the unit circle. The arrangement of these (n-1) roots (even in number) inside the unit circle is one of two alternatives. (1) The first alternative is that, because complex roots appear in conjugate, the total number of real roots between plus and minus one is either zero or even. Now if we impose the auxiliary constraint (1) and (2b) on F(z) we find that the last single real root from the constraint $|A_n| < |B_n|$ should lie inside the unit circle from Lemma 1. (2) The second alternative is when the auxiliary constraint is satisfied in addition to the first (n-1) constraints, then the number of real roots between plus and minus one is either one or odd, and thus in this arrangement there exists a single complex root inside the unit circle. Since complex roots appear in conjugate, the last constraint $|A_n| < |B_n|$ is necessarily satisfied. Similarly if the auxiliary constraint is not satisfied, then this indicates a single real root outside the unit circle and thus the last constraint is also not satisfied.

For the case where $|A_n| = |B_n|$ this indicates a real root on the unit circle which is also the condition of the auxiliary constraint when written in absolute values equated to zero. Therefore, we have shown for n odd that the auxiliary constraint is equivalent to the last constraint.

(b) n is even: Suppose we satisfy the constraint constants up to A_{n-1} and B_{n-1}, then this indicates that there exists (n-1) roots inside the unit circle. The arrangement of these (n-1) roots (odd in number) inside the unit circle is one of two alternatives. (1) The first alternative is that, because complex roots appear in conjugate, the total number of real roots between plus and minus one is either one or odd. Now if we impose the auxiliary constraint (1) and (2a) on F(z) we find that the last single real root from the constraint $|A_n| > |B_n|$ should lie inside the unit circle from Lemma 2. (2) The second alternative is where the auxiliary constraint is satisfied in addition to the first (n-1) constraints, then the number of real roots between plus and minus one is either zero or even, and thus in this arrangement there exists a single complex root inside the unit circle. Since complex roots appear in conjugate therefore the last constraint $|A_n| > |B_n|$ is necessarily satisfied. Similarly if the auxiliary constraint is not satisfied then this indicates that a single real root lies outside the unit circle and thus the last constraint is also not satisfied.

For the case when $|A_n| = |B_n|$ this indicates a real root on the unit circle which is also the condition of the auxiliary constraint (written in absolute values) when equated to zero. Therefore, we have shown for n even that the auxiliary constraint is also equivalent to the last constraint.

Therefore, for stability test it can be concluded that the first (n-1) constraints of the A's and B's should be satisfied and the auxiliary constraint is then equivalent to the last constraint $|A_n| \lesseqgtr |B_n|$. This equivalence has been checked for the examples discussed in this note. Furthermore in the next sections this equivalence will be demonstrated mathematically.

THE MODIFIED STABILITY CRITERION[9]

Combining the previous discussions we can restate the stability criterion in a modified form as follows:

A necessary and sufficient condition for the polynomial $F(z) = a_0 + A_1 z + a_2 z^2 + \ldots + a_k z^k + \ldots + a_n z^n$, to have all its roots inside the unit circle is represented by the constraints

$$|A_k| < |B_k| \text{ for k odd}$$

and

$$|A_k| > |B_k| \text{ for k even, } k = 1, 2, \ldots, n-1 \tag{22}$$

and by the following auxiliary constraint.

$$F(z)\Big|_{z=1} > 0 \text{ and } F(z)\Big|_{z=-1} \begin{matrix} >0 \text{ n is even} \\ <0 \text{ n is odd} \end{matrix}, \text{ for } a_n > 0 \tag{22a}$$

or

$$F(1) \,.\, F(-1) \begin{matrix} > 0 & \text{n even} \\ < 0 & \text{n odd} \end{matrix}, \text{ for any } a_n \tag{23}$$

MODIFIED SCHÜR-COHN CRITERION[9]

From the above consideration we can usefully modify the Schür-Cohn criterion as follows[4]:

If for the polynomial with real coefficients

$$F(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \ , \ a_n > 0 \tag{24}$$

satisfying the auxiliary constraint, all the stability constants A_k and B_k ($k = 1, \ldots, n-1$) are not equal, then $F(z)$ has no zeros on the circle $|z| = 1$ and $(\mu + 1)$ zeros inside the unit circle for n even and μ odd as well as for n odd and μ even. (μ is the number of variations of inequality sign in the stability constraints $[1, (A_1, B_1), \ldots, (A_{n-1}, B_{n-1})]$. Furthermore, when n and μ are even and when n and μ are odd the number of zeros inside the unit circle is μ.

REDUCTION IN THE NUMBER OF DETERMINANTS FOR OBTAINING THE STABILITY CONSTANTS A_k's and B_k's

In this section we will show that for stability test only about half the number of determinants for obtaining the A_k's and B_k's are required. This important reduction is based on certain properties that exist between the A_k's and B_k's. We will indicate these properties first and then show how they can be used for this major simplification

$$1) \quad A_k^2 \gtrless B_k^2 \rightleftarrows A_{k-1}A_{k+1} \gtrless B_{k-1}B_{k+1}, \ k = 2, 3, 4, \ldots, n-1 \tag{25}$$

The above equivalence is established by expansion for the first few values of "k" and can be generalized similarly for any other value of k up to n-1. When k = n-1, the above equivalence is reduced to:

$$A_{n-1}^2 \gtrless B_{n-1}^2 \rightleftarrows A_{n-2}A_n \gtrless B_{n-2}B_n \tag{26}$$

Equation (25) is the key identity for reducing the determinant of A_k and B_k, for it is noticed that by forcing certain restrictions on the A's and B's before and after a certain k we can dispense with $A_k^2 \gtrless B_k^2$. This will be best illustrated by the few examples to be discussed.

$$2) \quad A_n = (a_0 + a_2 + a_4 + \ldots)(A_{n-1} - B_{n-1}) \tag{27}$$

$$B_n = (a_1 + a_3 + a_5 + \ldots)(A_{n-1} - B_{n-1}) \ , \ n \geq 2 \tag{28}$$

The above identity can be easily verified for the first few values of "n" and can be generalized for any "n"+. The importance of this property lies in the mathematical proof of the previous section as follows:

$$A_n^2 \gtrless B_n^2 \Longrightarrow (a_0+a_2+a_4+\dots)^2 \gtrless (a_1+a_3+a_5\dots)^2 \qquad (29)$$

or

$$A_n^2 - B_n^2 \gtrless 0 \rightleftarrows F(1). \quad F(-1) \begin{matrix} > 0 \\ < 0 \end{matrix} \qquad (30)$$

which verifies the equivalence $|A_n| \gtrless |B_n|$ to the auxiliary constraint.

$$3)\ (A_{n-1}+B_{n-1}) = A_{n-2}(a_0+a_2+a_4+\dots) - B_{n-2}(a_1+a_3+a_5+\dots)$$

$$n \geq 3 \qquad (31)$$

From (26) we can write

$$(A_{n-1}-B_{n-1})(A_{n-1}+B_{n-1}) \begin{matrix} > 0 \\ < 0 \end{matrix} \rightleftarrows A_{n-2}A_n - B_{n-2}B_n \begin{matrix} > 0 \\ < 0 \end{matrix} \qquad (32)$$

using (27) and (28) in the right side of equation (32)

$$A_{n-2}(a_0+a_2+a_4+a_6\dots)(A_{n-1}-B_{n-1}) - B_{n-2}(a_1+a_3+a_5\dots)(A_{n-1}-B_{n-1}) \begin{matrix} > 0 \\ < 0 \end{matrix} \qquad (33)$$

or

$$(A_{n-1}-B_{n-1})\ [A_{n-2}(a_0+a_2+a_4\dots) - B_{n-2}(a_1+a_3+\dots)] \begin{matrix} > 0 \\ < 0 \end{matrix} \qquad (34)$$

Comparing equation (34) with the left side of equation (32) we obtain the following identity:

$$(A_{n-1}+B_{n-1}) \begin{matrix} > 0 \\ < 0 \end{matrix} \rightleftarrows A_{n-2}(a_0+a_2+a_4+\dots) - B_{n-2}(a_1+a_3+a_5\dots) \begin{matrix} > 0 \\ < 0 \end{matrix}, \quad n \geq 3 \qquad (35)$$

The above relationship can also be derived directly for the first few values of "n" and can be generalized for any "n". Furthermore the above property can also be shown as an equivalence relationship which can be written as a recurrence equation.

$$(A_{n-1}+B_{n-1}) = A_{n-2}(a_0+a_2+a_4\dots) - B_{n-2}(a_1+a_3+a_5\dots), \quad n \geq 3 \qquad (36)$$

The use of this property lies in the fact that one can obtain directly $(A_{n-1}+B_{n-1})$ from the previously obtained A_{n-2} and B_{n-2}, and also to verify the first property for the upper limit, i. e., $k=n-1$ (See Appendix).

The above three properties in combination with the preceding discussion will now be used in obtaining the new stability constraints for low order systems and then, by generalization, to obtain the constraint for an n-order system.

+ To establish the identity for the general case, it is easier to show the following equivalent conditions for (27) and (28):

$$A_n+B_n = F(1)(A_{n-1}-B_{n-1});\ A_n-B_n = F(-1)(A_{n-1}-B_{n-1})$$

Examples of Low Order Systems:

We will apply the reduction properties wherever they are applicable to n = 2, 3, 4, 5, and then obtain the stability constraint for any n. We will assume $a_n > 0$. This can be easily done by multiplying F(z) by minus one, if necessary.

(a) $\underline{n} = \underline{2}$, $F(z) = a_0 + a_1 z + a_2 z^2$, $a_2 > 0$ (37)

The stability constraints, using equations (22 and 22a) are given

$$|a_0| < a_2 \text{ or } |A_1| < |B_1| \tag{38}$$

$$a_0+a_1+a_2 > 0\,,\quad a_0-a_1+a_2 > 0\,,\quad \text{or } |A_2| > |B_2| \tag{39}$$

One could also remove the absolute sign from equation (38) for if a_0 is negative with magnitude larger than a_2, equation (39) will not be satisfied. However, we may leave the absolute sign in order to discontinue the stability test if (1) is violated.

(b) $\underline{n} = \underline{3}$, $F(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$, $a_3 > 0$ (40)

The stability constraints from the modified stability equations (22) and (22a) are given by the following inequalities:

$$|a_0| < a_3\,,\quad |A_1| < |B_1| \tag{41}$$

$$|a_0^2 - a_3^2| > |a_0 a_2 - a_1 a_3|\,,\quad |A_2| > |B_2| \tag{42}$$ +

$$F(z)\Big|_{z=1} > 0\,,\quad F(z)\Big|_{z=1} < 0\,,\quad \text{or } |A_3| < |B_3| \tag{43}$$

+ To show how to obtain A_2 and B_2 for this case:

1) Expand the determinant of Eq. (20a) for k=2 and n=3, as follows:

$$|X_2 + Y_2| = \begin{vmatrix} a_0 + b_2 & a_1 + b_3 \\ b_3 & a_0 \end{vmatrix} = a_0^2 + a_0 b_2 - a_1 b_3 - b_3^2$$

2) To identify A_2 and B_2, follow procedure (2) on page 6, to obtain

$$A_2 = a_0^2 - b_3^2$$

$$B_2 = a_0 b_2 - a_1 b_3$$

3) Replace all the b's of A_2 and B_2 by the a's to get:

$$A_2 = a_0^2 - a_3^2$$

$$B_2 = a_0 a_2 - a_1 a_3$$

A similar procedure is used for obtaining the A_k's and B_k'(s) for any "k" and "n".

Reduction of the constraint conditions:

From the first property of the A_k and B_k, we may write in th[illegible] case

$$A_2^2 > B_2^2 \Longrightarrow A_1A_3 > B_1B_3 \tag{44}$$

Since B_1 is positive (i.e., $a_3 > 0$), we may write for the right side

$$\frac{A_1}{B_1} A_3 > B_3 . \tag{42a}$$

Using conditions (41) and (43) in combination with (42a) the new stability constraints are:

$$|a_0| < a_3 \tag{45}$$

$$B_3 < 0 \tag{46}$$

$$|A_3| < |B_3| \text{ , or } F(1) > 0, \; F(-1) < 0 \tag{47}$$

From the second property in equation (28)

$$B_3 = (a_1 + a_3)(A_2 - B_2) \tag{48}$$

Since $a_1 + a_3$ is positive (from Eq. 47) because it is equal to $F(1) - F(-1) > 0$, therefore B_3 is negative only when $A_2 - B_2 < 0$. From the third property of the stability constants it is readily seen that $A_2 + B_2 < 0$ is identically satisfied from the first and third conditions. Thus the simplified form of the stability constraint for n=3 reduces to

$$|a_0| < a_3 \tag{49}$$

$$a_0^2 - a_3^2 < a_0a_2 - a_1a_3 \tag{50}$$

$$a_0 + a_1 + a_2 + a_3 > 0, \; a_0 - a_1 + a_2 - a_3 < 0 \tag{51}$$

Stability diagrams for a second and third-order case are presented in Figures 1 and 2.

(c) $\underline{n} = \underline{4}$, $F(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4$, $a_4 > 0$ (52)

The stability constraint

$$|a_0| < a_4 , \; |A_1| < |B_1| \tag{53}$$

$$|a_0^2 - a_4^2| > |a_0a_3 - a_1a_4| , \; |A_2| > |B_2| \tag{54}$$

$$|a_0^3 + a_0a_2a_4 + a_1a_3a_4 - a_0a_4^2 - a_2a_4^2 - a_0a_3^2| < |a_0^2a_4 + a_0^2 a_2 + a_1^2 a_4 - a_0a_2a_4 - a_4^3 - a_0a_1a_3| \tag{55}$$

or $$|A_3| < |B_3| \tag{55}$$

$$F(z)\Big|_{z=1} > 0\,,\ F(z)\Big|_{z=-1} > 0\,,\ \text{ or } |A_4| > |B_4| \tag{56}$$

Reduction of the constraint equations

Using property (1), we may write as in the previous case:

$$A_2^2 > B_2^2 \rightleftarrows A_1A_3 > B_1B_3\,,\ \text{or } \frac{A_1}{B_1}\ A_3 > B_3 \tag{57}$$

Equations (53), (54), and (55) are now equivalent to (53), $B_3 < 0$ and (55). Using $B_3 < 0$ with (55), we finally obtain the reduced constraints:

$$|a_0| < a_4,\ \text{or } |A_1| < B_1 \tag{58}$$

$$A_3 - B_3 > 0,\ A_3 + B_3 < 0 \tag{59}+$$

$$F(1) > 0,\quad F(-1) > 0 \tag{60}$$

It is noticed that for the fourth-order++ case only one determinant of third-order for obtaining A_3 and B_3 is required. All other conditions are very simple.

(d) $\underline{n} = \underline{5}$, $F(z) = a_0+a_1z+a_2z^2+a_3z^3+a_4z^4+a_5z^5,\ a_5 > 0$ (61)

The stability constraints in symbolic form

$$|A_1| < B_1\,,\ B_1 = a_5 > 0 \tag{62}$$

$$|A_2| > |B_2| \tag{63}$$

$$|A_3| < |B_3| \tag{64}$$

$$|A_4| > |B_4| \tag{65}$$

$$F(1) > 0\,,\ F(-1) < 0,\ \text{or } |A_5| < |B_5| \tag{66}$$

+Constraints in (59) can also be written as $B_3<0$, $|A_3| < |B_3|$. The latter has an advantage in numerical testing if $B_3 < 0$ is violated. Then the test could be discontinued without having to calculate A_3.

++An alternate form which is more advantageous for design can be obtained for the fourth order case. This form can be easily obtained by using properties (1) and (2). It is given as follows: (1) $A_2 < 0$, (2) $A_2 < -|B_2|$, (3) $A_3 - B_3 > 0$, (4) $F(1) > 0$, $F(-1) > 0$.

In this case only one third order equation in (3) and one second order equation in (2) are to be solved, while in the former case two third order equations are to be solved. It should be noted that when (2) is satisfied, relationship (1) becomes redundant.

Reduction of the constraint relationship:

We may keep in this case condition (63) but we eliminate (64) by using the first property, i. e.,

$$A_3^2 < B_3^2 \Rightarrow A_2A_4 < B_2B_4 \tag{64a}$$

With A_2 negative because of (62)[+], the stability constraint for (62), (64), (64a) and (65) becomes: (62), (63), $A_4 > 0$, and $A_4^2 > B_4^2$. Using $A_4 > 0$ in (65) i. e., $(A_4-B_4)(A_4+B_4) > 0$ we obtain $A_4 - B_4 > 0$ and $A_4 + B_4 > 0$. Furthermore, since $A_2 = A_1^2 - B_1^2$ is to be negative from (62), it is readily satisfied if the second constraint is equivalent to $A_2 - B_2 < 0$, $A_2 + B_2 < 0$. Finally, we obtain for the reduced form the following:

$$A_2 - B_2 < 0, \quad A_2 + B_2 < 0 \tag{67}$$

$$A_4 - B_4 > 0, \quad A_4 + B_4 > 0 \tag{68}$$

$$F(1) > 0, \quad F(-1) < 0 \tag{69}$$

It is noticed that in this case only the second and fourth order determinants are required. One may also use a different form of reduction by eliminating (68). However, this will not yield much simplification over the previous form because a fourth-order determinant with a third-order determinant is then required.

The above discussion can be generalized for an "n" which finally reduces to the following simplified criterion.

The New Stability Criterion:

A necessary and sufficient condition for the polynomial $F(z) = a_0 + a_1z + a_2z^2 + \ldots + a_kz^k + \ldots + a_nz^n$ with $a_n > 0$, to have all its roots inside the unit circle is represented by the following constraints for n even and n odd respectively:

[+] Note $A_2 = A_1^2 - B_1^2$

n even+	n odd+
$\lvert A_1 \rvert < B_1, \quad B_1 = a_n > 0$	$A_2 - B_2 < 0, \quad A_2 + B_2 < 0$
$A_3 - B_3 > 0, \quad A_3 + B_3 < 0$	$A_4 - B_4 > 0, \quad A_4 + B_4 > 0$
$A_5 - B_5 < 0, \quad A_5 + B_5 > 0$	$A_6 - B_6 < 0, \quad A_6 + B_6 < 0$
$A_7 - B_7 > 0, \quad A_7 + B_7 < 0$	. .
. .	. .
$A_{n-1} - B_{n-1}$ > 0 for n=4k, < 0 for any other n	$A_{n-1} - B_{n-1}$ > for n-1 = 4k, < 0 for any other (n-1)
$A_{n-1} + B_{n-1}$ < 0 for n = 4k, > 0 for any other n	$A_{n-1} + B_{n-1}$ > 0 for n-1=4k, < 0 for any other (n-1)
k = 1, 2, 3. . .	k = 1, 2, 3. . .
$F(1) > 0, \quad F(-1) > 0$	$F(1) > 0, \quad F(-1) < 0$

Alternate Forms:

An alternate equivalent method which is of advantage if the stability constants evaluation is carried out by methods other than a computer is hereby presented.

n even	n odd
$\lvert A_1 \rvert < B_1, \quad B_1 = a_n > 0$	$\lvert A_2 \rvert > \lvert B_2 \rvert, \quad A_2 < 0$
$\lvert A_3 \rvert < \lvert B_3 \rvert, \quad B_3 < 0$	$\lvert A_4 \rvert > \lvert B_4 \rvert, \quad A_4 > 0$
$\lvert A_5 \rvert < \lvert B_5 \rvert, \quad B_5 > 0$	$\lvert A_6 \rvert > \lvert B_6 \rvert, \quad A_6 < 0$
$\lvert A_7 \rvert < \lvert B_7 \rvert, \quad B_7 < 0$	. .
. .	. .
$\lvert A_{n-1} \rvert < \lvert B_{n-1} \rvert$, B_{n-1} <0, for n=4k; >0, for any other n	$\lvert A_{n-1} \rvert > \lvert B_{n-1} \rvert$, A_{n-1} >0, for n-1=4k; <0, for any other n
k=1, 2, 3. . .	k=1, 2, 3. . .
$F(1) > 0, \quad F(-1) > 0$	$F(1) > 0, \quad F(-1) < 0$

+Note that $A_k + B_k$ and $A_k - B_k$ can be obtained directly from equations (17) and (18)

In the above case the identification of $|A_k|$ and $|B_k|$ from the determinant $|X_k - Y_k|$ could be used. This criterion can be applied for design purposes when the a_k's of $F(z)$ are given other than numerically.

Illustrative Examples

To illustrate the stability test, we choose two problems, one involving design and the other a numerical test of a polynomial.

1. The design problem is concerned with obtaining the maximum allowable value of k (the gain) of a feedback sampled-data system[12] shown in Figure 3, to be stable.

The overall transfer function is given as:

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)} \tag{70}$$

where

G(z) = z-transform of the forward path transfer function G(s)

$$= \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \; e^{-0.5s} \; \frac{0.5k}{s + 0.5}\right] = \frac{0.118k}{z^2(z - 0.882)} \tag{71}$$

For stability we have to examine the denominator of Eq. (70) i.e., $1 + G(z)$

$$1 + G(z) = z^3 - 0.882\, z^2 + 0.118k \tag{72}$$

The above equation is a third-order polynomial in z, and thus we can apply the stability tests obtained earlier for n = 3. In this case $a_3 = 1$, $a_2 = -0.882$, $a_1 = 0$, $a_0 = 0.118k$, and k to be positive.

$$1)\quad |a_0| < a_3,\; 0.118k < 1,\; k \le 8.47 \tag{73}$$

2) $a_0^2 - a_3^2 - a_0 a_2 < 1$, yields $k \le 5.75$ or $k \ge -13.21$ (for positive feedback)

3) $a_0 + a_1 + a_2 + a_3 > 0$, $1 - 0.882 + 0.118\,k > 0$ is satisfied for any positive $k > 0$

$$a_0 - a_1 + a_2 - a_3 < 0,\; -1 - 0.882 \pm 0.118\,k < 0,\; k \le 15.9 \tag{74}$$

Therefore, the maximum allowable gain for stability is the lowest value which is in this case $k_{max} = 5.75$.

2. Test for stability the following polynomial:

$$F(z) = z^3+2z^2-0.5z-0.95 \tag{75}$$

The above polynomial is again of third order, i.e., $n = 3$, thus we apply the stability constraints for this case, $a_3=1$, $a_2 = 2$, $a_1=-0.5$, $a_0= -0.95$

1) $|a_0| < a_3$, $0.95 < 1$

2) $a_0^2 - a_3^2 < a_0a_2 - a_1a_3$, $0.95^2 - 1 > -1.90 + 0.5$ (76)

or, $-0.1 > -1.40$

The second condition is violated, thus there exists at least one root outside the unit circle and thus the system is unstable. To determine the number of roots outside the unit circle from the modified Schür-Cohn criterion, (see p. 10) we also have to examine the sign of the last condition

3) $a_0+a_1+a_2+a_3 > 0$, $-0.95 -0.5 + 2 + 1 > 0$

$a_0-a_1+a_2-a_3 < 0$, $-0.95 + 0.5 + 2 - 1 > 0$ (77)

The last condition $|A_3| < |B_3|$ is violated. Now the number of changes of sign of 1, A_k's and B_k's are the number of roots inside the unit circle. In this case, the sign changes are, 1, $A_1^2 - B_1^2 < 0$, $A_2^2 - B_2^2 < 0$, $A_3^2 - B_3^2 > 0$. There are two changes of sign, and since only three roots exist, therefore only a single real root exists outside the unit circle.

CONCLUSION

From the preceding discussion, it is shown first that in the original Schür-Cohn or the modified Routh-Hurwitz criterion, the number of determinants required for the stability is almost halved. The use of the criteria for both design of discrete systems as well as for testing roots of a polynomial inside the unit circle is illustrated. This criterion will now be useful in many applications such as the stability test of difference equations with constant and periodically varying coefficients, in nonlinear discrete systems for the stability study of limit cycles, in the design of digital computers, in the stability test of linear systems with randomly varying parameters and in many other applications. Thus it is hoped that this criterion will find many applications in various fields in addition to the above and its use by engineers, physicists and mathematicians will be greatly enhanced.

REFERENCES

1. A. Cohn, "Oberdie Anzahl der Wurzeln einer algebraichen Gleichurg in einen Kreise," Math. Zert., Vol. 14 (1922), pp. 110-148.
2. M Marden, Geometry of the Zeros, American Mathematical Society, 1949.
3. F. R. Gantmacher, Theory of Matrices, Chelsea Publishing Co., Vol. II, 1959, pp. 221-227.
4. E. I. Jury, Sampled-Data Control Systems, John Wiley and Sons, Inc. 1958.
5. Tsypkin, Y., Theory of Pulse Systems, 1959, Moscow, State Press for Physics and Mathematical Literature.
6. Van J. Tschauner, "Die Stabilitat von Impulssystemen," Regelungtechnik Heft 2, 1960, Germany.
7. R. C. Oldenbourg and H. Sartorious, The Dynamics of Automatic Controls American Society of Mechanical Engineers, New York, New York, 1948 Chap. 5.
8. E. I. Jury and B. H. Bharucha, "Notes on the Stability Criterion for Linear Discrete Systems," IRE Trans. on Automatic Control, Feb. 1961.
9. E. I. Jury, "Additions to Stability Criterion for Linear Discrete Systems," to be published by IRE, PGAC, Aug., 1961.
10. B. H. Bharucha, "Analysis of Integral-Square Error in Sampled-Data Control Systems," Electronics Research Laboratory Series 60, Issue No. 206, June 30, 1958, University of California, Berkeley.
11. John C. Truxal, Control System Synthesis, Ch. 9, McGraw-Hill Co. 1955.
12. D. K. Cheng, Analysis of Linear Systems, Addison Wesley, 1955.
13. E. Mishkin and L. Braun, Jr., Adaptive Control Systems, McGraw-Hill Co., 1960.

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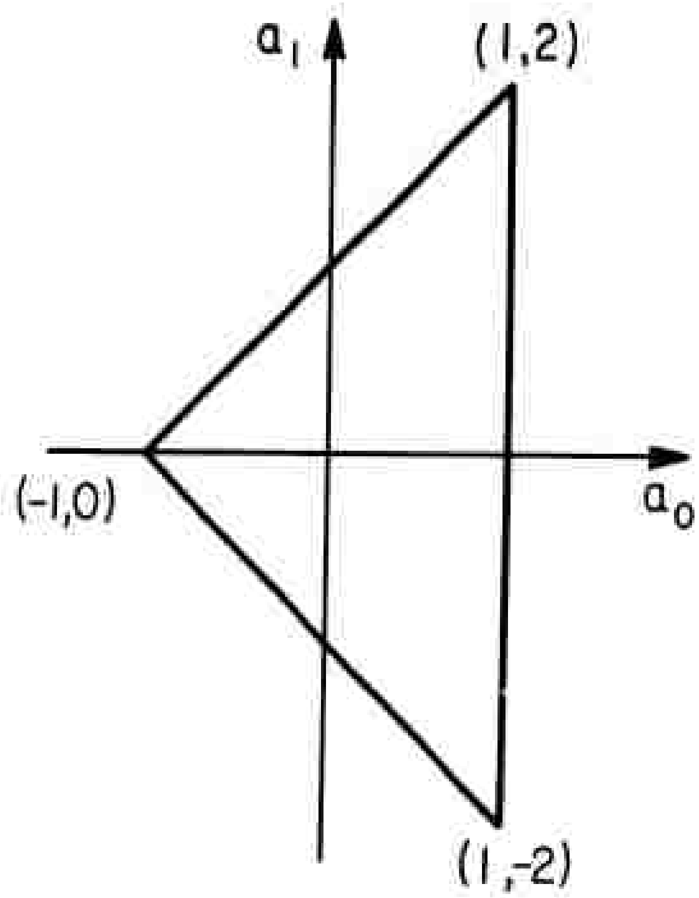


FIG. 1 STABILITY DIAGRAM FOR A SECOND ORDER CASE
$F(z) = a_0 z + a_1 z + a_2 z^2$, $a_2 = 1$

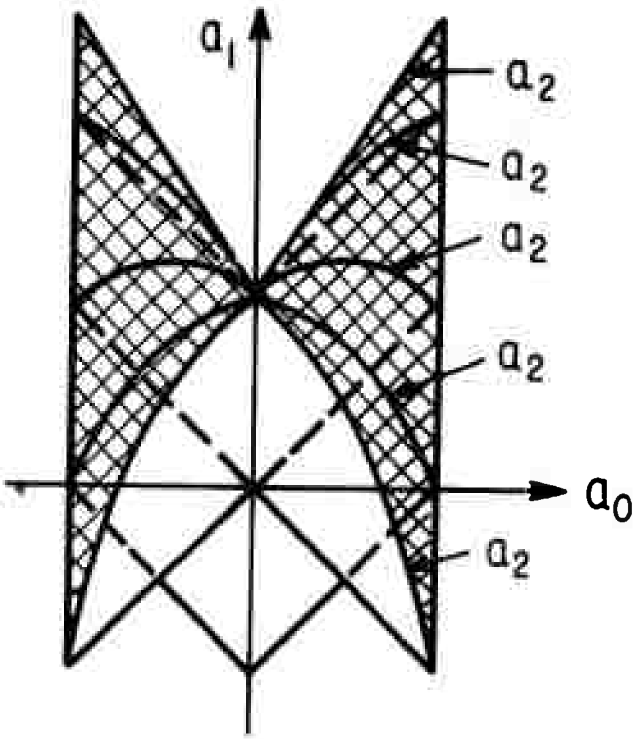
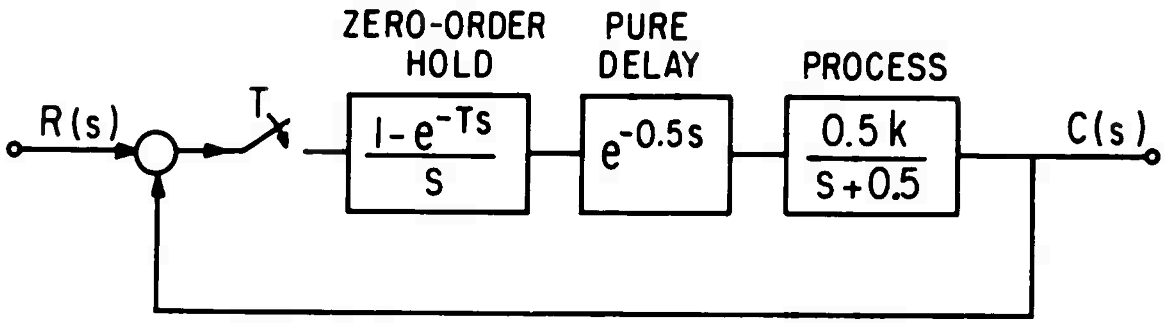


FIG. 2 STABILITY DIAGRAM FOR A THIRD-ORDER CASE

$$F(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 , \quad a_3 = 1$$



T = 0.25 sec

FIG. 3 A SAMPLED DATA FEEDBACK SYSTEM

APPENDIX+

General Proof of the Properties of the Stability Constants A_k's and B_k's

In this appendix the second and the third properties of the stability constants, i. e., Eqns. (27), (28) and (31), will be mathematically proven and from these two properties, the limiting case of the first property, i. e., Eqn. (26) will be demonstrated. A heuristic argument will be presented to indicate that relationship (25) is valid for all $k = 2, 3, \ldots, n-1$.

Proof of the Second Property:

The second property as indicated in Eqns. (27) and (28) is given as follows:

$$A_n = (a_o + a_2 + a_4 + a_6 + \ldots)(A_{n-1} - B_{n-1})$$

$$B_n = (a_1 + a_3 + a_5 + a_7 + \ldots)(A_{n-1} - B_{n-1}), \quad n \geqslant 2$$

To show this property it is simpler to manipulate the following equivalent relationship, which is obtained by adding and subtracting the above two equations.

$$A_n + B_n = (a_o + a_1 + a_2 \ldots\ldots)(A_{n-1} - B_{n-1})$$

$$A_n - B_n = (a_o - a_1 + a_2 - \ldots\ldots (-1)^n a_n)(A_{n-1} - B_{n-1})$$

The above can be also written as:

$$A_n + B_n = F(1)\,(A_{n-1} - B_{n-1}) \qquad (1)$$

$$A_n - B_n = F(-1)(A_{n-1} - B_{n-1}) \qquad (2)$$

+ The author acknowledges the aid of Mr. Jean Blanchard in the discussions of this appendix.

We will first demonstrate relationship (1) and following the same procedure relationship (2) can be similarly demonstrated. The proof consists ofdeterminant manipulation and in particular using the following property.

"The value of the determinant is unchanged if the elements of any row (column) are replaced by the sums or (differences) of the elements of that row and the corresponding elements of another row (column)."

To show relationship (1), we write first the determinant A_n+B_n as follows:

$$A_n+B_n = |X_n+Y_n| = \begin{vmatrix} a_o+a_1 & a_1+a_2 \cdots\cdots & a_{q-1}+a_q & \cdots\cdots\cdots & a_{n-1}+a_n \\ a_2 & a_1+a_3 \cdots\cdots & a_{q-2}+a_{q-1} & \cdots\cdots & a_{n-2} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ a_\ell & \cdots\cdots\cdots\cdots\cdots\cdots & a_{q-\ell+1}+a_{q+\ell-2} & \cdots\cdots & a_{n-\ell+1} \\ a_{\ell+1} & \cdots\cdots\cdots\cdots\cdots & a_{q-\ell+1}+a_{q+\ell-1} & \cdots\cdots & a_{n-\ell} \\ a_{\ell+2} & \cdots\cdots\cdots\cdots\cdots & a_{q-\ell-1}+a_{q+\ell} & \cdots\cdots & a_{n-\ell-1} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ a_n & 0 \quad 0 \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots 0 & 0\,a_o \end{vmatrix} \tag{3}$$

We will show the equivalence of (1) for the general row and column in the above matrix, by concentrating only on rows $\ell-1$,

ℓ, $\ell+1$ and columns q-1, q and q+1 as follows:

	column q-1	column q	column q+1
row $\ell-1$	$a_{q-\ell}+a_{q+\ell-3}$	$a_{q-\ell+1}+a_{q+\ell-2}$	$a_{q-\ell+2}+a_{q+\ell-1}$
row ℓ	$a_{q-\ell-1}+a_{q+\ell-2}$	$a_{q-\ell}+a_{q+\ell-1}$	$a_{q-\ell+1}+a_{q+\ell}$
row $\ell+1$	$a_{q-\ell-2}+a_{q+\ell-1}$	$a_{q-\ell-1}+a_{q+\ell}$	$a_{q-\ell}+a_{q+\ell+1}$

(4)

with $a_\mu = 0$

for $\mu > n$ or $\mu < 0$

Similarly we obtain the same rows and columns for the determinant $A_{n-1}-B_{n-1} = |X_{n-1}-Y_{n-1}|$

General Coefficients in Matrix $A_{n-1}-B_{n-1}$

	column q-1	column q	column q+1
row $\ell-1$	$a_{q-\ell} - a_{q+\ell-2}$	$a_{q-\ell+1} - a_{q+\ell-1}$	$a_{q-\ell+2} - a_{q+\ell-1}$
row ℓ	$a_{q-\ell-1} - a_{q+\ell-1}$	$a_{q-\ell} - a_{q+\ell}$	$a_{q-\ell+1} - a_{q+\ell+1}$
row $\ell+1$	$a_{q-\ell-2} - a_{q+\ell}$	$a_{q-\ell-1} - a_{q+\ell+1}$	$a_{q-\ell} - a_{q+\ell+2}$

(5)

To demonstrate (1) we perform the following matrix manipulations:
(1) In the determinant A_n+B_n , add up to the last row, all the previous rows. We then notice that all the coefficients of the last row after this adding become,

$$\sum_{p=0}^{p=n} a_p = F(1).$$

The new determinant after factorization of $F(1)$ becomes,

$$A_n+B_n = F(1)\;\underbrace{\begin{vmatrix} & & D & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix}}_{n\ \text{columns}}\left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n\ \text{rows} \qquad (6)$$

The elements in determinant D are the same as the initial determinant A_n+B_n.

(2) In (6) we subtract the columns 2 from 1, ... the column q from $q-1$, ... and the column n from $n-1$. By so doing, it is noticed that except for the coefficients of the column n and row n which is equal to 1, all other coefficients of the row n are equal to zero.

Therefore the matrix in A_n+B_n can be written as:

$$
A_n + B_n = F(1) \qquad \left| \begin{array}{ccccc|c}
 & & & & & a_{n-1}+a_n \\
 & & & & & \vdots \\
 & & D_1 & & \left.\begin{array}{c} \\ \\ \end{array}\right\} n-1 \text{ rows} & \vdots \\
 & & & & & a_2 \\
\hline
 & & & & & a_1 \\
0 & 0 & \cdots & 0 & 0 & 1
\end{array} \right|
$$

$$\underbrace{0 \quad 0 \quad \cdots \quad 0 \quad 0}_{n-1 \text{ columns}}$$

Now if we expand the determinant with respect to the last coefficient in rows n we obtain a determinant of order $n-1$ for D_1.

The general coefficients for D_1, are as follows:

	column $q-1$	column q
row $\ell-1$	$a_{q-\ell}-a_{q+\ell-2}$ + $a_{q+\ell-3}-a_{q-\ell+1}$	$a_{q-\ell+1}-a_{q+\ell-1}$ + $a_{q+\ell-2}-a_{q-\ell+2}$
row ℓ	$a_{q-\ell-1}-a_{q+\ell-1}$ + $a_{q+\ell-2}-a_{q-\ell}$	$a_{q-\ell}-a_{q+\ell}$ + $a_{q+\ell-1}-a_{q-\ell+1}$
row $\ell+1$	$a_{q-\ell-2}-a_{q+\ell}$ + $a_{q+\ell-1}-a_{q-\ell-1}$	$a_{q-\ell-1}-a_{q+\ell+1}$ + $a_{q+\ell}-a_{q-\ell}$

To identify D_1 with the matrix $A_{n-1}-B_{n-1}$, in (5), we have to eliminate the encircled coefficients in the determinant D_1.

(3) To show the above we rewrite for simplcity only the coefficients in the column q of the D_1 determinant as follows:

row 1	$a_{q-1}-a_{q+1}$ $+$ $\cancel{a_q}-\cancel{a_q}$
row 2	$a_{q-2}-a_{q+2}$ $+$ $a_{q+1}-a_{q-1}$
row 3	$a_{q-3}-a_{q+3}$ $+$ $a_{q+2}-a_{q-2}$
. . . .	. . . .
row ℓ	$a_{q-\ell}-a_{q+\ell}$ $+$ $a_{q+\ell-1}-a_{q-\ell+1}$
:	:
row n-1	$a_{q-n+1}-a_{q+n-1}$ = 0, if $q\neq 0$, or n-1, when q=0, it is equal to a_{n-1}, when q=n-1 it is equal to a_o $a_{q+n-2}-a_{q-n+2}$
	column q

To cancel the encircled coefficients in D_1 or in above, we perform the following operations: Add up row 1 to 2, we cancel out the encircled term in (2), then considering the new row 2 obtained and adding this row to row 3, we cancel out the encircled terms in row (3), we continue this process to cancel all the encircled terms. Finally we obtain the column q in D_1 as follows:

Column q in D_1

row 1	$a_{q-1}-a_{q+1}$
row 2	$a_{q-2}-a_{q+2}$
row 3	$a_{q-3}-a_{q+3}$
.	.
.	.
row ℓ	$a_{q-\ell}-a_{q+\ell}$
.	.
.	.
.	.
row n-1	0, if $q \neq 1$, when $q=1$, it is equal to $(-a_n)$ or $q \neq n-1$ $q=n-1$, it is equal to a_o

Comparing this column with column q in Eqn. (5), we readily establish the equivalence which is valid for all q columns and ℓ rows: Thus the identity between $A_{n-1}-B_{n-1}$ and D_1 is established. Therefore Eqn. (1) is verified.

Following the same procedure with the appropriate operations, Eqn. (2) can be similarly verified and thus the second property is demonstrated.

Proof of the Third Property of A_k's and B_k's:

The third property as given in Eqn (31) is represented as follows:

$$A_{n-1}+B_{n-1} = (a_o+a_2+a_4+ \dots a_{2p}+\dots)\ A_{n-2}-(a_1+a_3+ \dots +a_{2p+1}+\dots)B_{n-2}$$

The above relationship is also equivalent to the following,

$$A_{n-1} + B_{n-1} = 1/2 \left\{F(1)\ (A_{n-2}-B_{n-2}) + F(-1)(A_{n-2}+B_{n-2})\right\}$$

In this discussion a rigorous proof of the above relationship will be obtained from which the above property is established. Furthermore by combining the third property with the second, we will establish the limiting case of the first property, i. e., $k = n-1$.

The proof will be based on determinant manipulations, by using the same properties as in the previous case.

The determimant of $A_{n+1}+B_{n-1}$ can be written as:

$$
A_{n-1}+B_{n-1} = \left| X_{n-1}+Y_{n-1} \right| =
\begin{array}{r}
\\ \\ \\ \\ \text{row } 2\ell \\ \text{row } 2\ell+1 \\ \text{row } 2\ell+2 \\ \\ \\
\end{array}
\begin{array}{ccccc}
 & & \text{column } 2q+1 & \text{column } 2q+2 & \\
\end{array}
\left|
\begin{array}{ccccc}
a_o+a_2 & a_1+a_3 & a_{2q}+a_{2q+2} & a_{2q+1}a_{2q+3} & a_{n-2}+a_n \\
a_3 & a_0+a_4 & a_{2q-1}+a_{2q+3} & \cdot & a_{n-3} \\
a_4 & a_5 & a_{2q-2}+a_{2q+4} & \cdot & a_{n-4} \\
\vdots & & & \vdots & \\
a_{2\ell+1} & \cdots\cdots & a_{2q-2\ell+1}+a_{2q+2\ell+2} & & \vdots \\
a_{2\ell+2} & & a_{2q-2\ell}+a_{2q+2\ell+2} & & \vdots \\
a_{2\ell+3} & & a_{2q-2\ell-1}+a_{2q+2\ell+3} & & \vdots \\
\vdots & & & & \vdots \\
a_n & 0 \quad 0 & \cdots\cdots\cdots\cdots\cdots\cdots 0 & 0 \quad 0 & a_o
\end{array}
\right|
$$

For the determinant $A_{n-2} \pm B_{n-2}$, we can write in a similar fashion, however we concentrate on the general rows and columns as shown:

<u>Matrix of $A_{n-2} \pm B_{n-2}$</u>

$1 \leq \ell <$	column $2q+1$	column $2q+2$
row 2ℓ	$a_{2q-2\ell+1} \pm a_{2q+2\ell+2}$	$a_{2q-2\ell+2} \pm a_{2q+2\ell+3}$
row $2\ell+1$	$a_{2q-2\ell} \pm a_{2q+2\ell+3}$	$a_{2q-q\ell+1} \pm a_{2q+2\ell+4}$
row $2\ell+2$	$a_{2q-2\ell-1} \pm a_{2q+2\ell+4}$	$a_{2q-2\ell} \pm a_{2q+2\ell+5}$

with $a\mu = 0$ if $\mu < u$ or $\mu > n$

Procedure for the Proof:

1. Using the matrix $A_{n-1} + B_{n-1}$, we add to row 1, the rows $3, 5, 7 \ldots 2\ell+1, \begin{pmatrix} n-1 \text{ if } n=2p \\ n \quad \text{if } n=2p+1 \end{pmatrix}$. It is readily seen that the first row of the new matrix obtained becomes:

	column 1	column 2	....	column 2q+1	column 2q+2
row 1	$\sum a_{2p}$	$\sum 2p+1$	....	$\sum a_{2p}$	$\sum a_{2p+1}$

with $\sum a_{2p} = a_o + a_2 + a_4 + a_6 + \ldots + a_{2q} + \ldots$

$$\sum a_{2p+1} = a_1 + a_3 + a_5 + a_7 + \ldots + a_{2q+1} + \ldots$$

2. Then subtract from the columns $2q+1$, $\begin{matrix} o < q < p \text{ if } n=2p+1 \\ o \leq q < p-1 \text{ if } n=2p \end{matrix}$ the columns $2q-1$. Similarly we subtract from the columns $2q$, the columns $2q-2$, this operation being performed step by step. For instance if $n=6$, we first subtract the column 3 from the column 5. Then the column 1 from 3. Similarly we subtract the column 4 from 6 and then the column 2 from 4. By performing this operation, we notice that the first row contains all zeros except for the first and second columns, where the coefficients are now $\sum a_{2p}$ and $\sum a_{2p+1}$, and by noting that

$$\frac{1}{2}\,[F(1) + F(-1)] = \sum a_{2p}, \text{ and } \frac{1}{2}\,[F(1) - F(-1)] = \sum a_{2p+1},$$

the first row of $A_{n-1} + B_{n-1}$ becomes:

		column 1	column 2	
$A_{n-1}+B_{n-1}$	Row 1	$\frac{1}{2}\,[F(1)+F(-1)]$	$\frac{1}{2}\,[F(1)-F(-1)]$	000...000...00

3. Add column 2 to column 1, then in the new determinant multiply column 2 by 2, and divide the determinant by 2, and then subtract column 1 from column 2, we get for the determinant $A_{n+1}+B_{n-1}$:

$$
A_{n-1}+B_{n-1} = \left\{\frac{1}{2}\right\}
\begin{array}{c}
\begin{array}{cc} \text{column 1} & \text{column 2} \end{array} \\
\left|\begin{array}{cc|c|c|c}
F(1) & F(-1) & & 0 & 0 \\
\vdots & \underbrace{\Delta}_{} & \vdots & \vdots & \vdots
\end{array}\right|
\end{array}
$$

in the determinant Δ, the following coefficients in the general row and column appear:

		column 2q+1	column 2q+2
	Row 2ℓ	$a_{2q-2\ell+1}+a_{2q+2\ell+1}$ $-a_{2q-2\ell-1}-a_{2q+2\ell-1}$	$a_{2q-2\ell+2}+a_{2q+2\ell+2}$ $-a_{2q-2\ell}-a_{2q+2\ell}$
$1<\ell<$ $o<q<$	Row $2\ell+1$	$a_{2q-2\ell}+a_{2q+2\ell+2}$ $-a_{2q-2\ell-2}-a_{2q+2\ell}$	$a_{2q-2\ell+1}+a_{2q+2\ell+3}$ $-a_{2q-2\ell-1}-a_{2q+2\ell+1}$

We expand the previous determinant with respect to the first row to obtain:

$$
A_{n-1} + B_{n-1} = \frac{1}{2}\,[F(1)\,D_2 + F(-1)\,D_1]
$$

where D_2 and D_1 are the appropriate determinant obtained from Δ.

If we can show that $D_2 = A_{n-2}-B_{n-2}$ and $D_1 = A_{n-2}+B_{n-2}$, then we complete the proof of the third property.

4. We will demonstrate first the equivalence between $A_{n-2}+B_{n-2}$ and D_1 as follows: D_1 is obtained from Δ as,

$D_1 =$

	column 1	column 2	…	column 2q	column 2q+1
Row 1	$(a_0+a_3)+a_4$	$a_1+a_5-a_3$	…		
Row 2	$(a_4)+a_5$	$a_0+a_6-a_4$	…		
⋮	⋮	⋮	⋮	⋮	⋮
Row $2\ell-1$	$(a_{2\ell+1})+a_{2\ell+2}$	$a_{3\ell+2}-a_{2\ell+1}$	…	$a_{2q-2\ell+1}+a_{2q+2\ell+1}$ $-a_{2q-2\ell-1}-a_{2q+2\ell-1}$	$a_{2q-2\ell+2}+a_{2q+2\ell+2}$ $-a_{2q-2\ell}-a_{2q+2\ell}$
Row 2ℓ	$(a_{2\ell+2})+a_{2\ell+3}$	$a_{2\ell+4}-a_{2\ell+2}$	…	$a_{2q-2\ell}+a_{2q+2\ell+2}$ $-a_{2q-2\ell-2}-a_{2q+2\ell}$	$a_{2q-2\ell+1}+a_{2q+2\ell+3}$ $-a_{2q-2\ell-1}-a_{2q+2\ell+1}$
⋮	⋮	⋮	⋮	⋮	⋮
n-4	$(a_{n-2})+a_{n-1}$	a_n-a_{n-2}	…		
n-3	$-(a_{n-1})+a_n$	$-a_{n-1}$	…		
n-2	(a_n)	$-a_n$	…		

It should be noted that the elements in column q and row ℓ of D_1 are identical to the elements in column $q+1$ and row $\ell+1$ of Δ or of $A_{n-1}+B_{n-1}$.

5. We subtract row n-2 from row n-3, then after simplifying the new row n-3, we subtract it from row n-4, and proceed in the same fashion up to the first row. Then we add column 1 of the new determinant to column 2, then after simplification, we add the new column 2 to column 3 and repeat the same process. We finally obtain for D_1 the following:

$$D_1 = \begin{array}{c} \\ \text{row 1} \\ \text{row 2} \\ \\ \text{row} 2\ell-1 \\ \text{row } 2\ell \end{array} \begin{array}{c} \begin{array}{ccccc} \text{column 1} & \text{column 2} & & \text{column } 2q & \text{column } 2q+1 \end{array} \\ \left| \begin{array}{ccccc} a_o+a_3 & a_1+a_4 & \cdots & a_{2q-1}+a_{2q+2} & a_{2q}+a_{2q+3} \\ a_4 & a_o+a_5 & \cdots & a_{2q-2}+a_{2q+3} & a_{2q-1}+a_{2q+4} \\ & & \cdots & & \\ a_{2\ell+1} & a_{2\ell+2} & \cdots & a_{2q-2\ell+1}+a_{2q+2\ell} & a_{2q-2\ell+2}+a_{2q+2\ell+1} \\ a_{2\ell+2} & a_{2\ell+3} & \cdots & a_{2q-2\ell+}\,a_{2q+2\ell+1} & a_{2q-2\ell+1}+a_{2q+2\ell+2} \end{array} \right| \end{array}$$

It is noticed that the new expression of D_1 is identical to the expression of $A_{n-2}+B_{n-2}$, as shown earlier on p. (29). Therefore $D_1 = A_{n-2}+B_{n-2}$. Similarly we can show the identity of $D_2 = A_{n-2} - B_{n-2}$.

Therefore with this equivalence we have verified the following identity:

$$A_{n-1}+B_{n-1} = \frac{1}{2}\left\{F(1)(A_{n-2}-B_{n-2})+F(-1)(A_{n-2}+B_{n-2})\right\}$$

or equivalently the third property

$$A_{n-1}+B_{n-1} = (a_o+a_2+a_4+\ldots a_{2p}+\ldots) A_{n-2}-(a_1+a_3+a_5\ldots)B_{n-\ell}$$

is established.

<u>Discussion of the first property:</u>

The first property as given in Eqn. 25, can be also written as:

$$A_k^2 - B_k^2 = A_{k-1}A_{k+1} - B_{k-1}B_{k+1}, \quad k = 2, 3, 4, 5\ldots n-1$$

We can readily verify the above property for the limiting case, i.e., when $k = n-1$, by combining the second and third properties discussed earlier as follows:

The third property gives:

$$A_{n-1}+B_{n-1} = (a_o+a_2+a_4\ldots+a_{2p}+\ldots)A_{n-2}-(a_1+a_3\ldots) B_{n-2}$$

The second property gives:

$$A_n = (a_o+a_2+\ldots a_{2p}+\ldots)(A_{n-1}-B_{n-1})$$

and

$$B_n = (a_1+a_3+\ldots+a_{2p+1}+\ldots)(A_{n-1}-B_{n-1})$$

if we multiply the third property by $A_{n-1}-B_{n-1}$ and use the second property we obtain:

$$A_{n-1}^2 - B_{n-1}^2 = A_nA_{n-2} - B_nB_{n-2}$$

The above is exactly the first property for the limiting case, i.e., when $k = n-1$. By actual expansion, the first property has also been verified for $k = 2, 3, 4, 5$. In order to complete the proof it has to be shown that it is valid for any k between 5 and n-1. This proof could be achieved in one of the following two procedures:

a) By determinant manipulation as has been done for the other properties, if we write for the first property the following equivalent form:

$$(A_k-B_k)(A_k+B_k) = \frac{1}{2}\left[(A_{k-1}-B_{k-1})(A_{k+1}+B_{k+1}) + (A_{k-1}+B_{k-1})(A_{k+1}-B_{k+1})\right] \quad *$$

b) By induction method, i. e., to show if it is valid for k-1, it is also valid for k.

Both the above procedures involve difficult and complicated manipulations which were not attempted in this report. However, we may present a simple heuristic argument to indicate that the first property holds for all k. This is based on the following observation.

If we assume a certain n, i. e., $n = 5$, then the stability constraints are given as $1 > 0$, $\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$, $\Delta_4 > 0$, $\Delta_5 < 0$. Now if we assume any general $n > 5$, the stability constraints are given by $1 > 0$, $\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$, $\Delta_4 > 0$, $\Delta_5 < 0$, $\Delta_6 > 0$, $\Delta_7 < 0\ldots$ The Δ's for the general case up to Δ_5 are the same as the Δ's for $n = 5$ except for replacement of the specific $n = 5$, the general n in computating the determinants. Furthermore, any relationship that holds between the Δ's, i. e., $A_k^2-B_k^2 = A_{k-1}A_{k+1}-B_{k-1}B_{k+1}$, $k=2, 3, 4$, for $n= 5$, also holds for any n. Thus one may deduce that. "If the first property is verified for any specific n, it also holds for any n."

Based on the above deduction, we can use the limiting case of the first property to extend the range for $n=6, 7\ldots$ For instance if $n= 7$, then the first property verified for $k=n-1$, becomes also valid for $k=6$ and for all n. Similarly we may proceed step by step in the same fashion to cover all the intermediate cases of k.

---

*By showing that this equation which holds for $k=5$ and $n=6$, to be valid for $k=5$ and any "n", then a rigorous proof has been constructed by using the limiting case to extend the range of k.

Admittedly, the above argument doesn't constitute a rigorous proof but only indicates a convincing argument that the first property cannot be violated for any k between 5 and $(n-1)$. One can also use the expansion method to verify the results for higher "k", however, this again involves a complicated procedure.

In summary, the material of the appendix yields rigorous proofs for the second and the third property and from these properties a heuristic argument for the validity of the first property is indicated.

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