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DEPARTMENT OF CIVIL ENGINEERING AND ENGINEERING MECHANICS

ON THE CONDUCTION OF HEAT IN A
MELTING SLAB

by

Stephen J. Citron

XEROX

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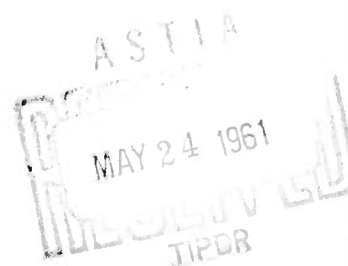On the Conduction of Heat in a Melting Slab

by

Stephen J. Citron*

Purdue University

Abstract: A new method for the solution of the problem of heat conduction
in a melting slab, where the molten material is immediately removed upon
formation, is presented. No restrictions are placed on the boundary con-
ditions which may be imposed on the slab and the material properties are
allowed to be temperature dependent. The problem of determining the tem-
perature distribution in the slab and the amount of material melted is re-
duced to finding the solution of an ordinary differential equation on the amount
of material melted. This reduction from a partial differential equation prob-
lem is accomplished by determining a Taylor's series expansion in space for
the temperature distribution. The equation so obtained for the determination
of the amount of material melted is of a form readily solved numerically.
Comparisons with known results for a slab insulated on one face and subjected
to a constant heat input on the other face are given.

INTRODUCTION

Problems involving free (moving) boundaries are of great current interest in heat conduction. However, due to the difficulties caused by the non-linearity of the problem when the motion of the boundary is unknown analytical solutions in general cannot be found. Work done by Landau [1], Dewey, Schlesinger, Sashkin,[2] and Lotkin [3] among others all utilize high speed computers for the solution of the partial differential equations involved. Because of the inherent complexity of obtaining numerical solutions to the non-linear partial differential equations it is of interest to consider if the problem can be further reduced before any numerical work is begun.

Recently Boley [4] developed a method in which the problem is reformulated so as to require the solution of two ordinary integro-differential equations. By expanding the equations in powers of the time after melting starts, an exact analytical solution is obtained which is particularly useful for small times. The author [5] devised a method of successive approximations by means of which a solution may be obtained by solving an ordinary differential equation on the amount of material melted.

The purpose of the present work is to present a new method of reducing the problem of determining the temperature distribution and the amount of material melted to one requiring the solution of an ordinary differential equation. It differs from the method of reference [5] in that it eliminates the need for the successive approximation procedure

used there. The main feature of the present method is the expansion
in space of the temperature distribution in a Taylor series. The one
boundary condition on the temperature which is not identically satisfied
by the expansion yields an ordinary differential equation for the
determination of the amount of material melted. The initial conditions
for this new equation are found by using the initial temperature dis-
tribution or an analytical starting solution as will be discussed later.
Once this equation has been solved (and hence the amount of material
melted as a function of time is known) the temperature distribution at
any time can be immediately calculated.

A comparison of solutions obtained by this method with certain
known results for a slab insulated on one face and subjected to a
constant heat input on the other are given. The example used to demonstrate
the method was chosen as it is an important limiting case encountered
in ablation problems. In addition, this problem has been studied by
others using techniques more complex than the method to be presented.
The problem thus furnishes, by comparison, an illustration of the
simplicity of the method developed here.

PROBLEM FORMULATION

Consider the slab shown in Fig. (1) with temperature-dependent
thermal properties subjected to an arbitrary heat input $Q(t)$ on one face.
No restrictions are placed on the boundary conditions which may be
prescribed on the other face of the slab. Once melting occurs at the
heated side the problem to be solved requires the determination of the
temperature distribution $T(x,t)$ in the slab and the amount of material
melted as a function of time $s(t)$. The molten material is taken to be
immediately removed upon formation.

For the determination of $T(x,t)$ and $s(t)$ the solution of the heat conduction equation

$$\frac{\partial}{\partial x} \left\{ k(T) \frac{\partial T}{\partial x} \right\} = \rho C(T) \frac{\partial T}{\partial t} \quad s(t) < x < \ell \tag{1}$$

must be found subject to two initial condition and three boundary conditions

$$
\begin{aligned}
&\text{a)} \quad T(x, t^*) = T_0(x) \\
&\text{b)} \quad T(s, t) = T^* \\
&\text{c)} \quad Q(t) = -k(T^*) \frac{\partial T(s,t)}{\partial x} + \rho L \dot{s} \\
&\text{d)} \quad G(T, \frac{\partial T}{\partial x}, \ldots) = g(t) \\
&\text{e)} \quad S(t^*) = 0
\end{aligned}
\tag{2}
$$

The first condition gives the temperature distribution at the start of melting $t = t*$. Condition b) requires that the melting face be maintained at the melting temperature $T*$ while condition c) specifies the division of the incident heat flux between the part entering the solid and the part going toward overcoming the latent heat of melting L^*. The function $G(T, \partial T/\partial x, \ldots)$ in condition d) represents an arbitrary boundary condition on the temperature at the face $x = 1$. Condition e) is an initial condition on the amount of material melted.

Before proceeding with the general solution it is convenient to fix the moving boundary and put Eq. (1) and Eq. (2) in non-dimensional form. We first define new non-dimensional in/dependent variables

---

* The condition of radiation incident on the melting boundary may be expressed in the same form as condition c) in Eq. (2).

$$z = \frac{\ell - x}{\ell - s} \qquad , \qquad \tau = \frac{K(T^*)}{\ell^2}[t - t^*] \qquad (3)$$

The slab is now of fixed unit length while τ measures the non-dimensional time after melting has begun. The following dimensionless quantities are also introduced*

$$\theta(z,\tau) = \frac{T(z,\tau)}{T^*} \qquad \tilde{Q}(\tau) = \frac{Q(\tau)}{Q_o} \qquad \tilde{R}(\tau) = \frac{R(\tau)}{R(T^*)} \qquad (4)$$

$$S(\tau) = \frac{s(\tau)}{\ell} \qquad \dot{S}(\tau) = \frac{d}{d\tau}S(\tau) \qquad \tilde{C}(\tau) = \frac{C(\tau)}{C(T^*)}$$

$$\delta(\tau) = [1 - S(\tau)] \qquad M = \frac{\pi}{2}^{1/2}\frac{C(T^*)T^*}{L} \qquad r = \frac{R(T^*)T^*}{\ell Q_o}$$

Eq. (1) then becomes

$$\frac{\partial}{\partial z}\left\{\tilde{R}\frac{\partial\theta}{\partial z}\right\} = \tilde{C}\,\delta^2\frac{\partial\theta}{\partial\tau} + \tilde{C}\,\delta\,\dot{S}\,z\,\frac{\partial\theta}{\partial z} \qquad 0 < z < 1 \qquad (5)$$

under the conditions corresponding to those of Eq. (2) given below

a) $\theta(z,0) = \dfrac{T_0(z)}{T^*}$

b) $\theta(1,\tau) = 1$ (6)

c) $\dfrac{\partial\theta(1,\tau)}{\partial z} = \dfrac{\delta\tilde{Q}}{r} - \dfrac{\pi}{2}^{1/2}\dfrac{\delta\dot{S}}{M}$

d) $G(\theta, \frac{\partial\theta}{\partial z}, ..) = g(\tau)$ at $z = 0$

e) $S(0) = 0$

It may also be noted from Eq. (6c) that continuity of $\partial\theta/\partial z$ at $\tau = 0$ requires that $\dot{S} = 0$ for $M \neq \infty$

* The quantity Q_o is a reference value of the heat imput.

PROBLEM SOLUTION

The method of solution depends on the assumption that it is possible to express the temperature distribution in the slab in a power series expansion in space about the melting face $z = 1$. This assumption is to be checked a posteriori by verifying that all equations and conditions are satisfied by a convergent series or, lacking that, that the expansion formally satisfies all differential equations and conditions and in addition checks some known results.

The Taylor series expansion of the temperature is given formally by

$$\Theta(\mathfrak{z},\tau) = \Theta(1,\tau) + \frac{\partial\Theta}{\partial\mathfrak{z}}(1,\tau)(\mathfrak{z}-1) + \frac{\partial^2\Theta}{\partial\mathfrak{z}^2}(1,\tau)\frac{(\mathfrak{z}-1)^2}{2!}+\dots \quad (7)$$

where the coefficients of the powers of $(z - 1)$ must be determined. These coefficients are found by utilizing the boundary conditions at $z = 1$ and the differential equation governing the problem Eq. (5). From Eq. (6 b,c) we have

$$\Theta(1,\tau) = 1$$

$$\frac{\partial\Theta}{\partial\mathfrak{z}}(1,\tau) = \frac{\delta\tilde{Q}}{\tau} - \frac{\pi^{1/2}}{2}\frac{\delta\dot{S}}{M} \quad (8)$$

The higher order coefficients are then obtained by using these expressions with Eq. (5). For this purpose we rewrite Eq. (5) in the form

$$\frac{\partial^2\Theta}{\partial\mathfrak{z}^2} = \frac{\tilde{c}}{\tilde{R}}\mathfrak{z}^2\frac{\partial\Theta}{\partial\tau} + \frac{\tilde{c}}{\tilde{R}}(\delta\dot{S}\mathfrak{z} - \frac{\partial\tilde{R}}{\partial\mathfrak{z}})\frac{\partial\Theta}{\partial\mathfrak{z}} \quad (9)$$

Thus the second derivative of θ with respect to z, evaluated at $z = 1$, is given in terms of the first derivative of θ with respect to z and of the value of θ at $z = 1$, both known quantities from Eq. (8). In a similar manner all higher derivatives of θ may be found from Eq. (9) by successively differentiating both sides. The n'th derivative of θ with respect to z will be a function of the n - 1 preceeding derivatives which will have already been evaluated. As examples the third and fourth differential coefficients are given below.*

$$\frac{\partial^2 \theta}{\partial z^2}(1,\tau) = \left[\delta \, \dot{S} - \frac{\partial}{\partial z}\tilde{R}(1,\tau) \right] \frac{\partial \theta}{\partial z}(1,\tau)$$

$$\frac{\partial^3 \theta}{\partial z^3}(1,\tau) = \delta^2 \frac{\partial}{\partial \tau} \frac{\partial \theta}{\partial z}(1,\tau) + \frac{\partial}{\partial z}\tilde{R}(1,\tau)\left[\delta\dot{S} - \frac{\partial}{\partial z}\tilde{R}(1,\tau) \right] \frac{\partial \theta}{\partial z}(1,\tau) \qquad (10)$$

$$+ \left[\delta\dot{S} - \frac{\partial^2}{\partial z^2}\tilde{R}(1,\tau) \right] \frac{\partial \theta}{\partial z}(1,\tau) + \left[\delta\dot{S} - \frac{\partial}{\partial z}\tilde{R}(1,\tau) \right] \frac{\partial^2 \theta}{\partial z^2}(1,\tau)$$

The term $\dfrac{\partial}{\partial \tau} \dfrac{\partial \theta}{\partial z}(1,\tau)$ occurring in the expression for $\dfrac{\partial^3 \theta}{\partial z^3}(1,\tau)$ above is found by differentiating Eq. (6c) with respect to τ.

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* It has been assumed that it is possible to interchange the order of τ and z differentiations. Note also that $\tilde{R}(1,\tau) = 1$ from its definition.

Once the coefficients of the expansion have been obtained, substitution of Eq. (7) into Eq. (6d) [the only boundary condition which has not been identically satisfied by the expansion] yields an ordinary differential equation for the determination of $S(\tau)$. If, for example, the slab were insulated at $z = o$ then Eq. (6d) becomes $\frac{\partial \theta}{\partial \zeta}(o,\tau) = o$ and substitution of Eq. (7) into this relation yields the equation

$$o = \frac{\partial \theta}{\partial \zeta}(1,\tau) - \frac{\partial^2 \theta}{\partial \zeta^2}(1,\tau) + \frac{1}{2!}\frac{\partial^3 \theta}{\partial \zeta^3}(1,\tau) - \cdots \quad (11)$$

for the determination of $S(\tau)$.

Note that the first term is the expansion of $\theta(z,\tau)$ is a constant, the second and third terms contain $S'(\tau)$ and $\dot{S}'(\tau)$, the fourth and fifth terms contain $S(\tau)$, $\dot{S}'(\tau)$ and $\ddot{S}(\tau)$. In general the $2n$ and $2n + 1$ terms of the expansion will contain $S(\tau)$ and its first n derivatives. It may also be seen from the expansion that the highest derivative of $S(\tau)$ occurring always appears in a linear manner.

INITIAL CONDITIONS

In practice the series for the temperature will be terminated after a finite number of terms, say $2n$ or $2n + 1$. The n'th order differential equation on $S(\tau)$ resulting from satisfying Eq. (6d) then requires n initial conditions of which only two are already known, namely $S(o) = \dot{S}(o) = o$. Two ways are now suggested of determining the additional initial conditions required to start the solution. The two methods given here will later be compared through examples.

1) If an analytical solution for $S'(\tau)$ is known in the neighborhood of $\tau = 0$ then $S(\tau)$ and the first $n - 1$ derivatives of $S(\tau)$ may be calculated at some time τ, within the range of validity of the solution. The values of $S(\tau_1), \frac{\partial S(\tau_1)}{\partial \tau}, \cdots \frac{\partial^{m-1} S'(\tau_1)}{\partial \tau^{m-1}}$ thus calculated are used as initial conditions for starting the numerical solution of Eq. (6d) at $\tau = \tau_1$. The solution proceeds by substituting these values into Eq. (6d) from which a value of $\frac{\partial^m S(\tau_1)}{\partial \tau^m}$ is calculated. Having a value of $\frac{\partial^m S(\tau_1)}{\partial \tau^m}$ the values of $S(\tau_2), \frac{\partial S}{\partial \tau}(\tau_2), \cdots \frac{\partial^{n-1} S'}{\partial \tau^{m-1}}(\tau_2)$ where

$\tau_2 = \tau_1 + \delta\tau$ are determined from

$$S(\tau_2) = S(\tau_1) + \delta\tau \, \frac{\partial S(\tau_1)}{\partial \tau}$$

$$\frac{\partial S'(\tau_2)}{\partial \tau} = \frac{\partial S(\tau_1)}{\partial \tau} + \delta\tau \, \frac{\partial^2 S(\tau_1)}{\partial \tau^2}$$

$$- - - - - - - - - - - - -$$

$$\frac{\partial^{m-1} S(\tau_2)}{\partial \tau^{m-1}} = \frac{\partial^{m-1} S(\tau_1)}{\partial \tau^{m-1}} + \delta\tau \, \frac{\partial^m S'(\tau_1)}{\partial \tau^m}$$

(12)

The process is then repeated so as to cover the time interval of interest. It should be noted that the initial condition on the temperature Eq. (6a) is automatically satisfied by using the exact starting solution.

A method of obtaining the analytical starting solution required by the above procedure has been given by Boley [4] who explicitly gives as an example the starting solution for the case of a semi-infinite solid of constant thermal properties under constant heat flux at the melting boundary. The analogous formula for the slab obtained by the method of

reference [4] is later given in the present paper for use as a starting
solution in the examples which follow. Boley's method, presented here
in brief for completeness, consists in dealing mathematically with a
fictitious solid of constant dimensions under an equivalent unknown
heat input. This results in two ordinary integro-differential equations
for $S'(\tau)$ and the fictitious heat input which must be solved simul-
taneously. By expanding these quantities about $\tau = o$ in the proper
manner it is possible to determine an analytical solution valid about
$\tau = o$.

2) When the thermal properties of the slab are variable it
may prove difficult to determine an analytical starting solution, for
the period after melting has begun, as described above. We therefore
require a different method of determining values of $S, \dfrac{\partial S'}{\partial \tau}, \cdots \dfrac{\partial^{m-1} S'}{\partial \tau^{m-1}}$
at $\tau = o$ to start the numerical solution of Eq. (6d) and we do this
by satisfying the one remaining condition, the initial condition on
the temperature Eq. (6a), in an approximate manner. If, for example,
2n or 2n + 1 terms are retained in the expansion of θ given by Eq. (7)
then there are n initial conditions to be determined, namely the values
of $S(o), \dfrac{\partial S'(o)}{\partial \tau}, \cdots \dfrac{\partial^{m-1} S(o)}{\partial \tau^{m-1}}$. We have already shown,
however, that $S(0) = \dot{S}(0) = o$ which leaves n - 2 initial values to
be found $\dfrac{\partial^2 S(o)}{\partial \tau^2}, \dfrac{\partial^3 S(o)}{\partial \tau^3}, \cdots \dfrac{\partial^{m-1} S(o)}{\partial \tau^{m-1}}$. These n - 2

values are used to match the initial temperature distribution at n - 2
points therefore approximately satisfying the only remaining condition.*
The numerical integration can now proceed exactly as given under method
1 above.

EXAMPLE

The general method of solution given above and the two methods
of starting the solution will now be illustrated by considering the
problem of a slab of constant thermal properties insulated at $x = 1$
and subjected to a constant heat flux Q on the other side. The accuracy
of the results will be demonstrated by comparison with the conditions
which are known exactly for the problem . [1],[4],[5] These conditions
are:

1) the total time for complete melting   given by

$$\tau_{\ell} = \frac{1}{3} + \frac{\pi^{1/2}}{2} \frac{\tau}{M} - \frac{2}{\pi^2} \sum_{1}^{\infty} e^{-\frac{n^2 \pi^2 \kappa t^*}{\ell^2}} \tag{13}$$

2) the melting rate when $s = \ell$; $S = 1$

$$\dot{S}_f = \frac{2M}{\pi^{1/2} \tau} \tag{14}$$

3) the temperature distribution at the start of melting
and the temperature distribution in the neighborhood of this time derived
by the method of reference [4].

---

* Note that since Eq. (6b, c,d) will be satisfied the approximate temper-
ature distribution at $\tau = o$ actually matches the exact distribution at
n - 1 points and has in addition the correct slope at $z = 1$ and the
correct condition at $z = o$.

The temperature distribution at the start of melting is given by [6]

$$\Theta(x, \mathcal{I}^*) = \frac{1}{r\alpha} \left\{ ierfc \frac{x\alpha}{\ell} + \sum_{1}^{\infty} \left[ierfc (2m + \frac{x}{\ell})\alpha \right. \right.$$

$$\left. \left. + ierfc (2m - \frac{x}{\ell})\alpha \right] \right\} \qquad (15)$$

where

$$\alpha = \frac{\ell}{2\sqrt{\kappa \mathcal{I}^*}}$$

Since $\Theta(0, \mathcal{I}^*) = 1$ we find for the relation between r and α

$$r = \frac{1}{\sqrt{\pi}\,\alpha} (1 + 2\sqrt{\pi} \sum_{1}^{\infty} ierfc\ 2m\alpha) \qquad (16)$$

In the numerical work which follows we shall use the numerical values

$$M = 2 \quad , \quad r = 2 \qquad (17)$$

Substitution of this value of r into Eq. (16) then yields $\alpha = .39$

We proceed by first giving the analytical solution for $S(\tau)$ valid in the neighborhood of $\tau = 0$ which is derived by the method of reference [4]

$$S(\tau) = b_0 \tau^{3/2} + b_1 \tau^2 + b_2 \tau^{5/2} + \cdots \qquad (18)$$

where

$$b_0 = -\frac{4}{3\sqrt{\pi}} \frac{M}{r} a_0$$

$$b_1 = -\frac{1}{\sqrt{\pi}} \frac{M}{r} a_1$$

$$b_2 = \frac{2M}{5\pi r} \left\{ \pi a_0 b_0 - 2\sqrt{\pi} a_2 + 4\alpha b_0 \left(1 + 2\sum_1^\infty e^{-4m^2\alpha^2}\right) \right\}$$

and

$$a_0 = -2\alpha \left\{ \frac{2}{\pi} + \frac{8}{\sqrt{\pi}} \sum_1^\infty m\alpha \, \text{erfc} \, 2m\alpha + \frac{4}{\sqrt{\pi}} \sum_1^\infty \text{erfc} \, 2m\alpha \right\} \tag{19}$$

$$a_1 = \frac{3}{4} \sqrt{\pi} \, b_0$$

$$a_2 = \frac{8}{3\pi} \left\{ \sqrt{\pi} a_0 b_0 + 2\alpha^3 + \sqrt{\pi} b_1 + 4\alpha^3 \sqrt{\pi} \sum_1^\infty 2m\alpha \, \text{erfc} \, 2m\alpha \right.$$
$$\left. - 8\alpha^3 \sum_1^\infty 4m^2\alpha^2 e^{-4m^2\alpha^2} + 4\alpha^3 \sum_1^\infty \text{erfc} \, 2m\alpha \right\}$$

The temperature distribution derived by the same method
of reference [4] is given by

$$\frac{T(x/\ell, \tau)}{T^*} = \frac{a_0}{r\sqrt{\pi}} \left\{ -\frac{\pi}{2} \frac{x}{\ell} \sqrt{\tau} \, \text{erfc} \, \frac{x/\ell}{2\sqrt{\tau}} + \frac{\pi}{2} \tau \, \text{erfc} \, \frac{x/\ell}{2\sqrt{\tau}} \right\}$$

$$+ \frac{a_1}{r\sqrt{\pi}} \left\{ 2\sqrt{\pi} \left(\tau^{3/2} + \frac{x^2/\ell^2 \sqrt{\tau}}{6} \right) \text{erfc} \, \frac{x/\ell}{2\sqrt{\tau}} - \frac{2}{3} \tau^{3/2} e^{-\frac{x^2}{4\ell^2\tau^2}} \right\}$$

$$+ \frac{a_2}{r\sqrt{\pi}} \left\{ \tau^2 \left[\frac{3\pi}{8} + \frac{3x^2}{8\ell^2\tau} + \frac{x^4}{32\ell^4\tau^2} \right] \text{erfc} \, \frac{x/\ell}{2\sqrt{\tau}} + \right.$$

$$-\tau^2\sqrt{\pi}\left[\frac{5}{8}\frac{x/\ell}{\sqrt{\tau}}+\frac{1}{16}\frac{x^3/\ell^3}{\tau^{3/2}}\right]e^{-\frac{x^2}{4\ell^2\tau^2}}\Bigg\}$$

$$+\frac{\sqrt{4\tau+1/\alpha^2}}{\tau}\Bigg\{\text{ierfc}\frac{x/\ell}{[4\tau+1/\alpha^2]^{1/2}} \tag{20}$$

$$+\sum_1^\infty\left[\text{ierfc}\frac{(2m+x/\ell)}{[4\tau+1/\alpha^2]^{1/2}}+\text{ierfc}\frac{(2m-x/\ell)}{[4\tau+1/\alpha^2]^{1/2}}\right]\Bigg\}$$

The expansion for the temperature distribution in this example obtained by retaining the first five terms of Eq. (7) is given by

$$\Theta(\mathfrak{z},\tau)=1+\frac{\delta}{\tau}(1-\frac{\pi^{1/2}}{2}\frac{\tau}{M}\dot{S})(\mathfrak{z}-1)+\frac{\dot{S}}{\tau}\delta^2(1-\frac{\pi^{1/2}}{2}\frac{\tau}{M}\dot{S})\frac{(\mathfrak{z}-1)^2}{2!}$$

$$+\left\{\frac{\dot{S}^2}{\tau}\delta^3(1-\frac{\pi^{1/2}}{2}\frac{\tau}{M}\dot{S})-\frac{\pi^{1/2}}{2M}\delta\ddot{S}\right\}\frac{(\mathfrak{z}-1)^3}{3!} \tag{21}$$

$$+\left\{\frac{\ddot{S}}{\tau}\delta^4(1-\frac{\pi^{1/2}}{2}\frac{\tau}{M}\dot{S})-\frac{\pi^{1/2}}{M}\delta^4\dot{S}\ddot{S}+\delta^4\dot{S}^3(1-\frac{\pi^{1/2}}{2}\frac{\tau}{M}\dot{S})\right\}\frac{(\mathfrak{z}-1)^4}{4!}$$

Utilizing this expansion the condition that the slab be insulated at $x = 1$ ($z = 0$) becomes

$$1=\delta\dot{S}-\frac{1}{2}\left\{\delta^2\dot{S}^2-\frac{\delta^2\ddot{S}}{\frac{2M}{\pi^{1/2}\tau}-\dot{S}}\right\} \tag{22}$$

$$+\frac{1}{6}\left\{\delta^3\dddot{S}+\delta^3\dot{S}^3-\frac{2\delta^3\dot{S}\ddot{S}}{\frac{2M}{\pi^{1/2}\tau}-\dot{S}}\right\}$$

The analytical starting solution of Eq. (18) was used to determine values of $S(\tau)$ and $\dot{S}(\tau)$ at $\tau = .04$, the point chosen to begin the numerical integration. The value of $\ddot{S}(.04)$ was then calculated from Eq. (22) when first four and then five terms of the temperature expansion were retained. Substitution of these values into Eq. (22) retaining the corresponding number of terms then yields expressions for the temperature distribution at $\tau = .04$. The temperature distributions at $\tau = .04$ calculated from Eq. (20) by retaining first four and then five terms in the expansion are compared in Fig. (2) with the exact temperature distribution at this time given by Eq. (20). The figure shows the good agreement obtained.

The solutions for $S(\tau)$ calculated by numerically integrating Eq. (22) after $\tau = .04$ are shown in Fig. (3). The error in the value of τ_{ℓ} is only 4.9% in the four-term expansion and is reduced to 2.5% in the five-term expansion. It may also be seen from Eq. (22) that the correct final melting rate $\dot{S}_{\ell} = \dfrac{2M}{\pi^2 \tau}$ is reached in both the four and five-term expansions.

It remains to illustrate the method of solution when an analytical starting solution is not known. For this purpose seven terms were retained in the expansion for the temperature. The expansion now contains terms up to $\dddot{S}(\tau)$ which enables the matching of one additional

point of the approximate temperature distribution with the exact
distribution at $\tau = 0$ as was previously discussed. The point chosen
was the insulated fact $x = \mathcal{L}$ ($z = 0$). A comparison of the approximate
distribution thus calculated with the exact distribution given by
Eq. (15) is shown in Fig. (4). The agreement between the two temperature
distributions is again good.

The results of numerically integrating Eq. (22) starting at
$\tau = 0$ with the additional terms retained is shown in Fig. (5). A
comparison is made in the figure with the solution obtained by retaining
only five terms of the expansion and starting the numerical integration
at $\tau = .04$ with the exact starting solution.

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Table of Figures

SYMBOLS

x	$=$	space variable
t	$=$	time variable
$k(T)$	$=$	thermal conductivity
$c(T)$	$=$	specific heat
ρ	$=$	density
$\kappa(T)$	$=$	$k/\rho c$, thermal diffusivity
L	$=$	latent heat of melting
l	$=$	slab length
$Q(t)$	$=$	incident heat flux
Q_o	$=$	reference value of the heat flux
$T(x,t)$	$=$	slab temperature distribution
$T_o(x)$	$=$	slab temperature distribution when melting begins
T^*	$=$	melting temperature
t^*	$=$	time at which melting begins
$s(t)$	$=$	position of solid boundary after melting has begun (amount of material melted)
S	$=$	s/l, non-dimensional form of s
z	$=$	$\dfrac{l-x}{l-s}$, non-dimensional space variable
τ	$=$	$\dfrac{\kappa(T^*)}{l^2}(t-t^*)$
τ_1	$=$	non-dimensional time after melting has begun at which slab is completely melted
θ	$=$	T/T^*, non-dimensional temperature variable
$\dfrac{2M}{\pi}$	$=$	$\dfrac{c(T^*)T^*}{L}$, ratio of heat content at melting to latent heat of fusion

$$r = \frac{k(T^*) \, T^*}{1Q_o}$$

$$\widetilde{Q}(\tau) = \frac{Q(\tau)}{\bar{Q}_o}$$

$$\widetilde{\mathcal{R}}(T) = \frac{\mathcal{R}(T)}{\mathcal{R}(T^*)}$$

$$\widetilde{C}(T) = \frac{C(T)}{C(T^*)}$$

FIG. I

FIG.2

TEMPERATURE DISTRIBUTION
$\tau = .04$

———————— EXACT SOLUTION

o o o o FOUR TERMS OF EXPANSION } EXACT STARTING SOLUTION

x x x x FIVE TERMS OF EXPANSION

$M = 2$
$r = 2$
$\alpha = .39$

FIG.3

TEMPERATURE DISTRIBUTION

$\tau = 0$

——— EXACT SOLUTION

x x x x x x SOLUTION USING SEVEN TERMS OF EXPANSION

(APPROXIMATE STARTING SOLUTION)

M = 2
r = 2
α = .39

FIG. 4

FIG.5