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CONT CONTRIBUTIONS TO THE THEORY OF PARTIAL COHERENCE

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ABSTRACT

A brief survey of previous developments in the theory of partial coherence is given in Chapter 1 and the relations between the complex degree of coherence (see Wolf 1955) and the measures of coherence introduced by earlier authors are given. Because of the sundry formulations of the theory to be found in the literature a detailed review of the formalism used here is also given in this chapter.

Coherence theory is formulated in terms of correlation functions and analytic signals and the developments of this thesis required several new theorems concerning the convolution and cross-correlation of analytic signals. These theorems are developed in Chapter 2. The form of the general unimodular analytic signal is also obtained in this chapter. It is shown that when considered as a function of a complex variable this function is a meromorphic function of order one with isolated poles in either the upper or the lower half plane only and with zeros at conjugate points.

In Chapter 3 these theorems are applied to the detailed analysis of the limiting forms of the mutual coherence function for both polychromatic and quasi-monochromatic fields. In the rigorous analysis, applicable to fields of arbitrary spectral width, it is shown that: 1.) an optical field is coherent if and only if it is monochromatic; 2.) the mutual coherence function for a coherent field may be expressed as the product of a simple wave function, evaluated at one point, with its complex conjugate, evaluated at a second point, multiplying a simple periodic factor, $e^{2\pi i \nu_0 \tau}$; 3.) an incoherent field cannot exist in free space though an incoherent source may be defined in a manner consistent with this result. The essential differences between these theorems and the corresponding ones for quasi-monochromatic fields are discussed.

In Chapter 4 the propagation of mutual coherence is studied. A new derivation of the wave equations for this quantity is given and the equations are solved for the field created by an arbitrary plane source, i.e., an extended partially coherent polychromatic source. Using the results of the previous chapter, the limiting forms of the general solution are examined in detail for both polychromatic and quasi-monochromatic illumination. In particular it is shown that : 1.) a coherent source always gives rise to a coherent field; and 2.) an incoherent source always creates a partially coherent field. These results are shown to be valid regardless of the spectral width of the illumination. By examining the general solution, the well-known van Cittert-Zernike theorem is found to be an approximate form of the incoherent limit of the quasi-monochromatic solution.

The results of the previous chapters are applied in Chapter 5 to a frequency domain analysis of the optical imaging problem. A general solution to this problem for partially coherent polychromatic light is obtained and generalized transfer functions are introduced. It is shown that, under the quasi-monochromatic approximation, these generalized transfer functions reduce to the familiar forms found in the literature. A transfer function is introduced for obtaining the mutual spectral density of the image, a function of two points, from the self spectral density of the object, a function of one point.

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CHAPTER 1

INTRODUCTION

It is customary to describe interference and diffraction phenomena in terms of the superposition of strictly coherent or strictly incoherent vibrations ignoring completely the possibility of intermediate states, partial coherence. That this practice is a considerably restrictive idealization is attested to by the fact that experience indicates that both of these extremes are unrealizable.

An example of the inadequacy of the concepts of complete coherence and complete incoherence for the description of physically interesting phenomena was known as early as about 1869. At that time Verdet (1869) demonstrated that the light from two pinholes in a screen illuminated by the sun will interfere in Young's interference experiment if the separation of the pinholes is less than about 1/20mm. Since interference is customarily a property of coherent oscillations, this result suggested the idea of an "area of coherence" surrounding a point in an incoherent field.

Another early example of the inadequacy of these limiting concepts can be found in the work of Michelson from about 1890 to 1920. The interferometric method, introduced by Michelson (1890-1921), of measuring stellar diameters essentially involves measuring the degree of coherence of the illumination produced by the stars. Although this work was not interpreted in terms of coherence theory until much later, the quantity, visibility, introduced by Michelson to describe the quality of interference fringes has proved to be an important key to understanding the concept of partial coherence. In fact, in Zernike's (1938) formulation of coherence theory, the degree of coherence between the vibrations at two points in an optical field is defined as the visibility of the fringes obtained by allowing them to interfere in a suitable experiment.

From the middle of the nineteenth century until the last two decades the theory of partial coherence received but little attention. The few papers that did appear on the subject are more or less disconnected since each investigator introduced his own apparently different formulation of the theory. The lack of interest in the subject during that period may be attributed to the fact that applications in which the theory is important were either unknown or involved measurements which were not refined enough to take account of the degree of coherence.

However, in more recent years the concept of partial coherence has become important in virtually every branch of physics which involves electromagnetic radiation regardless of the frequencies considered. In visible optics, for example, coherence theory is tantamount to the understanding of such topics as image formation and the effect of illumination on the resolution in a microscopic image. In spectroscopy the influence of the slit width on the degree of coherence of the illumination can produce measurable effects. In radio astronomy source diameters are measured by interferometric techniques, which effectively involves the measurement of the degree of coherence of the radiation. Similar problems arise in those applications of radar where questions of resolution and mapping are of central importance.

Even at the much longer wave lengths used in communication, coherence theory plays an important role. In scatter communications, for example, Beran (1958) has recently studied the propagation of the ensemble correlation in the scatter field and with the present author (1957) has shown that the reliability of a spatially diversified scatter system can be computed from a knowledge of the degree of coherence in the reflected field.

In the last few years as these applications became important and as measuring techniques became more refined, the theory of partial coherence has received increasing attention in the fields mentioned above. In spite of this attention several aspects of the theory have been misunderstood; and the strong emphasis on application has left many of the fundamental questions unanswered.

The aim of this thesis is the deduction of certain of the general mathematical and physical implications of coherence theory. For this purpose the most convenient and rigorous formulation of the subject is that introduced by Wolf (1955); and this formulism will be used exclusively in this thesis. However, before reviewing in detail the structure of the theory to be used here, it will prove useful to give a brief survey of the previous research on the subject, paying particular attention to the definitions of degree of coherence introduced by earlier authors. In this way some of the advantages of the formulation used here will become apparent.

1.1 Survey of Previous Researches

Early research on partial coherence is associated with the names of Verdet (1869), von Laue (1907), Berek (1926 a, b, c, d) and van Cittert (1934, 1939). The investigations of this subject in the last two decades are found primarily in the work of Zernike (1938), Hopkins (1951, 1957), Blanc-Lapierre and Dumontet (1955), and Wolf (1955). Since all of the earlier formulations of the theory are special cases of the one used here, it will prove convenient to introduce the principle functions of this general theory before discussing the contributions of the previous writers.

The study of partial coherence in optics is essentially the study of the complex cross-correlation of the disturbances at two typical points, P_1 and P_2, in the optical field. This function is defined as the mutual coherence function*, $\Gamma_{12}(\tau)$, i.e.,

$$\Gamma_{12}(\tau) = \langle V_1(t+\tau)\, V_2^*(t) \rangle \quad ,$$

where $V_1(t)$ and $V_2(t)$ are the "complex** disturbances" at the two points, the sharp brackets, $\langle\,\rangle$, denote time average, and τ is the time delay. The complex degree of coherence, $\gamma_{12}(\tau)$, is defined as the normalised mutual coherence function,

$$\gamma_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\,\Gamma_{22}(0)}} \quad .$$

It should be noted that the degree of coherence is a function not only of the coordinates of P_1 and P_2 but also of the time delay or path difference coordinate, τ. With these definitions in mind we may now direct our attention to the early contributions to the subject of partial coherence.

After the work of Verdet (1869), mentioned earlier, a quantitative measure for partial coherence was given in a paper by von Laue (1907). In this paper, in which was discussed the thermodynamical aspects of diffraction, the quantity γ_L, proportional to square of the time averaged product of the disturbances at two points in the field, proved to be of central importance. While this quantity was sufficient to characterise the optical field for the problems discussed by von Laue, the formulation is too restrictive for general application. However, we may mention that the recent interesting experiment of Hanbury-Brown and Twiss

* Precise definitions of this and the other functions referred to here are given in the next section.

** The precise meaning of the term complex disturbance is given in Section 2.

(1956 a, b) measured precisely the quantity γ_L .

It is shown in Appendix 1 of this thesis that

$$\gamma_L = \left| \gamma_{12}(0) \right|^2 . \qquad (1.1.1)$$

The next theoretical treatment of the subject appeared some twenty years later in the work of Berek (1926) in which was introduced the so-called degree of consonance as a measure of coherence. Berek's formulation was applied to some problems in the theory of image formation in the microscope. Some of his results, however, were contradicted by the experiments of Lakeman and Groosmuller (1928).

After another decade the subject was again reformulated. van Cittert (1938) showed that in a plane illuminated by an incoherent, nearly monochromatic source the optical disturbance is normally distributed. Three theorems concerning his "komplex korrelation" were also given in this paper. The best known of these theorems expresses the correlation in the illuminated plane in terms of the intensity distribution across the source plane. The analysis of this paper is in terms of ensemble averages, but by invoking the ergodic hypothesis, it can be shown that when the treatment is applicable (incoherent, quasi-monochromatic sources) the correlation function introduced by van Cittert is equal to the zero ordinate of the complex degree of coherence,

$$\gamma_c = \gamma_{12}(0) . \qquad (1.1.2)$$

A significant augmentation of the theory of partial coherence was given by Zernike (1938). In this paper the treatment of the subject is tightly bound to the interpretation of Young's interference experiment in terms of Michelson's visibility. In fact the degree of coherence between the disturbances at two points is defined in that paper as the visibility of the fringes obtained by allowing the light from these points to interfere in a suitable experiment (i. e. short path differences and equal intensities). So valuable has this experimental definition proved in understanding the physical aspects of partial coherence, that many later authors overlook the fact that Zernike formulated the subject analytically.

Zernike's formulation is applicable to quasi-monochromatic fields produced by any source (coherent, partially coherent, or incoherent) and in this sense is the most general treatment prior to the work of Blanc-Lapierre and Dumontet (1955), and Wolf (1955) to be discussed below. The fundamental quantity in his analysis is the mutual intensity function, defined as the time averaged product of the disturbance at one point with the complex conjugate of the disturbance at the second point. The degree of coherence is the mutual intensity function suitably normalized. Zernike's degree of coherence, γ_z, may be shown to be equal to the zero ordinate of the complex degree of coherence used here,

$$\gamma_z = \gamma_{12}(0) \quad . \tag{1.1.3}$$

In the same paper an approximate law for the propagation of the mutual intensity was also presented; and as a consequence of this law a theorem relating to the mutual intensity on a plane illuminated by an incoherent plane source was determined. This theorem is by virtue of the ergodic hypothesis the same as the theorem of van Cittert's mentioned earlier and is now termed the van Cittert-Zernike theorem (cf. Born, M. and Wolf, E. 1959, p. 507).

In 1951 H. H. Hopkins reformulated the theory of partial coherence. In this treatment the complex degree of coherence is defined in terms of an integral over the primary source of the radiation, assumed always to be incoherent. While the arguments of this paper have been the subject of considerable discussion, (cf. Wolf (1954, 1958), Hopkins (1956), Zucker (1957)), the techniques introduced have proved very powerful for many practical optical problems. When the formulation of this paper is applicable (i.e. incoherent quasi-monochromatic sources) it is equivalent to that of Zernike. Hence

$$\gamma_H = \gamma_{12}(0) \quad , \tag{1.1.4}$$

where γ_H is the degree of coherence as defined by H. H. Hopkins.

In spite of the usefulness of the results presented in the papers described above we must point out three unsatisfactory aspects of their formulations : 1.) the formulations are applicable only to quasi-monochromatic fields; 2.) apart from Zernike's work, the analyses are applicable only when the source is incoherent; 3.) apart from von Laue, each of the above formulations are in terms of complex functions the significance of which is obscure.

The first of these considerations is perhaps the most significant; for the restriction to incoherent sources is removed by Zernike's formulation; and the ambiguity as to complex representation may be removed either by dealing exclusively with real functions or by carefully defining the complex functions. The restriction to quasi-monochromatic light, however, is not simply removable. In fact the theory of partial coherence could be extended to fields of arbitrary spectral width only after the introduction of the cross-correlation function and a more detailed consideration of the statistical aspects of the subject.

These shortcomings were eliminated in the formulation of the subject by Wolf (1955) and that of Blanc-Lapierre and Dumontet (1955). Both of these formulations are rigorously applicable to polychromatic fields created by any type of source (coherent, partially coherent, or incoherent); and both define the degree of coherence in terms of the cross-correlation of the disturbance at two points in the field. The essential difference between these two formulations is the fact that Wolf treats the subject in terms of carefully defined complex functions, while Blanc-Lapierre and Dumontet deal in the main with real functions.

These two treatments are rigorous and general, but the several advantages of the complex representation (discussed by Born and Wolf (1959) and Parrent (1959) make it more suitable for an analysis in which the usual optical theorems for natural light are to be regarded as limiting forms. The degree of coherence as defined in the real function treatment, $\gamma_{BL+D}(\tau)$ is simply the real part of the complex degree of coherence, i.e.,

$$\gamma_{BL+D}(\tau) = R\left\{\gamma_{12}(\tau)\right\} . \qquad (1.1.5)$$

For the reasons stated above the formulation of coherence theory due to Wolf will be used exclusively in this thesis. In the next section this formulation is described in detail.

1.2 Review of the General Formulation of Coherence Theory

As originally introduced by Wolf (1954) the theory of partial coherence is formulated in terms of the electromagnetic field. The basic entities in this formulation are correlation matrices. the elements of which are the cross-correlations of the Cartesian components of the electromagnetic field vectors in an appropriate complex representation. While such a general treatment may be necessary for the description of partial polarization phenomena, a great many optical phenomena are adequately described by a scalar wave function.

There are at least two approaches to the justification of the use of a scalar theory, apart from the experimental fact that it is in excellent agreement with experience. The customary approach is to take as the basic physical quantity a single Cartesian component of the electric vector, reserving the right to include the additional components if necessary to describe some particular phenomenon (e.g. polarization effects).

An alternative approach is that adopted by Green and Wolf (1953). In this and subsequent papers by Wolf (1959) and Roman (1955, 1959) a rather fruitful attempt to completely describe optical phenomena in terms of a complex scalar function has been made. This theory is not as yet complete and hence will not be used here. In this thesis we adopt the first approach and represent by $V^r(t)$ a single (real) Cartesian component of the electric vector.

Since the concept of Gabor's (1946) analytic signal is fundamental to the understanding of the definitions to be given here, it will prove useful to review briefly the method of associating an analytic signal with a given real function in spite of the fact that Chapter 2 is devoted largely to analytic signals and their associated Hilbert transforms.

As will become clear in the following development, the analytic signal may be obtained by a simple generalization of the method of associating a complex exponential with a simple periodic function. Let $V^r(t)$ be a real function possessing a Fourier representation,

$$V^r(t) = \int_0^\infty a(\nu) \cos[\phi(\nu) - 2\pi\nu t] \, d\nu \quad . \tag{1.2.1}$$

We associate with $V^r(t)$ another real function, $V^i(t)$, obtained by changing the phase of each spectral component of $V^r(t)$ by $\pi/2$. Thus

$$V^i(t) = \int_0^\infty a(\nu) \sin[\phi(\nu) - 2\pi\nu t] \, d\nu \quad . \tag{1.2.2}$$

The analytic signal, $V(t)$, may then be defined as

$$V(t) = V^r(t) + iV^i(t) \quad . \tag{1.2.3}$$

The fundamental quantity in this study is the mutual coherence function, $\Gamma_{12}(\tau)$, which is defined as the complex cross-correlation of the analytic signals $V_1(t)$ and $V_2(t)$ associated with the real disturbance $V_1^r(t)$ and $V_2^r(t)$ at two typical points, P_1 and P_2, in the field, i. e.,

$$\Gamma_{12}(\tau) = \left\langle V_1(t+\tau)\ V_2^*(t) \right\rangle , \qquad (1.2.4)$$

where the sharp brackets denote time average.* Three functions derivable from $\Gamma_{12}(\tau)$ are of sufficient importance to have received separate names. The self coherence function, $\Gamma_{11}(\tau) \equiv I_1(\tau)$, is the complex auto-correlation of the analytic signal associated with the disturbance at a typical point, P_1, in the optical field,

$$\Gamma_{11}(\tau) \equiv I_1(\tau) = \left\langle V_1(t+\tau)\ V_1^*(t) \right\rangle . \qquad (1.2.5)$$

As will be proved in a later chapter, the intensity at a typical point in the field is the zero ordinate of the self coherence function, i. e.,

$$\Gamma_{11}(o) \equiv I_1(o) = \left\langle V_1(t)\ V_1^*(t) \right\rangle . \qquad (1.2.6)$$

This result may be regarded as a generalization of the familiar theorem for monochromatic light.

The Fourier transforms of the mutual coherence function and the self coherence function are termed the mutual spectral density, $\hat{\Gamma}_{12}(\nu)$, and the spectral density, $\hat{\Gamma}_{11}(\nu) \equiv \hat{I}_1(\nu)$, respectively;** i.e.,

$$\begin{aligned} \hat{\Gamma}_{12}(\nu) &= \int_{-\infty}^{\infty} \Gamma_{12}(\tau)\ e^{2\pi i \nu\tau}\ d\tau \qquad && \nu > 0 \\ &= 0 && \nu < 0 , \end{aligned} \qquad (1.2.7)$$

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* The precise form of this time average is discussed in Chapter 2.

** It will be proved in Chapter 2 that the cross-correlation of two analytic signals is itself an analytic signal, and hence $\hat{\Gamma}_{12}(\nu)$ contains only positive frequency.

and

$$\hat{\Gamma}_{11}(\nu) = \int_{-\infty}^{\infty} \Gamma_{11}(\tau)\ e^{2\pi i\nu\tau}\ d\tau \qquad \nu > 0$$

$$= 0 \qquad \nu < 0\ . \tag{1.2.8}$$

In terms of these functions the complex degree of coherence (function) is defined as the normalized mutual coherence function, where the normalization factor is the square root of the product of the intensities at P_1 and P_2:

$$\gamma_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\ \Gamma_{22}(0)}} \tag{1.2.9}$$

By appealing to the Schwarz inequality, it can be shown that the modulus of $\gamma_{12}(\tau)$ is bounded by zero and one,

$$0 \leq |\gamma_{12}(\tau)| \leq 1\ , \tag{1.2.10}$$

and these extreme values characterize by definition complete coherence and complete incoherence, respectively. It may be shown (see Wolf, 1955) that $\gamma_{12}(\tau)$ may be identified with the visibility of the fringes obtained by causing $V_1^r(t)$ and $V_2^r(t)$ to interfere with a path difference of $v\tau$, where v is the velocity of light in the medium, assumed to be homogeneous. It is, therefore, clear that the above definitions are in agreement with the consideration that coherent light interferes and incoherent light does not. In fact the definitions used here may be regarded as a rigorous generalization of those introduced by Zernike (1938).

The essential mathematical structure of coherence theory is contained in equations (1.2.1) through (1.2.10) except that the form of the time average has not been explicitly given. This omission will be remedied in Chapter 2 where alternative definitions of the time average are discussed.

CHAPTER 2

MATHEMATICAL DEVELOPMENTS

The definitions introduced in the previous chapter employ the concept of Gabor's analytic signals (see Gabor (1946)); and since the development of subsequent chapters makes extensive use of the properties of analytic signals, it will prove helpful to discuss in some detail their mathematical properties at this point.

The method given in Chapter 1 for associating an analytic signal with a given real function is patterned after the introduction by Wolf (1955) and is useful in understanding the significance of the analytic signals. However, for the purpose of the analysis given here it is more convenient to introduce an alternative definition in terms of Hilbert transforms. The equivalence of these two definitions will be demonstrated.

In Section 2.2 several convolution-type theorems for analytic signals are given. Section 2.3 presents a number of theorems concerning the cross-correlation of analytic signals. In the final section the form of the most general unimodular analytic signal is given.

2.1 Hilbert Transforms

The Hilbert transform, $V^i(t)$, of the real function $V^r(t)$ is defined by the relation

$$V^i(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{V^r(t')\,dt'}{t'-t} \text{ and } V^r(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{V^i(t')\,dt'}{t'-t} . \qquad \text{.........} (2.1.1)$$

Here the integral is the Cauchy principal value integral which may be defined as

$$\int_{-\infty}^{\infty} \frac{V^r(t')dt'}{t' - t} = \lim_{\epsilon \to 0} \left[\int_{-\infty}^{t-\epsilon} \frac{V^r(t')dt'}{t' - t} + \int_{t+\epsilon}^{\infty} \frac{V^r(t')dt'}{t' - t} \right] . \qquad \ldots\ldots (2.1.2)$$

However, in almost every instance it will prove more convenient to evaluate the principal value integral by contour integration. To rewrite (2.1.2) in terms of contour integrals we replace t by z, ($z = x + iy$ where x and y are real), and assume that the function $V^r(z')$ is continuous in some neighbourhood of $z' = z$.

The principal value integral may then be defined by

$$\int_{-\infty}^{\infty} \frac{V^r(z')dz'}{z' - z} = \frac{1}{2} \left[\int_{C_+} \frac{V^r(z')dz'}{z' - z} + \int_{C_-} \frac{V^r(z')\,dz'}{z' - z} \right], \qquad \ldots\ldots (2.1.3)$$

where the integrals on the right are line integrals along the open curves C_+ and C_-. The curves C_+ and C_- extend along the real axis from left to right curving in semi-circular arcs respectively above and below the point $z' = z$ as in Figure 1. The value of the line integrals in (2.1.3) are of course independent of the actual path of the curved part of C_+ and C_- as long as they do not include singularities. The line integrals, therefore, remain unchanged in the limit of vanishing radius for the semi-circular arcs. The continuity of $V^r(z')$ in the neighbourhood of $z' = z$ assures that in the limit the contributions from the upper and lower arcs will cancel each other; thus the

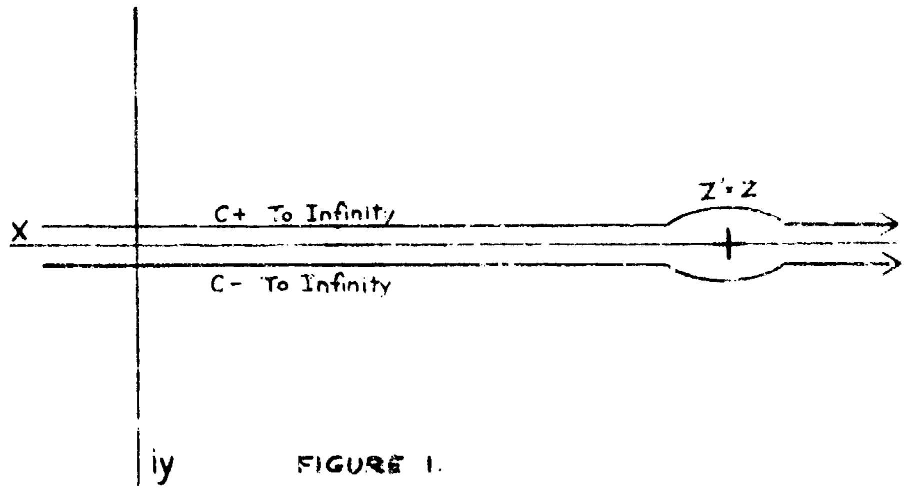

FIGURE 1.

equivalence of the definitions of principle value is demonstrated.*

The integrals in (2.1.3) are line integrals; the paths by which C_+ and C_- are closed will of course depend on the form of $V^r(z')$ in any given problem.

In terms of the Hilbert transform the analytic signal may be defined as follows: let $V^r(t)$ be a real function of t such that its Hilbert transform, $V^i(t)$, exists.** The analytic signal may then be defined as

$$V(t) = V^r(t) + iV^i(t) \quad . \tag{2.1.4}$$

* Actually (2.1.3) is a more general definition including (2.1.2) as a special case; for (2.1.2) will not exist if $V^r(z')$ has a pole at $z' = z$ while (2.1.3) will always exist except for the cases of an essential singularity or branch point at $z' = z$.

** A sufficient condition for the existence of the Hilbert transform is that $V^r(z)$ be square integrable. However, as with the Fourier transform the necessary conditions for the existence of the Hilbert transform are not known. Since many of the functions occurring in this chapter are not square integrable but do posess Hilbert transforms, we assume the weaker conditions that the transform exists and that the inversion theorem is valid

The equivalence of this and the previous definition, (1.2.1) - (1.2.3), may be demonstrated as follows: express $V^r(t)$ as the Fourier integral

$$V^r(t) = \int_0^\infty a(\nu) \cos [\phi(\nu) - 2\pi\nu t] \, d\nu \quad . \tag{2.1.5}$$

Take the Hilbert transform of both sides of (2.1.5) after setting $t = z$ and expressing the cosine in terms of complex exponentials. Thus

$$\begin{aligned} V^i(z) &= \frac{1}{2\pi}\int_0^\infty a(\nu) \int_{-\infty}^{\infty} \frac{e^{i[\phi(\nu) - 2\pi\nu z']}}{z' - z} \, dz' d\nu \\ &+ \frac{1}{2\pi}\int_0^\infty a(\nu) \int_{-\infty}^{\infty} \frac{e^{i[\phi(\nu) - 2\pi\nu z']}}{z' - z} \, dz' d\nu \end{aligned} \tag{2.1.6}$$

The order of integration has been interchanged and the inner integrals are interpreted as the line integrals in Figure 1. The contour is closed at infinity below the axis in the first integral and at infinity above the axis in the second integral. Since there are no poles within the contour in either integrand except at $z' = z$, we obtain by the residue theorem

$$V^i(t) = - \int_0^\infty a(\nu) \sin [\phi(\nu) - 2\pi\nu t] \, d\nu \quad , \tag{2.1.7}$$

where we have replaced z by t after performing the integration. Comparison of (2.1.7) and (1.2.2) establishes the equivalence of the two definitions.

It will be convenient for later developments to list here several theorems concerning Hilbert transforms (c.f. Bateman Manuscript Project 1954).

Table 1

Theorem	$f(t)$	$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(t')\,dt'}{t'-t}$
	$V^r(t)$	$V^i(t)$
I	$V^i(t)$	$-V^r(t)$
II	$V^r(t+a)$	$V^i(t+a)$
III	$V^r(\pm at)$	$\pm V^i(\pm at)$
IV	$(t+a)\ V^r(t)$	$(t+a)\ V^i(t) + \frac{1}{\pi}\int_{-\infty}^{\infty} V^r(t)dt$
V	$\frac{d}{dt}\ V^r(t)$	$\frac{d}{dt}\ V^i(t)$

2.2 Convolution Theorems for Analytic Signals

We shall require several theorems concerning the cross-correlation of analytic signals. Since these theorems do not seem to appear in the literature, we shall give a detailed derivation of them at this point. Because of the complication introduced by the time average in the cross-correlation function, we shall first demonstrate these theorems in terms of convolution integrals and later derive the correlation theorems from them.

Theorem VI: The convolution of two real functions $f_1(t)$ and $f_2(t)$ is equal to the convolution of their Hilbert transforms, $g_1(t)$ and $g_2(t)$ (in the same order), i.e.,

$$\int_{-\infty}^{\infty} f_1(t) \quad f_2(t + \tau) \quad dt \;=\; \int_{-\infty}^{\infty} g_1(t) \; g_2(t + \tau) \quad dt. \qquad (2.2.1)$$

The proof of this theorem can be obtained with the help of Theorems I and II of Table 1. Let

$$F(\tau) \;=\; \int_{-\infty}^{\infty} f_1(t) \quad f_2(t + \tau) \quad dt \; . \qquad (2.2.2)$$

Applying the inversion theorem (Theorem I, Table 1) twice to both sides of (2.2.2), we obtain

$$\pi^2 F(\tau'') \;=\; - \int_{-\infty}^{\infty} \frac{1}{\tau' - \tau''} \int_{-\infty}^{\infty} \frac{1}{\tau - \tau'} \int_{-\infty}^{\infty} f_1(t) \quad f_2(t + \tau) \, dt \, d\tau \, d\tau' \; . \qquad \ldots\ldots (2.2.3)$$

Using Theorem II of Table 1 after inverting the order of integration, we find that

$$\pi^2 F(\tau'') \;=\; - \int_{-\infty}^{\infty} \frac{1}{\tau' - \tau''} \int_{-\infty}^{\infty} f_1(t) \quad g_2(t + \tau) \quad dt \quad d\tau' \; . \qquad (2.2.4)$$

Introducing a new variable $\tau = t + \tau'$ we obtain

$$F(\tau'') \;=\; \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f_1(t) \quad g_2(\tau) dt \;\, d\tau}{t - (\tau - \tau'')} \; . \qquad (2.2.5)$$

Using Theorem II again we finally have the result that

$$F(\tau'') = \int_{-\infty}^{\infty} g_1(\tau - \tau'') \; g_2(\tau) d\tau . \qquad (2.2.6)$$

On setting $\tau - \tau'' = t$ (2.2.6) reduces to (2.2.1) and the theorem is demonstrated.

Two special cases of theorem (2.2.1) are of particular interest. The first case, $f_s(t) = V^r(t)$ $(s = 1,2)$, leads to

$$\int_{-\infty}^{\infty} V_1{}^r(t) \; V_2{}^r(t + \tau)dt = \int_{-\infty}^{\infty} V_1{}^i(t) \; V_2{}^i(t + \tau)dt. \qquad (2.2.7)$$

The second case, $f_1(t) = V_1{}^r(t)$ and $f_2(t) = V_2{}^i(t)$, leads to

$$\int_{-\infty}^{\infty} V_1{}^r(t) \; V_2{}^i(t + \tau)dt = - \int_{-\infty}^{\infty} V_1{}^i(t) \; V_2{}^r(t + \tau) dt. \qquad (2.2.8)$$

Using (2.2.7) and (2.2.8), we may obtain the convolution theorems for analytic signals from which some of the advantages of this complex formulation will be clear. Consider the integral

$$\int_{-\infty}^{\infty} V_1(t) \; V_2{}^* (t + \tau)dt . \qquad (2.2.9)$$

where $V_s(t)$ $(s = 1,2)$ is of the form of (2.1.4). Substituting from (2.1.4) into (2.2.9) and using (2.2.7) and (2.2.8) we obtain

$$\int_{-\infty}^{\infty} V_1(t)\ V_2^*(t+\tau)\,dt = 2\int_{-\infty}^{\infty} V_1^r(t)\ V_2^r(t+\tau)\,dt \tag{2.2.10}$$

$$+\ 2i\int_{-\infty}^{\infty} V_1^r(t)\ V_2^i(t+\tau)\,dt\ .$$

Equation (2.2.10) expresses the useful result that the real part of the convolution of two analytic signals is, apart from a factor of two, the convolution of the real functions with which they are associated. Another useful property of the analytic signals is seen by putting $V_1(t)$ equal to $V_2(t)$ in (2.2.10). Thus

$$\frac{1}{2}\int_{-\infty}^{\infty} V_1(t)\ V_1^*(t+\tau)\,dt = \int_{-\infty}^{\infty} V_1^r(t)\ V_1^r(t+\tau)\,dt$$

$$+\ i\int_{-\infty}^{\infty} V_1^r(t)\ V_1^i(t+\tau)\,dt. \tag{2.2.11}$$

We may now evaluate the second integral on the right as follows: using (2.2.8),

$$\int_{-\infty}^{\infty} V_1^r(t)\ V_1^i(t+\tau)\,dt = -\int_{-\infty}^{\infty} V_1^i(t)\ V_1^r(t+\tau)\,dt\ , \tag{2.2.12}$$

or evaluating (2.2.12) at $\tau = 0$ we find

$$\int_{-\infty}^{\infty} V_1^r(t)\ V_1^i(t)\,dt = 0\ , \tag{2.2.13}$$

i.e., $V_1^r(t)$ and $V_1^i(t)$ are orthogonal. Evaluating (2.2.11) at $\tau = 0$ and using (2.2.13), we obtain

$$\int_{-\infty}^{\infty} V_1(t)\ V_1^*(t)\,dt = 2\int_{-\infty}^{\infty} V_1^r(t)\ V_1^r(t)\,dt\ . \tag{2.2.14}$$

Equation (2.2.14) expresses a second important property of the analytic signals; namely, the integral over-all time of the squared modulus of an analytic signal is, apart from a factor of two, the integral over-all time of the square of the real function with which it is associated.

An additional theorem on the convolution of analytic signals, which will prove important in the next section is :

Theorem VII : The convolution of two analytic signals is itself an analytic signal. To demonstrate this theorem we write

$$F(\tau) = 2 \int_{-\infty}^{\infty} V_1(t) \; V_2^*(t + \tau)dt \; . \qquad (2.2.15)$$

Using (2.2.7) and (2.2.8), (2.2.15) can be rewritten as

$$F(\tau) = \int_{-\infty}^{\infty} V_1^r(t) \; V_2^r(t+\tau)dt + \int_{-\infty}^{\infty} V_1^i(t) \; V_2^i(t+\tau)dt$$
$$+ i\left\{\int_{-\infty}^{\infty} V_1^r(t) \; V_2^i(t+\tau)dt - \int_{-\infty}^{\infty} V_1^i(t) \; V_2^r(t+\tau)dt\right\}. \qquad \ldots\ldots (2.2.16)$$

Denoting by R and I the real and imaginary parts respectively and taking the Hilbert transform of the real part of $F(\tau)$ we obtain

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R\left\{F(\tau')\right\}}{\tau' - \tau} d\tau' = - I\left\{F(\tau)\right\}, \qquad (2.2.17)$$

and Theorem VII is demonstrated.

The results established in this section are, of course, applicable only to convolutions and the definitions introduced in Chapter 1 are in terms of correlations. We shall show in the next section that these theorems are also valid for the cross-correlation of analytic signals if the time average is suitably defined.

2.3 The Cross-Correlation of Analytic Signals

The complex cross-correlation function may be defined in several ways. The customary definition is (c.f. Davenport and Root (1958), page 70)

$$\psi_{12}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(t)\; f_2^*(t+\tau)dt. \tag{2.3.1}$$

However, for our present purpose the mutual coherence function, $\Gamma_{12}(\tau)$, is defined as

$$\Gamma_{12}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} V_1(T,t+\tau)\; V_2^*(T,t)dt, \tag{2.3.2}$$

where

$$V(T,t) = V^r(T,t) - i_T V^i(t) \tag{2.3.3}$$

and

$$V^r(T,t) = \left\{ \begin{matrix} V^r(t) & |t| < T \\ 0 & |t| > T \end{matrix} \right\}, \tag{2.3.4}$$

and

$${}_TV^i(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{V^r(T,t')dt'}{t'-t}. \tag{2.3.5}$$

The different notation, $V^r(T,t)$ and ${}_TV^i(t)$, is used since, while $V^r(T,t)$ vanishes for $|t| > T$, its Hilbert transform ${}_TV^i(t)$ will not in general vanish in this range. Since $V^r(t)$ is assumed to be everywhere finite, the function $V^r(T,t)$ is square integrable; its Hilbert transform ${}_TV^i(t)$, is therefore also square integrable (c.f. Titchmarsh (1948)). Consequently, all the required Hilbert transforms exist. The equivalence of the two definitions of an average, of the type (2.3.1) and (2.3.2) is discussed by Born and Wolf (see Born and Wolf (1959), page 496). For our purposes it is more convenient to employ the definition (2.3.2) and accordingly the sharp brackets are defined as

$$\left\langle V_1^r(t+\tau)\ V_2^*(t)\right\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-\infty}^{\infty} V_1(T,t+\tau)\ V_2^*(T,t)dt .$$

The presence of the parameter T, in $V(T,t)$ in no way affects the arguments of the previous section; and the operation $\lim\limits_{T\to\infty}$ will commute with the integrations involved, since T and t are independent variables. The theorems established in Section 2.2 may, therefore, be taken over mutatis mutandis for the cross-correlation function $\Gamma_{12}(\tau)$. Thus if we adopt the notation

$$\Gamma_{12}^{rr}(\tau) = \left\langle V_1^r(t+\tau)\quad V_2^r(t)\right\rangle . \tag{2.3.6}$$

and

$$\Gamma_{12}^{ri}(\tau) = \left\langle V_1^r(t+\tau)\quad V_2^i(t)\right\rangle . \tag{2.3.7}$$

We may summarize the principal theorems as follows :

$$\Gamma_{12}(\tau) = 2\left[\ \Gamma_{12}^{rr}(\tau) + i\ \Gamma_{12}^{ri}(\tau)\ \right] \tag{2.3.8}$$

$$\Gamma_{11}^{ri}(0) = 0 . \tag{2.3.9}$$

Setting $V_1(t)$ equal to $V_2(t)$ and $\tau = 0$ in (2.3.8) and using (2.3.9), we obtain the result anticipated in Chapter 1, Section 2, viz.,

$$\Gamma_{11}(0) = 2\,\Gamma_{11}^{rr}(0) = 2\,I(P_1)\ , \tag{2.3.10}$$

where $I(P_1)$ is the intensity at P_1. Further it follows from (2.2.19) and the arguments of this section that

$$\Gamma_{12}^{ri}(\tau) = -\,\Gamma_{12}^{ir}(\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Gamma_{12}^{rr}(\tau')d\tau'}{\tau' - \tau}\ . \tag{2.3.11}$$

Thus the mutual coherence function is an analytic signal.

For the sake of continuity in later arguments we include at this point two lemmas concerning complex cross-correlation functions. Both of these lemmas are well-known and follow immediately from the definition of $\Gamma_{12}(\tau)$ and the stationarity condition; they are, therefore, given here without proof:

Lemma I

$$\Gamma_{21}(-\tau) = \Gamma_{12}^{*}(\tau)\ , \tag{2.3.12}$$

and

Lemma II

$$\hat{\Gamma}_{21}(\nu) = \hat{\Gamma}_{12}^{*}(\nu)\ . \tag{2.3.13}$$

The hooked notation is used to denote temporal Fourier transforms, i.e.,

$$\hat{\Gamma}_{12}(\nu) = \int_{-\infty}^{\infty}\Gamma_{12}(\tau)e^{2\pi i\nu\tau}d\tau \qquad \nu > 0 \tag{2.3.14}$$

$$= 0 \qquad\qquad \nu < 0$$

The spectrum of $\Gamma_{12}(\tau)$ is zero for half the frequency range, since the mutual coherence function is an analytic signal.

2.4 The Most General Unimodular Analytic Signal

We shall be concerned in Chapter 3 with the determination of limiting forms of the mutual coherence function for coherent and incoherent fields. The form of $\Gamma_{12}(\tau)$ for coherent fields will be shown to be

$$\Gamma_{12}(\tau) = \sqrt{\Gamma_{11}(0)\ \Gamma_{22}(0)}\ e^{i\phi_{12}(\tau)} , \tag{2.4.1}$$

where $e^{i\phi_{12}(\tau)}$ is a unimodular analytic signal. The determination of $\phi_{12}(\tau)$ involves the solution of a singular integral equation; and since this solution is rather lengthy and purely mathematical in nature it will be included in this chapter.

Since $e^{i\phi_{12}(\tau)}$ is an analytic signal, its real and imaginary parts are Hilbert transforms, i.e.,

$$\sin \phi_{12}(\tau) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos\ \phi_{12}(\tau')\,d\tau'}{\tau' - \tau} . \tag{2.4.2}$$

Using the inversion theorem, Theorem I, Table 1, we may write

$$\cos\ \phi_{12}(\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\ \phi_{12}(\tau')\,d\tau'}{\tau' - \tau} . \tag{2.4.3}$$

Combining (2.4.2) and (2.4.3) we obtain

$$e^{i\phi_{12}(\tau)} = -\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{i\phi_{12}(\tau')}}{\tau' - \tau}\,d\tau' . \tag{2.4.4}$$

Here we have dropped the subscript 1, 2 since the development of this section is not concerned with the space dependence of $\phi_{12}(\tau)$.

Equation (2.4.4) is an integral equation for $\phi_{12}(\tau)$, and may be recognized as a singular form of a Carleman-type equation, i.e.,

$$f(\tau) = \lambda \int_{-\infty}^{\infty} \frac{f(\tau')}{\tau' - \tau} d\tau' + x(\tau) , \tag{2.4.5}$$

where λ is a generally complex constant. The general solution to (2.4.5) is (c.f. Tricomi (1957), page 175)

$$f(\tau) = \frac{1}{1 + \lambda^2 \pi^2} \left[x(\tau) + \lambda \int_{-\infty}^{\infty} \frac{x(\tau')}{\tau' - \tau} d\tau' \right] . \tag{2.4.6}$$

However, since in (2.4.4) $\lambda = i/\pi$ and $x(\tau) = 0$, the solution, (2.4.6) becomes in this case indeterminate. In fact it is clear that (2.4.5) can have no solution for $x(\tau) = 0$, except possibly for the eigenvalues, $\lambda = \pm\ i/\pi$. This singular form is solved here by utilizing the properties of analytic functions. To this end we replace the real variable τ by a complex variable, $z = x + iy$; and write (2.4.4) as

$$f(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{f(z')}{z' - z} dz' , \tag{2.4.7}$$

where

$$f(z) \equiv e^{i\phi(z)} . \tag{2.4.8}$$

Since $\phi(x)$ is a real function, we have

$$\overline{\phi(z)} = \phi(\bar{z}) ; \tag{2.4.9}$$

here and throughout the rest of this section we use the bar to denote complex conjugate. In the subsequent discussion we shall treat only the case $\lambda = -i/\pi$ since the argument is essentially the same for the positive eigenvalue.

It has been pointed out in Chapter 1 that the term analytic signals derives from the fact that these functions are, when considered as a function of a complex variable, analytic in half the complex plane. Since extensive use of this property is made in this section we shall digress briefly and demonstrate it.

Let U(t) be any function of the real variable t such that the integral

$$F(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{U(t)dt}{t - z} \tag{2.4.10}$$

exists. Then the function F(z) is analytic in the lower half of the complex plane (c.f. Whittaker and Watson (1950) page 92). Equation (2.4.10) may be rewritten as

$$F(z) = \frac{1}{\pi i} ⨍_{-\infty}^{\infty} \frac{U(t)dt}{t - z} + \pi i \;\; U(z) \quad , \tag{2.4.11}$$

where Cauchy's theorem has been used.

In the limit as $z \to x$ from below the real axis

$$F(z) \to F(x) = \frac{1}{\pi i} ⨍_{-\infty}^{\infty} \frac{U(t)dt}{t - x} + \pi i \;\; U(x) \quad ,$$

$$= U(x) - \frac{i}{\pi} ⨍_{-\infty}^{\infty} \frac{U(t)dt}{t - x} \; . \tag{2.4.12}$$

From (2.4.12) it is clear that the analytic signals may be regarded as the limit as the real axis is approached of a function analytic in half the complex plane. Or, conversely, if in an analytic signal the real variable is replaced by a complex one the resulting function is analytic in half the complex plane.

For our present problem, the solution of (2.4.7), the domain of analyticity of $f(z)$ may be extended to include the real axis since 1.) (2.4.9) excludes the possibility of poles or essential singularities on the real axis and 2.) physically we require $f(t)$ to be unique which excludes branch points on the real axis.

Apart from the trivial solution of $f(z)$ identically equal to zero, the function $f(z)$ can have at most isolated zeros in the lower half plane. This conclusion follows from the analyticity of $f(z)$ in that domain (c.f. Titchmarsh (1939) page 88).

Equation (2.4.9) expresses the value of $f(z)$ at all points in the lower half plane in terms of its values at conjugate points in the upper half plane. Further, (2.4.9) indicates that corresponding to every zero in the lower half plane there is a pole at the conjugate point in the upper half plane and conversely zeros in the upper half plane correspond to poles in the lower half plane. Hence, there are no zeros in the upper half plane and the singularities in the upper half plane are isolated. Furthermore, the singularities in the upper half plane are poles and not essential singularities; for by the Weierstrass theorem (c.f. Titchmarsh (1939), page 93) in every neighbourhood of an isolated essential singularity the function tends to any given limit an infinite number of times, and this behaviour would by (2.4.9) be reflected into the lower half plane. That is, an essential singularity in the upper half plane would necessarily correspond to an essential singularity in the lower half plane. The preceding argument does not exclude the possibility of essential singularities at infinity since infinity is not an isolated point. In the same way a branch point in the upper half plane is excluded since it would imply a branch point in the lower half plane contradicting the requirement of analyticity. We may conclude, therefore, that $f(z)$ is a meromorphic function, i. e., its only singularities for finite z are poles.

By a modification of Hadamard's theorem (c. f. Titchmarsh (1939), page 284g where a full discussion of the concepts employed below is given) any meromorphic function $f(z)$ may be expressed in the form

$$f(z) = \frac{\prod_{n=1}^{\infty} E\left(\frac{z}{a_n}, p_1\right)}{\prod_{n=1}^{\infty} E\left(\frac{z}{b_n}, p_2\right)} e^{iQ(z)} , \qquad (2.4.15)$$

where $Q(z)$ is a polynomial of order N ; $\prod_{n=1}^{\infty} E\left(\frac{z}{a_n}, p\right)$ is

the canonical product of the primary factors,

$$E(u,p) = (1 - u)\, e^{u + \frac{u^2}{2} + \cdots \frac{u^p}{p}} \qquad (2.4.16)$$

a_n and b_r are the zeros and poles respectively of $f(z)$.

The genus of the canonical product satisfies the inequality

$$p_j \leq \rho \qquad (j = 1,2) , \qquad (2.4.17)$$

where ρ is the order of the meromorphic function. The order, N, of the polynomial, Q, also satisfies the inequality, (2.4.17), i.e.,

$$N \leq \rho . \qquad (2.4.18)$$

From the previous argument it is clear that the poles and zeros occur at conjugate points, i. e.,

$$\bar{a}_n = b_n . \qquad (2.4.19)$$

Therefore, using (2.4.16), (2.4.15) may be written as

$$f(z) = \prod_{n=1}^{\infty}\left(\frac{1-\frac{z}{a_n}}{1-\frac{z}{\bar{a}_n}}\right) e^{\sum_{j=1}^{m}\left[ib_j + \sum_{n=1}^{\infty}\left(\frac{1}{a_{n_j}} - \frac{1}{\bar{a}_{n_j}}\right)\frac{z^j}{j!}\right]},$$

$$\ldots\ldots (2.4.20)$$

where we have set

$$Q(z) = \sum_{j=0}^{m} b_j z^j , \qquad (2.4.21)$$

and m is the largest of the integers, p_1, p_2, N. After setting

$$ic_j = \sum_{n=1}^{\infty}\left(\frac{1}{a_{n_j}} - \frac{1}{\bar{a}_{n_j}}\right) , \qquad (2.4.22)$$

where c_j is a real constant, (2.4.20) can be rewritten as

$$f(z) = \prod_{n=1}^{\infty}\left(\frac{1-\frac{z}{a_n}}{1-\frac{z}{\bar{a}_n}}\right) e^{i\sum_{j=0}^{m}(b_j + c_j)\frac{z^j}{j!}} . \qquad (2.4.23)$$

However,

$$f(z) \sim e^{i(b_m + c_m)z^m} \qquad (2.4.24)$$

when $|z| \to \infty$; but (2.4.24) has m poles and m zeros equally spaced on the circle at infinity; and, therefore, if $m > 1$, $f(z)$ has singularities in the lower half plane. We conclude, therefore, that the most general allowable solution is with $m = 1$. The expression (2.4.23) becomes finally

$$f(z) = \prod_{n=1}^{\infty} \left(\frac{1 - \frac{z}{a_n}}{1 - \frac{z}{\bar{a}_n}} \right) e^{i [\beta + \gamma z]}, \tag{2.4.25}$$

a meromorphic function of order 1. Thus we have established the following;

Theorem VIII : **The most general unimodular analytic signal is a meromorphic function of order one with zeros $z = a_n$ only in the lower half plane and with poles at conjugate points in the upper half plane and is given by the formula (2.4.25) where β and γ are real constants and the imaginary part of a_n is greater than zero.**

CHAPTER 3

SOME IMPLICATIONS OF COHERENCE AND INCOHERENCE

In Chapter 2 it was pointed out that the modulus of the degree of coherence is bounded by one and zero and that these extremes are by definition characteristic of coherence and incoherence respectively. For quasi-monochromatic light these definitions are in accord with the consideration that coherent light interferes and incoherent light does not. (See Wolf, 1954.) However, in this chapter we are concerned with the implications of coherence and incoherence on the form of $\Gamma_{12}(\tau)$, for fields of arbitrary spectral width. In a detailed analysis of such fields in which the terms "coherent limit" and "incoherent limit" are to have a clear and unambiguous meaning the above definitions must be more precisely stated.

The ambiguity which arises in the study of polychromatic fields stems from the fact that the modulus of the degree of coherence between the disturbances at two points is a function not only of the position of the two points but also of the time delay τ. Thus it is possible that for some values of τ, say $\tau = \tau_1$ and $\tau = \tau_2$, $|\gamma_{12}(\tau_1)| = 1$ and $|\gamma_{12}(\tau_2)| = 0$ for the same pair of disturbances in such fields, and the limiting concepts of coherence and incoherence become ambiguous.

This difficulty does not arise in the study of quasi-monochromatic light for the approximation characterizing such fields makes the modulus of the degree of coherence independent of τ for all values of τ for which the theory is applicable. This conclusion is evident from the following considerations. For quasi-monochromatic light the mutual coherence function for sufficiently small $|\tau|$ is given by (see Appendix 2)

$$\Gamma_{12}(\tau) \cong \Gamma_{12}(0)\, e^{-2\pi i \bar{\nu}\tau} \qquad \left(|\tau| << \frac{1}{\Delta\nu} \right).$$

Hence

$$\gamma_{12}(\tau) \simeq \gamma_{12}(0)\, e^{-2\pi i \bar{\nu}\tau} \qquad \left(|\tau| << \frac{1}{\Delta\nu} \right),$$

and

$$|\gamma_{12}(\tau)| \simeq |\gamma_{12}(0)| \qquad \left(|\tau| << \frac{1}{\Delta\nu} \right).$$

Here $\bar{\nu}$ is the mean frequency and $\Delta\nu$ is the spectral width of the illumination.

The ambiguity which arises in discussing polychromatic light will be removed if for the limiting cases of coherence and incoherence we demand that the modulus of the degree of coherence be τ-independent. Accordingly, the following definitions are introduced :

I. The DISTURBANCES $V_1(t)$ and $V_2(t)$ will be described as coherent if $|\gamma_{12}(\tau)| = 1$ for all τ and incoherent if $|\gamma_{12}(\tau)| = 0$ for all τ.

II. An OPTICAL FIELD will be said to be $\begin{Bmatrix}\text{coherent}\\ \text{incoherent}\end{Bmatrix}$ if the disturbances at all pairs of points in the field are $\begin{Bmatrix}\text{coherent}\\ \text{incoherent}\end{Bmatrix}$.

Thus $|\gamma_{12}(\tau)| = 1$ for a coherent field, and

$|\gamma_{12}(\tau)| = 0$ for an incoherent field.

Since, in the past, detailed analysis has been limited to quasi-monochromatic light, it is clear from the above discussion that this modification of the definitions cannot lead to contradiction with the work of earlier writers.

The analysis of the general properties of partially coherent wave fields is conveniently performed in terms of the mutual coherence function, $\Gamma_{12}(\tau)$, rather than the degree of coherence, $\gamma_{12}(\tau)$. We shall, therefore, examine the implications of the above definitions concerning the form of the mutual coherence function in coherent and incoherent fields. These limiting forms of $\Gamma_{12}(\tau)$ will prove useful in examining the extremes of the propagation law obtained in Chapter 4 and in recovering the familiar forms of elementary optical wave theory from the generalized transfer functions introduced in Chapter 5. Apart from these immediate applications, the limiting forms of the mutual coherence function will prove useful in their own right since they will provide insight into the structure and questions of existence of coherent and incoherent fields.

3.1 Wave Equations for the Propagation of $\Gamma_{12}(\tau)$

The determination of the form of $\Gamma_{12}(\tau)$ for coherent and incoherent fields will be seen to be intimately connected with the fact that in vacuum $\Gamma_{12}(\tau)$ is propagated (rigorously) according to the wave equations

$$\nabla_s^2 \Gamma_{12}(\tau) = \frac{1}{c^2} \frac{\partial^2 \Gamma_{12}(\tau)}{\partial \tau^2} \quad (s = 1,2). \tag{3.1.1}$$

Here ∇_s^2 is the Laplacian operator in the coordinates of P_s and c is the velocity of light in vacuum. For this reason it will prove helpful to digress briefly and derive these equations.*

As pointed out in Chapter 1 we consider the optical field to be characterized by a real scalar function, $V^r(t)$, which in vacuum satisfies the equation

* These equations were originally derived by Wolf (1955) by different arguments.

$$\nabla^2 V^r(t) = \frac{1}{c^2} \frac{\partial^2 V^r(t)}{\partial t^2} . \tag{3.1.2}$$

We recall that

$$V^i(t) = -\int_{-\infty}^{\infty} \frac{V^r(t')dt'}{t' - t} \tag{3.1.3}$$

Operating on both sides of (3.1.3) with the Laplacian, ∇^2, we obtain

$$\nabla^2 V^i(t) = \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{\nabla^2 V^r(t')dt'}{t' - t} = \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{1}{c^2} \frac{\frac{\partial^2 V^r(t')}{\partial t'^2}}{t' - t} dt' ;$$

and using Theorem V of Table 1, we have

$$\nabla^2 V^i(t) = \frac{1}{c^2} \frac{\partial^2 V^i(t)}{\partial t^2} . \tag{3.1.4}$$

Next we multiply both sides of (3.1.4) by i and then add (3.1.2) and (3.1.4) ; we then obtain

$$\nabla^2 V(t) = \frac{1}{c^2} \frac{\partial^2 V(t)}{\partial t^2} , \tag{3.1.5}$$

where $V(t) = V^r(t) + i\, V^i(t)$ is the analytic signal associated with $V^r(t)$. Thus, not only the real disturbance but also its associated analytic signal satisfies the wave equation.

We recall that the mutual coherence function is defined as

$$\Gamma_{12}(\tau) = \langle V_1(t + \tau)\, V_2^*(t) \rangle . \tag{3.1.6}$$

Differentiating both sides of (3.1.6) with respect to P_1 and P_2 separately, formally interchanging the order of differentiation and integration on the right hand side, using (3.1.5) and the stationarity of $V(t)$, we obtain the two wave equations (3.1.1).

3.2 Polychromatic Fields

In this section we shall study the limiting forms of the mutual coherence function without making any approximations as to the spectral width of the illumination. It will be shown that an optical field is coherent if and only if it is monochromatic.* Further it will be shown that a coherent field is completely described, including its coherence properties, by a simple wave function (depending on one point only).

It will also be shown in this section that an incoherent field cannot exist in free space even if the illumination is polychromatic. It is possible, however, to define an incoherent source and the definition will be seen to be consistent with the above consideration namely, that an incoherent source, as defined here, always gives rise to a partially coherent field.

3.2.1 The Form of $\Gamma_{12}(\tau)$ for a Coherent Field

Coherence is characterised by

$$\left| \gamma_{12}(\tau) \right| = 1 , \tag{3.2.1}$$

which implies (c.f. Chapter 1 (1.2))

$$\Gamma_{12}(\tau) = A_1 A_2 e^{i\phi_{12}(\tau)} . \tag{3.2.2}$$

---

* In much of the current literature the term monochromatic is erroneously used to describe light with a small but finite spectral width. However, the term is used here in its strict sense.

Here we have set

$$A_s = \sqrt{\Gamma_{ss}(0)} \qquad (s = 1,2) , \tag{3.2.3}$$

and $\phi_{12}(\tau)$ is a real function of the time delay τ, and the coordinates of P_1 and P_2. Since $\Gamma_{12}(\tau)$ is an analytic signal, the function $e^{i\phi_{12}(\tau)}$ is a unimodular analytic signal. Hence, according to Theorem VIII (Chapter 2) it must be of the form

$$\Gamma_{12}(\tau) = A_1 \ A_2 \ e^{i(\beta + 2\pi\nu_0\tau)} \prod_{n=1}^{\infty} \frac{a_n^*}{a_n} \left(\frac{a_n - \tau}{a_n^* - \tau}\right).$$

..... (3.2.4)

Here the a_n are complex constants with complex conjugates a_n^*; and the imaginary part of each a_n is greater than zero. In (3.2.4) $e^{i\phi_{12}(\tau)}$ is, when considered as a function of a complex variable, z, a meromorphic function; and the product is taken over all the poles $z = a_n^*$. The a_n are all finite and non-zero. The constants β and ν_0 are real. While (3.2.4) represents the most general unimodular analytic signal, it can, as will now be shown, be interpreted as a mutual coherence function only in the degenerate case where

$$\Gamma_{12}(\tau) = A_1 \ A_2 \ e^{i(\beta + 2\pi\nu_0\tau)} . \tag{3.2.5}$$

By definition $\Gamma_{12}(\tau)$ represents the complex cross correlation of the disturbances at two points, P_1 and P_2, in the field. Hence, when P_1 coincides with P_2, the corresponding mutual coherence function, $\Gamma_{11}(\tau)$, is a complex auto-correlation function. The real part of $\Gamma_{11}(\tau)$, is the real auto-correlation function

$$\Gamma_{11}(\tau) = 2\, \Gamma_{11}^{rr}(\tau) = 2 < V_1^{r}(t+\tau)\; V_1^{r}(t) > , \qquad (3.2.6)$$

(c.f. Chapter 2, section 2). Therefore, using also the fact that the field is stationary,

$$\Gamma_{11}^{rr}(\tau) = \Gamma_{11}^{rr}(-\tau) \quad , \qquad (3.2.7)$$

(c.f. Chapter 2, section 2). This condition, (3.2.7), may be expressed by the statement that

$$\int \Gamma_{11}^{rr}(\tau) \;\sin 2\pi\nu\tau \;\; d\tau \equiv 0 \qquad (3.2.8)$$

for all ν. In Appendix 2 of this thesis it is shown that (3.2.8) can be satisfied if and only if (3.2.4) assumes the degenerate form given in (3.2.5), i.e. if

$$\Gamma_{11}(\tau) = A_1^{\,2}\; e^{-2\pi i\,(\nu_0\tau + \beta)} \quad . \qquad (3.2.9)$$

The physical significance of (3.2.9) is that all the energy in the field at P_1 is contained in the single spectral component ν_0. Since (3.2.9) is valid for all P_1, it is clear that the field is everywhere monochromatic. Further, since (3.2.9) is deduced solely from the time dependence, the "constant" ν_0 could in principle depend on the coordinates of P_1; but the cross-correlation of two monochromatic disturbances of different frequencies is identically zero, and therefore ν_0 must, in fact, be a true constant characteristic of the entire field. Since the field is everywhere monochromatic of frequency ν_0, the τ dependence of $\Gamma_{12}(\tau)$ must be of the form

$$\Gamma_{12}(\tau) = F_{12}\; e^{2\pi i\nu_0\tau} , \qquad (3.2.10)$$

where F_{12} is a function of the coordinates of P_1 and P_2 only. From the requirement that the field be coherent, (3.2.1), and the fact that the field is monochromatic, (3.2.10), it follows that

$$\Gamma_{12}(\tau) = A_1 A_2 e^{i(\beta_{12} + 2\pi \nu_0 \tau)} , \qquad (3.2.11)$$

where β_{12} is a function of the coordinates of P_1 and P_2.

The form of the function β_{12} can be deduced from the fact that $\Gamma_{12}(\tau)$ satisfies the two wave equations, (3.1). Substituting (3.2.11) into (3.4) we obtain

$$[\nabla_s^2 + k^2] A_1 A_2 e^{i\beta_{12}} = 0 \quad (s = 1,2), \qquad (3.2.12)$$

where $k = 2\pi \nu_0/c$. Performing the operations indicated in (3.2.12), we find that

$$\frac{\nabla_1^2 A_1}{A_1} - (\nabla_1 \beta_{12})^2 + k^2 + i\left[\nabla_1^2 \beta_{12} + \frac{2\nabla_1\beta_{12} \cdot \nabla_1 A_1}{A_1}\right] = 0; \qquad \ldots\ldots (3.2.13)$$

and a similar equation involving ∇_2 and A_2 holds. Equating the real and imaginary parts of (3.2.13) separately to zero, we obtain the two equations

$$\nabla_1^2 \beta_{12} = - \frac{2(\nabla_1 \beta_{12}) \cdot (\nabla_1 A_1)}{A_1} \qquad (3.2.14)$$

and

$$(\nabla_1 \beta_{12})^2 = \frac{\nabla_1^2 A_1}{A_1} + k^2 . \qquad (3.2.15)$$

The right-hand side of (3.2.15) is a function of the coordinates of P_1 only and we can therefore write (3.2.15) in the form

$$(\nabla_1 \beta_{12})^2 = f^2 (x_1, y_1, z_1) . \qquad (3.2.16)$$

On the left side of (3.2.16) the coordinates of P_2 (contained in β_{12}) may be regarded as parameters. Equation (3.2.16) is therefore, of the form of the eikonal equation of geometrical optics or the Hamilton-Jacobi equation of dynamics. The general solution of this equation is well known (c.f. Born and Wolf (1959), page 721). Let β^0 be the value of β_{12} on a surface over which it is constant (independent of x_1, y_1, z_1). Then the solution is

$$\beta_{12} = \beta^0(p,q, x_2,y_2,z_2) + \int_{P_1^0}^{P_1} f(x,y,z)ds ; \qquad (3.2.17)$$

where the integral is taken along the extremal of the variational problem

$$\delta \int_{P_1^0}^{P_1} f(x_1,y_1,z_1) \; ds = 0 ; \qquad (3.2.18)$$

and P_1^0 is a typical point on the surface $\beta_{12} = \beta^0$. Further p and q are two free parameters which may be regarded as characterizing the orientation of the surface $\beta_{12} = \beta^0$.

It is clear from equation (3.2.17) that β_{12} may be expressed as

$$\beta_{12} = \beta_1(P_1) + \beta_2(P_2) . \qquad (3.2.19)$$

where β_1 and β_2 depend only on the coordinates of P_1 and P_2, respectively. Hence (3.2.11) becomes

$$\Gamma_{12}(\tau) = \left[A_1 \, e^{i\beta_1} \right] \left[A_2 \, e^{i\beta_2} \right] e^{-2\pi i \nu_0 \tau} . \qquad (3.2.20)$$

Interchanging the roles of P_1 and P_2 in (3.2.20), and using Lemma I (Chapter 2), we find

$$A_1(P_1) = A_2(P_2) , \tag{3.2.21}$$

and

$$\beta_1(P_1) = -\beta_2(P_1) + 2n\pi , \tag{3.2.22}$$

where n is any integer. If we now introduce the function

$$U(P) = A(P)e^{i\beta(P)} \tag{3.2.23}$$

we may rewrite (3.2.20) in the form

$$\Gamma_{12}(\tau) = U(P_1) \quad U^*(P_2)\,e^{-2\pi i\nu_0\tau} . \tag{3.2.24}$$

Thus in a coherent field $\Gamma_{12}(\tau)$ is of the form given by (3.2.24). That the converse is true, namely that a mutual coherence function of the form given by (3.2.24) always characterizes a coherent field, may be seen by substituting (3.2.24) into the definition of $\Gamma_{12}(\tau)$, (1.2.10).

Thus with no approximation on the spectral width of the illumination the following theorems have been established :

Theorem IX : An optical field is coherent if and only if it is monochromatic.

Theorem X: The mutual coherence function for a coherent optical field can be expressed in the form given by (3.2.24), i.e., as a simple periodic factor, $e^{-2\pi i\nu_0\tau}$, multiplying the product of a wave function, U, evaluated at P_1 with its complex conjugate, U^*, evaluated at P_2.

3.2.2 Form of $\Gamma_{12}(\tau)$ for an Incoherent Field

Incoherence is characterized by

$$\left| \gamma_{12}(\tau) \right| \equiv 0 \, , \qquad (3.2.25)$$

which implies (since the intensity, $\Gamma_{ss}(0)$, is assumed to be finite)

$$\Gamma_{12}(\tau) \equiv 0 \, . \qquad (3.2.26)$$

However, by definition

$$\Gamma_{11}(\tau) = I(P_1) = \langle V_1(t - \tau) \; V_1^*(t) \rangle \, . \qquad (3.2.27)$$

The auto-correlation function, $\Gamma_{11}(\tau)$, is called the self intensity; and it is clear from (3.2.27) that, apart from the trivial case of an identically zero field, the self intensity is not identically zero. From (3.2.26) and (3.2.27) it follows that the mutual coherence function in an incoherent field should be of the form

$$\Gamma_{12}(\tau) = \begin{Bmatrix} 0 & P_1 \neq P_2 \\ & \\ I(P_1, \tau) & P_1 = P_2 \end{Bmatrix} . \qquad (3.2.28)$$

By a simple generalization of the argument used by the present author for quasi-monochromatic illumination* (see Parrent (1959b)) it will be shown that $\Gamma_{12}(\tau)$ cannot be of the form given by (3.2.28) and at the same time satisfy the wave equation (3.1.1). This theorem is most easily demonstrated in the frequency domain. Accordingly, we introduce the mutual spectral density, $\hat{\Gamma}_{12}(\nu)$, which may be defined as the Fourier transform of $\Gamma_{12}(\tau)$, i. e.,

* Actually the argument used in that paper is slightly different from the argument given here but the general structure of the development is the same.

$$\hat{\Gamma}_{12}(\nu) = \int_{-\infty}^{\infty} \Gamma_{12}(\tau)\, e^{2\pi i \nu \tau}\, d\tau . \tag{3.2.29}$$

It has been shown (Chapter 2) that, since $\Gamma_{12}(\tau)$ is an analytic signal, $\hat{\Gamma}_{12}(\nu) = 0$ for negative ν. By the Fourier inversion theorem it then follows that

$$\Gamma_{12}(\tau) = \int_{0}^{\infty} \hat{\Gamma}_{12}(\nu)\, e^{-2\pi i \nu \tau}\, d\nu . \tag{3.2.30}$$

Substituting from (3.2.30) into (3.1) we obtain

$$\left[\nabla_s^{\,2} + \left(\frac{2\pi\nu}{C}\right)^2 \right] \hat{\Gamma}_{12}(\nu) = 0 \quad (s = 1,2). \tag{3.2.31}$$

In terms of $\hat{\Gamma}_{12}(\nu)$ (3.2.28) can be expressed as

$$\hat{\Gamma}_{12}(\nu) = \begin{Bmatrix} 0 & P_1 \neq P_2 \\ \hat{I}\,(P,\nu) & P_1 = P_2 \end{Bmatrix} , \tag{3.2.32}$$

where $\hat{I}(P,\nu)$ is the Fourier transform of $I(P,\tau)$.

Let V be a finite volume of space throughout which the field is assumed to be incoherent (i.e. (3.2.32) satisfied throughout V). Let Σ be any closed surface contained in V. By repeated application of Green's theorem (see Parrent (1959b), or Chapter 4) we obtain the formal solution to the Helmholtz equations, (3.2.31), as

$$\hat{\Gamma}_{12}(\nu) = \frac{1}{(2\pi)^2} \int_{\Sigma}\int_{\Sigma} \hat{\Gamma}(S_1,S_2,\nu)\, \frac{\partial G_1}{\partial n}\, \frac{\partial G_2}{\partial n}\, d\Sigma_1\, d\Sigma_2$$

. . . . . (3.2.33)

where G_1 and G_2 are Green's functions satisfying (3.2.31) and vanishing over Σ; S_1 and S_2 are points on Σ; and P_1 and P_2 are points within Σ.

Since in (3.2.33) we allow S_1 and S_2 to explore the surface Σ independently, the integral is four-dimensional. The integrand is, however, only two-dimensional since it is non-zero only when the two points S_1 and S_2 coincide. Therefore, if $\hat{I}(S_1, S_2, \nu)$ is assumed to be everywhere finite the integral is identically zero, i.e.,

$$\hat{\Gamma}_{12}(\nu) \equiv 0 , \tag{3.2.34}$$

for all P_1 and P_2 (including $P_1 = P_2$). This conclusion, however, contradicts the assumption that (3.2.32) is satisfied throughout V. We thus find,

Theorem XI : An incoherent field cannot exist in free space.

However, following Blanc-Lapierre and Dumontet (1954), we may define an incoherent source as one for which the mutual coherence function is of the form

$$\Gamma_{12}(\tau) = I(P_2, \tau)\,\delta(P_2 - P_1) \tag{3.2.35}$$

for all pairs of points on the source, where δ is the Dirac delta function.

3.3 Quasi-Monchromatic Fields

The theorems derived in the previous section are valid for illumination of arbitrary spectral width. However, in most of the current applications of coherence theory one deals with quasi-monochromatic illumination. Further, as pointed out in Chapter 1, most of the results of coherence theory established in the literature are applicable only to this special case. Therefore, to relate the results obtained in this thesis to application and to investigations reported in the literature, it will prove useful to examine in some detail the limiting forms of some of our formulae for quasi-monochromatic light.

An optical field is said to be quasi-monochromatic if the spectral width, $\Delta\nu$, of the light is small compared to the mean frequency, $\bar{\nu}$ i.e., if

$$\frac{\Delta\nu}{\bar{\nu}} << 1. \tag{3.3.1}$$

More precisely, $\bar{\nu}$ and $\Delta\nu$, may be defined in terms of the mutual spectral density as*

$$\bar{\nu} = \frac{\int_0^\infty \nu \left| \hat{\Gamma}_{12}(\nu) \right|^2 d\nu}{\int_0^\infty \left| \hat{\Gamma}_{12}(\nu) \right|^2 d\nu}, \tag{3.3.2}$$

and

$$\Delta\nu = \frac{\int_0^\infty (\nu-\bar{\nu})^2 \left| \hat{\Gamma}_{12}(\nu) \right|^2 d\nu}{\int_0^\infty \left| \hat{\Gamma}_{12}(\nu) \right|^2 d\nu}. \tag{3.3.3}$$

It can be shown that when (3.3.1) is satisfied the mutual coherence function may be expressed in the form (c.f. Appendix 2)

$$\Gamma_{12}(\tau) \simeq \Gamma_{12}(0)\, e^{-2\pi i\bar{\nu}\tau} \qquad \left(|\tau| << \frac{1}{\Delta\nu}\right), \tag{3.3.4}$$

provided we restrict our attention to sufficiently small time differences, as indicated on the right in (3.3.4). Quasi-monochromatic fields may be said to be coherent if the condition

$$\left|\gamma_{12}(\tau)\right| = 1 \qquad \left(|\tau| << \frac{1}{\Delta\nu}\right), \tag{3.3.5}$$

is satisfied. Adoption of this considerably weaker condition is equivalent to

* Following Wolf (1958b).

examining only the central fringes in a Young's interference experiment used to measure the degree of coherence. Such a "modified definition" is reasonable for quasi-monochromatic fields since monochromatic illumination is an unrealizable idealization, and light of sufficiently narrow spectral width behaves as coherent light under suitable conditions.*
Substituting from (3.3.4) into the wave equations (3.1.1), we obtain

$$\left[\nabla_s^2 + \bar{k}^2\right] \Gamma_{12}(0) = 0, \quad (s = 1,2) \quad (3.3.6)$$

where $\bar{k}$ is the mean wave number. Combining (3.3.4) and (3.3.5) we find the mutual intensity for coherent quasi-monochromatic fields to be of the form

$$\Gamma_{12}(0) = A_1 \ A_2 \ e^{i\phi_{12}} , \quad (3.3.7)$$

where as before $\sqrt{\Gamma_{ss}(0)} = A_s$. From (3.3.6) and (3.3.7) and the arguments of Section 2 of this chapter we obtain the form of Theorems IX and X for the quasi-monochromatic approximation.

Theorem XII . A quasi-monochromatic field is coherent if and only if the mutual intensity can be expressed as the product of a wave function, U, evaluated at P_1 with its complex conjugate, U^*, evaluated at P_2, i.e., if

$$\Gamma_{12}(0) = U(P_1)U^*(P_2) . \quad (3.3.8)$$

U(P) is the function defined in Section 2.

This result is in agreement with the observation that, for the purpose of calculating the intensity in diffraction phenomena involving short path differences

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* We point out that quasi-monochromatic light may also be incoherent or partially coherent, c.f. Chapter 4 of this thesis; on the other hand, monochromatic light is always coherent.

one may treat coherent quasi-monochromatic fields as monochromatic. That is a coherent quasi-monochromatic field is completely specified by a single monochromatic wave function $Ue^{i\bar{\omega}t}$.

The theorem of the preceding section, which states that an incoherent optical field cannot exist in free space, is valid in general and hence is obviously true for quasi-monochromatic fields. The theorem may, however, be demonstrated directly for the case considered here (this result was originally obtained for quasi-monochromatic fields in this manner (see Parrent (1959b)). An incoherent quasi-monochromatic source may be defined, by analogy with the general case, as a source the mutual intensity of which is of the form

$$\Gamma_{12}(0) = I(S_2)\,\delta\,(S_2 - S_1) \quad . \tag{3.3.9}$$

CHAPTER 4

THE PROPAGATION OF PARTIALLY COHERENT LIGHT

The laws governing the propagation of partially coherent optical fields have been of central importance in the development of coherence theory. The most important of the earlier contributions to this aspect of the theory is a theorem discovered separately by van Cittert (1934, 1939) and Zernike (1938). Under suitable conditions on the spectral width of the illumination (quasi-monochromatic light), this theorem expresses the mutual intensity on a plane illuminated by an incoherent plane source in terms of the intensity distribution across the source. In the same paper Zernike also derived an approximate law for the propagation of the mutual intensity. Later Hopkins (1951) derived these theorems in a different way.

In the formulation of coherence theory used here these theorems will be seen to be limiting or approximate forms of a Green's function solution to the wave equations which were derived in Chapter 3. In the present chapter we will obtain the general solution for the propagation of <u>mutual coherence</u> from a plane polychromatic partially coherent source. The limiting forms of this general solution will be examined in some detail both for sources of wide and small spectral ranges, and the van Cittert-Zernike theorem will be shown to represent an approximate form of the incoherent limit of the quasi-monochromatic solution.

4.1 General Solution for the Distribution of Mutual Coherence from a Plane Source

In this section we shall determine the mutual coherence function for a field created by a plane polychromatic source. In Figure 2, which serves to define the coordinates, σ is the plane containing the extended polychromatic source with a known distribution of mutual coherence. P_1 and P_2 are points in the illuminated field and S_1 and S_2 are points in the plane of the source. P_1' and P_2' are the mirror images of the points P_1 and P_2 respectively in the plane σ.

As shown in Chapter 3 the propagation of the mutual coherence function in vacuum is governed by the two wave equations

$$\Delta_s^2 \Gamma_{12}(\tau) = \frac{1}{c^2} \frac{\partial^2 \Gamma_{12}(\tau)}{\partial \tau^2} \quad (s = 1,2). \qquad (4.1.1)$$

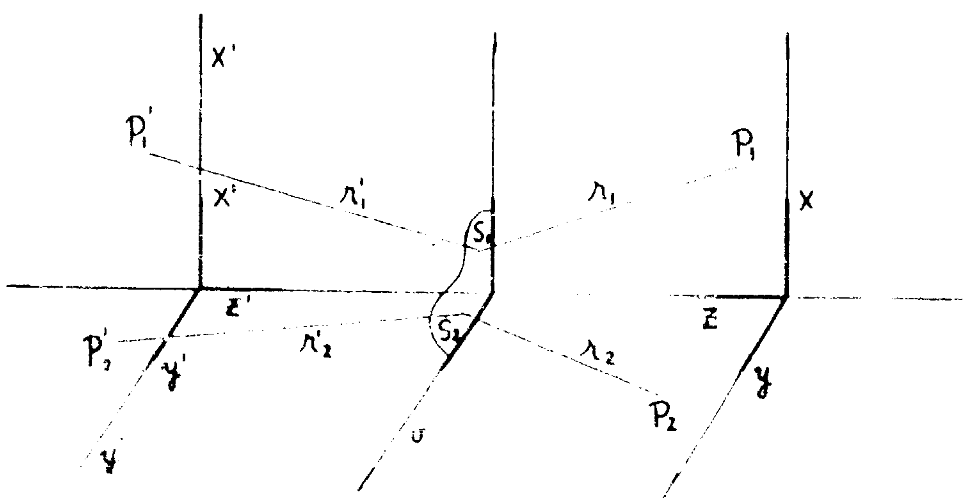

Figure 2

We assume that $\Gamma_{12}(\tau)$ is known for all pairs of points S_1 and S_2 in the plane σ. Let $\hat{\Gamma}_{12}(\nu)$ be the Fourier time transform of $\Gamma_{12}(\tau)$. Since $\Gamma_{12}(\tau)$ is an analytic signal (see Chapter 2), it contains only positive frequencies, i.e.,

$$\Gamma_{12}(\tau) = \int_0^\infty \hat{\Gamma}_{12}(\nu)\, e^{-2\pi i\nu\tau}\, d\nu, \qquad (4.1.2)$$

and by the inversion theorem

$$\hat{\Gamma}_{12}(\nu) = \int_{-\infty}^{\infty} \Gamma_{12}(\tau) e^{2\pi i\nu\tau}\, d\tau \qquad \nu > 0, \qquad (4.1.3)$$

$$= \quad 0 \qquad\qquad \nu < 0.$$

Substituting from (4.1.2) into (4.1.1) and interchanging the order of integration and differentiation we obtain

$$\int_0^\infty [\nabla_s^2 + k^2(\nu)] \quad \hat{\Gamma}_{12}(\nu) e^{-2\pi i\nu\tau}\, d\nu = 0 \quad (s = 1,2)$$

$$\ldots\ldots (4.1.4)$$

since (4.1.4) must hold for all τ, we have

$$[\nabla_s^2 + k^2(\nu)] \quad \hat{\Gamma}_{12}(\nu) = 0, \quad (s = 1,2) \qquad (4.1.5)$$

where $k(\nu) = 2\pi\nu/c$. Thus, each spectral component of $\Gamma_{12}(\tau)$ satisfies the two scalar Helmholtz equations, (4.1.5).

Equation (4.1.5) can be formally solved by employing Green's functions. To this end we integrate first over the coordinates of S_1 and obtain*

$$\hat{\Gamma}(P_1, S_2, \nu) = -\frac{1}{2\pi} \int_\sigma \hat{\Gamma}(S_1, S_2, \nu) \frac{\partial G_1}{\partial n}\, dS_1 . \qquad (4.1.6)$$

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* $\Gamma_{12}(\tau)$ and $\hat{\Gamma}_{12}(\nu)$ will be written as $\Gamma(P_1, P_2, \tau)$ and $\hat{\Gamma}(P_1, P_2, \nu)$ when necessary to stress the space dependence.

Here G_1 is a Green's function satisfying the equation

$$[\nabla_S^2 + k^2(\nu)] \quad G_1 = 0 \tag{4.1.7}$$

with the boundary condition

$$G(S_1) = 0. \tag{4.1.8}$$

Equation (4.1.6) provides the boundary condition for the solution of the second Helmholtz equation; and employing the same theorem again we obtain

$$\hat{\Gamma}(P_1, P_2, \nu) = -\frac{1}{2\pi} \int_\sigma \hat{\Gamma}(P_1, S_2, \nu) \quad \frac{\partial G_2}{\partial n} \quad dS_2 , \tag{4.1.9}$$

by integrating over the coordinates of S_2, where G_2 is a second Green's function satisfying the same conditions as G_1. Substituting from (4.1.6) into (4.1.9) we obtain

$$\hat{\Gamma}(P_1, P_2, \nu) = \frac{1}{(2\pi)^2} \int_\sigma \int_\sigma \hat{\Gamma}(S_1, S_2, \nu) \; \frac{\partial G_1}{\partial n} \; \frac{\partial G_2}{\partial n} \; dS_1 \; dS_2 .$$

$$\ldots\ldots (4.1.10)$$

In order to determine the exact form of G_1 and G_2 for cases of physical interest, we impose on $\hat{\Gamma}_{12}(\nu)$ the radiation condition of Sommerfeld. In essence this condition implies that the distant field is essentially that of a diverging spherical wave. By applying the radiation condition to $V_1(\nu)$ and $V_2^*(\nu)$ and appealing to the definition of $\hat{\Gamma}_{12}(\nu)$ in terms of these functions, it is readily seen that $\hat{\Gamma}_{12}(\nu)$ must behave asymptotically, for $kr_1 >> 1$ and $kr_2 >> 1$ as

$$\hat{\Gamma}_{12}(\nu) \sim f(\theta_1, \theta_2, \phi_1, \phi_2) \; \frac{e^{ik(r_1 - r_2)}}{r_1 r_2} \quad , \tag{4.1.11}$$

where θ_s and ϕ_s are the spherical polar coordinates of P_s. To satisfy this condition and those defining G_1 and G_2 the required Green's functions are

$$G_1 = \frac{e^{ikr_1}}{r_1} - \frac{e^{ikr_1'}}{r_1'} , \tag{4.1.12}$$

and

$$G_2 = \frac{e^{-ikr_2}}{r_2} - \frac{e^{-ikr_2'}}{r_2'} , \tag{4.1.13}$$

where r_1, r_2, r_1' and r_2' are defined in Figure 2. That these are the required Green's functions can be seen by direct substitution. Before substituting into (4.1.10) we obtain the normal derivatives; we have

$$\frac{\partial G_1}{\partial n} = (ikr_1 - 1) \frac{e^{ikr_1}}{r_1} \frac{\partial r_1}{\partial n} + (1 - ikr_1') \frac{e^{ikr_1'}}{r_1'} \frac{\partial r_1'}{\partial n} . \qquad \ldots\ldots (4.1.14)$$

Noting that $r_1 \big|_\sigma = r_1' \big|_\sigma$ and that $\frac{\partial r_1}{\partial n}\Big|_\sigma = z = - \frac{\partial r_1'}{\partial n}\Big|_\sigma$,

and setting $\cos \theta_s = z_s/r_s$, we may rewrite (4.1.14) as

$$\frac{\partial G_1}{\partial n}\Big|_\sigma = 2(ikr_1 - 1) \cos\theta_1 \frac{e^{ikr_1}}{r_1} , \tag{4.1.15}$$

and similarly

$$\frac{\partial G_2}{\partial n}\Big|_\sigma = - 2(ikr + 1) \cos\theta_2 \frac{e^{-ikr_2}}{r_2} . \tag{4.1.16}$$

On substituting from (4.1.15) and (4.1.16) into (4.1.10), we obtain

$$\hat{\Gamma}(P_1,P_2,\nu) = \frac{1}{(2\pi)^2} \iint_{\sigma\,\sigma} \hat{\Gamma}(S_1,S_2,\nu)\,(1-ikr_1)(1+ikr_2)\,\cos\theta_1\,\cos\theta_2\,\frac{e^{ik(r_1-r_2)}}{r_1 r_2}\,dS_1\,dS_2 \qquad \ldots\ldots (4.1.17)$$

Equation (4.1.17) is the contribution from a single spectral component, ν. The complete solution is obtained from (4.1.17) by taking the Fourier transform of both sides of (4.1.17), i.e.

$$\Gamma_{12}(\tau) = \frac{1}{(2\pi)^2} \int_0^\infty \iint_{\sigma\,\sigma} \hat{\Gamma}(S_1,S_2,\nu)\,(1-ikr_1)(1+ikr_2)\cos\theta_1\cos\theta_2\,\frac{e^{ik(r_1-r_2)}}{r_1 r_2}\,e^{-2\pi i\nu\tau}\,dS_1\,dS_2\,d\nu \qquad \ldots\ldots (4.1.18)$$

Since dS_1, dS_2, and $d\nu$ are independent, we may invert the order of integration and obtain

$$\Gamma_{12}(\tau) = \frac{1}{(2\pi)^2} \iint_{\sigma\,\sigma} \cos\theta_1\,\cos\theta_2\,\Omega_{12}(\tau)\,dS_1\,dS_2 \,, \qquad \ldots\ldots (4.1.19)$$

where

$$\Omega_{12}(\tau) = \int_0^\infty \hat{\Gamma}(S_1,S_2,\nu)\,e^{-2\pi i\nu\left(\tau - \frac{r_1-r_2}{c}\right)}\,d\nu$$

$$+\, i\frac{(r_2-r_1)}{c} \int_0^\infty 2\pi\nu\,\hat{\Gamma}(S_1,S_2,\nu)\,e^{-2\pi i\nu\left(\tau - \frac{r_1-r_2}{c}\right)}\,d\nu$$

$$+\, \frac{r_1 r_2}{c^2} \int_0^\infty 4\pi^2\nu^2\,\hat{\Gamma}_{12}(\nu)\,e^{-2\pi i\nu\left(\tau - \frac{r_1-r_2}{c}\right)}\,d\nu \,.$$

We recall the following well-known theorem from Fourier analysis. If

$$f(t) = \int_0^\infty g(\nu) e^{-2\pi i \nu t} d\nu \ : \tag{4.1.20}$$

then

$$\frac{\partial^n f(t)}{\partial t^n} = \int_0^\infty (-2\pi i \nu)^n \ g(\nu) e^{-2\pi i \nu t} d\nu \ . \tag{4.1.21}$$

Using (4.1.21) $\Omega_{12}(\tau)$ may be evaluated giving

$$\Omega_{12}(\tau) = \Gamma\left(S_1, S_2, \tau - \frac{r_1 - r_2}{c}\right) + \frac{r_1 - r_2}{c} \frac{\partial}{\partial \tau} \Gamma\left(S_1 S_2, \tau - \frac{r_1 - r_2}{c}\right) - \frac{r_1 r_2}{c^2} \frac{\partial^2}{\partial \tau^2} \Gamma\left(S_1, S_2, \tau - \frac{r_1 - r_2}{c}\right)$$

$$\ldots\ldots \ (4.1.22)$$

and the final solution is

$$\Gamma_{12}(\tau) = \iint_{\sigma\ \sigma} \cos\ \theta_1 \ \cos\ \theta_2 \ \mathcal{F}\left[\Gamma\left(S_1, S_2, \tau - \frac{r_1 - r_2}{c}\right)\right] dS_1, dS_2,$$

$$\ldots\ldots \ (4.1.23)$$

where $\mathcal{F}$ is the differential operator

$$\mathcal{F} = \left[1 + \frac{r_1 - r_2}{c} \frac{\partial}{\partial \tau} - \frac{r_1 r_2}{c^2} \frac{\partial^2}{\partial \tau^2}\right]. \tag{4.1.24}$$

Equation (4.1.23) is the general solution for the mutual coherence in a field produced by a plane polychromatic source.

4.2 Coherent and Incoherent Polychromatic Sources

In this section we examine the limiting forms of (4.1.23) for coherent and incoherent sources. No approximation on the spectral width of the illumination will be made here. We will show that 1.) a coherent source always gives rise to a coherent field and 2.) an incoherent source always creates a partially coherent field. By examining the incoherent limit we will obtain a rigorous generalization of the van Cittert-Zernike theorem to polychromatic sources.

4.2.1 Coherent Source

It was shown in Chapter 3 that coherence is characterized by a mutual coherence function of the form

$$\Gamma(S_1,S_2,\tau) = U_\sigma(S_1)\ U_\sigma^*(S_2)\, e^{-2\pi i\nu_0\tau} . \tag{4.2.1}$$

Taking the Fourier transform of both sides of (4.2.1) we obtain

$$\hat{\Gamma}(S_1,S_2,\nu) = U_\sigma(S_1)\ U_\sigma^*(S_2)\ \delta(\nu-\nu_0) . \tag{4.2.2}$$

Substituting from (4.2.2) into (4.1.17) we obtain

$$\hat{\Gamma}(P_1,P_2,\nu) = U(P_1)\ U^*(P_2)\ \delta(\nu-\nu_0) , \tag{4.2.3}$$

where

$$U(P_1) = \int_\sigma U_\sigma(S_1) \cos\vartheta\ (1 - ikr_1)\ \frac{e^{ikr_1}}{r_1}\, dS_1 . \tag{4.2.4}$$

Taking the inverse transform of both sides of (4.2.3), we obtain

$$\Gamma(P_1,P_2,\tau) = U(P_1)\, U^*(P_2) e^{-2\pi i\nu_0\tau} . \tag{4.2.5}$$

From (4.2.5) and Theorem IX, p. 57, it is clear that the following theorem holds :

Theorem XIII : In vacuum a coherent source will always give rise to a coherent field.

4.2.2 Incoherent Source

In Chapter 3 it was shown that an incoherent field cannot exist in free space although an incoherent source may be defined. In the present section we will show that the definition of such a source is consistent with the result that an incoherent source will give rise to a partially coherent field.

By definition, an incoherent source is characterized by a mutual coherence function of the form

$$\Gamma(S_1,S_2,\tau) = I(S_2,\tau)\delta\,(S_2 - S_1)\,. \tag{4.2.6}$$

Hence $\hat{\Gamma}(S_1,S_2,\nu)$ is given by

$$\hat{\Gamma}(S_1,S_2,\nu) = \hat{I}(S_1,\nu)\delta\,(S_2 - S_1)\,. \tag{4.2.7}$$

We substitute (4.2.7) into (4.1.17); and, after integrating over S_2, we obtain

$$\hat{\Gamma}(P_1,P_2,\nu) = \int_\sigma \hat{I}(S_1,\nu)\,(1-ikr_1)\,(1+ikr_2)\cos\theta_1\cos\theta_2\,\frac{e^{ik(r_1-r_2)}}{r_1 r_2}\,dS_1 \tag{4.2.8}$$

where r_1 and r_2 are now interpreted as the distance from S to P_1 and P_2 respectively.

By substituting from (4.2.6) into (4.1.18), and integrating over S_2, we obtain the mutual coherence function for the field of an extended, incoherent, polychromatic, source ,

$$\Gamma_{12}(\tau) = \int_\sigma \cos\theta_1\,\cos\theta_2\,\mathcal{F}[\,I(S_1\,\tau - \frac{r_1 - r_2}{c})\,]\,dS\,. \tag{4.2.9}$$

Equation (4.2.9) is the generalization of the van Cittert-Zernike theorem for polychromatic sources. It expresses the mutual coherence function under the conditions stated, in terms of the self coherence across the source.

Since an optical field is coherent if and only if it is monochromatic, it is clear that, while an incoherent source gives rise, by (4.2.9), to a partially coherent field, an incoherent source cannot create a coherent field.

4.3 Quasi-Monochromatic Sources

As mentioned earlier, most of the current applications of coherence theory involve quasi-monochromatic light. For this reason we shall treat the quasi-monochromatic limits as separate problems deducing them directly from the general solution, (4.1.23), rather than obtaining them as special cases of the results of the previous section.

4.3.1 Coherent Source

Since in the quasi-monochromatic approximation a field is described as coherent if the condition $|\Gamma_{12}(\tau)| = 1$ is satisfied only for sufficiently small τ, coherence is a considerably weaker condition for the class of problems considered than in the general case.

A coherent quasi-monochromatic source has the property that for $|\tau| << \frac{1}{\Delta\nu}$ its mutual coherence function is of the form

$$\Gamma(S_1,S_2,\tau) = U_\sigma(S_1)\; U_\sigma(S_2)\, e^{-2\pi i\bar{\nu}\tau} \quad . \tag{4.3.1}$$

Substituting from (4.3.1) into (4.1.23) yields

$$\Gamma(P_1,P_2,\tau) = U(P_1)\; U^*(P_2)\, e^{-2\pi i\bar{\nu}\tau} \quad (|\tau| << \frac{1}{\Delta\nu}) \; , \tag{4.3.2}$$

where

$$U(P_1) = \int_\sigma U_\sigma(S_1)(1 - ikr_1)\cos\theta_1 \frac{e^{ikr_1}}{r_1}\, dS \; . \tag{4.3.3}$$

Equations (4.3.2) and (4.3.3) constitute the mathematical restatement of Theorem XIII for quasi-monochromatic fields.

This result is not simply a special case of the previous result even though in Section 4.2 we were able to establish the theorem with no approximations. The result is perhaps more interesting and certainly of more immediate practical importance in the present case. In the general treatment where no approximations on the spectral width of the light were made we concluded that an optical field is coherent if and only if it is monochromatic. It is certainly to be expected that such a field will remain coherent as it propagates in free space. It is also to be expected that such a field can be completely specified (apart from polarization effects) by a simple wave function, depending on the coordinates of one point only.

The fact that these results hold under the narrow spectral width approximation, however, indicates something more. It is evident from (4.3.2) and (4.3.3) (and in fact from the results of Chapter 3), that if we restrict ourselves to phenomena involving sufficiently small path differences, $(|\tau| << \frac{1}{\Delta\nu})$, a quasi- monochromatic field may behave in some respects like a monochromatic field. But there is an essential difference between these two types of illumination :

A monochromatic field is everywhere coherent for all τ, while a quasi-monochromatic field cannot be coherent in the strict sense, $|\gamma_{12}(\tau)| = 1$, but only in terms of the weaker condition, $|\gamma_{12}(\tau)| \approx 1 \quad (|\tau| << \frac{1}{\Delta\nu})$. Further a quasi-monochromatic field may, as will be discussed in Section 4.3.2, be incoherent.

In spite of these considerations the similarity between these two types of fields for a large class of diffraction phenomena has led to a loose usage of the term monochromatic and hence to such meaningless statements frequently found in the literature "partially coherent (or incoherent) monochromatic light" and the phrase "to extend the concepts of partial coherence to light with finite spectral width".

4.3.2 Incoherent Source

Under the conditions justifying the quasi-monochromatic approximation we may write (see Appendix 2)

$$\Gamma_{12}(\tau) \simeq \Gamma_{12}(0)\, e^{-2\pi i\bar{\nu}\tau} \qquad \left(|\tau| \ll \frac{1}{\Delta\nu}\right) . \qquad (4.3.4)$$

On substituting from (4.3.4) into (4.1.8) and integrating over ν, we obtain

$$\Gamma_{12}(\tau) = \frac{e^{-2\pi i\bar{\nu}\tau}}{(2\pi)^2} \int_\sigma\int_\sigma (1 - ikr_1)\,(1 + ikr_2)\,\cos\theta_1\,\cos\theta_2\,\Gamma(S_1,S_2,0)\,\frac{e^{ik(r_1-r_2)}}{r_1 r_2}\,dS_1\,dS_2 .$$

. . . . . (4.3.5)

Equation (4.3.5) expresses the mutual coherence function for small $|\tau|$ in the field created by a plane partially coherent quasi-monochromatic source.

We now consider the limiting form of (4.3.5) when the source is incoherent. By definition an incoherent quasi-monochromatic source may be specified by a mutual coherence function of the form

$$\Gamma_{12}(\tau) = I(S_2)\,\delta(S_2 - S_1)\, e^{-2\pi i\bar{\nu}\tau} \qquad \left(|\tau| \ll \frac{1}{\Delta\nu}\right) .$$

. . . . . (4.3.6)

Substituting (4.3.6) into (4.3.5) and integrating over S_2 yields

$$\Gamma_{12}(\tau) = \frac{e^{-2\pi i\bar{\nu}\tau}}{(2\pi)^2} \int_\sigma I(S)\,(1 - ikr_1)\,(1 + ikr_1)\,\cos\theta_1\,\cos\theta_2\,\frac{e^{ik(r_1-r_2)}}{r_1 r_2}\,dS \qquad \left(|\tau| \ll \frac{1}{\Delta\nu}\right) .$$

. . . . . (4.3.7)

Here r_1 and r_2 are now interpreted as the distance from a typical point S on the source to the field points P_1 and P_2 respectively, and $\cos\theta_s = z_s/r_s$, $(s = 1, 2)$. Equation (4.3.7) expresses the mutual coherence function, $\Gamma_{12}(\tau)$, for sufficiently small $|\tau|$ in terms of the intensity distribution across the source.

If (4.3.7) is evaluated at $\tau = 0$, the obliquity factors are ignored and attention is limited to field points on a plane parallel to σ we obtain the theorem due to van Cittert (1934) and Zernike (1938) already referenced, viz.,

$$\Gamma_{12}(0) \simeq \int_{\sigma} I(S)\frac{e^{ik(r_1-r_2)}}{r_1 r_2}\, dS. \tag{4.3.8}$$

This theorem expresses the mutual intensity on the illuminated plane in terms of the intensity distribution of the incoherent source.

In most applications one is interested in the form of (4.3.8) in the Fraunhofer approximation. The right-hand side of (4.3.8) then reduces to a Fourier transform of the intensity distribution.* If the intensity distribution is suitably normalized (4.3.8) becomes

$$\gamma_{12}(0) = \iint I(\xi, \eta)\, e^{ik(p\xi + q\eta)}\, d\xi d\eta \ , \tag{4.3.9}$$

where $\qquad p = \frac{x_1}{R_1} - \frac{x_2}{R_2}, \quad q = \frac{y_1}{R_1} - \frac{y_2}{R_2},$

and $\qquad R_s = \sqrt{x_s^2 + y_s^2 + z_s^2}$. These coordinates are defined in Figure 2, p. 67. In the region where (4.3.9) is valid R_1 and R_2 may be taken as equal. Equation (4.3.9) is the most commonly used form of the van Cittert-Zernike theorem. We see that formula (4.3.5) is a generalization of this theorem to partially coherent (but quasi-monochromatic) sources.

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* Actually, the validity of the Fourier transform relation, (4.3.8), does not depend on the Fraunhofer approximations [illegible] tion (4.3.9) applies whenever the obliquity factors and the variations [illegible] in (4.3.7) can be ignored while the Fraunhofer approximation requires in addition to these conditions that $R \gg (p^2 + q^2)$ [illegible]

CHAPTER 5

IMAGING OF EXTENDED POLYCHROMATIC SOURCES AND GENERALIZED TRANSFER FUNCTIONS

In this chapter we shall apply the theorems and results of the earlier chapters to the determination of the relation between object and image for systems which image extended polychromatic objects. We shall treat the problem primarily in the spatial frequency domain, an approach introduced by Duffieux (1946) in 1946. Since its introduction, the frequency domain analysis has proved very powerful in the study of imaging systems. In this analysis the imaging system is described by a transfer function (also called modulation function, transmission factor, transmission function, contrast rendition function, frequency response function). The imaging problem is then solved as follows : the object and image are described in terms of the distribution of a suitable physical characteristic of the optical disturbance, which characteristic is determined by the degree of coherence of the object illumination. For example, an incoherently illuminated object is described in terms of the intensity distribution across it. The spatial spectrum of the image is then obtained as the product of the transfer function with the spatial spectrum of the object, i.e., the spatial Fourier transform of the above-mentioned distribution. That is, the optical system is treated as a spatial frequency filter. After its introduction into the study of optical systems, the transfer function analysis was applied to the study of the mapping problems of radio astronomy (see Bracewell and Roberts (1954)) and radar.

This analysis is particularly promising in the study of cascaded systems as exemplified by Schade's (1948) treatment of television systems. In cascaded systems the final image in the frequency domain is obtained by multiplying the spectrum of the object with the product of the transfer functions describing each stage of the system.

The essential step in this approach is the recognition that many optical systems may to a good approximation be treated as linear stationary systems in terms of their spatial as well as their temporal dependence. All the advantages, familiar to electronics engineers, of performing linear system analysis in the frequency domain may then be realized in optical imaging problems. In optics there are, however, some difficulties confronting this approach.

For example, the functions, describing objects and images, met in the analysis of optical systems, depend on spatial as well as temporal coordinates; some optical systems of practical interest are not "stationary" in their spatial variation. The most significant difficulty is the fact that the form of the transfer function is determined by the degree of coherence of the object illumination. As we shall see in the following development, some of these difficulties do not appear in analogous problems in radio astronomy and radar, though these fields present other problems.

The realization that imaging systems can, under suitable conditions, be analyzed as linear stationary systems suggests strongly the application of the techniques of information and communication theory. Here again, however, several basic difficulties are encountered.

Apart from the considerations already mentioned, an imaging device is not in general a communication system (since no opportunity of encoding the input exists) but rather an observation system. The difficulties confronting the analysis of such a system in terms of information and communication theory are discussed and illustrated by Woodward (1953) and lie outside the domain of our present discussion.

In spite of these considerations some results have been obtained from the application of information theory to imaging systems. The chief contributions of this theory to the study of image formation are: 1.) the demonstration that an optical image has a finite number of degrees of freedom (see Fellgett and Linfoot (1955) and Gabor (1956)) ; and 2.) the demonstration that the criteria for judging the quality of an imaging device must take account of the objects that the system is to image (see Fellgett and Linfoot (1955) and Schade (1948)). The [illegible] of these conclusions follows immediately from Shannon's Sampling Theorem (c.f. Woodward, 1953) and the fact that an imaging system behaves as a low pass filter with a finite cut off frequency. The second consideration will become evident from the subsequent discussion of this chapter.

Some of the various quality criteria for imaging devices which were introduced by Fellgett and Linfoot and Schade have been evaluated for aberrated optical systems (see O'Neill (1956), Fukui (1957) and Parrent and Drane (1956)) and for antenna systems employing Dolph-Tchebbycheff apodization* (see Drane (1957)). The results obtained in each case were in good qualitative agreement with experience.

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* It can be shown (see Dolph (1946)) that if the currents in the elements of a linear (antenna) array are proportional to the coefficients of the Tchebbycheff polynomials the resulting diffraction pattern has the minimum possible side-lobe level for a given beam width. These polynomials provide a means of varying the apodization continuously from edge illumination (which gives the cosine squared diffraction pattern of a simple interferometer and hence the minimum beam width for a given aperture size) to a binomial distribution of currents which gives a diffraction pattern consisting of a main lobe with no side lobes.

In spite of the difficulties mentioned, the transfer function analysis has contributed to the understanding of the problems of image formation. Among the interesting consequences of the application of the transfer function analysis to optical systems are the spatial filtering techniques developed by O'Neill (1956) and Marechal and Croce (1953) for sharpening blurred edges and recovering wanted detail from an image containing noise (e.g. photographic grain).

The computation of the transfer function of any given system often leads to numerical integration; and for this reason detailed analysis has been limited to relatively simple systems, utilizing strictly coherent or strictly incoherent illumination, with small aberrations (see Steel (1953) or with a single aberration (Parrent (1955), Hopkins (1955), De (1955), O'Neill (1956)).

The limitation to coherent or incoherent illumination is, however, more basic and stems from the following considerations : 1.) A system imaging an incoherently illuminated object may be regarded as linear in <u>intensity</u>; 2.) A system imaging a coherently illuminated object may be regarded as linear in <u>amplitude</u>; 3.) Systems using partially coherent illumination are linear in neither of these quantities. H. H. Hopkins (1956) and Dumontet (1954) extended the transfer function analysis to systems imaging partially coherent objects by showing that such systems may be regarded as linear in <u>mutual intensity</u>. The transfer functions for systems with small aberrations and partially coherent objects have been computed by Steele (1957) using the Hopkins formulation.

In every case, coherent, partially coherent or incoherent, the analysis has, however, been limited to quasi-monochromatic light. It is our aim in this chapter to provide the framework for the analysis in terms of the transfer function of systems employing polychromatic illumination; and further using the theorems and results of the preceding chapters to verify that the familiar solutions to the quasi-monochromatic imaging problems follow from our general solutions as approximate forms. It is hoped that, in addition to providing the mathematical framework for the solutions to these general problems, this approach will eliminate the frequently encountered confusion concerning the significance and interpretation of the various transfer functions.*

5.1 General Formulation of the Imaging Problem

In this section it will be shown that the general scalar imaging problem, involving partially coherent polychromatic objects, can be completely solved in terms of the observables, $\Gamma_{12}(\tau)$, in object and image space with no recourse to the random disturbance, $V^r(t)$, itself. By dealing solely with the mutual coherence function, and functions simply derivable from it, our entire analysis, apart from the limiting forms, will involve only square-integrable functions. The advantages of such an approach are immediately obvious since we deal extensively with both Fourier and Hilbert transforms and their respective inversion theorems.

It will be shown here that using this general solution it is possible to define generalized transfer functions whose properties are formally similar to those of the quasi-monochromatic transfer functions. Further it will be shown that these new functions are simply derivable from the aperture illumination function (pupil function) of the imaging system.

* This confusion is discussed and illustrated by F. J. Zucker in his summary comments published in the Electronics Research Directorate, Air Force Cambridge Research Center, ARDC, January 1957.

5.1.1 Mathematical Conventions and Notations

In the development that follows extensive use is made of multi-dimensional Fourier transforms. To prevent the equations from becoming too unwieldy the following conventions and condensed notations will be used.

Cartesian coordinates will be denoted by (ξ, η) in object space, (x, y) in image space, and (α, β) in the exit pupil. The coordinates in image space are normalized by the lateral magnification of the imaging system. This is done to make the coordinates of a given object point equal in magnitude to those of the corresponding image point. The conventions regarding functional representation are :

$$f(\underline{x}) \equiv f(x, y) , \tag{5.1.1}$$

$$f(\underline{x}_1 - \underline{x}_2) \equiv f(x_1 - x_2, \; y_1 - y_2) , \tag{5.1.2}$$

$$d\underline{x}_1 \equiv dx_1 \, dy_1 , \tag{5.1.3}$$

and

$$\underline{x}_1 \equiv x_1 \underline{x}^0 + y_1 \underline{y}^0 . \tag{5.1.4}$$

Here the subscripts 1 and 2 denote the point, P_1 or P_2, whose coordinates are used; and $\underline{x}^0$ and $\underline{y}^0$ are unit vectors in the direction of the x and y axis respectively. The same conventions of course apply in the aperture and object planes (see Figure 3).

We shall be concerned with the transmission of distributions from object space to image space and the subscripts 0 and i will denote that the distribution is an object or image respectively, i.e.,

$f_0(\xi)$	distribution in object space	(5.1.5)
$f_i(\underline{x})$	Corresponding distribution in image space.	

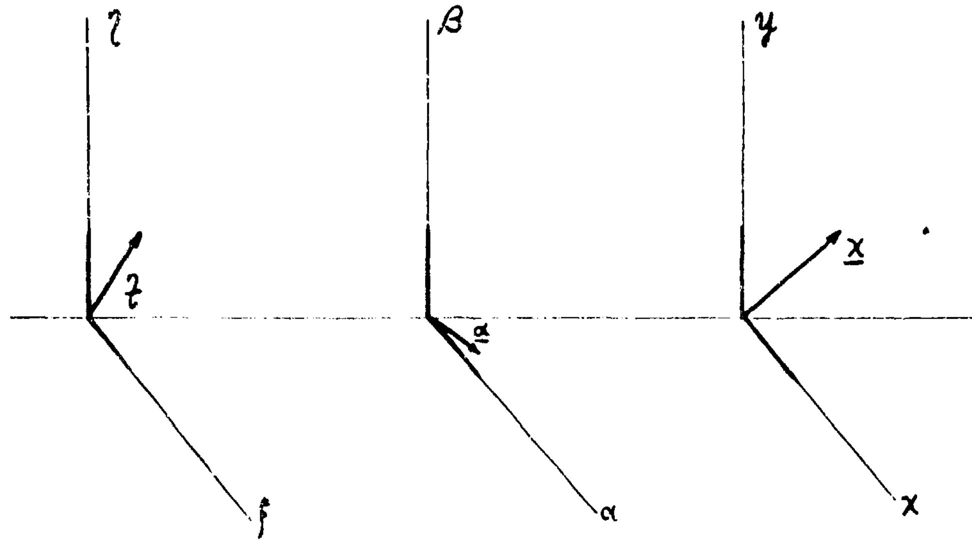

FIGURE 3

Since we shall require the Fourier transform of space functions, we associate with each space coordinate a spatial frequency coordinate using the following convention : μ_{1x} is the spatial frequency associated with the Cartesian coordinate x_1 , and μ_{1y} is associated with y_1. The functional conventions introduced for the space functions will of course also be used for the spatial frequency functions; i.e.,

$$f(\underline{\mu}_1) \equiv f(\mu_{1x}, \mu_{1y}) \quad , \tag{5.1.6}$$

$$\underline{\mu}_1 = \mu_{1x}\underline{x}^0 + \mu_{1y}\,\underline{y}^0 \quad , \tag{5.1.7}$$

and

$$\underline{d}\mu_1 = d\mu_{1x}\, d\mu_{1y} \quad . \tag{5.1.8}$$

The time coordinate is denoted by t and the time delay coordinate by τ. The associated temporal frequency is denoted by ν.

Contrary to the convention of the preceding chapters, the mutual coherence function will be written here with its full argument rather than with the subscript 12.

Associated with every function of the space time coordinates, $F(\underline{x}_1, \underline{x}_2, \tau)$, will be three other functions; namely, its "spatial" Fourier transform, $\tilde{F}(\underline{\mu}_1, \underline{\mu}_2, \tau)$; its "temporal" Fourier transform, $\hat{F}(\underline{x}_1, \underline{x}_2, \nu)$; and its total Fourier transform $\mathring{F}(\underline{\mu}_1, \underline{\mu}_2, \nu)$; i.e.,

$$\tilde{F}(\underline{\mu}_1, \underline{\mu}_2, \tau) = \iiiint_{-\infty}^{\infty} F(\underline{x}_1, \underline{x}_2, \tau)\, e^{2\pi i(\underline{\mu}_1 \cdot \underline{x}_1 + \underline{\mu}_2 \cdot \underline{x}_2)}\, d\underline{x}_1\, d\underline{x}_2 \qquad \ldots\ldots \quad (5.1.9)$$

$$\hat{F}(\underline{x}_1, \underline{x}_2, \nu) = \int_{-\infty}^{\infty} F(\underline{x}_1, \underline{x}_2, \tau)\, e^{2\pi i \nu \tau}\, d\tau, \qquad (5.1.10)$$

$$\mathring{F}(\underline{\mu}_1, \underline{\mu}_2, \nu) = \iiiint_{-\infty}^{\infty} F(\underline{x}_1, \underline{x}_2, \tau)\, e^{2\pi i(\underline{\mu}_1 \cdot \underline{x}_1 + \underline{\mu}_2 \cdot \underline{x}_2 + \nu\tau)}\, d\underline{x}_1\, d\underline{x}_2\, d\tau. \qquad \ldots\ldots \quad (5.1.11)$$

Throughout the rest of the chapter all integrals will be written with a single integral sign without limits. The order of the integration will be implied by the differentials.

5.1.2 A Generalized Transfer Function

In this section will be discussed the problem of determining the image of an extended polychromatic object and the inverse problem, viz., that of determining the object from a knowledge of the image. We shall show that the solution of this later problem is fundamentally impossible with systems of finite aperture. The object is considered to be planar and is specified by its mutual coherence function, $\Gamma_0(\xi_1, \xi_2, \tau)$ The central problem is to formulate the relation between $\Gamma_0(\xi_1, \xi_2, \tau)$ and $\Gamma_1(x_1, x_2, \tau)$ in terms of the aperture illumination function. No assumptions concerning the spectral width or degree of coherence will be made in this section. We shall show that, while the evaluation of the integrals might prove somewhat formidable in certain practical applications, a solution in closed form can be obtained and a transfer function defined which is simply related to the pupil function.

From the linearity of Maxwell's equation it follows (see Appendix 4) that if there are no non-linear devices in the imaging system the mutual spectral density will be propagated through the system in accordance with two linear differential equations, i.e.,

$$D_s \left[\hat{\Gamma}_{12}(\nu) \right] = 0 \qquad (s = 1,2) , \tag{5.1.12}$$

where D_s is a linear differential operator in the coordinates of P_s. Solving the first of these equations we obtain the mutual spectral density, $\hat{\Gamma}_{10}(x_1, x_2, \nu)$, between the oscillations at a typical object point ξ_1 and those at a typical image point x_1 . Here the subscript 10 denotes that the function depends on a point in the object space and a point in the image space. This partial solution may be obtained by using only the linearity of (5.1.12) as follows : Let the contribution to the "complex disturbance" at x_1 due to the "disturbance" from an element $d\xi_1$ of the source around ξ_1 be $\hat{\Gamma}_0(\xi_1, \xi_2, \nu)$ $K(\xi_1, x_1, \nu)\, d\xi_1$. The function $K(\xi_1, x_1, \nu)$ describes the optical imaging system. Then, since (5.1.12) is a linear differential equation, the total "disturbance" at $\underline{x}_1$ is given by

$$\hat{\Gamma}_{10}(\underline{x}_1, \underline{\xi}_2, \nu) = \int \hat{\Gamma}_0(\underline{\xi}_1, \underline{\xi}_2, \nu)\, K(\underline{x}_1, \underline{\xi}_1, \nu) d\underline{\xi}_1 \qquad (5.1.13)$$

Next we repeat the argument used above solving this time the second of the equations (5.1.12). For the sake of generality we assume the optical system to be characterized by a second function, $J(\underline{x}_2, \underline{\xi}_2, \nu)$. The relation between J and K will be determined below. Using the linearity of the remaining equation we obtain the image as

$$\hat{\Gamma}_1(\underline{x}_1, \underline{x}_2, \nu) = \int \hat{\Gamma}_{10}(\underline{x}_1, \underline{\xi}_2, \nu) \quad J(\underline{x}_2, \underline{\xi}_2, \nu)\, d\underline{\xi}_2 . \qquad (5.1.14)$$

Substituting from (5.1.13) into (5.1.14) we obtain finally

$$\hat{\Gamma}_1(\underline{x}_1, \underline{x}_2, \nu) = \iint \hat{\Gamma}_0(\underline{\xi}_1, \underline{\xi}_2, \nu) \quad J(\underline{x}_2, \underline{\xi}_2, \nu) K(\underline{x}_1, \underline{\xi}_1, \nu) d\underline{\xi}_1\, d\underline{\xi}_2$$

$$\ldots\ldots \quad (5.1.15)$$

Before discussing the physical significance of the functions J and K, we shall show that there is a simple relation between them. To this end we interchange the roles of $\underline{x}_1$ and $\underline{x}_2$ of $\underline{\xi}_1$ and $\underline{\xi}_2$ in (5.1.15) obtaining

$$\hat{\Gamma}_1(\underline{x}_2, \underline{x}_1, \nu) = \iint K(\underline{x}_2, \underline{\xi}_2, \nu) J(\underline{x}_1, \underline{\xi}_1, \nu) \quad \hat{\Gamma}_0(\underline{\xi}_2, \underline{\xi}_1, \nu) d\underline{\xi}_1\, d\underline{\xi}_2$$

$$\ldots\ldots \quad (5.1.16)$$

However, by Lemma I (Chapter 1)

$$\hat{\Gamma}_i(\underline{x}_2,\underline{x}_1,\nu) = \hat{\Gamma}_i^*(\underline{x}_1,\underline{x}_2,\nu) ,$$

and

$$\hat{\Gamma}_0(\underline{\xi}_2,\underline{\xi}_1,\nu) = \hat{\Gamma}_0^*(\underline{\xi}_1,\underline{\xi}_2,\nu) .$$

Using these two relations, comparison of (5.1.16) and (5.1.15) shows that

$$[K(\underline{x}_1,\underline{\xi}_1,\nu)\, J(\underline{x}_2,\underline{\xi}_2,\nu)]^* = K(\underline{x}_2,\underline{\xi}_2,\nu)\, J(\underline{x}_1,\underline{\xi}_1,\nu) \qquad \ldots\ldots (5.1.17)$$

From (5.1.17) it follows immediately that

$$J(\underline{x}_1,\underline{\xi}_1,\nu) = K^*(\underline{x}_1,\underline{\xi}_1,\nu) . \qquad (5.1.18)$$

Using this result, (5.1.18), we may rewrite (5.1.15) as

$$\hat{\Gamma}_i(\underline{x}_1,\underline{x}_2,\nu) = \iint K(\underline{x}_1,\underline{\xi}_1,\nu)\, K^*(\underline{x}_2,\underline{\xi}_2,\nu)\, \hat{\Gamma}_0(\underline{\xi}_1,\underline{\xi}_2,\nu)\, d\underline{\xi}_1\, d\underline{\xi}_2 \qquad \ldots\ldots (5.1.19)$$

Equation (5.1.19) is the basic relation of this analysis. It expresses the temporal mutual spectral density of the image in terms of the temporal mutual spectral density of the object and a function K which characterizes the imaging system.

Before continuing the development, it is useful to consider the physical interpretation of the function $K(\underline{x}_1,\underline{\xi}_1,\nu)$. The demonstration of the interpretation of $K(\underline{x}_1,\underline{\xi}_1,\nu)$ is straightforward but requires some results from a later section. Therefore, to preserve the continuity of the present discussion, this demonstration is given in Appendix 3 and we simply state here that the function K

represents the complex amplitude at x_1, due to a monochromatic point source at ξ_1 of frequency, ν.

Until now we have not specified the position of the image plane. Since there are by definition no imaging elements between the exit pupil and the image plane it is clear that equation (5.1.19) is valid throughout this entire region. However, since the form of K in the exit pupil varies significantly from its form in Gaussian image plane, and since its behaviour in each of these planes is of particular physical importance, it will prove convenient to designate by two different symbols the form of $K(\underline{x}_1, \underline{\xi}_1, \nu)$ on these two surfaces. Accordingly, we shall denote by $A(\underline{\xi}, \underline{\alpha}, \nu)$ the form of K in the plane of the exit pupil. Thus $A(\underline{\xi}, \underline{\alpha}, \nu)$ is the complex disturbance at a point $\underline{\alpha}$ in exit pupil due to the monochromatic point source of frequency ν at a point in the object plane. We retain the symbol $K(\underline{x}, \underline{\xi}, \nu)$ to denote the function in the image plane. Using this convention, $K(\underline{x}, \underline{\xi}, \nu)$ may be thought of as the distribution in the image plane due to a monochromatic distribution of complex amplitude in the plane of the exit pupil remembering of course that the distribution in the exit pupil is determined by the object. The relation between these two functions may then be expressed as

$$K(\underline{x}, \underline{\xi}, \nu) = \int A(\underline{\xi}, \underline{\alpha}, \nu) \frac{\partial G}{\partial n} (\underline{x}, \underline{\alpha}, \nu) \, d\underline{\alpha} \tag{5.1.20}$$

Here $\underline{\alpha}$ is a point in the aperture of the system; G is a Green's function satisfying the Helmholtz equation and vanishing over the plane of the exit pupil.

Under the conditions characterizing most imaging systems (5.1.20) takes a particularly simple form; but before discussing this point, we shall continue the development up to the introduction of the general transfer function. This is done to avoid the erroneous impression that the transfer function analysis involves the Fraunhofer (or far field) approximations, as is sometimes believed to be the case.

Equation (5.1.19) assumes a convenient and useful form if the system under consideration is "spatially stationary", i.e., if the function $K(\underset{\sim}{x}, \underset{\sim}{\xi}, \nu)$ is a function of the difference of the spatial coordinates,

$$K(\underset{\sim}{x}, \underset{\sim}{\xi}, \nu) \equiv K(\underset{\sim}{\xi} - \underset{\sim}{x}, \nu) \quad . \tag{5.1.21}$$

This condition is satisfied by scanning systems,* which includes most antenna systems and also by many important (visible) optical systems. For systems which do not scan but form the entire spatial image simultaneously, K will not in general be a function of the difference $(\underset{\sim}{\xi} - \underset{\sim}{x})$ only. However, for most optical systems the form of the diffraction pattern varies slowly across the image plane. Hence, the image space may be divided into "isoplanatic" areas over which K may be assumed to be a function of $(\underset{\sim}{\xi} - \underset{\sim}{x})$ to any desired accuracy. This possibility is discussed at length by Fellgett and Linfoot (1955) and by Dumontet (1955).

Throughout the rest of this discussion we shall only be concerned with systems for which the condition (5.1.21) is satisfied.

* By this we mean systems in which the image forming device (e.g. antenna) scans while the position of the detector (e.g. feed) relative to the aperture remains fixed, i.e., the antenna and feed move together. The above consideration is not valid for a system in which the image forming device remains fixed while the detector scans the aerial image. A camera with a focal plane shutter is a simple example of such a system; the lens creates the entire image simultaneously and the shutter slit then scans the image.

Using the stationarity condition, (5.1.19) may be rewritten as

$$\hat{\Gamma}_1(x_1,x_2,\nu) = \iint K(\xi_1 - x_1,\nu)\, K^*(\xi_2 - x_2,\nu)\, \hat{\Gamma}_0(\xi_1,\xi_2,\nu)\, d\xi_1\, d\xi_2 \qquad \ldots\ldots (5.1.22)$$

Taking the "space-type" Fourier transform of both sides of (5.1.22) and using the convolution theorem yields

$$\overset{0}{\Gamma}_1(\mu_1,\mu_2,\nu) = \tilde{K}(\mu_1,\nu)\ \tilde{K}^*(-\mu_2,-\nu)\ \overset{0}{\Gamma}_0(\mu_1,\mu_2,\nu) \ . \qquad (5.1.23)$$

At this point we introduce the transfer function, $L(\mu_1, \mu_2,\nu)$, defined by

$$\mathcal{L}(\mu_1,\mu_2,\nu) = \tilde{K}(\mu_1,\nu)\ \tilde{K}^*(-\mu_2,-\nu) \ . \qquad (5.1.24)$$

In terms of L (5.1.2) becomes simply

$$\overset{0}{\Gamma}_1(\mu_1,\mu_2,\nu) = \mathcal{L}(\mu_1,\mu_2,\nu)\ \overset{0}{\Gamma}_0(\mu_1,\mu_2,\nu) \ . \qquad (5.1.25)$$

Equation (5.1.2) may be regarded as the basic equation in the frequency domain analysis of imaging systems. It is clear from the foregoing discussion that the transfer function analysis is rigorously applicable to any "spatially stationary" system. That is no approximation need be made concerning the relation between the aperture illumination function, A , and the diffraction pattern, K. However, in most optical applications and in fact in most antenna applications, the diffraction pattern is characterized by the Fraunhofer approximations (or the formally equivalent far field approximations); and since under these conditions (5.1.20) assumes a particularly simple form, we shall introduce these approximations at this point and retain them throughout the subsequent sections.

Under the usual approximations which characterize Fraunhofer diffraction, (5.1.20) reduces to

$$K(\underline{x}_1, \underline{\xi}_1, \nu) = \int A(\underline{x}, \underline{\alpha}, \nu)\, e^{-2\pi i\left(\frac{\underline{\alpha}}{\lambda R}\cdot \underline{\xi}\right)}\, d\underline{\alpha}, \qquad (5.1.26)$$

where λ is the wave length of the spectral component belonging to frequency ν and R is the radius of the Gaussian reference sphere. However, $K(\underline{x}, \underline{\xi}, \nu)$ may be expressed as

$$K(\underline{x}, \underline{\xi}, \nu) = \int \widetilde{K}(\underline{x}, \underline{\mu}, \nu)\, e^{-2\pi i(\underline{\mu}\cdot \underline{\xi})}\, d\underline{\mu}. \qquad (5.1.27)$$

By comparing (5.1.26) and (5.1.27) we identify the spatial frequency, $\underline{\mu}$, as the reduced aperture coordinate, i.e.,

$$\underline{\mu} = \frac{\underline{\alpha}}{\lambda R}, \text{ i.e. } \mu_{1x} = \frac{\alpha}{\lambda R}, \ \mu_{1y} = \frac{\beta}{\lambda R}.$$

Further we note the important relation

$$\widetilde{K}(\underline{x}, \underline{\mu}, \nu) = \widetilde{K}(\underline{x}, \frac{\underline{\alpha}}{\lambda R}, \nu) = A(\underline{x}, \underline{\mu}, \nu).$$

Our subsequent analysis will deal solely with K and its transform $\widetilde{K}$. From (5.1.24), (5.1.26) and (5.1.27) it is clear that under the conditions stated the transfer function, $L(\underline{\mu}_1, \underline{\mu}_2, \nu)$, for a system utilizing partially coherent polychromatic illumination is the product of the frequency dependent aperture illumination function considered as a function of spatial frequency and evaluated at $\underline{\mu}_1 = \underline{\alpha}_1/\lambda R$ with its complex conjugate evaluated at $\underline{\mu}_2 = \underline{\alpha}_2/\lambda R$.

It is clear from the above considerations that the inverse problem, namely that of determining the object from a knowledge of the image, cannot be solved if the imaging system has a finite aperture. This conclusion can be understood by formally inverting (5.1.25). We then obtain

$$\overset{0}{\Gamma}_0(\mu_1,\mu_2,\nu) = \frac{\overset{0}{\Gamma}_i(\mu_1,\mu_2,\nu)}{\mathcal{L}(\mu_1,\mu_2,\nu)} \qquad \mathcal{L} \neq 0 \ . \tag{5.1.28}$$

However, from (5.1.27) and (5.1.24) it follows that if the aperture is finite $\mathcal{L}$ is identically zero beyond some maximum frequency $|\underline{\mu}|$ max, and hence, (5.1.28) is indeterminate. Thus the inverse problem is soluble only up to an arbitrary function, f ,

$$\overset{0}{\Gamma}_0(\mu_1,\mu_2,\nu) = \frac{\overset{0}{\Gamma}_i(\mu_1,\mu_2,\nu)}{\mathcal{L}(\mu_1,\mu_2,\nu)} + f \ , \tag{5.1.29}$$

where f is any function of frequencies greater than $|\underline{\mu}|_{max}$.

Equation (5.1.25) gives the relation between the total spectral densities of the object and image and is thus the solution sought in this section. However, in many applications one is interested in a much less general solution, namely, the intensity distribution in the image. Obtaining the intensity distribution in general, from (5.1.22), is somewhat involved and not very helpful. However, in the limiting cases of coherently and incoherently illuminated objects the problem is tractable, and in the subsequent sections we shall discuss these limits in detail for both polychromatic and quasi-monochromatic illumination.

5.2 The Limiting Forms of the Transfer Function

Following the pattern established in the earlier chapters we first examine the limiting forms for polychromatic light and in a later section examine the extremes under the quasi-monochromatic approximation.

We shall show in this section that the transfer function for coherent objects is the frequency dependent aperture illumination function $\widetilde{K} = A(\underline{x}, \frac{\underline{\rho}}{\lambda R}, \nu)$

evaluated at ν_0 and considered as a function of spatial frequency. (Here ν_0 is the frequency of the illumination. See Chapter 3)

The analysis of the systems imaging incoherent objects is complicated by the fact that as explained in Chapter 4 the image is partially coherent. Thus, if one seeks a complete solution (i.e. the mutual coherence of the image), the transfer function must operate on the spectral density of the object, $\dot{I}(\mu, \nu)$ (a function of one point only) to produce the mutual spectral density of the image, $\overset{0}{\Gamma_1}(\mu_1, \mu_2, \nu)$, (a function of two points). This consideration, overlooked or omitted in the literature on the imaging problem, is important in the treatment of cascaded systems. The required transfer function will be shown to be the function L introduced in the previous section.

If, on the other hand, one requires only the intensity distribution in the image, $I(x, 0)$, the entire analysis may be performed with functions of one point only, the spectral densities of the object and image. For this problem, the transfer function, M, will be shown to be the convolution of the frequency dependent aperture illumination with its complex conjugate.

5.2.1 The Coherent Limit

It was shown in Chapter 3 that in a coherent field the mutual coherence function is of the form

$$\Gamma_0(\xi_1, \xi_2, \tau) = U_0(\xi_1) U_0^*(\xi_2) e^{-2\pi i \nu_0 \tau} \qquad (5.2.1)$$

with a mutual spectral density

$$\hat{\Gamma}_0(\xi_1, \xi_2, \nu) = U_0(\xi_1) U_0^*(\xi_2)\, \delta(\nu - \nu_0). \qquad (5.2.2)$$

Substituting from (5.2.2) into the general solution, (5.1.19), and taking the temporal Fourier transform on both sides we obtain

$$\Gamma_i(\underline{x}_1, \underline{x}_2, \tau) = U_i(\underline{x}_1)\, U_i^*(\underline{x}_2)\, e^{-2\pi i \nu_0 \tau} , \tag{5.2.3}$$

where

$$U_i(\underline{x}) = \int U_0(\underline{\xi})\, K\,(\underline{\xi} - \underline{x})\, d\underline{\xi} \tag{5.2.4}$$

From (5.2.3) and the theorems of Chapter 3 it follows immediately that:

Theorem XIV : The image of a coherent object is coherent.

Taking the spatial Fourier transform of both sides of (5.2.4) and using the convolution theorem we obtain

$$\tilde{U}_i(\underline{\mu}_1) = \tilde{U}_0(\underline{\mu}_1)\, \tilde{K}\,(\underline{\mu}_1, \nu_0) . \tag{5.2.5}$$

The appropriate transfer function is the frequency dependent aperture illumination function evaluated at ν_0 and considered as a function of spatial frequency, $\tilde{K}(\underline{\mu}_1, \nu_0) = A(\underline{\alpha}/\lambda_R, \nu_0)$. The coherent image is completely determined by (5.2.5), and the intensity distribution is obtained as a special case of (5.2.3) by setting $\underline{x}_1 = \underline{x}_2$, and $\tau = 0$.

5.2.2 The Incoherent Limit

An incoherent object is described by a mutual coherence function of the form (see Chapter 3)

$$\Gamma_0(\underline{\xi}_1, \underline{\xi}_2, \tau) = I_0(\underline{\xi}_2, \tau)\, \delta\,(\underline{\xi}_2 - \underline{\xi}_1) , \tag{5.2.6}$$

where $I_0(\xi_2, \tau)$ is the self coherence function at ξ_2 defined by the relation

$$I(\xi, \tau) = \langle V(\xi, t + \tau) \; V^*(\xi, t) \rangle . \qquad (5.2.7)$$

The mutual spectral density is, therefore,

$$\hat{\Gamma}_0(\xi_1, \xi_2, \nu) = \hat{I}(\xi_2, \nu) \; \delta \; (\xi_2 - \xi_1) . \qquad (5.2.8)$$

Substituting from (5.2.8) into the general solution, (5.1.19), we obtain the image, $\hat{\Gamma}_1(x_1, x_2, \nu)$, i.e.

$$\hat{\Gamma}_1(x_1, x_2, \nu) = \iint K(\xi_1 - x_1, \nu) \; K^*(\xi_2 - x_2, \nu) \; \hat{I}_0(\xi_2, \nu) \, \delta \, (\xi_2 - \xi_1) \; d\xi_1 \; d\xi_2$$

. . . . . (5.2.9)

We may now integrate over $d\xi_2$ and obtain

$$\hat{\Gamma}_1(x_1, x_2, \nu) = \int K(\xi_1 - x_1, \nu) \; K^* \, (\xi_1 - x_2, \nu) \; \hat{I}(\xi_1, \nu) \, d\xi_1 \; ; \qquad (5.2.10)$$

and taking the spatial Fourier transform of both sides and using again the convolution theorem yields

$$\overset{0}{\Gamma}_1(\mu_1, \mu_2, \nu) = \overset{\circ}{I}(\mu_1 + \mu_2, \nu) \, \mathcal{L}(\mu_1, \mu_2, \nu) , \qquad (5.2.11)$$

where $\mathcal{L} \, (\mu_1, \mu_2, \nu)$ is the generalized transfer function defined in Section 5.1 ,

$$\mathcal{L} \, (\mu_1, \mu_2, \nu) = \tilde{K}(+\mu_1, +\nu) \, \tilde{K}^*(-\mu_2, -\nu) \, . \qquad (5.2.12)$$

It is convenient to rewrite (5.2.11) in the form

$$\overset{0}{\Gamma}_i(\mu_1, \mu-\mu_1, \nu) = \hat{I}(\mu, \nu) \; \mathcal{L}(\mu_1, \mu-\mu_1, \nu) . \tag{5.2.13}$$

Equation (5.2.13) expresses the fact that the energy contained in the object at μ is distributed among all pairs of frequency, μ_1, and $\mu - \mu_1$, in the image.

While equation (5.2.13) is the complete solution (for an incoherent object) for the total mutual spectral density in the image and hence for the mutual coherence function, one is often interested in the more restrictive solution, the intensity in the image. This is obtained at once by setting $x_1 = x_2$ in (5.2.10), which gives

$$\hat{I}_i(x, \nu) = \hat{\Gamma}_i(x_1, x_1, \nu) = \int \left| K(\xi_1 - x_1, \nu) \right|^2 I(\xi_1, \nu) \, d\xi_1 . \tag{5.2.14}$$

Taking the spatial Fourier transform of both sides of (5.2.14) we obtain

$$\hat{I}_i(\mu, \nu) = \mathcal{M}(\mu, \nu) \; I_0(\mu_1, \nu) , \tag{5.2.15}$$

where

$$\mathcal{M}(\mu, \nu) = \int \left| K(x, \nu) \right|^2 e^{2\pi i \mu \cdot x} \, dx . \tag{5.2.16}$$

From the interpretation of K given in Appendix 4 it is clear that $|K|^2$ is the frequency dependent intensity diffraction pattern of the imaging system. The transfer function for determining the total spectral density of the image of an incoherent object is thus the transform of the "intensity diffraction pattern" of the system. The formula (5.2.16) is simplified further by again using the convolution theorem and (5.1.2) which gives

$$\mathcal{M}(\underline{\mu},\nu) = \int \tilde{K}(\underline{\sigma} - \underline{\mu},\nu)\ \tilde{K}^*(\underline{\sigma},\nu)\, d\sigma . \tag{5.2.17}$$

The spatial spectral density may now be obtained by taking the Fourier transform of both sides of (5.1.15) and evaluating at $\tau = 0$; thus

$$\tilde{I}_1(\underline{\mu},0) = \int \mathcal{M}(\underline{\mu},\nu)\ \tilde{I}_0(\underline{\mu},\nu)\, d\nu \quad . \tag{5.2.18}$$

The formula (5.2.18) expresses the fact that each temporal spectral component contributes separately and independently to the energy in the spatial frequency component $\underline{\mu}$. The intensity distribution in the image is then given by the spatial transform of (5.2.18).

5.3 Imaging with Quasi-Monochromatic Light

While many imaging systems of practical and theoretical interest deal with polychromatic light, only the problem of imaging with quasi-monochromatic or monochromatic illumination has been extensively discussed in the literature. This omission of the more general problem is easily understood since without a rigorous and general formulation of coherence theory the mathematical analysis is prohibitive; and the earlier formulations of this theory discussed in Chapter 1, are not well suited for extension to this general case. Therefore, the transfer function analysis as found in the available literature is applicable only to quasi-monochromatic light, and in order to compare the results given here with those of earlier writers, we examine in this section the form of the transfer functions under the quasi-monochromatic approximation.

A partially coherent quasi-monochromatic object will be described by a mutual coherence function of the form (see Chapter 3)

$$\Gamma_0(\xi_1,\xi_2,\tau) \simeq \Gamma_0(\xi_1,\xi_2,0)\,e^{-2\pi i\bar{\nu}\tau} \quad (|\tau| << \frac{1}{\Delta\nu}); \quad (5.3.1)$$

its temporal transform is

$$\hat{\Gamma}_0(\xi_1,\xi_2,\nu) = \Gamma_0(\xi_1,\xi_2,0)\,\delta\,(\nu-\bar{\nu}) \quad (|\tau| << \frac{\nu}{\Delta\nu}). \quad (5.3.2)$$

Substituting from (5.3.2) into the general solution (5.1.19) and taking the inverse Fourier transform, we obtain

$$\hat{\Gamma}_1(x_1,x_2,\nu) = e^{-2\pi i\bar{\nu}\tau}\int K(\xi_1-x_1,\bar{\nu})\;K^*(\xi_2-x_2,\nu)\,\Gamma_0(\xi_1,\xi_2,0)\;d\xi_1\,d\xi_2 \;. \quad \ldots\ldots (5.3.3)$$

Equation (5.3.3) provides the starting point for the analysis of partially coherent images. Beginning from (5.3.3) the entire analysis of the two preceding sections may be taken over mutatis mutandis for the quasi-monochromatic imaging problems considered here. Denoting by the suffix q that the functions are applicable to quasi-monochromatic light, the various transfer functions and frequency domain image equations are :

Coherent Object

transfer function $\mathcal{X}_q = \tilde{K}(-\mu,-\bar{\nu})$

imaging equation $\mathring{U}_1(\mu,\bar{\nu}) = \mathcal{X}_q(\mu,\bar{\nu})\;\mathring{U}_0(\mu,\bar{\nu})$

Partially Coherent Object

transfer function $\mathcal{L}_q = K(-\mu_1,-\bar{\nu})\;K^*(\mu_2,\bar{\nu})$

imaging equation $\mathring{\Gamma}_1(\mu_1,\mu_2,\bar{\nu}) = \mathcal{L}_q(\mu_1,\mu_2,\bar{\nu})\,\mathring{\Gamma}_0(\mu_1,\mu_2,\bar{\nu})$

Incoherent Object *

transfer function $\mathcal{X}_q = \tilde{K}(-\mu_1,\bar{\nu})\,\tilde{K}^*(\mu_2,\bar{\nu})$

imaging equation $\mathring{\Gamma}_i(\mu_1,\mu_2,\bar{\nu}) = \mathcal{X}_q(\mu_1,\mu_2,\bar{\nu})\ \mathring{I}_0(\mu_1,\bar{\nu})$

transfer function $\mathcal{M}_q = \int K(\underline{\alpha}-\underline{\mu},\bar{\nu})\ K^*(\underline{\alpha},\bar{\nu})\,d\underline{\alpha}$

imaging equation $\dot{I}_i(\underline{\mu},\bar{\nu}) = \mathcal{M}_q(\underline{\mu},\bar{\nu})\cdot\ \mathring{I}_0(\underline{\mu},\bar{\nu})$

The transfer functions for quasi-monochromatic light are thus seen to be simply the generalized transfer functions evaluated at the mean frequency. This result is to have been expected since mathematically the quasi-monochromatic approximation is characterized by an approximately monochromatic mutual coherence function. However, it should be emphasized that while the transfer functions obtained in this section may be formally obtained by taking a single spectral component of the general solutions the inverse procedure (integrating the quasi-monochromatic solution over frequency to obtain the general solutions) is not justifiable. This conclusion is evident from the fact that the solutions obtained in this section are only approximate ($|\tau| << \frac{1}{\Delta\nu}$) and accordingly the transfer functions depend on the mean frequency, $\bar{\nu}$, not on an isolated frequency, ν_0. It is thus logically impossible to obtain the general solutions from the quasi-monochromatic solutions.

The transfer functions defined in this chapter are summarized in Table 2.

* N.B. Two imaging equations are required for systems involving incoherent objects. $\mathcal{M}_q(\underline{\mu},\bar{\nu})$ is used to determine spatial spectral density in the image, while the more general problem of determining the mutual spectral density involves the transfer function $\mathcal{X}_q(\underline{\mu}_1,\underline{\mu}_2,\bar{\nu})$. However, both of these functions operate on the spatial spectral density of the object.

TABLE 2

$\lvert\gamma_{12}(\tau)\rvert$	Transfer Function	Imaging Equation
1	$\mathbf{K}(\underline{\mu}) = \widetilde{\mathbf{K}}(\underline{\mu}) = \widetilde{\mathbf{K}}(\underline{\alpha}/\lambda R)$	$\widetilde{U}_i(\underline{\mu}) = \mathbf{K}(\underline{\mu}, \nu_0)\, \widetilde{U}_0(\underline{\mu})$
$0 < \lvert\gamma_{12}(\tau)\rvert < 1$	$\mathcal{L}(\underline{\mu}_1, \underline{\mu}_2, \nu) = \widetilde{\mathbf{K}}(\underline{\mu}_1, \nu)\, \widetilde{\mathbf{K}}^*(-\underline{\mu}_2, -\nu)$	$\overset{0}{\Gamma}_i(\underline{\mu}_1, \underline{\mu}_2, \nu) = \mathcal{L}(\underline{\mu}_1, \underline{\mu}_2, \nu)\, \overset{0}{\Gamma}_0(\underline{\mu}_1, \underline{\mu}_2, \nu)$
0	$\mathcal{L}(\underline{\mu}_1, \underline{\mu}_2, \nu) = \widetilde{\mathbf{K}}(\underline{\mu}_1, \nu)\, \widetilde{\mathbf{K}}^*(-\underline{\mu}_2, \nu)$	$\overset{0}{\Gamma}_i(\underline{\mu}_1, \underline{\mu} - \underline{\mu}_1, \nu) = \mathcal{L}(\underline{\mu}_1, \underline{\mu} - \underline{\mu}_1, \nu)\, \overset{0}{I}_i(\underline{\mu}, \nu)$
0	$\mathcal{M}(\underline{\mu}, \nu) = \int \mathbf{K}(\alpha - \underline{\mu}, \nu)\, \mathbf{K}^*(\underline{\alpha}, \nu)\, d\underline{\alpha}$	$\overset{0}{I}_i(\underline{\mu}, \nu) = \mathcal{M}(\underline{\mu}, \nu)\, \overset{0}{I}_0(\underline{\mu}, \nu)$

The corresponding equation and transfer functions for quasi-monochromatic light may be obtained formally by setting $\nu = \bar{\nu}$ in these relations.

APPENDIX 1

RELATION BETWEEN VON LAUE'S MEASURE OF COHERENCE AND THE COMPLEX DEGREE OF COHERENCE

The measure of coherence, γ_L, introduced by von Laue (1907) is defined by

$$\gamma_L = \frac{\langle f_1(t) f_2(t)\rangle^2 + \langle f_1(t) g_2(t)\rangle^2}{\langle f_1^2(t)\rangle\langle f_2^2(t)\rangle}. \qquad \text{(A.1.1)}$$

Here $f_1(t)$ and $f_2(t)$ are real functions of time which describe the (scalar) optical disturbance at the two points P_1 and P_2, respectively. The functions $f_i(t)$ $(i = 1,2)$ have, therefore, the Fourier representation

$$f_i(t) = \int_0^\infty F_i(\nu) \cos[\phi_i(\nu) - 2\pi\nu t]\, d\nu , \qquad \text{(A.1.2)}$$

and the functions $g_i(t)$ are defined as

$$g_i(t) = \int_0^\infty F_i(\nu) \sin[\phi(\nu) - 2\pi\nu t]\, d\nu . \qquad \text{(A.1.3)}$$

This formulation strongly suggests the introduction of the analytic signal; and accordingly, we may rewrite (A.1.1) as

$$\gamma_L = \frac{\langle V_1^{i}(t) V_2^{r}(t)\rangle^2 + \langle V_1^{r}(t) V_2^{i}(t)\rangle^2}{\langle V_1^{r\,2}(t)\rangle\langle V_2^{r\,2}(t)\rangle}, \qquad \text{(A.1.4)}$$

where the functions V_1^r and V_2^r are identified as f and g, respectively.

In order to obtain the relation between γ_L and $\gamma_{12}(\tau)$ we have but to recall the definition of $\gamma_{12}(\tau)$. Equation (1.2.10) of Chapter 1 may be formally expanded to give

$$\gamma_{12}(\tau) = \frac{(A + iB)}{\sqrt{\langle V_1(t)V_1^*(t+\tau)\rangle \langle V_2(t)V_2^*(t+\tau)\rangle}}$$

where

$$A = \langle V_1^r(t)V_2^r(t+\tau)\rangle + \langle V_1^i(t)V_2^i(t+\tau)\rangle$$

$$B = i\left[\langle V_1^i(t)V_2^r(t+\tau)\rangle - \langle V_1^r(t)V_2^i(t+\tau)\rangle\right] .$$

$$\ldots\ldots \text{(A.1.5)}$$

but by the theorems established in Chapter 2

$$\langle V_1^r(t)\ V_2^r(t+\tau)\rangle = \langle V_1^i(t)\ V_2^i(t+\tau)\rangle ,$$

and

$$\langle V_1^r(t)\ V_2^i(t+\tau)\rangle = -\langle V_2^r(t)\ V_1^i(t+\tau)\rangle ,$$

whence Equation (A.1.5) becomes

$$\gamma_{12}(\tau) = \frac{2\left[\langle V_1^r(t)\ V_2^r(t+\tau)\rangle + i\langle V_1^i(t)\ V_2^r(t+\tau)\rangle\right]}{\sqrt{\langle V_1(t)\ V_1^*(t+\tau)\rangle \langle V_2(t)\ V_2^*(t+\tau)\rangle}} \quad \text{(A.1.6)}$$

Multiplying both sides of (A.1.6) by $\gamma_{12}^*(\tau)$ and noting from (2.2.14) that

$$\langle V_1(t)\ V_1^*(t+\tau)\rangle = 2\ \langle V_1^r(t)\ V_2^r(t+\tau)\rangle ,$$

we obtain

$$\left|\gamma_{12}(\tau)\right|^2 = \frac{\left\langle V_1^r(t)\, V_2^r(t+\tau)\right\rangle^2 + \left\langle V_1^i(t)\, V_2^r(t+\tau)\right\rangle^2}{\left\langle V_1^r(t)\, V_1^r(t+\tau)\right\rangle \left\langle V_2^r(t)\, V_2^r(t+\tau)\right\rangle}, \qquad \text{...... (A.1.7)}$$

or at $\tau = 0$

$$\left|\gamma_{12}(0)\right|^2 = \frac{\left\langle V_1^r(t)\, V_2^r(t)\right\rangle^2 + \left\langle V_1^i(t)\, V_2^r(t)\right\rangle^2}{\left\langle V_1^{r^2}(t)\right\rangle \left\langle V_2^{r^2}(t)\right\rangle} \qquad \text{...... (A.1.8)}$$

Comparison of (A.1.8) and (A.1.4), yields the relation between von Laue's measure of coherence and Wolf's complex degree of coherence function as

$$\gamma_L = \left|\gamma_{12}(0)\right|^2 . \qquad \text{(A.1.9)}$$

APPENDIX 2

THE QUASI-MONOCHROMATIC APPROXIMATION

We may define quasi-monochromatic illumination by the following property : the mutual spectral density, $\hat{\Gamma}_{12}(\nu)$, is appreciably different from zero only for those spectral components, ν, which satisfy the inequality

$$\left| \nu - \bar{\nu} \right| \ll \Delta\nu ,$$

where $\bar{\nu}$ is the mean frequency and $\Delta\nu$ is the spectral width of the light. Physically this condition implies that most of the energy in the field is contained in the spectral region $\bar{\nu} - \Delta\nu < \nu < \bar{\nu} + \Delta\nu$.

To obtain the form of the mutual coherence function for quasi-monochromatic fields we first recall that $\Gamma_{12}(\tau)$ may always be expressed in the form

$$\Gamma_{12}(\tau) = \int_0^\infty \hat{\Gamma}_{12}(\nu)\, e^{-2\pi i\nu\tau}\, d\nu . \tag{A.2.1}$$

We may now factor out the mean frequency term of the integrand in (A.2.1) and obtain

$$\Gamma_{12}(\tau) = e^{-2\pi i\bar{\nu}\tau} \int_0^\infty \hat{\Gamma}_{12}(\nu)\, e^{-2\pi i(\nu-\bar{\nu})\tau}\, d\nu . \tag{A.2.2}$$

If we limit our attention to sufficiently small $|\tau|$, more precisely if $|\tau| \ll \frac{1}{\Delta\nu}$, the frequency dependent factor of the exponent of the integrand satisfies the inequality

$$\left| (\nu-\bar{\nu})\,\tau \right| \ll 1 ,$$

for all values of ν for which $\left| \hat{\Gamma}_{12}(\nu) \right|$ is significant. Thus, the variations in $e^{-2\pi i(\nu-\bar{\nu})\tau}$ may to a good approximation be ignored, and (A.2.2) may be rewritten as

$$\Gamma_{12}(\tau) \simeq e^{-2\pi i \bar{\nu}\tau} \int_0^\infty \hat{\Gamma}_{12}(\nu)\, d\nu \qquad \left(|\tau| \ll \frac{1}{\Delta\nu} \right) . \quad \text{(A.2.3)}$$

The integral in (A.2.3) may be formally evaluated to give

$$\int_0^\infty \hat{\Gamma}_{12}(\nu)\, d\nu = \Gamma_{12}(0) \quad ,$$

and we have finally

$$\Gamma_{12}(\tau) \simeq \Gamma_{12}(0)\, e^{-2\pi i \bar{\nu}\tau} . \qquad \left(|\tau| \ll \frac{1}{\Delta\nu} \right) . \qquad \text{(A.2.4)}$$

APPENDIX 3

THE GENERAL UNIMODULAR ANALYTIC SIGNAL AS AN AUTOCORRELATION FUNCTION

We will show in this appendix that the general unimodular analytic signal can be interpreted as an autocorrelation function only in the degenerate case that it can be written as

$$e^{i\phi_{11}(\tau)} = e^{2\pi i(\nu_0 \tau + \beta)}$$

The constants a_n^* determine the position of the poles of the meromorphic function

$$A_1^2 e^{i\phi_{12}(z)} = \left[\prod_{n=1}^{\infty} \frac{a_n^*}{a_n} \left(\frac{a_n - z}{a_n^* - z}\right)\right] e^{i(\beta - 2\pi\nu_0 z)} \quad \text{. . . . . (A.3.1)}$$

In (A.3.1) the imaginary part of a_n has the same sign as ν_0, i.e.,

$$I\left\{a_n\right\} < 0 , \qquad \text{(A.3.2)}$$

and β and ν_0 are real constants.

The most general unimodular analytic signal is obtained by setting $z = \tau$ in (A.3.1) (see Chapter 2). The requirement that (A.3.1) represent an autocorrelation function is equivalent (see Chapter 3) to

$$R\left\{\int_{-\infty}^{\infty}\Gamma_{11}(\tau)\sin 2\pi\nu\tau\, d\tau\right\} \equiv 0\,, \tag{A.3.3}$$

where

$$\Gamma_{11}(\tau) = A_1^{\,2}\, e^{\,i\phi_{11}(\tau)} \tag{A.3.4}$$

We shall show here that (A.3.3) can be satisfied if and only if the meromorphic function, (A.3.1), has no poles.

The integral in (A.3.3) is conveniently evaluated in the complex plane. Equation (A.3.1) can be written as

$$R\left\{F_{+} + F_{-}\right\} = 0\,, \tag{A.3.5}$$

where

$$F_{\pm} = \frac{1}{2i}\int_{-\infty}^{\infty}\Gamma_{11}(z)\, e^{\pm 2\pi i\nu_0 z}\, dz\,. \tag{A.3.6}$$

In (A.3.5) and (A.3.6) the identity,

$$\sin 2\pi\nu_0 z = \frac{e^{2\pi i\nu_0 z} - e^{-2\pi i\nu_0 z}}{2i} \tag{A.3.7}$$

was used. The function F_+ in (A.3.6) is given by an integral along the real axis, i.e.,

$$F_{+} = \frac{A^2}{2i}\int_{-\infty}^{\infty}\left[\prod_{n=1}^{\infty}\frac{a_n^{\,*}}{a_n}\left(\frac{a_n - z}{a_n^{\,*} - z}\right)\right] e^{\,i[\beta+2\pi(\nu_0+\nu)z]}\, dz.$$

$$\ldots\ldots \quad \text{(A.3.8)}$$

The integral in (A.3.8) can be evaluated by contour integration closing the contour at infinity above or below the real axis accordingly as $\nu_0 + \nu$ is less than or greater than zero. Thus by Cauchy's residue theorem

$$F_+ = \begin{cases} 0 & \nu < -\nu_0 \\ \dfrac{A^2}{2i\pi} \displaystyle\sum_{m=1}^{\infty} \prod_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{a_n^*}{a_n} \left(\frac{a_n - a_m^*}{a_n^* - a_m^*} \right) e^{i\beta} e^{2\pi i(\nu_0+\nu)a_m^*} & \nu > -\nu_0 \end{cases}, \qquad \text{.....(A.3.9)}$$

since by (A.3.2) the poles are all in the lower half plane. Similarly,

$$F_- = \begin{cases} 0 & \nu < \nu_0 \\ \dfrac{A^2}{2i\pi} \displaystyle\sum_{m=1}^{\infty} \prod_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{a_n^*}{a_n} \left(\frac{a_n - a_m^*}{a_n^* - a_m^*} \right) e^{i\beta} e^{2\pi i(\nu_0-\nu)a_m^*} & \nu > \nu_0 \end{cases} \qquad \text{.....(A.3.10)}$$

The product

$$\prod_{n=1}^{\infty} \frac{a_n^*}{a_n} \left(\frac{a_n - z}{a_n^* - z} \right), \qquad \text{(A.3.11)}$$

converges for all z (c.f. Chapter 2). It therefore converges for $z = a_m^*$; and we may write

$$\frac{e^{i\beta}}{2i\pi} \prod_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{a_n^*}{a_n} \left(\frac{a_n - a_m^*}{a_n^* - a_m^*} \right) = B_m + iC_m \qquad \text{(A.3.12)}$$

Using (A.3.12) we may write

$$F_+ + F_- = \sum_{m=1}^{\infty} (B_m + iC_m)\left[\cos 2\pi(\nu+\nu_0)b_m + i\sin 2\pi(\nu+\nu_0)b_m\right] e^{2\pi(\nu+\nu_0)C_m}$$

$$+ \sum_{m=1}^{\infty} (B_m + i\,C_m)\left[\cos 2\pi(\nu_0-\nu)\,b_m + i\sin 2\pi(\nu_0+\nu)\,b_m\right] e^{2\pi(\nu_0-\nu)C_m} \quad \nu > \nu_0 .$$

..... (A.3.13)

In (A.3.13) b_m and c_m are the real and imaginary parts respectively of a_m; and c_m is greater than zero (see Chapter 2) The physical requirement, (A.3.1), is that $F_+ + F_-$ vanish identically for all ν. Equation (A.3.13) can be written in a more convenient form as

$$R\left\{F_+ + F_-\right\} = \sum_{m=1}^{\infty} R_m\left[\left(e^{2\pi(\nu_0+\nu)C_m}\right)\cos[\phi_m + 2\pi(\nu_0+\nu)b_m] + \left(e^{2\pi(\nu_0-\nu)C_m}\right)\cos[\phi_m + 2\pi(\nu_0-\nu)b_m]\right] = 0 \quad \nu > \nu_0 .$$

..... (A.3.14)

where

$$R_m^{\ 2} = B_m^{\ 2} + C_m^{\ 2} \quad \text{and} \quad \phi_m = \tan^{-1}\left[\frac{C_m}{B_m}\right] . \qquad (A.3.15)$$

That this equation cannot be satisfied for $R_m \neq 0$ is clear from the asymptotic behaviour of the sums. The second sum vanishes for large ν while the first sum diverges term by term as e^{ν}. Thus we must have

$$R_m = 0 \qquad (A.3.16)$$

However, since in the solution of the integral equation (3.7) poles at the origin were excluded and the product was taken only over the poles occurring for finite z, the R_m cannot vanish if there are poles or zeros in the meromorphic function. Consequently there can be no poles for finite z if (A.3.1) is satisfied. Thus

$$\Gamma_{11}(\tau) = A_1 A_1 e^{i(\beta + 2\pi\nu_0\tau)} . \qquad (A.3.17)$$

APPENDIX 4

EXISTENCE OF A LINEAR DIFFERENTIAL EQUATION FOR THE MUTUAL SPECTRAL DENSITY

From Maxwell's equations it follows that each spectral component of the optical disturbance, $V^r(t)$, satisfies a linear differential equation, in particular the Helmholtz equation in each medium comprising the imaging system. Hence we may write

$$D\left[\hat{V}^r(\nu)\right] = D\left[\lim_{T\to\infty} \hat{V}^r(T,\nu)\right] = 0 , \qquad \text{(A.4.1)}$$

where $\hat{V}^r(T,\nu)$ is the spectrum of the truncated function introduced in Chapter 2 and D is a linear differential operator. Using (A.4.1) and the convolution theorem of Fourier analysis we may write

$$\lim_{T\to\infty} D\left[\hat{V}^r(T,\nu)\right] = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{\sin 2\bar{\nu}(\nu-T)}{2\bar{\nu}(\nu-T)} D\left[\hat{V}^r(\nu)\right] d\nu = 0,$$

or simply

$$D\left[\hat{V}^r(T,\nu)\right] = 0. \qquad \text{(A.4.2)}$$

Thus each spectral component of the truncated function satisfies the same differential equation. That the function ${}_T\hat{V}^i(\nu)$, likewise introduced in Chapter 2, also satisfies the same equation can be seen as follows. By definition

$$\int_{-\infty}^{\infty} {}_T\hat{V}^i(\nu)\, e^{2\pi i\nu t}\, d\nu = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t'-t} \int_{-\infty}^{\infty} \hat{V}^r(T,\nu)\, e^{2\pi i\nu t'}\, d\nu dt' .$$

Operating on both sides with D and using (A.4.2) we find

$$D[\,{}_T\hat{V}^i(\nu)\,] = 0. \tag{A.4.3}$$

From the linearity of D and equations (A.4.2) and (A.4.3) it follows that

$$D[\,\hat{V}(T|\nu)\,] = 0, \tag{A.4.4}$$

where $\hat{V}(T|\nu) = \hat{V}^r(T,\nu) + i\;{}_T\hat{V}^i(\nu)$ is the spectrum of the analytic signal associated with the truncated function $\hat{V}^r(T,\nu)$. Equation (A.4.4) expresses a particularly convenient property of the analytic signals, viz., that an analytic signal satisfies the same linear differential equation as the real function with which it is associated.

We recall from Chapter 2 that

$$\hat{\Gamma}_{12}(\nu) = \lim_{T\to\infty}\left\{\frac{\hat{V}_1(T|\nu)\,\hat{V}_2^*(T|\nu)}{2T}\right\}$$

and operating on both sides with D_s, $(s = 1,2)$, where the subscripts denote that the operation is in the coordinates of P_s we obtain finally

$$D_s\,[\,\hat{\Gamma}_{12}(\nu)\,] = 0 \qquad (s = 1,2) \tag{A.4.5}$$

Thus the mutual spectral density will be propagated through the system by the same equations which govern the optical disturbance itself.

APPENDIX 5

INTERPRETATION OF THE RESPONSE FUNCTION $K(\underline{x}, \underline{\xi}, \nu)$

It was shown in Chapter 4 that a coherent (scalar) field may be completely described by a single complex wave function U. In Chapter 5 it is shown that this complex function is of central importance in treating the image of coherent objects. To determine the physical significance of $K(\underline{x}, \underline{\xi}, \nu)$ we consider a coherent point object as the input to the imaging system. For this object the function U is of the form

$$\hat{U}_0(\underline{\xi}, \nu) = \delta(\underline{\xi} - \underline{\xi}_0)\ \delta(\nu - \nu_0) \qquad \text{(A.5.1)}$$

Substituting from (A.5.1) into (5.2.4) and integrating over d we obtain

$$\hat{U}_1(\underline{\xi}, \nu) = K(\underline{\xi}_0 - \underline{x}, \nu)\ \delta(\nu - \nu_0) \quad , \qquad \text{(A.5.2)}$$

or

$$U_1(\underline{\xi}, \tau) = K(\underline{\xi}_0 - \underline{x}, \nu_0) \quad . \qquad \text{(A.5.3)}$$

The physical interpretation of $K(\underline{\xi}, \underline{x}, \nu)$ is clear from (A.5.3); that is $K(\underline{\xi}, \underline{x}, \nu)$ is the complex amplitude in a diffraction pattern due to a point source of frequency ν located at

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