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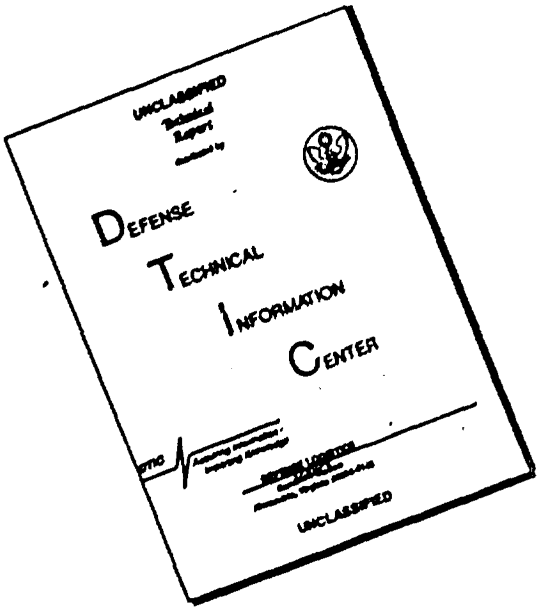

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STATISTICAL TECHNIQUES IN LIFE TESTING

CHAPTER III

PROBLEMS OF ESTIMATION

by

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AUTHOR'S NOTE

CHAPTER II "Problems of Estimation," is part of the [illegible] in connection with a contemplated handbook on Statistical [illegible] Life Testing [illegible] preliminary report [illegible] undertaking.

Earlier reports [illegible]

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Technical Report No, 3, " Testing of Hypotheses", October 1, 1958.

Further material dealing with other aspects of life testing is in preparation.

Comments and suggestions [illegible]

Benjamin Epstein

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CHAPTER III

Problems of Estimation

It is frequently important that one make estimates of mean life, rates of failure, probability of survival for a given time, etc, on the basis of data arising from life tests. The data may be generated in many ways; e.g., they may arise from truncated, censored, sequential, replacement, non-replacement, interrupted, or combined experiments; we may or may not know the exact times to failure. We shall try in what follows to give rules and procedures which enable us to give point and interval estimates which are in some sense optimum.

Section 1.

Estimation in the Censored One Sample Case. (Number of failures is fixed. Items which fail may or may not be replaced).

Basic Considerations. Point and Interval Estimates for θ .

Let us make the following assumptions:

(i) n items are drawn at random from a density function of the form $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $x > 0$, $\theta > 0$;

(ii) the n items are placed on life test at time zero and failure times become available in order. That is to say, $x_{1,n} \leq x_{2,n} \leq \cdots \leq x_{r,n} \leq \cdots \leq x_{n,n}$, where by $x_{i,n}$ is meant the time when the i^{th} failure occurs, (measured from the beginning of the life test).

(iii) the experiment is discontinued as soon as $x_{r,n}$ has become available (i.e., after the first r observations are made).

We wish under assumptions (i), (ii), and (iii) to find a "good" estimate of θ and to give the distribution of this estimate in both the non-replacement case (where failed items are not replaced) and in the replacement case (where failed items are replaced immediately by new items). This is given by the following theorem:

Theorem: Under (i), (ii), and (iii) an estimate based on the first r ordered observations which is "best" in the sense that it is maximum likelihood, unbiased, minimum variance, efficient, and sufficient is given by

$$(1) \qquad \hat{\theta}_{r,n} = T_r/r$$

where T_r is the total life of items on test observed up to the time of the r^{th} failure. In the non-replacement case:

$$(2) \quad T_r = nx_1 + (n-1)(x_2-x_1) + \dots + (n-i+1)(x_i-x_{i-1}) + \dots + (n-r+1)(x_r-x_{r-1})$$

$$= \sum_{i=1}^{r} x_i + (n-r)x_r ,$$

and so the "best" estimate (1) becomes

$$(3) \qquad \hat{\theta}_{r,n} = \left[\sum_{i=1}^{r} x_i + (n-r)x_r\right] /r .$$

In the replacement case:

(4) $$T_r = nx_1 + n(x_2-x_1) + \dots + n(x_r-x_{r-1}) = nx_r$$

and so the "best" estimate (1) becomes

(5) $$\hat{\theta}_{r,n} = nx_r/r \ .$$

The probability density function of $\hat{\theta}_{r,n}$ in either the replacement or non-replacement case is given by

(6) $$f_r(y) = \frac{1}{(r-1)!}(r/\theta)^r y^{r-1} e^{-ry/\theta}, \ y > 0$$

$$= 0 \ , \text{ elsewhere.}$$

The proof of this theorem is given in Appendix 3B.

From (6) it follows at once that $W = 2r\,\hat{\theta}_{r,n}/\theta = 2T_r/\theta$ is distributed as $\chi^2(2r)$. Consequently if the constant $\chi^2_\gamma(2k)$ is defined as $\Pr(\chi^2(2k) > \chi^2_\gamma(2k)) = \gamma$, then a $100(1-\alpha)$ percent two-sided confidence interval for θ is given by

(7) $$\left(\frac{2r\hat{\theta}_{r,n}}{\chi^2_{\frac{\alpha}{2}}(2r)} \ , \ \frac{2r\hat{\theta}_{r,n}}{\chi^2_{1-\frac{\alpha}{2}}(2r)} \right) ,$$

equivalently $\left(\dfrac{2T_r}{\chi^2_\alpha(2r)}, \infty\right)$ will cover the true but unknown value of θ, $100(1-\alpha)$ percent of the time.

Let $c_3(r,\alpha) = 2r/\chi^2_\alpha(2r)$, then a $100(1-\alpha)$ percent one-sided confidence interval for θ can be written as $[c_3(r,\alpha)\hat{\theta}_{r,n}, \infty]$. In Table 2 we give the values of $c_3(r,\alpha)$ for $\alpha = .01, .05, .10, .20, .25,$ and $.50$ and $r = 1(1)\ 20(5)\ 30(10)\ 50(25)\ 100$.

For large r (say ≥ 50) $\chi^2(2r)$ is approximately normally distributed with mean $2r$ and variance $4r$. Consequently, the two-sided $100(1-\alpha)$ percent confidence interval becomes (for large r)

(10) $$\left(\frac{\hat{\theta}}{1+\frac{c_\alpha}{\sqrt{r}}}, \frac{\hat{\theta}}{1-\frac{c_\alpha}{\sqrt{r}}}\right)$$

where	$c_\alpha = 2.576$	if	$\alpha = .01$
	$= 1.960$		$= .05$
	$= 1.645$		$= .10$
	$= 1.282$		$= .20$
	$= .674$		$= .50$

In the one-sided case the $100(1-\alpha)$ percent confidence interval becomes

(11) $$\left(\frac{\hat{\theta}}{1+\frac{d_\alpha}{\sqrt{r}}}, \infty\right)$$

where $d_\alpha = 2.326$ if $\alpha = .01$
$= 1.645$ $= .05$
$= 1.282$ $= .10$
$= .674$ $= .25$
$= 0$ $= .50$

Estimation of Other Quantities:

(a) In many practical problems one does not wish to find point or interval estimates for the mean life θ, but rather for a quantile x_p, where x_p is that life such that

(12) $$\Pr(X \geq x_p) = p .$$

For the exponential p.d.f. this means that

(13) $$e^{-x_p/\theta} = p \quad \text{or} \quad x_p = \theta \log \frac{1}{p} .$$

It is therefore clear that the maximum likelihood estimate of x_p is given by $\hat{\theta} \log \frac{1}{p}$. Furthermore, two-sided and one-sided $100(1-\alpha)$ percent confidence intervals for x_p are:

$$\left(\frac{2r\hat{\theta}_{r,n} \log \frac{1}{p}}{\chi^2_{\frac{\alpha}{2}}(2r)} , \frac{2r\hat{\theta}_{r,n} \log \frac{1}{p}}{\chi^2_{1-\frac{\alpha}{2}}(2r)} \right)$$

(14) or equivalently

$$\left(\frac{2T_r \log \frac{1}{p}}{\chi^2_{\frac{\alpha}{2}}(2r)} \; , \; \frac{2T_r \log \frac{1}{p}}{\chi^2_{1-\frac{\alpha}{2}}(2r)} \right)$$

and

(15) $$\left(\frac{2r\hat{\theta} \log \frac{1}{p}}{\chi^2_{\alpha}(2r)} , \infty \right) \text{ or } \left(\frac{2T_r \log \frac{1}{p}}{\chi^2_{\alpha}(2r)} , \infty \right)$$

respectively.

In Table 3 we give values of $\log \frac{1}{p}$ for various useful values of p. Two-sided and one-sided confidence intervals for x_p can be found by using Tables 1, 2, and 3 and substituting appropriately in equations (14) and (15).

Remark 1: Formula (15) can be interpreted as follows. On the basis of the estimate $\hat{\theta}_{r,n}$ we can be $100(1-\alpha)$ percent confident of the assertion that the probability of surviving

$$\tau = \frac{2r\hat{\theta}_{r,n}}{\chi^2_{\alpha}(2r)} \log \frac{1}{p} \text{ time units}$$

is $\geq p$. This is a tolerance interval statement in the sense that if we observe $\hat{\theta}_{r,n}$ for a sample we can be $100(1-\alpha)$ percent confident of the correctness of the assertion that the fraction of items in the population surviving τ or more time units is $\geq p$.

Remark 2: It should also be noted that if we observe $\hat{\theta}_{r,n}$ then formulae (14) and (15) give one and two-sided $100(1-\alpha)$ percent confidence bands for the entire distribution.

(b) Frequently we wish to make confidence statements about the proportion of items surviving some preassigned time t^*, on the basis of the first r failure times. Since the probability of surviving for a fixed time t^* is given by

$$p_{t^*} = \Pr(X > t^*) = e^{-t^*/\theta} \tag{16}$$

it is clear that the maximum likelihood estimate of p_{t^*} is given by

$$\hat{p}_{t^*} = e^{-t^*/\hat{\theta}_{r,n}} . \tag{17}$$

From (7) it follows immediately that a $100(1-\alpha)$ percent two-sided confidence interval for p_{t^*} is given by

$$\left(e^{-\chi^2_{\frac{\alpha}{2}}(2r)t^*/2r\hat{\theta}_{r,n}} , e^{-\chi^2_{1-\frac{\alpha}{2}}(2r)t^*/2r\hat{\theta}_{r,n}} \right)$$

(18) or equivalently

$$\left(e^{-\chi^2_{\frac{\alpha}{2}}(2r)t^*/2T_r} , e^{-\chi^2_{1-\frac{\alpha}{2}}(2r)t^*/2T_r} \right) .$$

One-sided $100(1-\alpha)$ percent confidence intervals for $e^{-\frac{t^*}{\theta}}$ are particularly important. It is an immediate consequence of (9) that this confidence interval is given by

$$(19) \quad \left(e^{-\chi^2_\alpha(2r)t^*/2r\hat{\theta}_{r,n}}, 1 \right) \text{ or equivalently } \left(e^{-\chi^2_\alpha(2r)t^*/2T_r}, 1 \right) .$$

The question may be asked: how large should the observed $\hat{\theta}_{r,n}$ (or equivalently T_r) be in order that we be 100(1-α) percent confident that

$$p_{t^*} = e^{-t^*/\theta} \geq \gamma \text{ ?}$$

From (19) this implies that

$$(20) \qquad e^{-\chi^2_\alpha(2r)t^*/2r\hat{\theta}_{r,n}} \geq \gamma .$$

This is equivalent to

$$\hat{\theta}_{r,n} \geq \chi^2_\alpha(2r)t^*/2r \log \frac{1}{\gamma} \text{ or } T_r \geq \chi^2_\alpha(2r)t^*/2 \log \frac{1}{\gamma} .$$

The meaning of the inequality is as follows:

If the total life observed in getting r failures exceeds $\chi^2_\alpha(2r)t^*/2 \log \frac{1}{\gamma}$, then we can be 100(1-$\alpha$) percent confident that the probability of surviving the time t^* is $\geq \gamma$. These values are readily computed from Tables 2 and 3.

Numerical Examples

Note: It is assumed throughout that the underlying distribution of life is exponential.

Example 1: 20 electron tubes are placed on test. A tube which fails is replaced at once by a new tube. The fifth failure is observed to occur 407 hours after the start of the life test.

(a) Estimate the mean life θ and give one and two-sided 95% confidence intervals for θ.

(b) Estimate $x_{.9}$, where $x_{.9}$ is such that

$$\Pr(X \geq x_{.9}) = .9$$

Give one and two-sided 95% confidence intervals for $x_{.9}$.

(c) Make a one and two-sided 95% confidence statement for the probability of surviving 100 hours.

Solution:

(a) We are dealing with a replacement situation with $n = 20$, $r = 5$, $x_5 = 407$. The total life observed is given by $T_5 = 20x_5 = 20(407) = 8140$. Thus it follows from (3) that

$$\hat{\theta} = T_5/5 = 1628 .$$

To find a two-sided 95% confidence interval we use (7) with $\chi^2_{.025}(10) = 20.483$ and $\chi^2_{.975}(10) = 3.247$. This gives the two-sided interval (795,5014). To find a one-sided 95% confidence interval we use (9) with $\chi^2_{.05}(10) = 18.307$. This gives the one-sided interval $(889,\infty)$. The values can also be obtained directly from Tables 1 and 2.

(b) The solution is found by multiplying through by $\log \frac{1}{p} = \log \frac{10}{9} = .1054$. Thus we get $\hat{x}_{.9} = (1628)(.1054) = 172$.

A 95% two-sided confidence interval is given by (83.8,528) and a 95% one-sided confidence interval is given by $(93.7,\infty)$.

(c) The maximum likelihood estimate of p_{t^*} , the probability of surviving $t^* = 100$ hours is given by $\hat{p}_{t^*} = e^{-(100)/1628} = e^{-.0614} = .9404$. Similarly a two-sided 95% confidence interval for p_{t^*} is given by $(e^{-100/795}, e^{-100/5014}) = (e^{-.1258}, e^{-.0199}) = (.8817,.9802)$ and a one-sided 95% confidence interval for p_{t^*} is given by $(e^{-100/889},1) = (e^{-.1125},1) = (.8936,1)$.

Example 2: 20 electron tubes are placed on test. Tubes which fail are not replaced. The first five observations to failure were $x_{1,20} = 26$, $x_{2,20} = 64$, $x_{3,20} = 119$, $x_{4,20} = 145$, and $x_{5,20} = 182$. Estimate the mean life θ and give a one and two-sided 90% confidence interval for θ based on the data.

Solution: This is a non-replacement situation with $n = 20$ and $r = 5$. The total observed life is given by $T_5 = \sum_{i=1}^{5} x_i + 15x_5 = 536 + 2730 = 3266$. Thus it follows from (3) that $\hat{\theta} = T_5/5 = 3266/5 = 653$. A two-sided 90% confidence interval for θ is given by (357,1657) and a one-sided 90% confidence interval for θ is given by $(409,\infty)$. These values are obtained using Tables 1 and 2 .

Example 3: An extensive life test has been run and a $\hat{\theta}$ based on $r = 100$ failures has been computed. Suppose that $\hat{\theta} = 1000$. Give one and two-sided 95% confidence intervals for θ .

Solution: From (10) the two-sided 95% confidence interval for θ is given by

$$\left(\frac{1000}{1+\frac{1.96}{\sqrt{100}}}, \frac{1000}{1-\frac{1.96}{\sqrt{100}}}\right) = \left(\frac{1000}{1.196}, \frac{1000}{.804}\right)$$

$$= (836, 1244) .$$

From (11) the one-sided 95% confidence interval for θ is given by

$$\left(\frac{1000}{1+\frac{1.645}{\sqrt{100}}}, \infty\right) = \left(\frac{1000}{1.1645}, \infty\right) = (859, \infty) .$$

Example 4: The total life observed in obtaining 5 failures is 9205 hours. On the basis of this information, can we be 95% confident that the probability of surviving for a time $t^* = 100$ is $\geq .90$?

Solution: From (20) it is known that in order to be 95% confident that the probability of surviving for a time $t^* = 100$ is $\geq .9$, it is necessary that the total observed life

$$T_5 \geq \chi^2_{.05}(10)100/2 \log \frac{1}{.9} = 8689 .$$

Since the total life observed in obtaining 5 failures is 9205 hours, we can answer in the affirmative.

Example 5: Suppose that we want to keep a mechanism containing 1000 tubes in continuous operation for 1000 hours. Suppose that all we know about the tube life is based on the data contained in Example 1. Based on these data, how many tubes should we expect to put in as replacements for those which fail during the 1000 hour period? Find a two-sided and one-sided 95% confidence interval for the expected number of replacements needed.

Solution: We are in effect observing a Poisson process with failure rate $\lambda_\theta = 1000/\theta$. The maximum likelihood estimate of λ is, from the solution to (1), given by $\hat{\lambda} = 1000/\hat{\theta} = 1000/1628 = .614$. Therefore the expected number of replacements over 1000 hours is given by $1000\hat{\lambda} = 614$.

In example (1), we computed (795,5014) as the two-sided 95% confidence interval for θ . This gives the two-sided 95% confidence interval for the expected number of replacements:

$$\left(\frac{10^6}{5014}, \frac{10^6}{795}\right) = (199,1258) .$$

In example (1), we computed $(889,\infty)$ as the one-sided 95% confidence interval for θ. Therefore a one-sided 95% confidence interval for the expected number of replacements is given by

$$\left(0, \frac{10^6}{889}\right) = (0,1125) .$$

The limits are very wide, because the data are of course very inadequate, but they do give us some idea of what we may expect to get.

__Remark__: More generally suppose we want to keep a mechanism containing N tubes in continuous operation. To do this N good tubes must be in operation at all times. Suppose that we want this condition to hold for a time interval of length T. How many tubes can we expect to insert as replacements in a time T, basing our estimates on one or more previous life tests?

As indicated in example 5 we are in effect observing a Poisson process with parameter $\lambda_\theta = N/\theta$. Therefore, the expected number of replacements if we wish to keep N items functioning at all times in an interval of length T is given by $\hat{\lambda}_\theta T = NT/\hat{\theta}$. If $(\theta_1 \leq \theta \leq \theta_2)$ is a $100(1-\alpha)$ percent two-sided confidence interval for θ, then a $100(1-\alpha)$ percent confidence interval for the expected number of replacements is given by $(NT/\theta_2, NT/\theta_1)$. If (θ_3, ∞), is a $100(1-\alpha)$ percent one-sided confidence interval for θ, then a $100(1-\alpha)$ percent confidence interval for the expected number of replacements is given by $\left(0, \frac{NT}{\theta_3}\right)$.

In example 5, $\alpha = .05$, $N = 1000$, $T = 1000$, $\hat{\theta} = 1628$, $\theta_1 = 795$, $\theta_2 = 5014$, and $\theta_3 = 889$.

__Example 6__: Given the data in problem 1, find a number τ such that we can assert with 95% confidence that at least 90% of the population survives τ. (Note that this is a tolerance statement).

__Solution__: We noted in Remark (1) following our discussion of interval estimates for the quantile x_p that one-sided $100(1-\alpha)$ percent confidence statements regarding x_p are also tolerance statements in which we can have $100(1-\alpha)$ percent confidence. Hence using the solution to 1(b) we can assert that $\tau = 93.7$. Based on the data we can assert with 95% confidence that at least 90% of the population survives $\tau = 93.7$ hours.

To solve (b) graphically in the two-sided case we see where the horizontal line $x = 1000$ meets the two lines $x = 795 \log \frac{1}{p}$ and $5014 \log \frac{1}{p}$. The two values of p obtained are .28 and .82 . Thus a 95% two-sided confidence interval for $p_{t^* = 1000} = e^{-1000/\theta}$ (i.e., the probability of surviving 1000 hours) is given by (.28, .82) . In the one-sided case the horizontal line $x = 1000$ intersects the line $x = 889 \log \frac{1}{p}$ at $p = .32$. Hence we can state that (.32,1) is a 95% confidence interval for $p_{t^* = 1000} = e^{-1000/\theta}$.

Section 2.

An Estimation Problem (Fixed time of truncation. Items which fail are replaced by new items.)

Problem: n items are placed on life test at time $t = 0$. As the test proceeds, items which fail are replaced by new items. Life testing is terminated at time t^*. It is assumed that the underlying p.d.f. of life is given by

$$f(t;\theta) = \frac{1}{\theta} e^{-t/\theta} \, , \; t > 0 \, , \; \theta > 0 \, .$$

We wish to do the following:

(i) Estimate θ.

(ii) Make one and two-sided confidence statements about θ.

(iii) Make probability statements about the proportion of items having life greater than t^*.

Solution: In what follows let r = number of items which fail in $(0,t^*)$, then the solutions are as follows:

(i) The maximum likelihood estimate for θ is given by nt^*/r.

(ii) A one-sided $100(1-\alpha)$ percent confidence interval for θ is given by

$$\left(\frac{2nt^*}{\chi^2_\alpha(2r+2)} \, , \, \infty \right) . \tag{1}$$

A two-sided $100(1-\alpha)$ percent confidence interval for θ is given by

$$\left(\frac{2nt^*}{\chi^2_{\frac{\alpha}{2}}(2r+2)} \, , \, \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2}}(2r)} \right) . \tag{2}$$

(iii) From the results in (ii) regarding the one-sided $100(1-\alpha)$ percent confidence intervals for θ we can be $100(1-\alpha)$ percent confident that at-least 100 b% of the population survives t^* hours, with

(3) $$b = e^{-\chi^2_\alpha (2r+2)/2n} .$$

In other words a 100(1-α) percent one-sided confidence interval for $b = e^{-t^*/\theta}$ is given by

(4) $$\left(e^{-\chi^2_\alpha (2r+2)/2n}, 1\right).$$

From the results in (ii) regarding the two-sided 100(1-α) percent confidence intervals for θ, we can say that if we observe r failures in $(0,t^*)$ then a two-sided 100(1-α) percent confidence interval for $b = e^{-t^*/\theta}$ is given by

(5) $$\left(e^{-\chi^2_{\frac{\alpha}{2}} (2r+2)/2n}, e^{-\chi^2_{1-\frac{\alpha}{2}} (2r)/2n}\right).$$

<u>Proof</u>: Essentially we are observing a Poisson process with parameter $\lambda' = n\lambda$, where $\lambda = \frac{1}{\theta}$. If we observe r failures in $(0,t^*)$ then the maximum likelihood estimate for λ' is given by

(6) $$\hat{\lambda}' = \frac{r}{t^*} .$$

Thus

(7) $$\hat{\lambda} = \frac{\hat{\lambda}'}{n} = \frac{r}{nt^*}$$

Therefore

(8) $$\hat{\theta} = \frac{1}{\hat{\lambda}} = \frac{nt^*}{r}$$

and this establishes (i) .

It can be shown that the probability of observing r or fewer failures in $(0,t^*)$ is given by

(9) $$\Pr(k \leq r|\theta) = \sum_{k=0}^{r} e^{-nt^*/\theta}(nt^*/\theta)^k/k!$$
$$= \int_{\frac{nt^*}{\theta}}^{\infty} \frac{x^r}{r!} e^{-x} dx$$
$$= \Pr\left(\chi^2(2r+2) > \frac{2nt^*}{\theta}\Big|\theta\right) .$$

Thus, if $\theta \leq 2nt^*/\chi^2_{\alpha}(2r+2)$ then $\Pr(k \leq r|\theta) \leq \alpha$. This implies that if we observe the event $k = r$, then we are $100(1-\alpha)$ percent confident of the correctness of the assertion that $\theta > 2nt^*/\chi^2_{\alpha}(2r+2)$.

In a similar way it can be shown that if $\theta < 2nt^*/\chi^2_{\frac{\alpha}{2}}(2r+2)$ then $\Pr(k \leq r|\theta) \leq \frac{\alpha}{2}$ and if $\theta > 2nt^*/\chi^2_{1-\frac{\alpha}{2}}(2r)$ then $\Pr(k \geq r|\theta) \leq \frac{\alpha}{2}$.

From this it follows that if we observe the event $k = r$, then we are $100(1-\alpha)$ percent confident of the correctness of the assertion that

$$\left(\frac{2nt^*}{\chi^2_{\frac{\alpha}{2}}(2r+2)} < \theta < \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2}}(2r)}\right).$$

Remark 1: Define $\tilde{\theta}$ as $\tilde{\theta} = \hat{\theta}\left(\frac{r}{r+1}\right) = nt^*/r+1$. Then one can write the one-sided $100(1-\alpha)$ percent confidence interval for θ as $\left(\frac{2(r+1)\,\tilde{\theta}}{\chi^2_{\alpha}[2r+2]}, \infty\right)$ and the two-sided $100(1-\alpha)$ percent confidence interval for θ as

$$\left(\frac{(2r+2)\,\tilde{\theta}}{\chi^2_{\frac{\alpha}{2}}(2r+2)}, \frac{2\,r\,\hat{\theta}}{\chi^2_{1-\frac{\alpha}{2}}(2r)}\right).$$

Thus $\tilde{\theta}$ is involved in computing the one-sided interval and in finding the left-hand end point of the two-sided interval. $\hat{\theta}$ is involved in finding the right-hand end-point of the two-sided interval. It is now clear that we can use Tables 1 and 2 in order to compute the confidence intervals.

Remark 2: If $r = 0$, only the estimator $\tilde{\theta}$ makes sense and only one-sided intervals of the form (1) should be used.

Remark 3: The two-sided confidence intervals for θ given by formula (2) are direct consequences of formulae for two-sided confidence intervals for the parameter λ in a Poisson process given by F. Garwood in Biometrika 28, 437-442, 1936. This question is also treated in E.S. Pearson and H.O. Hartley, Biometrika Tables for Statisticians, Vol. I, pp. 74-77, Cambridge University Press, 1954.

$$(13) \qquad \left(\frac{2nt^* \log \frac{1}{p}}{\chi^2_{\frac{\alpha}{2}}(2r+2)} , \ \frac{2nt^* \log \frac{1}{p}}{\chi^2_{1-\frac{\alpha}{2}}(2r)} \right)$$

$$= \left(\frac{(2r+2)\ \tilde{\theta} \log \frac{1}{p}}{\chi^2_{\frac{\alpha}{2}}(2r+2)} , \ \frac{2r\ \hat{\theta} \log \frac{1}{p}}{\chi^2_{1-\frac{\alpha}{2}}(2r)} \right) .$$

Formulae (12) and (13) given $100(1-\alpha)$ percent one and two-sided confidence bands for the entire distribution.

Remark 7: It follows from (12) that if

$$(14) \qquad \tau = \frac{2nt^*}{\chi^2_\alpha(2r+2)} \log \frac{1}{p}$$

then we can assert with $100(1-\alpha)$ percent confidence that (τ, ∞) is a 100 p percent tolerance interval. More precisely, if one observes r failures in $(0, t^*)$ (where n items are constantly kept on test) then we can be $100(1-\alpha)$ percent confident that the probability of surviving for at least time τ is $\geq p$ (or that the fraction of the population surviving τ or more hours is $\geq p$). In terms of $\tilde{\theta}$, (12) can also be written as

$$(15) \qquad \tau = \frac{(2r+2)\ \tilde{\theta}}{\chi^2_\alpha(2r+2)} \log \frac{1}{p} \ .$$

This again makes it easy to compute τ from Tables 2 and 3.

Remark 8: In the section devoted to the testing of hypotheses we studied a truncated replacement life test procedure of the following kind. n items are placed on test, and it is decided in advance that the experiment will be terminated at min $(\tau_{r_o,n};t^*)$, where $\tau_{r_o,n}$ is a random variable equal to the time at which the r_oth failure occurs and t^* is a truncation time, beyond which the experiment will not be run. Both r_o and t^* are assigned in advance before life testing starts. If the experiment is terminated at $\tau_{r_o,n}$ (i.e., if r_o failures occur before time t^*), then the action in terms of hypothesis testing is the rejection of some specified null-hypothesis. If, however, the experiment is terminated at time t^* (i.e., the r_oth failure does not occur before time t^*), then the action in terms of hypothesis testing is the acceptance of some specified null-hypothesis.

Suppose now that such a test has been run and that we would like to use the data obtained not only for testing, but also for estimation. It is generally recognized that there are difficulties associated with using such data, since the stopping rule usually affects the estimates which can be obtained. It is interesting to point out that for the truncated life test under discussion the following rule gives $100(1-\alpha)$ percent one-sided confidence intervals:

(i) If $\tau_{r_o} > t^*$, i.e., if the number of observed failures k in $(0,t^*)$ is $0,1,2, \ldots ,r_o -1$, then a one-sided $100(1-\alpha)$ percent confidence interval is given by

(16) $$\left(\frac{2nt^*}{\chi^2_\alpha(2k+2)} , \infty \right)$$

(ii) If $\tau_{r_o} \leq t^*$, a one-sided 100(1-α) percent confidence interval is given by

(17) $$\left(\frac{2n\tau_{r_o}}{\chi^2_\alpha(2r_o)} , \infty \right)$$

In Appendix 3 E we prove that equations (16) and (17) generate 100(1-α) percent one-sided confidence intervals.

One might conjecture that two-sided 100(1-α) percent confidence intervals can be defined in an analogous way as:

(18) $$\left(\frac{2nt^*}{\chi^2_\alpha(2)} , \infty \right) \text{ if } k = 0$$

$$\left(\frac{2nt^*}{\chi^2_{\frac{\alpha}{2}}(2k+2)} , \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2}}(2k)} \right) \text{ if } k = 1,2, \ldots , r_o-1$$

and

(19) $$\left(\frac{2n\tau_{r_o}}{\chi^2_{\frac{\alpha}{2}}(2_{r_o})} , \frac{2n\tau_{r_o}}{\chi^2_{1-\frac{\alpha}{2}}(2_{r_o})} \right) \text{ if } \tau_{r_o} \leq t^* .$$

We have, up to now, not been able to establish this conjecture rigorously.

Numerical Examples

Problem 1: 30 items are placed on test. Items which fail are replaced. The life test is stopped after 100 hours have elapsed. Five failures are observed in the course of the experiment. Assuming that the underlying distribution of life is exponential, find

(a) An estimate of the mean life θ . Give one and two-sided 95% confidence intervals for θ .

(b) Make one and two-sided 95% confidence statements for the probability of surviving 100 hours.

(c) Make one and two-sided 95% confidence statements for the probability of surviving 50 hours.

Solution:

(a) In this problem $n = 30$, $t^* = 100$, the observed number of failures is $r = 5$. Thus the maximum likelihood estimate for θ is given by $\hat{\theta} = nt^*/r = 3000/5 = 600$. Substituting in formula (1) and using $\chi^2_{.05}(12) = 21.026$, one gets the one-sided 95% confidence interval $(285, \infty)$. Substituting in formula (2) and using $\chi^2_{.025}(12) = 23.337$ and $\chi^2_{.975}(10) = 3.247$ one gets a 95% two-sided confidence interval (257,1848).

(b) A one-sided 95% confidence interval for surviving $t^* = 100$ hours is given by $(e^{-21.026/60}, 1) = (e^{-.3504}, 1) = (.704, 1)$.

A two-sided 95% confidence interval for surviving $t^* = 100$ hours is given by

$$(e^{-23.337/60}, e^{-3.247/60}) = (e^{-.3889}, e^{-.0541}) = (.6778, .9473) .$$

(c) One and two-sided 95% confidence intervals for the probability of surviving $\tau = 50$ hours are given by $(e^{-.1752},1) = (.8393,1)$ and $(e^{-.1945}, e^{-.0271}) = (.8232, .9733)$ respectively.

Problem 2: Given the data in example 1. Estimate τ so that we will be 95% confident that the probability of surviving τ hours is at least .9 .

Solution:

$n = 30$, $t^* = 100$, $r = 5$, $\alpha = .05$, and $p = .9$. Thus substituting in (14) we have

$$\tau = \frac{60(100)}{21.026}(1.054) = 30.1.$$

More directly, using tables 2 and 3 and noting that $\tilde{\theta} = \frac{r}{r+1}\hat{\theta} = \frac{5(600)}{6} = 500$, one gets $\tau = (500)(.571)(.1054) = 30.1$.

On the basis of the data we can be 95% confident that the probability of surviving $\tau = 30.1$ hours is $\geq .9$.

Problem 3: A truncated replacement test consists of placing 30 items on test for at most 100 hours. If 3 failures occur before 100 hours, the life test is stopped at once and the lot is rejected. If, however, 3 items have not yet failed by the time 100 hours have elapsed, the test is terminated at 100 hours with acceptance. Items which fail are replaced at once by new items. Give a 95% one-sided confidence interval for θ if one observes exactly one failure.

Solution: We use formula (16) with $k = 1$, hence a one-sided 95% confidence interval is given by

$$\left(\frac{2nt^*}{\chi^2_{.05}(4)}, \infty\right) = \left(\frac{6000}{9.488}, \infty\right) = (632, \infty) .$$

Problem 4: Suppose it happened that the third failure was observed to occur at 50 hours. Give 95% one and two-sided confidence intervals in this case.

Solution: We use formula (17) with $k = 3$, $\tau_{3,30} = 50$. Hence a one-sided 95% confidence interval is given by

$$\left(\frac{2n\tau_3}{\chi^2_{.05}(6)}, \infty\right) = \left(\frac{3000}{12.592}, \infty\right) = (238, \infty) .$$

Substituting in formula (19) we get

$$\left(\frac{3000}{\chi^2_{.025}(6)}, \frac{3000}{\chi^2_{.975}(6)}\right) = \left(\frac{3000}{14.449}, \frac{3000}{1.237}\right) = (208, 2425)$$

as a two-sided 95% confidence interval.

SECTION 3

AN ESTIMATION PROBLEM
(Fixed time of truncation t*; failed items are not replaced)

Problem: n items are placed on life test for a time t^*. At the end of this time one counts the number of items that have failed in the time interval $[0, t^*]$. Items that fail are not replaced. We wish to do the following:

(i) Give an estimate for the probability of surviving for a length of time t^* and further estimate the mean life θ, if the underlying distribution is exponential.

(ii) Make one and two-sided confidence statements about the probability of living for more than t^*. Stated in reliability language we wish to make probability statements about the reliability of items in $[0, t^*]$.

(iii) Make one and two-sided confidence statements about the mean life θ in the case where the underlying distribution is exponential.

Solution: In what follows let r = number of items which fail in $[0, t^*]$, then the solutions are as follows:

(i) The maximum likelihood estimate of the probability of surviving more than t^* time units is given by

(1) $$\hat{p} = \left(\frac{n - r}{n}\right) .$$

If the underlying distribution is exponential, then $\hat{p} = e^{-t^*/\hat{\theta}}$ and hence

(2) $$\hat{\theta} = t^*/\log\left(\frac{n}{n - r}\right) .$$

(ii) There is a confidence of $100(1-\alpha)$ percent attached to the statement that at least $100\ b\%$ of the population survives for a length of time t^* with b given by the formula

$$(3) \qquad \frac{1}{1+\left(\frac{r+1}{n-r}\right) F_\alpha(2r+2,\ 2n-2r)} .$$

In other words the one-sided $100(1-\alpha)$ percent confidence interval for the probability of surviving t^* time units is given by

$$(3') \qquad \left(\frac{1}{1+\left(\frac{r+1}{n-r}\right) F_\alpha(2r+2,\ 2n-2r)},\ 1\right).$$

$F_\alpha(n_1, n_2)$ is defined in such a way that $\Pr(F(n_1, n_2) \geq F_\alpha(n_1, n_2)) = \alpha$, where $F(n_1, n_2)$ is the F distribution with n_1 degrees of freedom in the numerator and n_2 degrees of freedom in the denominator.

A two-sided $100(1-\alpha)$ percent confidence interval for the probability of surviving t^* time units is given by

$$(4) \qquad \left(\frac{1}{1+\left(\frac{r+1}{n-r}\right) F_{\frac{\alpha}{2}}(2r+2,\ 2n-2r)},\ \frac{1}{1+\left(\frac{r}{n-r+1}\right) F_{1-\frac{\alpha}{2}}(2r,\ 2n-2r+2)}\right).$$

These results are completely distribution free.

(iii) In the case where the underlying distribution is exponential, one obtains

$$(5) \qquad \left(\frac{t^*}{\log\left\{1+\left(\frac{r+1}{n-r}\right) F_\alpha(2r+2,\ 2n-2r)\right\}},\ \infty\right)$$

as a one-sided $100(1-\alpha)$ percent confidence interval for θ and

$$(6) \quad \left(\frac{t^*}{\log \left\{ 1 + \left(\frac{r+1}{n-r}\right) F_{\frac{\alpha}{2}}(2r+2,\ 2n-2r) \right\}},\ \frac{t^*}{\log \left\{ 1 + \left(\frac{r}{n-r+1}\right) F_{1-\frac{\alpha}{2}}(2r, 2n-2r+2) \right\}} \right)$$

as a two-sided $100(1 - \alpha)$ percent confidence interval for θ.

Proof: (i) Formulae (1) and (2) are obvious.

(ii) Suppose that we observe r failures in the time interval $[0, t^*]$ and that p = probability of failing in $[0, t^*]$ and $q = 1 - p$ = probability of surviving in $[0, t^*]$. Suppose that p_o is such that

$$(7) \qquad \sum_{k=0}^{r} \binom{n}{k} p_o^k q_o^{n-k} = \alpha,$$

then we can state that $\Pr(k \leq r \mid p) \leq \alpha$ if $p \geq p_o$. Hence if we observe $k = r$, we can be $100(1 - \alpha)$ percent confident that $p < p_o$ or that $q = 1 - p > 1 - p_o = q_o$. The question arises as to how one computes q_o. This can be done very easily by expressing (7) as an incomplete Beta Function and then using the well-known relationship between the Beta and F distributions. If this is done, one discovers that

$$(8) \qquad q_o = \frac{1}{1 + \left(\frac{r+1}{n-r}\right) F_{\alpha}(2r + 2,\ 2n - 2r)},$$

where $F_{\alpha}(n_1, n_2)$ is defined in such a way that $\Pr\left(F(n_1, n_2) \geq F_{\alpha}(n_1, n_2)\right) = \alpha$, and where n_1 and n_2 are the number of degrees of freedom in the numerator and denominator respectively. Thus (3) is established. In this connection one should also read S. Takada and S. Shimada, Part 1, July 1954, pp. 147 and 151. See bibliography given in the Appendix.

In Table 4 we give the values of q_o for $n = 1(1)20(5)30(10)50(25)100(50)200(100)500$; for $\alpha = .01, .05, .10, .25, .50$

and $r = 1(1) \min(n, 20)$. In Table 5 we tabulate q_o for n = 1000, 5000, 10000, 50000, 100000, 500000, 1000000; for α = .01, .05, .10, .25, .50 and $r = 1(1)20(10)100, 200, 500$.

Two-sided confidence results are obtained by finding $p = p_1$ and $p = p_2$ such that

(9) $$\sum_{k=0}^{r} \binom{n}{k} p_1^{\,k} q_1^{\,n-k} = \frac{\alpha}{2}$$

and

(10) $$\sum_{k=r}^{n} \binom{n}{k} p_2^{\,k} q_2^{\,n-k} = \frac{\alpha}{2} .$$

Hence if $k = r$ is observed we can be $100(1 - \alpha)$ percent confident that

$$p_2 < p < p_1 \quad \text{or that} \quad q_1 < q < q_2 .$$

The computation of q_1 and q_2 involves expressing (9) and (10) as an incomplete Beta Function and then using the well-known relationship between the Beta and F distributions. If this is done, it turns out that

(11) $$q_1 = \frac{1}{1 + \left(\frac{r+1}{n-r}\right) F_{\frac{\alpha}{2}}(2r+2,\ 2n-2r)}$$

and

$$q_2 = \frac{1}{1 + \left(\frac{r}{n-r+1}\right) F_{1-\frac{\alpha}{2}}(2r,\ 2n-2r+2)} .$$

Thus (4) is established.

Tables for q_1 and q_2 are being computed for the values of n, α, r used in Tables 4 and 5.

In the particular case where the underlying distribution happens to be exponential, (8) implies that (5) is a one-sided $100(1 - \alpha)$ percent confidence interval for θ and (11) implies that (6) is a two-sided $100(1 - \alpha)$ percent confidence interval for θ.

__Remark__: One and two-sided $100(1 - \alpha)$ percent confidence intervals for the probability of surviving an arbitrary time τ not necessarily = t^* are given, in the exponential case, by

$$(12) \qquad \left[1 + \left(\frac{r+1}{n-r}\right) F_{\alpha}(2r+2,\ 2n-2r)\right]^{-\frac{\tau}{t^*}}$$

and

$$(13) \left(\left[1 + \left(\frac{r+1}{n-r}\right) F_{\frac{\alpha}{2}}(2r+2,\ 2n-2r)\right]^{-\frac{\tau}{t^*}},\ \left[1 + \left(\frac{r}{n-r+1}\right) F_{1-\frac{\alpha}{2}}(2r,\ 2n-2r+2)\right]^{-\frac{\tau}{t^*}} \right)$$

respectively.

__Remark__: It happens sometimes that n is very large and r is very small. It is useful to note that in this case (3) becomes

$$(14) \qquad b \sim \frac{1}{1 + \left(\frac{r+1}{n}\right) F_{\alpha}(2r+2,\ \infty)} .$$

In other words, the one-sided $100(1 - \alpha)$ percent confidence interval for the probability of surviving time t^* is given by

$$(15) \left(\frac{1}{1 + \left(\frac{r+1}{n}\right) F_{\alpha}(2r+2,\ \infty)},\ \infty \right) = \left(\frac{1}{1 + \frac{\chi^2_{\alpha}(2r+2)}{2n}},\ \infty \right) .$$

Similarly a two-sided $100(1 - \alpha)$ percent confidence interval for the probability of survival is given by

$$(16) \quad \left(\frac{1}{1 + \left(\frac{r+1}{n}\right) F_{\frac{\alpha}{2}}(2r+2, \infty)} , \frac{1}{1 + \left(\frac{r}{n}\right) F_{1-\frac{\alpha}{2}}(2r, \infty)} \right)$$

$$= \left(\frac{1}{1 + \left[\chi^2_{\frac{\alpha}{2}}(2r+2)/2n\right]} , \frac{1}{1 + \left[\chi^2_{1-\frac{\alpha}{2}}(2r)/2n\right]} \right).$$

In (15) and (16) we use the fact that $F_{\alpha}(m, \infty) = \chi^2_{\alpha}(m)/m$.

In all of the results obtained up to this point in this section, we have not made any use of the failure times of those items which did indeed fail. Because of this we were able to state a certain number of non-parametric results. However, in the event that the underlying distribution of life really is exponential we are clearly losing some information, at least when n, the number of items originally placed on test is small. We say this because of the fact that if n were very large, then we would be effectively dealing with a replacement situation. In this case, the knowledge of actual times to failure is irrelevant if the underlying distribution is really exponential. It is only for small or moderate sizes of n that it would make a difference whether or not we use our knowledge of the actual times to failure of those items which did fail.

Throughout we assume, as before, that we start the life test with n items and that we do not replace failed items. Let r = number of items which fail in $(0, t^*)$ and let $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_r$ be the failure times. We assume further that the underlying distribution is exponential. An exact solution to the problem of finding $100(1 - \alpha)$ percent confidence intervals for θ is easy in principle, but difficult to carry out. Hence we give, without proof, some approximate procedures which are good enough in most practical problems. The first result is that approximate one-sided $100(1-\alpha)$ percent confidence intervals for θ are given by

$$(17) \qquad \left(\frac{2T(t^*)}{\chi^2_\alpha(2r+2)}, \infty\right) \quad \text{for } r = 0, 1, 2, \ldots, n-1$$

where $T(t^*) = \sum_{i=1}^{r} \tau_i + (n-r)t^*$

and

$$\left(\frac{2T(\tau_n)}{\chi^2_\alpha(2n)}, \infty\right)$$

where

$$T(\tau_n) = \sum_{i=1}^{n} \tau_i, \quad \text{if } r = n.$$

Approximate two-sided $100(1-\alpha)$ percent confidence intervals for θ are given by

$$(18) \qquad \left(\frac{2nt^*}{\chi^2_\alpha(2)}, \infty\right) \qquad \text{if } r = 0$$

by

$$\left(\frac{2T(t^*)}{\chi^2_{\frac{\alpha}{2}}(2r+2)}, \frac{2T(t^*)}{\chi^2_{1-\frac{\alpha}{2}}(2r)}\right) \quad \text{if } r = 1, 2, \ldots, n-1$$

and by

$$\left(\frac{2T(\tau_n)}{\chi^2_{\frac{\alpha}{2}}(2n)}, \frac{2T(\tau_n)}{\chi^2_{1-\frac{\alpha}{2}}(2n)}\right) \quad \text{if } r = n.$$

Formulae (17) and (18) should be compared with (5) and (6) respectively.

Remark: It is convenient to define $\tilde{\theta}$ as $\tilde{\theta} = \hat{\theta}\frac{r}{r+1} = \frac{T(t^*)}{r+1}$, for $r = 0, 1, 2, \ldots, n-1$ and as $\tilde{\theta} = \hat{\theta} = T(\tau_n)/n$ for $r = n$. Formula (17) then becomes

$$(17') \quad \left(\frac{2(r+1)\tilde{\theta}}{\chi^2_{\alpha}(2r+2)}, \infty \right) \quad \text{for} \quad r = 0, 1, 2, \ldots, n-1 \quad \text{and}$$

$$\left(\frac{2n\hat{\theta}}{\chi^2_{\alpha}(2n)}, \infty \right) \quad \text{for} \quad r = n$$

and formula (18) becomes

$$(18') \quad \left(\frac{2\tilde{\theta}}{\chi^2_{\alpha}(2)}, \infty \right) \quad \text{for} \quad r = 0$$

$$\left(\frac{2(r+1)\tilde{\theta}}{\chi^2_{\frac{\alpha}{2}}(2r+2)}, \frac{2r\hat{\theta}}{\chi^2_{1-\frac{\alpha}{2}}(2r)} \right) \quad \text{for} \quad r = 1, 2, \ldots, n-1$$

and

$$\left(\frac{2n\hat{\theta}}{\chi^2_{\frac{\alpha}{2}}(2n)}, \frac{2n\hat{\theta}}{\chi^2_{1-\frac{\alpha}{2}}(2n)} \right) \quad \text{for} \quad r = n.$$

As was done before, let us define the quantile x_p as that life such that $\Pr(X \geq x_p) = p$, i.e., $x_p = \theta \log \frac{1}{p}$. Then approximate one- and two-sided $100(1-\alpha)$ percent confidence intervals for x_p are obtained by multiplying the formulae in (17), (17'), (18), (18') by $\log \frac{1}{p}$. Furthermore it follows that if

$$(19) \quad t_p = \frac{(2r+2)\tilde{\theta}}{\chi^2_{\alpha}(2r+2)} \log \frac{1}{p}, \quad r = 0, 1, 2, \ldots, n-1$$

and

$$= \frac{2n\hat{\theta}}{\chi^2_{\alpha}(2n)} \log \frac{1}{p} \quad \text{for} \quad r = n$$

then we can be approximately $100(1-\alpha)$ percent confident of the correctness of the assertion that the fraction of the population surviving t_p or more hours is $\geq$ p. As before, one-sided confidence intervals on quantiles are equivalent to one-sided tolerance statements about the population with the same confidence.

If we want to place approximate one and two-sided $100(1 - \alpha)$ percent confidence intervals on $p_\tau = e^{-\tau/\theta}$, the probability of surviving τ hours, the results are as follows:

$$(20) \quad \left(e^{-\frac{\tau\chi^2_\alpha(2r+2)}{2(r+1)\hat{\theta}}}, 1\right) \text{ or } \left(e^{-\tau\chi^2_\alpha(2r+2)/2T(t^*)}, 1\right) \quad \text{for } r = 0, 1, 2, \ldots, n-1$$

and

$$\left(e^{-\frac{\tau\chi^2_\alpha(2n)}{2n\hat{\theta}}}, 1\right) \text{ or } \left(e^{-\tau\chi^2_\alpha(2n)/2T(\tau_n)}, 1\right) \quad \text{for } r = n$$

are approximate $100(1 - \alpha)$ percent one-sided confidence intervals on p_τ and

$$(21) \quad \left(e^{-\frac{\tau\chi^2_\alpha(2)}{2nt^*}}, 1\right) \quad \text{for } r = 0$$

$$\left(e^{-\frac{\tau\chi^2_{\frac{\alpha}{2}}(2r+2)}{2(r+1)\hat{\theta}}}, e^{-\frac{\tau\chi^2_{1-\frac{\alpha}{2}}(2r)}{2r\hat{\theta}}}\right) \quad \text{for } r = 1, 2, \ldots, n-1$$

and

$$\left(e^{-\frac{\tau\chi^2_{\frac{\alpha}{2}}(2n)}{2n\hat{\theta}}}, e^{-\frac{\tau\chi^2_{1-\frac{\alpha}{2}}(2n)}{2n\hat{\theta}}}\right) \quad \text{for } r = n$$

are approximate $100(1 - \alpha)$ percent two-sided confidence intervals on p_τ.

Suppose that the data are originally obtained in the course of running a truncated non-replacement life test with preassigned truncation time t^* and maximum allowable number of failures r_o. The stopping rule is $\min(\tau_{r_o,n}; t^*)$ where $\tau_{r_o,n}$ is a random variable equal to the time at which the r_o'th failure occurs. Then the following rule gives approximate $100(1 - \alpha)$ percent confidence intervals for θ.

(i) If $\tau_{r_o} > t^*$, i.e., if the number of observed failures k in $(0, t^*)$ is $0, 1, 2, \ldots, r_o - 1$ then an approximate one-sided $100(1 - \alpha)$ percent confidence interval is given by

$$\left(\frac{2T(t^*)}{\chi^2_\alpha(2k + 2)}, \infty \right), \quad k = 0, 1, 2, \ldots, r_o - 1 \tag{22}$$

where

$$T(t^*) = \sum_{i=1}^{k} \tau_i + (n_o - k)t^*$$

or equivalently as

$$\left(\frac{2(k + 1)\tilde{\theta}}{\chi^2_\alpha(2k + 2)}, \infty \right)$$

where $\tilde{\theta} = T(t^*)/k + 1$.

(ii) If $\tau_{r_o} < t^*$, then the appropriate interval is

$$\left(\frac{2T(\tau_{r_o})}{\chi^2_\alpha(2r_o)}, \infty \right) = \left(\frac{2r_o\hat{\theta}}{\chi^2_\alpha(2r_o)}, \infty \right) \tag{23}$$

where

$$T(\tau_{r_o}) = \sum_{i=1}^{r_o} \tau_i + (n - r_o)t^*$$

and $\hat{\theta} = T(\tau_{r_o})/r_o$.

In an analogous way, we can obtain approximate two-sided $100(1 - \alpha)$ percent confidence intervals for θ.

Remark: We wish to re-emphasize that in the last few pages we have given results which have not been and probably cannot be rigorously established. However, they can be used as good approximations to true results. Further discussion of this point is deferred to the Appendix.

Numerical Examples

Problem 1: 20 items are placed on life test for 100 hours. Two items fail before this time. Items which fail are not replaced.

(a) Make non-parametric 95% confidence statements (one and two-sided) about the probability of items surviving 100 hours.

(b) If the underlying distribution is exponential find one and two-sided 95% confidence intervals for θ, the mean life.

(c) If the underlying distribution is exponential, give one and two-sided 95% confidence intervals for surviving $\tau = 50$ hours.

Solution: (a) In this problem $n = 20$, $r = 2$, $\alpha = .05$, $t^* = 100$. Since $F_{.05}(6, 36) = 2.36$, it follows from (3') that a one-sided 95% confidence interval for the probability of surviving $t^* = 100$ hours is given by $(.718, 1)$. Since $F_{.025}(6, 36) = 2.79$ and $F_{.975}(4, 38) = 1/8.42$, it follows from (4) that a two-sided 95% confidence interval for the probability of surviving $t_o = 100$ hours is given by $(.683, .988)$;

(b) From (5) a one-sided 95% confidence interval for θ is given by $(302, \infty)$ and from (6) a two-sided 95% confidence interval is given by $(262, 805)$.

(c) From (12) a one-sided 95% confidence interval for the probability of surviving $\tau = 50$ hours is given by $(.847, 1)$ and from (13) a two-sided

95% confidence interval for surviving $\tau = 50$ hours is given by (.826, .994).

Problem 2: Out of 10,000 items tested, no items were observed to fail. Give a one-sided 95% confidence interval for the probability of survival.

Solution: In this case $n = 10{,}000$, $r = 0$. Since n is very large, $F_{.05}(2, 20000) \sim F_{.05}(2, \infty) = 3.00$, and so the one-sided 95% confidence interval is given by (.9997, 1). In other words, we have 95% confidence in the assertion that the true probability of survival is $\geq .9997$, if no items are observed to fail in a sample of 10,000. The answer can also be found very easily by using Table 5.

Problem 3: Out of 10,000 items tested, 10 items were observed to fail. Give one and two-sided 95% confidence intervals for the probability of survival.

Solution: In this case $n = 10{,}000$, $r = 10$. Since n is very large $F_{.05}(22, 19980) \sim F_{.05}(22, \infty) = 1.54$, and so the one-sided 95% confidence interval for the probability of survival is (.9983, 1). In other words, we have 95% confidence in the assertion that the true probability of survival is $\geq .9983$, if ten items are observed to fail in a sample of 10,000.

In the two-sided case $F_{.025}(22, 19980) = 1.67$ and $F_{.975}(20, 19982) = 1/2.00 = .500$ and so the two-sided 95% confidence interval for the probability of survival is given by (.9982, .9995).

Problem 4: A sample of 20 tubes is placed on test. Experimentation is truncated at time $t^* = 500$. Items which fail are not replaced. In this particular sample 6 items fail before $t^* = 500$ hours. The total life of the 6 items which failed before $t^* = 500$ was 956 hours. Estimate the mean life θ and give one and two-sided 95% confidence statements for θ, the mean life

Solution: (1) Let us solve the problem ignoring the information that the total life of the 6 failed items = 956. Thus we use formula (5) with $t^* = 500$, $n = 20$, $r = 6$, $\alpha = .05$. This gives the one-sided 95% confidence interval

$$\left(\frac{500}{\log\left\{1 + \frac{7}{14}\, F_{.05}(14,\ 28)\right\}},\ \infty\right) = \left(\frac{500}{\log\left\{1 + \frac{1}{2}(2.06)\right\}},\ \infty\right)$$

$$= \left(\frac{500}{\log\ (2.03)},\ \infty\right) = \left(\frac{500}{.7080},\ \infty\right) = (706,\ \infty).$$

Similarly using formula (6) we get the two-sided 95% confidence interval

$$\left(\frac{500}{\log\left\{1 + \frac{7}{14}\, F_{.025}(14,\ 28)\right\}},\quad \frac{500}{\log\left\{1 + \frac{6}{15}\, F_{.975}(12,\ 30)\right\}}\right)$$

$$= \left(\frac{500}{\log\left\{1 + \frac{1}{2}(2.37)\right\}},\quad \frac{500}{\log\left\{1 + \frac{2}{5}\left(\frac{1}{2.96}\right)\right\}}\right)$$

$$= \left(\frac{500}{\log\ (2.185)},\ \frac{500}{\log\ (1.135)}\right) = \left(\frac{500}{.7816},\ \frac{500}{.1266}\right) = (640,\ 3950).$$

(2) If we use the fact that the total life of the 6 observed failures = 956 we can use (17) or (17') to find a one-sided 95% confidence interval. In this problem

$$T(t^*) = \sum_{i=1}^{6} \tau_i + 14t^* = 956 + 7000 = 7956.$$

Further $\tilde{\theta} = \frac{T(t^*)}{7} = 1137$. Using (17') and Table 2 we get the one-sided 95% confidence interval $\left((1137)(.591),\ \infty\right) = (672,\ \infty)$. Substituting in (18') and using Table 1 we get the two-sided 95% confidence interval $\left((1137)(.536),\ (1326)(2.725)\right) = (609,\ 3613)$. The confidence intervals obtained by the two methods, one of which ignores the actual failure times are surprisingly close considering the smallness of the sample and the fact that we are dealing with a specific experiment.

Problem 5: Given the data of Problem 5, find one and two-sided 95% confidence intervals for $x_{.9}$, the time which is such that 90 percent of the items in the population live longer than $x_{.9}$.

Solution: We multiply the numbers obtained as 95% confidence limits for θ by $\log \frac{1}{p}$. Thus the one-sided 95% confidence interval for $x_{.9}$ is given by $(71, \infty)$ and the two-sided 95% confidence interval is given by $(64, 381)$.

Remark: We can interpret the one-sided 95% confidence interval for $x_{.9}$ as a one-sided tolerance statement, namely on the basis of the data we can be 95% confident in making the assertion that at least 90% of the population survives $x_{.9} = 71$ hours.

Problem 6: A certain company guarantees a television tube for the first month of use. Out of 1000 tubes sold, 50 are returned under this guarantee.

(i) Make a non-parametric one and two-sided confidence statement about the proportion of tubes lasting at least one month.

(ii) Assuming the exponential distribution to be valid, estimate the mean life θ, and give one and two-sided 95% confidence intervals for θ.

(iii) Assuming the exponential distribution to be valid, estimate $x_{.5}$, the time when we may expect 50% of the tubes to have failed. Place one and two-sided 95% confidence intervals on $x_{.5}$.

Solution: Clearly this problem can be considered as a truncated without replacement situation with $n = 1000$, $t^* = 1$, and $r = 50$. The problem can also be considered as consisting of 1000 non-replacement truncated life tests, where each life test consists of testing $n = 1$ item for at most $t^* = 1$ hour. The customer carries out this life test and in a sense accepts (keeps) the tube if it survives for one month and rejects the tube (is given a new tube) if the failure occurs before one month. We will assume that accurate records have not been kept and that we must base our estimate on the number of failures reported. From (i) the maximum likelihood estimate of the proportion of tubes surviving $t^* = 1$ month is given by $\hat{p} = \frac{n - r}{n} = \frac{1000 - 50}{1000} = .950$. A one-sided 95% confidence interval for the probability of surviving $t^* = 1$ month is given by substituting in (3') with $r = 50$, $n = 1000$. This confidence interval is

$$\left(\frac{1}{1 + \frac{51}{950} F_{.05}(102, 1900)}, 1 \right) = \left(\frac{1}{1 + \frac{51}{950}(1.25)}, 1 \right) = (.937, 1).$$

A two-sided 95% confidence interval is given by substituting in (9). This gives

$$\left(\frac{1}{1 + \frac{51}{950} F_{.025}(102, 1900)}, \frac{1}{1 + \frac{50}{951} F_{.975}(100, 1902)} \right)$$

$$= \left(\frac{1}{1 + \frac{51}{950}(1.31)}, \frac{1}{1 + \frac{50}{951} \frac{1}{1.35}} \right) = (.934, .963).$$

This gives the solution to (i).

To solve (ii) we substitute in (5) and (6) respectively. This gives the one-sided 95% confidence interval for θ,

$$\left(\frac{1}{\log \frac{1}{.937}}, \infty \right) = (15.4, \infty)$$

and the two-sided 95% confidence interval for θ,

$$\left(\frac{1}{\log \frac{1}{.934}} , \frac{1}{\log \frac{1}{.963}} \right) = (14.6, 26.5).$$

The best estimate for θ is

$$\hat{\theta} = \frac{1}{\log \frac{1}{.95}} = 19.5 \text{ months.}$$

To solve (iii) we multiply the answers in (ii) by $\log 2$. Hence the maximum likelihood estimate for $x_{.5}$ is (19.5)(.693) = 13.5 months. One and two-sided 95% confidence intervals for $x_{.5}$ are given by $(10.7, \infty)$ and (10.1, 18.4) respectively. One can interpret $(10.7, \infty)$ as a one-sided tolerance interval in the following sense: Based on the data and assuming the exponential distribution we can assert with 95% confidence that at least 50% of the items survive 10.7 months.

It is interesting to raise the question: Suppose one knew the actual failure times of the 50 tubes which fail. How much would our estimates and confidence intervals change? A reasonable assumption is that the total life of the failed items is about 25 months. This amounts roughly to assuming that the 50 failures are uniformly distributed over one month. Thus $T(t^*) = 25 + 950 = 975$. As a good estimate of θ with very little bias we take $\hat{\theta} = \frac{T(t^*)}{k+1} = \frac{975}{51} = 19.1$ months. One and two-sided 95% confidence intervals for θ are given by substituting in (17) and (18). Thus the one-sided 95% confidence interval for θ is given by

$$\left(\frac{2T(t^*)}{\chi^2_{.05}(102)} , \infty \right) = \left(\frac{1950}{126.5} , \infty \right) = (15.4, \infty)$$

and the two-sided 95% confidence interval for θ is given by

$$\left(\frac{2T(t^*)}{\chi^2_{.025}(102)}, \frac{2T(t^*)}{\chi^2_{.975}(100)}\right) = \left(\frac{1950}{131.8}, \frac{1950}{74.22}\right) = (14.8, 26.3).$$

The best estimate for $x_{.5}$ is $(19.5)(.693) = 13.5$ months. One and two-sided 95% confidence intervals for $x_{.5}$ are $(10.7, \infty)$ and $(10.3, 18.2)$ respectively. It would appear that little is gained from actual knowledge of the failure times. More will be said about this later.

<u>Problem 7</u>: 20 items are tested one at a time. If the item fails before 1000 time units have elapsed, the experiment is stopped. If the item is still living after 1000 time units have elapsed, the experiment is also stopped. 5 items are observed to fail with failure times 100, 400, 600, 800, 900 and 15 items are still living at 1000 hours. Give 95% one-sided confidence intervals for $Pr(T > t^* = 1000)$, the probability of surviving $t^* = 1000$ time units.

<u>Solution 1</u>: In the notation of this section, $n = 20$, $t^* = 1000$, $\alpha = .05$. A non-parametric solution is given by substituting in formula (3). Thus we are 95% confident of the validity of the assertion that the probability of surviving $t^* = 1000$ time units is

$$\geq \frac{1}{1 + \frac{6}{15} F_{.05}(12, 30)} = \frac{1}{1 + \frac{2}{5}(2.09)} = \frac{5}{9.18} = .544.$$

Put in reliability language, we are 95% confident of the correctness of the assertion that the reliability is $\geq .544$ over the time interval $t^* = 1000$ units.

<u>Solution 2</u>: Another solution is obtained by assuming that the underlying distribution is exponential and applying (20). We first calculate

$$T(t^*) = \sum_{i=1}^{5} \tau_i + 15t^* = 100 + 400 + 600 + 800 + 900 + 15(1000) = 17800.$$

Substituting in formula (20) we can be 95% confident of the correctness of the assertion that the probability of surviving time $t^* = 1000$ is

$$\geq e^{-t^* \chi^2_{.05}(12)/2T(t^* = 1000)} = e^{-1000\,(21.026)/35600} = .554.$$

Put in reliability language, we are 95% confident of the correctness of the assertion that the reliability is $\geq .554$ over the time interval $t^* = 1000$ time units.

It should be noted how close the two results (non-parametric and exponential) are. Because of its validity under much more general conditions, one would normally prefer the non-parametric solution 1.

SECTION 4

ESTIMATES OF BOUNDED RELATIVE ERROR FOR θ

Problem: To give an estimation procedure for the mean life θ having a small relative error. Put more precisely, give a procedure which will yield an estimate which is, with some preassigned confidence $1 - \alpha$, within a certain percentage (100 δ percent) of the true, but unknown mean life θ. In practice, α and δ will usually be small.

Approximate Solution: In the exponential case, the answer involves finding r, the number of failures, such that

$$(1) \qquad \Pr\left(\left|\frac{\hat{\theta}_r - \theta}{\theta}\right| \leq \delta\right) \geq 1 - \alpha .$$

Such a requirement will in general make it necessary that r be large. Let T_r be the total life associated with observing r failures and let $\hat{\theta}_r = T_r/r$. Then it can be assumed safely that $\sqrt{r}(\hat{\theta}_r - \theta)/\theta$ is approximately distributed as $N(0, 1)$. Thus to meet the conditions imposed by equation (1), means that r must be chosen in such a way that

$$(2) \qquad r \geq c_\alpha^2/\delta^2$$

where

$$c_\alpha = 2.576 \quad \text{if} \quad \alpha = .01$$
$$= 1.960 \quad \alpha = .05$$
$$= 1.645 \quad \alpha = .10 .$$

If δ = .01, .05, .10 and α = .01, .05, .10 the values of r required are tabulated in Table 6.

TABLE 6

Values of r

δ \ α	.01	.05	.10
.01	66,400	38,400	27,100
.05	2654	1537	1082
.10	664	384	271

Remark: The exact solution to this problem involves considerations analogous to those in the paper "Estimates of Bounded Relative Error in Particle Counting" by M.A. Girshick, H. Rubin, and R. Sitgreaves in the ANNALS OF MATHEMATICAL STATISTICS 26, 276-285, 1955. The values of r obtained in the range $0 < \alpha \leq .10$, $0 < \delta \leq .10$ are almost identical with those tabulated above. Further the "best" estimator of θ in a minimax sense for fixed r corresponding to the loss function,

$$L(\theta, a) = 0 \quad \text{if } 1 - \delta \leq \frac{a}{\theta} \leq 1 + \delta \tag{3}$$

$$= 1, \quad \text{otherwise}$$

is given by the estimator

$$a = \frac{2\delta T_r}{r \log \frac{1+\delta}{1-\delta}} . \tag{4}$$

However, for $0 < \delta \leq .1$, $a \sim \hat{\theta}_r = \frac{T_r}{r}$, since

$$a = \frac{T_r}{r}\left(1 - \frac{\delta^2}{3} + o(\delta^2)\right).$$

Remark: One can show that our confidence in the validity of the assertion that $1 - \delta \leq \frac{a}{\theta} \leq 1 + \delta$ where $a = 2\delta T_r / r \log \frac{1+\delta}{1-\delta}$, is given for any preassigned r by

$$(5) \qquad \Pr(1 - \delta \leq \frac{a}{\theta} \leq 1 + \delta) = \int_{(1-\delta)\frac{r}{2\delta}\log(\frac{1+\delta}{1-\delta})}^{(1+\delta)\frac{r}{2\delta}\log(\frac{1+\delta}{1-\delta})} \frac{x^{r-1}e^{-x}}{(r-1)!}\,dx$$

$$= \sum_{k=r}^{\infty} p\left[k;\ (1+\delta)\frac{r}{2\delta}\log(\frac{1+\delta}{1-\delta})\right] - \sum_{k=r}^{\infty} p\left[k;\ (1-\delta)\frac{r}{2\delta}\log(\frac{1+\delta}{1-\delta})\right]$$

For example, choose $r = 10$ and $\delta = .10$, then it is readily verified that

$$(1 + \delta)\frac{r}{2\delta}\log(\frac{1+\delta}{1-\delta}) = (1.1)\frac{10}{.2}\log(\frac{11}{9}) \approx 11$$

and

$$(1 - \delta)\frac{r}{2\delta}\log(\frac{1+\delta}{1-\delta}) \approx 9.$$

Hence (5) becomes

$$\Pr(1 - \delta \leq \frac{a}{\theta} \leq 1 + \delta) = \sum_{k=10}^{\infty} p(k;\ 11) - \sum_{k=10}^{\infty} p(k;\ 9)$$

$$= .6595 - .4126 = .2469.$$

In other words, we can have approximately 25% confidence in our assertion that $a \sim T_{10}/10$ is within 10% of the true mean life θ.

Numerical Examples

1. How many tubes should be tested in order that there is a probability of at least .90 that the estimate is within 10% of the true mean life?

Solution: In the notation of (1) and (2), $\delta = .1$, $\alpha = .1$, and $c_\alpha = 1.645$. Therefore the number of tubes tested should be $\geq (100)(1.645)^2 = 271$. If the underlying distribution is exponential this means that we must observe at least 271 failures in order to get an estimate such that we can be 90% confident that the estimator is within 10% of the true but unknown mean life.

2. We have available information from a life test in which 5 failures occurred with associated total life $T_r = 1000$. Assuming an exponential distribution, find the minimax estimator $\underline{a}$ associated with the loss function

$$L(\theta,a) = 0 \quad \text{if} \quad .8 \leq \frac{a}{\theta} \leq 1.2$$
$$= 1, \text{ otherwise.}$$

Also compute the confidence that we will have in the correctness of the assertion that $.8 \leq \frac{a}{\theta} \leq 1.2$.

Solution: From (4), the minimax estimator of θ based on the 5 failures is given by a $a = 2(.2)(200)/\log(\frac{1.2}{.8}) = 80/\log(1.5) = 197$. To find the confidence in our assertion that $.8 \leq a/\theta \leq 1.2$, we use formula (5). This gives us

$$\text{Confidence} = \sum_{k=5}^{\infty} p(k;6.08) - \sum_{k=5}^{\infty} p(k;4.06) = .7255 - .3829 = .3426.$$

SECTION 5

THE TWO PARAMETER EXPONENTIAL DISTRIBUTION

It has been found in many problems of life testing that there are occasions when a two parameter exponential distribution is more appropriate for fitting life test data than is a one parameter distribution. By a two parameter exponential distribution we mean a density function $f(x; \theta, A)$ such that

$$(1) \qquad f(x; \theta, A) = \frac{1}{\theta} e^{-(x - A)/\theta}, \; x \geq A \geq 0, \; \theta > 0.$$

A can be thought of as a guarantee period within which no failures can occur or as a minimum life. If $A = 0$, equation (1) reduces to the one parameter exponential.

Problem: A sample of n items is drawn at random from a population whose p.d.f. is described by (1). The experiment is terminated as soon as the first r failure times $x_1 \leq x_2 \leq \cdots \leq x_r$ become available. Items which fail are not replaced. Give "best" estimates for the unknown parameters A and θ.

Solution: It can be shown that x_1, the time to observe the first failure, and $T(x_r - x_1)$, the total life observed in the interval (x_1, x_r) are mutually independent and jointly sufficient for estimating A and θ. Sufficiency means roughly that x_1 and $T(x_r - x_1)$ jointly contain all of the relevant information for estimating A and θ that can be obtained from the first r failure times, $x_1 \leq x_2 \leq \cdots \leq x_r$. Best estimates for A and θ in the sense that they are unbiased and minimum variance are given by

(2) $$\hat{A} = x_1 - \frac{\hat{\theta}}{n}$$

and

(3) $$\hat{\theta} = T(x_r - x_1)/r - 1,$$

where

(4) $$T(x_r - x_1) = (n - 1)(x_2 - x_1) + (n - 2)(x_3 - x_2) + \dots$$

$$+ (n - r + 1)(x_r - x_{r-1})$$

$$= -(n - 1)x_1 + x_2 + x_3 + \dots + x_{r-1} + (n - r + 1)x_r.$$

It is often convenient in (3) and (4) to use the fact that

$$T(x_r - x_1) = T(x_r) - T(x_1) = T(x_r) - nx_1,$$

where

$$T(x_r) = \sum_{i=1}^{r} x_i + (n - r)x_r$$

Confidence limits for θ are easy to obtain from the fact that $2(r - 1)\hat{\theta}/\theta = 2T(x_r - x_1)/\theta$ is distributed as $\chi^2(2r - 2)$. Thus for $r \geq 2$, one and two-sided $100(1 - \alpha)$ percent confidence intervals for θ are given respectively by

(5) $$\left(\frac{2(r - 1)\hat{\theta}}{\chi^2_{\alpha}(2r - 2)}, \infty\right) \quad \text{or} \quad \left(\frac{2T(x_r - x_1)}{\chi^2_{\alpha}(2r - 2)}, \infty\right)$$

and

(6) $$\left(\frac{2(r - 1)\hat{\theta}}{\chi^2_{\frac{\alpha}{2}}(2r - 2)}, \frac{2(r - 1)\hat{\theta}}{\chi^2_{1-\frac{\alpha}{2}}(2r - 2)}\right)$$

or

$$\left(\frac{2T(x_r - x_1)}{\chi^2_{\frac{\alpha}{2}}(2r - 2)}, \frac{2T(x_r - x_1)}{\chi^2_{1-\frac{\alpha}{2}}(2r - 2)}\right).$$

it follows that the desired 100 γ percent confidence interval for A is given by

$$(10) \qquad \left(x_1 - z_\gamma \frac{\hat{\theta}(r-1)}{n},\, x_1 \right) = \left(x_1 - z_\gamma \frac{T(x_r - x_1)}{n},\, x_1 \right).$$

Since $z_\gamma = w_\gamma / r - 1$ we have, of course, the same confidence interval as before. However, z_γ is computable for any r and any γ.

<u>Remark 2</u>:

$$Z = \frac{n(x_1 - A)}{(r-1)\hat{\theta}} = \frac{n(x_1 - A)}{T(x_r - x_1)}$$

can be interpreted as the ratio between the total life between time A and x_1, the time when the first failure occurs, and the total life between x_1 and x_r inclusive. Clearly one wants to reject the hypothesis that $A = 0$, if $nx_1/T(x_r - x_1)$ is too large. It should be noted that under the hypothesis that $A = 0$, $nx_1/[T(x_r - x_1)/r - 1] = \frac{nx_1}{\hat{\theta}} \sim F(2, 2r-2)$.

<u>Remark 3</u>: Either formula (7) or its equivalent, formula (10), can be used to test whether or not A differs significantly from zero. If $x_1 - w_\gamma \frac{\hat{\theta}}{n}$ or equivalently $x_1 - z_\gamma \frac{T(x_r - x_1)}{n}$ are > 0, then A is significantly greater than zero at the $(1 - \gamma)$ level.

Remark 4: The 100 γ percent confidence interval for A can be interpreted as a one-sided tolerance interval. More precisely we can make the statement that all items live longer than $x_1 - z_\gamma \hat{\theta} \frac{(r-1)}{n}$ (or $x_1 - z_\gamma \frac{T(x_r - x_1)}{n}$) with confidence γ. 100 γ percent of these assertions will be correct.

Numerical Examples

1. 20 items are placed on test. Testing is terminated after one has observed the first 10 failures. Suppose that the first failure occurs 520 hours after the experiment starts. The total life observed between the time when the first failure occurs and the time when the tenth failure occurs is 12000 item hours. Assuming that the underlying distribution is exponential do the following:

(i) Test whether $A > 0$ at the .05 level.

(ii) If $A > 0$, find the shortest 95% confidence interval for A and an unbiased estimate for A.

(iii) Find an unbiased estimate for θ and one and two-sided confidence intervals for θ.

Solution: (i) Suppose that $A = 0$, then $\dfrac{nx_1}{T(x_{10} - x_1)/9}$ is distributed as $F(2, 18)$. From the data

$$\frac{nx_1}{T(x_{10} - x_1)/9} = \frac{(20)(520)}{12000/9} = \frac{(520)9}{600} = 7.80\,.$$

But the upper 5% point for $F(2, 18)$ is 3.55. Hence A is significantly different from zero on the .05 level. As a matter of fact, since the upper .5% point for $F(2, 18)$ is 7.21 and the upper .1% point for $F(2, 18)$ is 10.39, A is significantly different from zero at between the .001 and .005 levels.

(ii) From the data $\hat{\theta} = T(x_{10} - x_1)/9 = 12000/9 = 1333$. Hence an unbiased estimate for A is given by $x_1 - \hat{\theta}/n = 520 - 1333/20 = 520 - 67.7 = 452.3$.

The shortest 95% confidence interval can be computed from (7). Since $w_{.05} = 3.55$ in this case, the interval is $\left(520 - \frac{(3.55)(12000)}{9(20)}, 520\right)$ $= (283, 520)$.

(iii) In (ii) we saw that the best estimate for θ is given by $\hat{\theta} = 1333$. From (5) and (6), best one and two-sided 95% confidence intervals for θ are given by

$$\left(\frac{24000}{\chi^2_{.05}(18)}, \infty\right) = \left(\frac{24000}{28.87}, \infty\right) = (831, \infty)$$

and

$$\left(\frac{24000}{\chi^2_{.025}(18)}, \frac{24000}{\chi^2_{.975}(18)}\right) = \left(\frac{24000}{31.53}, \frac{24000}{7.906}\right) = (761, 3036)$$

respectively.

Remark: The tolerance interval in (2) can also be interpreted as follows: we are 95% confident of the assertion that all items survive 283 hours.

References

1. B. Epstein and M. Sobel, "Some Theorems Relevant to Life Testing from an Exponential Distribution," Annals of Mathematical Statistics 25, 373-381, 1954.

2. B. Epstein, "Simple Estimators of the Parameters of Exponential Distributions when Samples are Censored," Annals of the Institute of Statistical Mathematics 8, 15-25, 1956.

Appendix 3A

The material in Section 1 of Chapter 3 dealing with point and interval estimates for the mean life θ is given in detail in:

> B. Epstein and M. Sobel, "Some tests based on the first r ordered observations drawn from an exponential distribution," Stanford University Technical Report No. 6, Wayne University Technical Report No. 1, March 1952

and

> B. Epstein and M. Sobel, "Life Testing," Journal of the American Statistical Association 48, 486-502, 1953.

Appendix 3B

Proof of the Theorems in Chapter 3, Section 1

In order to show that $\hat{\theta}_{r,n}$ as given by (7) is the maximum likelihood estimate we write down the p.d.f. of the first r out of n ordered observations $x_{1,n}, x_{2,n}, \ldots, x_{r,n}$. This is given by:

$$f(x_{1,n}, x_{2,n}, \ldots, x_{r,n}) = \frac{n!}{(n-r)!} \frac{1}{\theta^r} e^{-\left[\sum_{i=1}^{r} x_{i,n} + (n-r)x_{r,n}\right]/\theta}$$

$$= \frac{n!}{(r-r)!} \frac{1}{\theta^r} e^{-T_r/\theta}$$

$$0 \leq x_{1,n} \leq x_{2,n} \leq \cdots \leq x_{r,n} < \infty$$

in the non-replacement case and

$$f(x_{1,n}, x_{2,n}, \ldots, x_{r,n}) = \left(\frac{n}{\theta}\right)^r e^{-nx_r/\theta} = \left(\frac{n}{\theta}\right)^r e^{-T_r/\theta}$$

in the replacement case. It is very easy to verify that the maximum of f occurs at $\hat{\theta} = T_r/r$ and this proves (1).

The sufficiency of the estimate can be verified at once by using a result in Cramér [p. 488] since $f(x_{1,n}, x_{2,n}, \dots, x_{r,n}; \theta)$ can be factored as

$$f(x_{1,n}, x_{2,n}, \dots, x_{r,n}; \theta) = g(\hat{\theta}_{r,n}, \theta)\, h(x_{1,n}, x_{2,n}, \dots, x_{r,n})$$

where

$$h(x_{1,n}, x_{2,n}, \dots, x_{r,n}) = 1 \quad \text{if} \quad 0 \leq x_{1,n} \leq \dots \leq x_{r,n} < \infty$$

$$= 0 \quad \text{otherwise.}$$

We next show that the p.d.f. of $\hat{\theta}_{r,n}$ is given by (6). To do this we introduce r new random variables defined as

$$y_1 = nx_1 \quad \text{and} \quad y_i = (n - i + 1)(x_i - x_{i-1}),\ 2 \leq i \leq r$$

in the non-replacement case and

$$y_1 = nx_1 \quad \text{and} \quad y_i = n(x_i - x_{i-1}),\ \ 2 \leq i \leq r$$

in the replacement case. We can now state the following lemma.

<u>Lemma</u>: The random variables $y_{i,n}$ defined above are mutually independent with common p.d.f. $\frac{1}{\theta} e^{-x/\theta}$, $x > 0$.

<u>Proof</u>: In both the replacement and non-replacement case the joint p.d.f. $f(x_{1,n}, x_{2,n}, \dots, x_{r,n})$ becomes

$$g(y_{1,n}, y_{2,n}, \dots, y_{r,n}) = e^{-\sum_{i=1}^{r} y_i/\theta} = \prod_{i=1}^{r} e^{-y_i/\theta},\ 0 \leq y_i < \infty,$$
$$i = 1, 2, \dots, r$$

and clearly the lemma is established.

Rewriting $\hat{\theta}_{r,n}$ in terms of the $y_{i,n}$, (1) becomes

$$\hat{\theta}_{r,n} = \sum_{i=1}^{r} y_i/r.$$

Since the characteristic function of the p.d.f. $\frac{1}{\theta} e^{-x/\theta}$, $x > 0$ is given by $\phi_x(t) = (1 - it\theta)^{-1}$ it follows at once from the independence proved in the lemma that

$$\phi_{\hat{\theta}}(t) = \prod_{i=1}^{r} \phi_{y_i}(t/r) = (1 - \frac{it\theta}{r})^{-r}.$$

From the uniqueness theorem for characteristic functions one gets by inversion that the p.d.f. of $\hat{\theta}_{r,n}$ is given by (6),

(6) $$f_r(y) = \frac{1}{(r-1)!} (\frac{r}{\theta})^r y^{r-1} e^{-ry/\theta}, \quad y > 0$$

$$= 0, \text{ elsewhere}.$$

To complete the proof of the theorem in Section 1 we show that $\hat{\theta}_{r,n}$ is unbiased, efficient, and minimum variance.

Unbiasedness is immediate, since

$$E(\hat{\theta}) = E(\sum_{i=1}^{r} y_i/r) = r\theta/r = \theta.$$

For efficiency and minimum variance let us compute the Cramér-Rao lower bound

$$\frac{1}{E(\frac{\partial \log f}{\partial \theta})^2}, \quad \text{where} \quad f = \frac{C}{\theta^r} e^{-T_r/\theta}$$

with $C = \frac{n!}{(n-r)!}$ in the non-replacement case and $C = n^r$ in the replacement case. Thus $\log f = \log C - r \log \theta - T_r/\theta$.

$$\frac{\partial}{\partial\theta}\log f = -r/\theta + T_r/\theta^2.$$

Thus

$$E\left(\frac{\partial \log f}{\partial\theta}\right)^2 = E\left(-\frac{r}{\theta}+\frac{T_r}{\theta^2}\right)^2$$

$$= E\left[\frac{r^2}{\theta^2} - \frac{2rT_r}{\theta^3} + \frac{T_r^2}{\theta^4}\right]$$

$$= \frac{r^2}{\theta^2}\left[1 - 2 + \frac{1}{\theta^2}(\theta^2/r + \theta^2)\right] = r/\theta^2.$$

Hence the Cramér-Rao lower bound is θ^2/r.

But Var $\hat{\theta}_{r,n} = \frac{\text{Var } y}{r} = \theta^2/r$ and since the assumptions needed for the derivation of Cramér-Rao lower bound are clearly met in the present problem, $\hat{\theta}_{r,n}$ is minimum variance and efficient since any other estimate has variance at least equal to θ^2/r. Thus the theorem in Section 1 is completely established.

Remark: It is of interest to note that while $\hat{\theta}$ is "best" among all unbiased estimators, it is not "best" or "admissible" if one uses other criteria. Using the language of decision theory, let us consider the loss function

$$L(\theta, a) = \frac{(\theta - a)^2}{\theta^2},$$

where θ is the true but unknown value we are trying to estimate and where a is our estimate of θ based on knowing the first r failure times. We would like to choose the estimate a in such a way that $E_\theta\left[L(\theta, a)\right]$ is made as small as possible in the minimax sense. It can be verified readily from results in Chapter 11 of Blackwell and Girshick's book, "Theory of Games and Statistical Decisions," that the best choice for a is given by

$$a = \tilde{\theta} = \frac{r\hat{\theta}}{r+1} = \frac{T(x_r)}{r+1} .$$

If the estimate $a = \tilde{\theta}$ is used then

$$E_\theta L(\theta, \tilde{\theta}) \underset{\theta}{\equiv} \frac{1}{r+1} ,$$

whereas if $\hat{\theta}$ is used as the estimate of θ, then

$$E_\theta L(\theta, \hat{\theta}) \underset{\theta}{\equiv} \frac{1}{r} .$$

Hence

$$E_\theta L(\theta, \tilde{\theta}) / E_\theta L(\theta, \hat{\theta}) \underset{\theta}{\equiv} \frac{r}{r+1}$$

and one always gets a smaller expected loss by using the estimate $\tilde{\theta}$ rather than $\hat{\theta}$. Stated in the language of decision theory, $\tilde{\theta}$ is an "admissible minimax" estimate for the above loss function, while $\hat{\theta}$ is not admissible. Here is a case where one does better using the "biased" estimate $\tilde{\theta}$ rather than the "unbiased" estimate $\hat{\theta}$.

Appendix 3C

We have seen that a $100(1 - \alpha)$ percent one sided confidence interval for the quantile x_p, where x_p is the solution to $Pr(X \geq x_p) = p$ (i.e., $x_p = \theta \log \frac{1}{p}$) is given by

$$\left(\frac{2r\hat{\theta}_{r,n} \log \frac{1}{p}}{\chi^2_\alpha(2r)} , \infty \right)$$

and that this implies the tolerance statement that we can be $100(1-\alpha)$ percent confident of the assertion that the fraction of items surviving

$$\tau = \frac{2r\,\hat{\theta}\,\log\frac{1}{p}}{\chi^2_{\alpha}(2r)} \text{ is } \geq p.$$

The proof of this assertion is now given. We can be $100(1-\alpha)$ percent confident of the assertion that $(\tau \leq x_p < \infty)$. But $\Pr(X \geq \tau) \geq \Pr(X \geq x_p) = p$. Combining the last two statements we can say that we are $100(1-\alpha)$ percent confident that the fraction of items surviving τ time units is $\geq p$. And this is what we wanted to prove.

Appendix 3D

It is interesting to compare the material in Section 1 and Section 2 of Chapter 3 in the replacement case. We assume that one starts the life test at time $t_o = 0$ with n items and replaces failed items at once by new items. In the situation treated in Section 1, the life test is continued until a prescribed number, r, of failures have occurred, and one stops testing at the random time $x_{r,n}$ (measured from the beginning of time). The total life observed up to and including $x_{r,n}$ is the random variable $T_r = nx_{r,n}$. In Section 2, the life test is terminated at a preassigned time t^* and the number of failures r that occur is a random variable. The total life observed is preassigned and given by $T^* = nt^*$. To sum up: in Section 1, the number of failures is fixed in advance and it is the waiting time (and hence total life) until the r'th failure which is random; in Section 2, the time (and hence total life) allotted to the life test is fixed in advance and it is the number of observed failures that is random.

Point estimation of θ in the case where the number of failures, r, is fixed in advance, is very simple. The estimator $\hat{\theta}_{r,n} = nx_{r,n}/r = T_r/r$ is among other things a maximum likelihood and unbiased estimator of θ. However in the case where t^* is fixed as in Section 2, $nt^*/r = T^*/r$ is a maximum likelihood estimator of θ, but it is biased (and in fact meaningless when $r = 0$). As a matter of fact it can be proved that no unbiased estimator of θ exists in this case. If we know apriori that $nt^* = T^* >> \theta$, then an almost unbiased estimator for θ is given by

$$\theta^* = \frac{nt^*}{r+1} = \frac{T^*}{r+1}.$$

This arises from the fact that

$$E(\theta^*) = \theta[1 - e^{-\frac{T^*}{\theta}}],$$

since

$$E(\theta^*) = \sum_{r=0}^{\infty} \frac{T^*}{r+1} \quad \left(\frac{T^*}{\theta}\right)^r e^{-T^*/\theta} \frac{1}{r!}$$

$$= e^{-T^*/\theta} \, \theta \sum_{r=0}^{\infty} \left(\frac{T^*}{\theta}\right)^{r+1}/(r+1)!$$

$$= \theta \, e^{-T^*/\theta} \left[e^{T^*/\theta} - 1\right]$$

$$= \theta\left[1 - e^{-T^*/\theta}\right].$$

In any case, one can find a point estimate of θ by solving the equation $\theta^* = \theta\left[1 - e^{-T^*/\theta}\right]$ numerically.

When one is dealing with confidence interval estimation, the situations in Sections 1 and 2 compare as follows:

	Fixed r, random T_r	Fixed T^*, random r
$100(1-\alpha)$ percent confidence interval, One sided	$\left(\frac{2T_r}{\chi^2_\alpha(2r)}, \infty\right)$	$\left(\frac{2T^*}{\chi^2_\alpha(2r+2)}, \infty\right)$
Two sided	$\left(\frac{2T_r}{\chi^2_{\frac{\alpha}{2}}(2r)}, \frac{2T_r}{\chi^2_{1-\frac{\alpha}{2}}(2r)}\right)$	$\left(\frac{2T^*}{\chi^2_{\frac{\alpha}{2}}(2r+2)}, \frac{2T^*}{\chi^2_{1-\frac{\alpha}{2}}(2r)}\right)$

It is very interesting to see that there is a striking similarity even though the two situations are radically different. It is curious that only the degrees of freedom for χ^2 need to be changed as indicated above when one goes from the situation in Section 1 to the situation in Section 2.

<u>Remark</u>: It is interesting to note that in the case where T^* is fixed, Cox [1] has given

$$\left(\frac{2T^*}{\chi^2_{\frac{\alpha}{2}}(2r+1)}, \frac{2T^*}{\chi^2_{1-\frac{\alpha}{2}}(2r+1)}\right)$$

as an approximate two sided $100(1-\alpha)$ percent confidence interval.

Appendix 3E

We should like to verify that equations (16) and (17) in Chapter 3, Section 2, Remark 7 generate $100(1-\alpha)$ percent one sided confidence intervals when data arise from a truncated replacement procedure $\min(\tau_{r_o,n}; t^*)$. Let us first consider the case where $r_o = 1$. In this case if no failures occur by time t^*, we stop life testing and according to (16) give $\left(\frac{2nt^*}{\chi^2_\alpha(2)}, \infty\right)$ as the one-sided $100(1-\alpha)$ percent confidence interval. If, however, a failure occurs at time $\tau_1 < t^*$, we stop the life test and according to (17) give $\left(\frac{2n\tau_1}{\chi^2_\alpha(2)}, \infty\right)$ as the one-sided $100(1-\alpha)$ percent confidence interval. We wish to verify that this is true. This means that we wish to prove that our assertion that θ is contained in the system of confidence intervals (16) and (17) is correct with probability $\geq 1-\alpha$ no matter what θ is. This is particularly easy to do for the case $r_o = 1$. In this case one can summarize the results in the following table.

$r_o = 1$

Value of θ	Probability that confidence statements based on (16) and (17) are correct
$\theta > \frac{2nt^*}{\chi^2_\alpha(2)}$	1
$\theta \leq \frac{2nt^*}{\chi^2_\alpha(2)}$	$1-\alpha$

If $\theta > \frac{2nt^*}{\chi^2_\alpha(2)}$, then no matter what happens our assertion is correct. If $\theta \leq \frac{2nt^*}{\chi^2_\alpha(2)}$, then our confidence interval will not include θ if the failure occurs after time $t = \frac{\theta\chi^2_\alpha(2)}{2n} \leq t^*$. But $\text{Prob}(\tau_1 > t \mid \theta) = e^{-\chi^2_\alpha(2)/2} = \alpha$. Hence the probability that our confidence interval does not include θ is equal to α, and the probability that our confidence statement is correct is equal to $1-\alpha$. If $r_o = 2$, (16) and (17) give the following: If no failures occur in $(0, t^*)$, stop the life test and give $\left(\frac{2nt^*}{\chi^2_\alpha(2)}, \infty\right)$ as the one sided $100(1-\alpha)$ percent confidence interval; if only one failure occurs in $(0, t^*)$ stop the life test and give $\left(\frac{2nt^*}{\chi^2_\alpha(4)}, \infty\right)$ as the one sided $100(1-\alpha)$ percent confidence interval; if 2 failures occur at time $\tau_2 < t^*$ then the appropriate $100(1-\alpha)$ percent confidence interval is given by $\left(\frac{2n\tau_2}{\chi_\alpha^2(4)}, \infty\right)$. Again we wish to prove that our system of confidence intervals is correct with probability $\geq 1-\alpha$ no matter what θ is. It can be verified that this is so and the results can best be summarized in the following table.

$r_o = 2$

Value of θ	Probability that confidence statements based on (16) and (17) are correct
$\theta > \frac{2nt^*}{\chi_\alpha^2(2)}$	1
$\theta = \frac{2nt^*}{\chi_\alpha^2(2)}$	$1-\alpha$
$\frac{2nt^*}{\chi_\alpha^2(4)} < \theta \leq \frac{2nt^*}{\chi_\alpha^2(2)}$	$1 - e^{-nt^*/\theta}$, where $e^{-\chi_\alpha^2(4)/2} < e^{-nt^*/\theta} \leq \alpha$
$\theta \leq \frac{2nt^*}{\chi_\alpha^2(4)}$	$1-\alpha$

If $r_o = 3$, one gets

Value of θ	Probability that confidence statements based on (16) and (17) are correct
$\theta > \frac{2nt^*}{\chi_\alpha^2(2)}$	1
$\theta = \frac{2nt^*}{\chi_\alpha^2(2)}$	$1-\alpha$

$\frac{2nt^*}{\chi^2_\alpha(4)} < \theta \le \frac{2nt^*}{\chi^2_\alpha(2)}$	$1 - e^{-nt^*/\theta}$, where $e^{-\frac{\chi^2_\alpha(4)}{2}} < e^{-nt^*/\theta} \le \alpha$
$\theta = \frac{2nt^*}{\chi^2_\alpha(4)}$	$1 - \alpha$
$\frac{2nt^*}{\chi^2_\alpha(6)} < \theta \le \frac{2nt^*}{\chi^2_\alpha(4)}$	$1 - e^{-nt^*/\theta} - (\frac{nt^*}{\theta}) e^{-nt^*/\theta}$, where $e^{-\frac{\chi^2_\alpha(6)}{2}} + \frac{\chi^2_\alpha(6)}{2} e^{-\frac{\chi^2_\alpha(6)}{2}} < e^{-nt^*/\theta} + \frac{nt^*}{\theta} e^{-nt^*/\theta} \le \alpha$
$\theta \le \frac{2nt^*}{\chi^2_\alpha(6)}$	$1 - \alpha$

For general r_o, one gets

Value of θ	Probability that confidence statements based on (16) and (17) are correct
$\theta > \frac{2nt^*}{\chi^2_\alpha(2)}$	1
$\theta = \frac{2nt^*}{\chi^2_\alpha(2)}$	$1 - \alpha$
$\frac{2nt^*}{\chi^2_\alpha(2k+2)} < \theta \le \frac{2nt^*}{\chi^2_\alpha(2k)}$	$\sum_{r=k}^{\infty} p(r; \frac{nt^*}{\theta})$, where $\sum_{r=0}^{k-1} p(r; \frac{\chi^2_\alpha(2k+2)}{2}) < \sum_{r=0}^{k-1} p(r; \frac{nt^*}{\theta}) \le \alpha$ and $1 \le k \le r_o - 1$
$\theta \le \frac{2nt^*}{\chi^2_\alpha(2r_o)}$	$1 - \alpha$

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Appendix 3F

In Chapter III, Section 3, we dealt with life test situations in which n items are placed on test, where testing is discontinued after a fixed time t^* has elapsed, and where items which fail are not replaced. In the first part of this section we gave estimation procedures which depended only on r, the observed number of items failing in $[0, t^*]$, and not on $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_r \leq t^*$, the actual failure times. More precisely, we gave non-parametric one and two sided confidence intervals for the probability of surviving for a length of time t^*, and in the special case where the underlying distribution is exponential we were able to translate these intervals into confidence statements about the mean life θ.

Since the sufficient statistic for estimating θ in this problem is given by the pair $(r, T(t^*)) = \sum_{i=1}^{r} \tau_i + (n - r)t^*)$ we know that we can make better estimates and better confidence statements about θ, if we use not only r but also $T(t^*)$. To carry this out in practice, however, is not easy since the c.d.f. of $T(t^*)$ is expressible only in a series of many terms. The c.d.f. is given in S. Takada and S. Shimada, "Statistical Analysis of Life of Vacuum Tubes," Hitachi Review, pp. 143-154, July, 1954. See particularly page 153. The c.d.f. is given by equation 2.6.16 in the paper. Thus one is virtually forced to use approximate confidence intervals and to use a certain amount of heuristic reasoning.

One approach to the problem of finding approximate confidence intervals is given in the paper by Takada and Shimada to which we have just referred. Essentially, their idea is as follows. If we place n items on test and truncate life testing at time t^*, we can treat $T(t^*) = \sum_{i=1}^{r} \tau_i + (n - r)t^*$ as the sum $\sum_{i=1}^{n} X_i$, where the X_i are identically and independently distributed random variables, each possessing the c.d.f.

$$F(t) = 1 - e^{-t/\theta}, \quad t < t^*$$

$$= 1 \qquad\qquad , \quad t \geq t^*.$$

Takada and Shimada apply the central limit theorem to obtain an approximation to the c.d.f. of $T(t^*)$ by the normal distribution. From this approximation they obtain appropriate confidence limits for θ. They claim that this approximation is a very good one and give a table which states, for example, that if $n = 20$, and $t^*/\theta = .05$, then an error of 5% is made. They further state that if $n \geq 30$, and $t^*/\theta \geq .1$ or $n \geq 50$ and $t^*/\theta \geq .05$ then the error associated with the approximation is less than 1%.

We have given another approximation in equations (17) and (18) of Chapter 3. These formulae are certainly excellent approximations for n large and even for small n, they should be quite good. There are a number of reasons why we believe that this statement is correct. Among these are:

(i) If t^*/θ is small, then the number of failures will be small and the non-replacement case becomes virtually a replacement case. One can then act as if we were observing a Poisson process with rate $\lambda = \frac{1}{\theta}$ for a length of time $T(t^*)$;

$$\sigma_{\hat{\lambda}}^2 = \frac{1}{\mathcal{E}\left(-\frac{\partial^2 \log f}{\partial \lambda^2}\right)} \cdot$$

But it is readily verified that

$$\frac{\partial^2 \log f}{\partial \lambda^2} = -r/\lambda^2.$$

Hence

$$\sigma_{\hat{\lambda}}^2 = \frac{1}{\mathcal{E}(\frac{r}{\lambda^2})} = \frac{\lambda^2}{n(1 - e^{-\lambda t^*})}$$

As another way of estimating λ, we note that

$$\Pr(T \leq t^*) = 1 - e^{-\lambda t^*} = \mathcal{E}(\frac{r}{n}).$$

Hence the statistic r/n is an unbiased estimate of $1 - e^{-\lambda t^*}$ and thus an estimate of λ is given by

$$\tilde{\lambda} = \log\left(\frac{n}{n - r}\right)/t^*$$

As was the case with $\hat{\lambda}$, $\tilde{\lambda}$ is also biased for finite n. However as $n \longrightarrow \infty$ $\tilde{\lambda} \xrightarrow[P]{} \lambda$ also. Let us now compute the asymptotic variance of $\tilde{\lambda}$. It can be shown that as $n \longrightarrow \infty$,

$$\sigma_{\tilde{\lambda}}^2 = \frac{\text{Var } n}{\left[\mathcal{E}(n - r)\right]^2 (t^*)^2} = \frac{n e^{-\lambda t^*}(1 - e^{-\lambda t^*})}{n^2 e^{-2\lambda t^*}(t^{*2})}$$

$$= \frac{e^{\lambda t^*}(1 - e^{-\lambda t^*})}{n(t^*)^2}.$$

Let us now compute the ratio $\sigma_{\hat{\lambda}}^2/\sigma_{\tilde{\lambda}}^2$. It is easy to verify that

$$\sigma_{\hat{\lambda}}^2/\sigma_{\tilde{\lambda}}^2 = \frac{(\lambda t^*)^2 e^{-\lambda t^*}}{(1 - e^{-\lambda t})^2} .$$

Expanding the numerator, we let

$$(\lambda t^*)^2 e^{-\lambda t^*} = (\lambda t^*)^2 - (\lambda t^*)^3 + \frac{(\lambda t^*)^4}{2!} - \ldots$$

And expanding the denominator we get

$$(1 - e^{-\lambda t^*})^2 = (\lambda t^*)^2 \left[1 - \lambda t^* + (\lambda t^*)^2 \frac{7}{12} - \ldots\right]$$

Neglecting higher order terms, $\sigma_{\hat{\lambda}}^2/\sigma_{\tilde{\lambda}}^2$ becomes

$$\frac{1 - \lambda t^* + \frac{(\lambda t^*)^2}{2} - \ldots}{1 - \lambda t^* + (\lambda t^*)^2 \frac{7}{12} - \ldots}$$

It is interesting that the ratio is close to one, (i.e., $\tilde{\lambda}$ is almost as efficient as $\hat{\lambda}$) particularly if λt^* is $\leq \frac{1}{2}$. Indeed, if $\lambda t^* = \frac{1}{2}$, it is readily verified that $\sigma_{\hat{\lambda}}^2/\sigma_{\tilde{\lambda}}^2 \simeq \frac{30}{31}$. Although what we have just done is for point estimates, clearly similar results will hold for confidence intervals. Also, it is trivially noted that although we were discussing estimation of λ, the conclusions obviously apply to the parameter $\theta = 1/\lambda$ as well. The upshot of the preceding discussion is that, in case the underlying distribution is exponential, then the confidence intervals (5) and (6) given in section 3, which depend only on the number of failures r in $(0, t^*)$, are almost as short as those based on using both r and $T(t^*)$.

Appendix 3G

It is interesting to note that if life testing is terminated not after a preassigned time t^*, but after a preassigned total life T^*, then the problem becomes one of making appropriate estimates of λ or $\theta = 1/\lambda$ when observing a Poisson process having rate $\lambda = 1/\theta$ for a length of time T^*. Thus the considerations in Section 2 and Appendix 3E can be used, the only difference being that we replace nt^* in Section 2 by T^*. We now state a number of results without proof.

Suppose that life testing stops after a total life T^* has been observed. If the underlying distribution is exponential with mean life θ, then the number of observed failures, r, is a Poisson random variable distributed with the probability law

$$\Pr(r = k \mid \theta) = p(k; \tfrac{T^*}{\theta}) = e^{-\frac{T^*}{\theta}} (\tfrac{T^*}{\theta})^k / k!, \quad k = 0, 1, 2, \ldots$$

Using precisely the same arguments as in Section 2, it can be asserted that if r = number of items which fail in $(0, T^*)$, then a one-sided $100(1 - \alpha)$ percent confidence interval for θ is given by

$$\left(\frac{2\,T^*}{\chi^2_{\alpha}(2r + 2)}, \infty\right)$$

and a two-sided $100(1- \alpha)$ percent confidence interval for θ is given by

$$\left(\frac{2\,T^*}{\chi^2_{\frac{\alpha}{2}}(2r + 2)}, \frac{2\,T^*}{\chi^2_{1 - \frac{\alpha}{2}}(2r)}\right)$$

Note that for $r = 0$, only one-sided confidence limits make sense.

Another kind of situation is where data become available as the result of the following rule of action: Reject if r_o failures occur before total life T^* has been used up; accept if fewer than r_o failures occur by the time one has observed a total life of T^* (it is assumed that r_o and T^* are preassigned). In the event that one rejects, experimentation stops at $T(\tau_{r_o})$, the total life observed up to and including τ_{r_o}, the r_o'th failure time. In the event that one accepts the total life observed will be T^*.

Using precisely the same considerations as in Section 2 and in Appendix 3E, we can assert that if the number of observed failures in $(0, T^*)$ is $0 \leq k \leq r_o - 1$, then a one-sided $100(1 - \alpha)$ percent confidence interval is given by

$$\left(\frac{2\,T^*}{\chi^2_{\alpha}(2k+2)} \, , \, \infty \right)$$

When $r = r_o$, i.e., if $T(\tau_{r_o}) \leq T^*$ then the appropriate $100(1 - \alpha)$ percent confidence interval is given by

$$\left(\frac{2\,T(\tau_{r_o})}{\chi^2_{\alpha}(2r_o)} \, , \, \infty \right) .$$

Similar results can be conjectured for two sided $100(1 - \alpha)$ percent confidence intervals. The results are

$$\left(\frac{2\,T^*}{\chi^2_{\alpha}(2)} \, , \, \infty \right), \quad \text{if} \quad k = 0$$

$$\left(\frac{2\,T^*}{\chi^2_{\frac{\alpha}{2}}(2k+2)} \, , \, \frac{2\,T^*}{\chi^2_{1-\frac{\alpha}{2}}(2k)} \right), \quad \text{if} \quad k = 1, 2, \ldots, r_o - 1$$

$$\left(\frac{2\, T(\tau_{r_o})}{\chi^2_{\frac{\alpha}{2}}(2r_o)}, \frac{2\, T(\tau_{r_o})}{\chi^2_{1-\frac{\alpha}{2}}(2r_o)} \right) \quad \text{if} \quad T(\tau_{r_o}) \leq T^*.$$

TABLE 1

Values of $c_2(r,\alpha)$ and $c_1(r,\alpha)$ such that $[c_1(r,\alpha)\hat{\theta}_{r,n}\ ,\ c_2(r,\alpha)\hat{\theta}_{r,n}]$ are $100(1-\alpha)$ percent two-sided confidence intervals for the mean life θ based on the first r failures from an exponential distribution

$$c_1(r,\alpha) = \frac{2r}{\chi^2_{\frac{\alpha}{2}}(2r)} \quad \text{and} \quad c_2(r,\alpha) = \frac{2r}{\chi^2_{1-\frac{\alpha}{2}}(2r)}$$

	$\alpha = .01$		$\alpha = .05$		$\alpha = .10$		$\alpha = .20$		$\alpha = .50$	
r	$c_2(r,\alpha)$	$c_1(r,\alpha)$	$c_2(r,\alpha)$	$c_1(r,\alpha)$	$c_2(r,\alpha)$	$c_1(r,\alpha)$	$c_2(r,\alpha)$	$c_1(r,\alpha)$	$c_2(r,\alpha)$	$c_1(r,\alpha)$
1	200.	.189	39.216	.271	19.417	.334	9.479	.434	3.478	.721
2	19.324	.269	8.264	.359	5.626	.422	3.759	.514	2.086	.743
3	8.876	.323	4.850	.415	3.670	.476	2.722	.564	1.737	.765
4	5.952	.364	3.670	.456	2.927	.516	2.292	.599	1.578	.783
5	4.638	.397	3.080	.488	2.538	.546	2.055	.626	1.484	.797
6	3.904	.424	2.725	.514	2.296	.571	1.904	.647	1.422	.808
7	3.436	.447	2.487	.536	2.131	.591	1.797	.665	1.377	.818
8	3.112	.467	2.316	.555	2.010	.608	1.718	.680	1.343	.826
9	2.873	.484	2.187	.571	1.917	.624	1.657	.693	1.316	.833
10	2.690	.500	2.085	.585	1.843	.637	1.607	.704	1.294	.839
11	2.545	.514	2.003	.598	1.783	.649	1.567	.714	1.276	.845
12	2.428	.527	1.935	.610	1.733	.659	1.533	.723	1.261	.850
13	2.330	.538	1.878	.620	1.691	.669	1.504	.731	1.247	.854
14	2.247	.549	1.829	.630	1.654	.677	1.478	.738	1.236	.858
15	2.176	.559	1.787	.639	1.622	.685	1.456	.745	1.226	.862

TABLE 2

Values of $c_3(r,\alpha)$ such that $[c_3(r,\alpha)\hat{\theta}_{r,n},\infty]$ are $100(1-\alpha)$ percent one-sided confidence intervals for θ based on the first r failures from an exponential distribution. $c_3(r,\alpha)=2r/\chi^2_\alpha(2r)$.

$c_3(r,\alpha)$

r \ α	.01	.05	.10	.20	.25	.50
1	.217	.334	.434	.621	.721	1.443
2	.301	.422	.514	.668	.743	1.192
3	.357	.476	.564	.701	.765	1.122
4	.398	.516	.599	.725	.783	1.089
5	.431	.546	.626	.744	.797	1.070
6	.458	.571	.647	.759	.808	1.058
7	.480	.591	.665	.771	.818	1.050
8	.500	.608	.680	.782	.826	1.043
9	.517	.624	.693	.791	.833	1.038
10	.532	.637	.704	.799	.839	1.034
11	.546	.649	.714	.806	.845	1.031
12	.558	.659	.723	.812	.850	1.028
13	.570	.669	.731	.818	.854	1.026
14	.580	.677	.738	.823	.858	1.024
15	.589	.685	.745	.828	.862	1.023
16	.598	.693	.751	.832	.865	1.021

TABLE 2 (continued)

r \ α	.01	.05	.10	.20	.25	.50
17	.606	.700	.757	.836	.869	1.020
18	.614	.706	.763	.840	.872	1.019
19	.621	.712	.767	.843	.874	1.018
20	.628	.717	.772	.846	.877	1.017
25	.657	.741	.792	.860	.888	1.013
30	.679	.759	.806	.870	.896	1.011
40	.712	.785	.828	.885	.908	1.008
50	.736	.804	.844	.896	.916	1.007
75	.776	.835	.869	.913	.930	1.004
100	.802	.855	.885	.923	.939	1.003

Example: A life test is discontinued after $r = 10$ failures have occurred. The observed mean life is $\hat{\theta}_{10,n} = 1000$. Find one-sided 90% and 50% confidence limits on θ.

Solution: From the table it is clear that the desired one-sided 90% confidence interval is $[704,\infty]$ (i.e., we can be 90% confident of the correctness of our assertion that $\theta > 704$). Similarly the desired one-sided 50% confidence interval is $[1034,\infty]$ (i.e., we can be 50% confident of our assertion that $\theta > 1034$).

TABLE 4(a)

Values of $\dfrac{1}{1+\left(\frac{r+1}{n-r}\right)F_{\alpha}(2r+2,2n-2r)}$ for $\alpha = .01$

	0	1	2	3	4	5	6	7	8	9
1	.0100									
2	.1000	.0050								
3	.2158	.0588	.0033							
4	.3162	.1408	.0420	.0025						
5	.3981	.2220	.1056	.0327	.0020					
6	.4640	.2945	.1731	.0847	.0268	.0017				
7	.5181	.3567	.2362	.1422	.0708	.0226	.0014			
8	.5622	.4098	.2933	.1981	.1210	.0608	.0197	.0013		
9	.5996	.4561	.3435	.2500	.1709	.1052	.0534	.0173	.0011	
10	.6309	.4956	.3883	.2971	.2182	.1503	.0932	.0475	.0155	.0010
11	.6579	.5302	.4280	.3396	.2622	.1938	.1344	.0836	.0428	.0141
12	.6814	.5607	.4627	.3775	.3025	.2349	.1747	.1215	.0759	.0390
13	.7016	.5871	.4937	.4195	.3390	.2730	.2128	.1589	.1108	.0694
14	.7198	.6109	.5215	.4435	.3724	.3080	.2488	.1947	.1457	.1018
15	.7357	.6323	.5469	.4717	.4029	.3404	.2822	.2288	.1793	.1345
16	.7498	.6510	.5693	.4969	.4309	.3701	.3134	.2607	.2117	.1663
17	.7627	.6683	.5903	.5201	.4569	.3976	.3423	.2907	.2421	.1970
18	.7742	.6838	.6086	.5419	.4803	.4226	.3691	.3187	.2709	.2261
19	.7848	.6982	.6257	.5610	.5017	.4459	.3937	.3448	.2979	.2538
20	.7943	.7111	.6417	.5790	.5220	.4682	.4175	.3689	.3234	.2799
25	.8317	.7624	.7042	.6509	.6017	.5562	.5117	.4697	.4294	.3902
30	.8576	.7985	.7480	.7025	.6596	.6194	.5803	.5430	.5077	.4730
40	.8913	.8453	.8058	.7698	.7360	.7042	.6732	.6434	.6141	.5862
50	.9121	.8744	.8422	.8127	.7853	.7583	.7326	.7079	.6838	.6596
75	.9404	.9147	.8927	.8722	.8529	.8342	.8166	.7991	.7822	.7655
100	.9550	.9356	.9187	.9030	.8883	.8742	.8605	.8471	.8332	.8212
150	.9697	.9566	.9452	.9346	.9246	.9151	.9056	.8965	.8878	.8788
200	.9772	.9672	.9586	.9506	.9430	.9358	.9286	.9216	.9150	.9081
300	.9847	.9780	.9722	.9668	.9617	.9569	.9520	.9473	.9428	.9382
400	.9885	.9835	.9791	.9751	.9712	.9675	.9639	.9603	.9570	.9535
500	.9908	.9868	.9833	.9800	.9770	.9740	.9711	.9683	.9655	.9627

TABLE 4(a) - cont.

Values of $\dfrac{1}{1+(\frac{r+1}{n-r})\,F_\alpha(2r+2,2n-2r)}$ for $\alpha = .01$

	10	11	12	13	14	15	16	17	18	19	20
11	.0009										
12	.0128	.0008									
13	.0358	.0118	.0008								
14	.0640	.0331	.0109	.0007							
15	.0944	.0594	.0307	.0102	.0007						
16	.1249	.0878	.0554	.0287	.0085	.0006					
17	.1553	.1168	.0821	.0519	.0269	.0076	.0006				
18	.1842	.1453	.1096	.0772	.0488	.0253	.0069	.0006			
19	.2126	.1733	.1367	.1033	.0727	.0460	.0237	.0063	.0005		
20	.2387	.2000	.1634	.1292	.0976	.0688	.0436	.0223	.0056	.0005	
25	.3520	.3165	.2817	.2479	.2155	.1846	.1553	.1276	.1007	.0765	.0541
30	.4383	.4056	.3738	.3436	.3131	.2834	.2555	.2281	.2013	.1757	.1509
40	.5592	.5313	.5059	.4802	.4541	.4290	.4052	.3814	.3580	.3344	.3122
50	.6372	.6146	.5923	.5700	.5489	.5274	.5066	.4861	.4657	.4457	.4261
75	.7494	.7335	.7173	.7020	.6868	.6711	.6558	.6408	.6258	.6112	.5970
100	.8082	.7958	.7840	.7720	.7598	.7478	.7362	.7247	.7132	.7019	.6906
150	.8674	.8617	.8531	.8448	.8368	.8283	.8201	.8120	.8043	.7967	.7888
200	.9017	.8953	.8888	.8825	.8762	.8770	.8635	.8575	.8515	.8457	.8397
300	.9339	.9295	.9254	.9211	.9165	.9123	.9082	.9041	.9001	.8962	.8920
400	.9502	.9469	.9437	.9405	.9370	.9338	.9307	.9277	.9246	.9216	.9185
500	.9601	.9575	.9549	.9523	.9496	.9470	.9445	.9420	.9396	.9372	.9347

TABLE 4(b)

Values of $\dfrac{1}{1 + \left(\frac{r+1}{n-r}\right) F_{\alpha}(2r+2, 2n-2r)}$ for $\alpha = .05$

	0	1	2	3	4	5	6	7	8	9
1	.0500									
2	.2237	.0254								
3	.3686	.1353	.0170							
4	.4728	.2488	.0977	.0127						
5	.5495	.3425	.1894	.0765	.0102					
6	.6067	.4181	.2714	.1531	.0629	.0085				
7	.6518	.4792	.3411	.2252	.1288	.0546	.0073			
8	.6879	.5295	.4000	.2894	.1928	.1111	.0464	.0064		
9	.7171	.5706	.4502	.3448	.2513	.1689	.0977	.0411	.0057	
10	.7413	.6057	.4932	.3933	.3038	.2226	.1499	.0873	.0397	.0051
11	.7618	.6353	.5300	.4357	.3500	.2710	.1998	.1389	.0787	.0333
12	.7792	.6611	.5618	.4727	.3912	.3156	.2451	.1808	.1230	.0719
13	.7941	.6834	.5898	.5051	.4276	.3552	.2874	.2239	.1656	.1126
14	.8074	.7035	.6144	.5340	.4598	.3906	.3253	.2640	.2058	.1529
15	.8188	.7209	.6369	.5597	.4889	.4223	.3596	.3003	.2440	.1899
16	.8294	.7360	.6557	.5835	.5161	.4512	.3915	.3333	.2787	.2266
17	.8383	.7498	.6739	.6045	.5394	.4785	.4200	.3644	.3106	.2597
18	.8467	.7623	.6897	.6229	.5611	.5019	.4459	.3923	.3407	.2913
19	.8543	.7739	.7042	.6410	.5814	.5240	.4705	.4178	.3679	.3205
20	.8610	.7838	.7177	.6559	.5993	.5447	.4926	.4422	.3941	.3470
25	.8872	.8236	.7692	.7180	.6709	.6250	.5807	.5383	.4959	.4558
30	.9050	.8512	.8046	.7615	.7200	.6812	.6434	.6059	.5705	.5344
40	.9279	.8867	.8510	.8174	.7855	.7553	.7248	.6964	.6684	.6394
50	.9418	.9087	.8794	.8523	.8260	.8013	.7770	.7531	.7300	.7071
75	.9608	.9384	.9184	.8999	.8822	.8648	.8481	.8321	.8156	.8003
100	.9705	.9534	.9385	.9244	.9107	.8977	.8850	.8727	.8601	.8483
150	.9802	.9688	.9586	.9491	.9401	.9313	.9227	.9140	.9057	.8974
200	.9851	.9765	.9688	.9616	.9548	.9482	.9417	.9352	.9290	.9226
300	.9901	.9843	.9791	.9743	.9697	.9653	.9609	.9567	.9525	.9482
400	.9925	.9882	.9843	.9807	.9772	.9739	.9706	.9674	.9643	.9611
500	.9940	.9906	.9874	.9845	.9818	.9791	.9765	.9739	.9714	.9688

TABLE 4(b) - cont.

Values of $\dfrac{1}{1 + \left(\frac{r+1}{n-r}\right) F_\alpha(2r+2, 2n-2r)}$ for $\alpha = .05$

	10	11	12	13	14	15	16	17	18	19	20
11	.0046										
12	.0304	.0043									
13	.0658	.0281	.0039								
14	.1041	.0611	.0260	.0036							
15	.1418	.0965	.0568	.0242	.0034						
16	.1779	.1407	.0903	.0531	.0227	.0032					
17	.2119	.1662	.1239	.0846	.0499	.0213	.0030				
18	.2441	.1989	.1563	.1163	.0797	.0492	.0201	.0028			
19	.2738	.2296	.1876	.1473	.1099	.0753	.0444	.0190	.0027		
20	.3021	.2586	.2170	.1773	.1394	.1037	.0712	.0422	.0181	.0026	
25	.4166	.3792	.3413	.3053	.2703	.2354	.2021	.1707	.1396	.1099	.0822
30	.5011	.4666	.4334	.4028	.3695	.3390	.3085	.2790	.2497	.2209	.1931
40	.6132	.5859	.5595	.5339	.5075	.4822	.4572	.4329	.4083	.3846	.3615
50	.6840	.6627	.6409	.6182	.5976	.5763	.5556	.5349	.5143	.4943	.4745
75	.7842	.7694	.7542	.7383	.7237	.7086	.6938	.6792	.6647	.6503	.6360
100	.8361	.8245	.8127	.8014	.7898	.7784	.7672	.7558	.7443	.7330	.7224
150	.8893	.8817	.8737	.8659	.8576	.8499	.8424	.8349	.8272	.8196	.8121
200	.9166	.9106	.9045	.8987	.8925	.8866	.8809	.8751	.8694	.8638	.8580
300	.9442	.9400	.9359	.9319	.9280	.9239	.9199	.9160	.9121	.9083	.9043
400	.9580	.9548	.9517	.9487	.9458	.9427	.9397	.9367	.9337	.9309	.9280
500	.9664	.9638	.9613	.9589	.9566	.9541	.9516	.9492	.9469	.9446	.9423

TABLE 4(c)

Values of $\dfrac{1}{1+\left(\frac{r+1}{n-r}\right)F_\alpha(2r+2,2n-2r)}$ for $\alpha = .10$

	0	1	2	3	4	5	6	7	8	9
1	.1000									
2	.3165	.0513								
3	.4644	.1957	.0345							
4	.5626	.3205	.1426	.0260						
5	.6313	.4158	.2469	.1124	.0209					
6	.6810	.4892	.3331	.2011	.0926	.0174				
7	.7194	.5474	.4039	.2786	.1695	.0787	.0149			
8	.7498	.5942	.4619	.3444	.2395	.1450	.0686	.0131		
9	.7745	.6319	.5102	.4011	.3012	.2100	.1295	.0608	.0116	
10	.7943	.6627	.5502	.4487	.3540	.2671	.1873	.1159	.0546	.0105
11	.8112	.6897	.5848	.4890	.4000	.3171	.2405	.1695	.1047	.0495
12	.8253	.7124	.6146	.5245	.4408	.3611	.2881	.2189	.1541	.0955
13	.8376	.7326	.6403	.5556	.4762	.4007	.3303	.2639	.2002	.1418
14	.8485	.7497	.6623	.5826	.5076	.4355	.3686	.3043	.2427	.1852
15	.8576	.7642	.6831	.6073	.5366	.4673	.4026	.3413	.2820	.2256
16	.8658	.7780	.7000	.6286	.5607	.4959	.4344	.3744	.3176	.2632
17	.8731	.7897	.7163	.6481	.5830	.5208	.4623	.4058	.3497	.2974
18	.8798	.8004	.7303	.6661	.6034	.5435	.4878	.4331	.3804	.3285
19	.8858	.8101	.7430	.6814	.6224	.5645	.5106	.4587	.4071	.3584
20	.8913	.8190	.7547	.6956	.6390	.5841	.5319	.4815	.4324	.3846
25	.9121	.8531	.8004	.7516	.7054	.6596	.6160	.5734	.5313	.4911
30	.9262	.8765	.8322	.7906	.7508	.7114	.6743	.6381	.6021	.5673
40	.9441	.9060	.8723	.8404	.8097	.7794	.7510	.7232	.6953	.6685
50	.9549	.9245	.8972	.8713	.8465	.8215	.7984	.7757	.7530	.7311
75	.9697	.9491	.9305	.9129	.8960	.8790	.8629	.8473	.8317	.8168
100	.9772	.9617	.9477	.9344	.9216	.9086	.8967	.8848	.8728	.8613
150	.9847	.9743	.9649	.9559	.9473	.9386	.9305	.9225	.9143	.9065
200	.9885	.9807	.9735	.9669	.9604	.9538	.9477	.9416	.9355	.9295
300	.9924	.9871	.9823	.9779	.9735	.9691	.9650	.9610	.9568	.9528
400	.9943	.9903	.9867	.9834	.9801	.9768	.9737	.9706	.9675	.9645
500	.9954	.9922	.9893	.9867	.9840	.9814	.9789	.9764	.9739	.9715

TABLE 4(c) - cont.

Values of $\dfrac{1}{1 + \left(\frac{r+1}{n-r}\right) F_{\alpha}(2r+2, 2n-2r)}$ for $\alpha = .10$

	10	11	12	13	14	15	16	17	18	19	20
11	.0095										
12	.0452	.0087									
13	.0879	.0417	.0081								
14	.1311	.0814	.0386	.0075							
15	.1719	.1220	.0756	.0360	.0070						
16	.2102	.1605	.1136	.0709	.0337	.0066					
17	.2460	.1969	.1500	.1068	.0667	.0317	.0062				
18	.2789	.2312	.1852	.1413	.1008	.0628	.0299	.0058			
19	.3092	.2628	.2181	.1750	.1337	.0951	.0595	.0283	.0055		
20	.3381	.2923	.2486	.2024	.1660	.1264	.0903	.0564	.0268	.0053	
25	.4513	.4119	.3740	.3311	.3014	.2643	.2302	.1963	.1631	.1310	.1001
30	.5325	.4982	.4648	.4323	.4006	.3689	.3373	.3068	.2768	.2472	.2178
40	.6409	.6137	.5873	.5616	.5367	.5112	.4855	.4603	.4356	.4114	.3877
50	.7085	.6863	.6646	.6433	.6224	.6008	.5796	.5587	.5383	.5181	.4984
75	.8012	.7860	.7711	.7566	.7424	.7274	.7127	.6982	.6839	.6698	.6559
100	.8494	.8378	.8265	.8155	.8047	.7932	.7818	.7706	.7595	.7486	.7378
150	.8984	.8906	.8830	.8755	.8683	.8605	.8528	.8451	.8376	.8304	.8228
200	.9234	.9175	.9117	.9061	.9006	.8946	.8888	.8831	.8774	.8717	.8662
300	.9488	.9448	.9409	.9371	.9334	.9295	.9255	.9217	.9179	.9141	.9104
400	.9614	.9584	.9554	.9526	.9498	.9468	.9438	.9409	.9380	.9352	.9324
500	.9690	.9666	.9642	.9619	.9597	.9573	.9549	.9526	.9503	.9480	.9457

TABLE 4(d)

Values of $\dfrac{1}{1 + \left(\frac{r+1}{n-r}\right) F_{\alpha}(2r+2, 2n-2r)}$ for $\alpha = .25$

	0	1	2	3	4	5	6	7	8	9
1	.2500									
2	.5000	.1340								
3	.6303	.3268	.0915							
4	.7067	.4559	.2427	.0694						
5	.7576	.5464	.3597	.1938	.0559					
6	.7937	.6112	.4469	.2964	.1613	.0469				
7	.8206	.6593	.5133	.3788	.2532	.1382	.0403			
8	.8412	.6972	.5666	.4448	.3292	.2203	.1208	.0354		
9	.8571	.7273	.6087	.4983	.3922	.2915	.1949	.1073	.0315	
10	.8703	.7525	.6446	.5418	.4444	.3511	.2607	.1756	.0965	.0283
11	.8814	.7728	.6742	.5797	.4895	.4016	.3178	.2358	.1592	.0877
12	.8909	.7914	.6983	.6114	.5263	.4459	.3662	.2900	.2163	.1456
13	.8990	.8065	.7208	.6378	.5590	.4825	.4093	.3366	.2676	.1990
14	.9056	.8186	.7394	.6627	.5882	.5172	.4465	.3783	.3120	.2475
15	.9119	.8304	.7545	.6834	.6128	.5453	.4799	.4153	.3517	.2899
16	.9170	.8408	.7692	.7019	.6349	.5723	.5093	.4477	.3877	.3286
17	.9216	.8496	.7825	.7172	.6549	.5952	.5354	.4775	.4195	.3636
18	.9257	.8575	.7937	.7324	.6731	.6161	.5588	.5035	.4489	.3947
19	.9295	.8647	.8039	.7454	.6897	.6352	.5803	.5271	.4755	.4237
20	.9328	.8712	.8132	.7573	.7039	.6510	.6000	.5488	.4991	.4508
25	.9459	.8961	.8490	.8036	.7599	.7179	.6748	.6331	.5923	.5525
30	.9548	.9130	.8733	.8355	.7985	.7622	.7262	.6908	.6560	.6219
40	.9659	.9340	.9040	.8755	.8475	.8195	.7924	.7656	.7390	.7128
50	.9726	.9469	.9228	.8998	.8771	.8547	.8327	.8110	.7893	.7679
75	.9817	.9643	.9482	.9326	.9173	.9025	.8877	.8730	.8584	.8440
100	.9862	.9731	.9609	.9491	.9375	.9263	.9151	.9039	.8927	.8818
150	.9908	.9819	.9737	.9658	.9580	.9504	.9429	.9353	.9278	.9204
200	.9930	.9864	.9802	.9743	.9684	.9627	.9570	.9513	.9456	.9400
300	.9954	.9909	.9868	.9828	.9788	.9750	.9712	.9674	.9635	.9598
400	.9965	.9932	.9901	.9871	.9841	.9812	.9783	.9755	.9726	.9697
500	.9972	.9945	.9921	.9896	.9873	.9850	.9826	.9803	.9780	.9758

TABLE 4(d) - cont.

Values of $\dfrac{1}{1 + \left(\frac{r+1}{n-r}\right) F_{\alpha}(2r+2, 2n-2r)}$ for $\alpha = .25$

	10	11	12	13	14	15	16	17	18	19	20
11	.0258										
12	.0801	.0237									
13	.1342	.0742	.0219								
14	.1852	.1250	.0689	.0203							
15	.2302	.1724	.1165	.0643	.0190						
16	.2706	.2151	.1563	.1091	.0602	.0178					
17	.3095	.2551	.2019	.1515	.1026	.0567	.0168				
18	.3419	.2912	.2402	.1913	.1429	.0968	.0535	.0158			
19	.3731	.3241	.2792	.2281	.1817	.1351	.0916	.0507	.0150		
20	.4015	.3538	.3074	.2577	.2162	.1715	.1282	.0870	.0482	.0143	
25	.5129	.4730	.4335	.3949	.3571	.3181	.2802	.2449	.2072	.1714	.1362
30	.5888	.5557	.5218	.4885	.4557	.4232	.3904	.3582	.3266	.2957	.2622
40	.6866	.6605	.6350	.6097	.5845	.5589	.5336	.5085	.4837	.4591	.4348
50	.7465	.7253	.7046	.6842	.6641	.6433	.6227	.6022	.5819	.5618	.5418
75	.8295	.8151	.8011	.7873	.7736	.7594	.7453	.7313	.7168	.7028	.6889
100	.8707	.8597	.8491	.8385	.8281	.8172	.8064	.7957	.7850	.7744	.7638
150	.9129	.9054	.8982	.8910	.8840	.8766	.8692	.8619	.8546	.8473	.8401
200	.9343	.9287	.9232	.9178	.9124	.9068	.9012	.8957	.8902	.8847	.8792
300	.9559	.9522	.9485	.9448	.9413	.9375	.9337	.9300	.9262	.9225	.9188
400	.9669	.9640	.9612	.9585	.9558	.9530	.9501	.9473	.9445	.9417	.9389
500	.9734	.9712	.9689	.9667	.9646	.9623	.9600	.9577	.9555	.9532	.9510

TABLE 4(e) - cont.

Values of $\dfrac{1}{1 + \left(\frac{r+1}{n-r}\right) F_{\alpha}(2r+2, 2n-2r)}$ for $\alpha = .50$

	10	11	12	13	14	15	16	17	18	19	20
11	.0614										
12	.1288	.0562									
13	.2016	.1266	.0521								
14	.2572	.1866	.1171	.0482							
15	.3062	.2392	.1747	.1096	.0451						
16	.3484	.2860	.2317	.1630	.1031	.0424					
17	.3865	.3289	.2681	.2123	.1538	.0973	.0400				
18	.4186	.3638	.3094	.2538	.1995	.1456	.0921	.0379			
19	.4490	.3976	.3455	.2938	.2410	.1894	.1382	.0874	.0360		
20	.4757	.4266	.3772	.3268	.2797	.2277	.1803	.1316	.0832	.0342	
25	.5791	.5400	.5000	.4599	.4207	.3799	.3462	.2935	.2635	.2239	.1834
30	.6486	.6156	.5826	.5495	.5164	.4836	.4511	.4186	.3838	.3514	.3183
40	.7356	.7109	.6861	.6612	.6363	.6117	.5870	.5622	.5375	.5125	.4880
50	.7882	.7684	.7485	.7286	.7085	.6888	.6690	.6492	.6293	.6095	.5896
75	.8585	.8452	.8320	.8186	.8053	.7921	.7789	.7657	.7524	.7391	.7259
100	.8937	.8838	.8738	.8638	.8538	.8439	.8340	.8240	.8141	.8041	.7941
150	.9291	.9224	.9158	.9091	.9024	.8958	.8892	.8826	.8759	.8693	.8626
200	.9468	.9418	.9368	.9318	.9268	.9218	.9168	.9119	.9069	.9019	.8969
300	.9645	.9612	.9579	.9545	.9512	.9479	.9445	.9412	.9379	.9346	.9312
400	.9734	.9709	.9684	.9659	.9634	.9609	.9584	.9559	.9534	.9509	.9484
500	.9787	.9767	.9747	.9727	.9707	.9687	.9667	.9647	.9627	.9607	.9587

Numerical Example for Tables 4(a) through 4(e).

20 items are drawn at random from a lot and placed on life test. The test runs for 100 hours. Suppose that no failures occur, then we can be:

(a) 99% confident of the assertion that at least 79.4% of the items in the lot survive 100 hours,

(b) 95% confident of the assertion that at least 86.1% of the items in the lot survive 100 hours,

(c) 90% confident of the assertion that at least 89.1% of the items in the lot survive 100 hours,

(d) 75% confident of the assertion that at least 93.3% of the items in the lot survive 100 hours,

(e) 50% confident of the assertion that at least 96.6% of the items in the lot survive 100 hours.

These are non parametric statements.

TABLE 5(a)

Values of $\dfrac{1}{1 + \left(\frac{r+1}{n-r}\right) F_{\alpha}(2r+2, 2n-2r)}$ for $\alpha = .01$

r \ n	1000	5,000	10,000	50,000	100,000	500,000	1,000,000
0	.99542	.99917	.99954	.99992	.99995	.999992	.999995
1	.99338	.99867	.99933	.99987	.99993	.999987	.999993
2	.99162	.99832	99916	.99983	.99992	.999983	.999992
3	.98999	.99799	.99899	.99980	.99990	.999980	.999990
4	.98844	.99767	.99884	.99977	.99988	.999977	.999988
5	.98696	.99738	.99869	.99974	.99987	.999974	.999987
6	.98549	.99708	.99854	.99971	.99985	.999971	.999985
7	.98405	.99679	.99839	.99968	.99984	.999968	.999984
8	.98270	.99651	.99826	.99965	.99983	.999965	.999983
9	.98129	.99623	.99811	.99962	.99981	.999962	.999981
10	.97997	.99596	.99798	.99960	.99980	.999960	.999980
11	.97863	.99569	.99784	.99957	.99978	.999957	.999978
12	.97730	.99542	.99771	.99954	.99977	.999954	.999977
13	.97605	.99517	.99758	.99952	.99976	.999952	.999976
14	.97465	.99488	.99744	.99949	.99974	.999949	.999974
15	.97334	.99462	.99731	.99946	.99973	.999946	.999973
16	.97209	.99436	.99718	.99944	.99972	.999944	.999972
17	.97085	.99411	.99705	.99941	.99971	.999941	.999971
18	.96961	.99386	.99693	.99938	.99969	.999938	.999969
19	.96841	.99362	.99680	.99936	.99968	.999936	.999968
20	.96717	.99336	.99668	.99933	.99967	.999933	.999967
30	.95490	.99087	.99543	.99908	.99954	.999908	.999954
40	.94318	.98848	.99423	.99884	.99942	.999884	.999942
50	.93149	.98618	.99308	.99860	.99931	.999860	.999931
60	.91962	.98364	.99180	.99836	.99918	.999836	.999918
70	.90807	.98126	.99061	.99812	.99906	.999812	.999906
80	.89695	.97897	.98946	.99789	.99894	.999789	.999894
90	.88621	.97676	.98835	.99766	.99883	.999766	.999883
100	.87568	.97488	.98726	.99745	.99874	.999745	.999874
200	.76750	.95267	.97628	.99525	.99762	.999525	.999762
500	.46461	.89242	.94616	.98923	.99461	.998923	.999461

TABLE 5(b)

Values of $\dfrac{1}{1+\left(\frac{r+1}{n-r}\right)F_{\alpha}(2r+2,2n-2r)}$ for $\alpha = .05$

r \ n	1000	5000	10,000	50,000	100,000	500,000	100,000
0	.99701	.99940	.99970	.99994	.99997	.999994	.999997
1	.99527	.99905	.99953	.99991	.99995	.999991	.999995
2	.99371	.99874	.99937	.99987	.99994	.999987	.999994
3	.99226	.99845	.99922	.99984	.99992	.999984	.999992
4	.99087	.99817	.99908	.99982	.99991	.999982	.999991
5	.98953	.99790	.99895	.99979	.99989	.999979	.999989
6	.98820	.99763	.99881	.99976	.99988	.999976	.999988
7	.98692	.99737	.99868	.99974	.99987	.999974	.999987
8	.98565	.99711	.99856	.99971	.99986	.999971	.999986
9	.98436	.99685	.99843	.99969	.99984	.999969	.999984
10	.98312	.99661	.99830	.99966	.99983	.999966	.999983
11	.98183	.99635	.99817	.99963	.99982	.999963	.999982
12	.98058	.99609	.99804	.99961	.99980	.999961	.999980
13	.97937	.99585	.99792	.99958	.99979	.999958	.999979
14	.97820	.99561	.99780	.99956	.99978	.999956	.999978
15	.97697	.99536	.99768	.99954	.99977	.999954	.999977
16	.97578	.99512	.99756	.99951	.99976	.999951	.999976
17	.97459	.99488	.99744	.99949	.99974	.999949	.999974
18	.97341	.99464	.99732	.99946	.99973	.999946	.999973
19	.97225	.99441	.99720	.99944	.99972	.999944	.999972
20	.97108	.99417	.99708	.99942	.99971	.999942	.999971
30	.95951	.99183	.99591	.99918	.99959	.999918	.999951
40	.94828	.98955	.99477	.99895	.99948	.999895	.999948
50	.93711	.98738	.99369	.99874	.99936	.999874	.999936
60	.92563	.98498	.99248	.99849	.99925	.999849	.999925
70	.91476	.98273	.99135	.99827	.99913	.999827	.999913
80	.90397	.98052	.99024	.99805	.99902	.999805	.999902
90	.89337	.97852	.98925	99785	.99892	.999785	.999892
100	.88297	.97642	.98819	.99764	.99882	.999764	.999882
200	.77703	.95474	.97733	.99546	.99773	.999546	.999773
500	.47456	.86439	.94717	.98943	.99471	.998943	.999471

TABLE 5(c)

Values of $\dfrac{1}{1+\left(\frac{r+1}{n-r}\right) F_{\alpha}(2r+2,2n-2r)}$ for $\alpha = .10$

r \ n	1000	5000	10,000	50,000	100,000	500,000	1,000,000
0	.99771	.99954	.99977	.99995	.99998	.999995	.999997
1	.99612	.99922	.99961	.99992	.99996	.999992	.999996
2	.99469	.99894	.99947	.99989	.99995	.999989	.999995
3	.99333	.99866	.99933	.99987	.99993	.999987	.999993
4	.99202	.99840	.99920	.99984	.99992	.999984	.999992
5	.99069	.99813	.99907	.99981	.99991	.999981	.999991
6	.98945	.99788	.99894	.99978	.99989	.999979	.999989
7	.98822	.99764	.99882	.99976	.99988	.999976	.999988
8	.98676	.99734	.99867	.99973	.99987	.999973	.999987
9	.98582	.99716	.99858	.99971	.99986	.999972	.999986
10	.98459	.99691	.99846	.99969	.99985	.999969	.999985
11	.98329	.99666	.99834	.99968	.99983	.999968	.999983
12	.98205	.99642	.99823	.99966	.99982	.999966	.999982
13	.98087	.99617	.99811	.99965	.99981	.999965	.999981
14	.97970	.99593	.99799	.99964	.99980	.999964	.999980
15	.97851	.99570	.99787	.99962	.99979	.999962	.999979
16	.97736	.99547	.99776	.99961	.99978	.999961	.999978
17	.97621	.99523	.99764	.99960	.99977	.999960	.999977
18	.97509	.99501	.99752	.99958	.99975	.999958	.999975
19	.97401	.99479	.99740	.99957	.99974	.999957	.999974
20	.97296	.99458	.99729	.99956	[illegible]9973	.999946	.999973
30	.96154	.99231	.99615	.99923	[illegible]962	.999923	.999962
40	.95131	.99002	.99513	.99900	.99951	.999900	.999951
50	.94010	.98798	.99399	.99880	.99940	.999880	.999940
60	.92937	.98563	.99294	.99856	.99930	.999856	.999930
70	.91878	.98342	.99188	.99834	.99919	.999834	.999919
80	.90762	.98136	.99076	.99813	.99908	.999813	.999908
90	.89693	.97929	.98969	.99793	.99897	.999793	.999897
100	.88575	.97721	.98860	.99772	.99886	.999772	.999886
200	.78227	.95603	.97799	.99559	.99780	.999559	.999780
500	.48201	.89591	.94792	.98958	.99479	.998958	.999479

TABLE 5(d)

Values of $\dfrac{1}{1+\left(\frac{r+1}{n-r}\right)F_{\alpha}(2r+2,2n-2r)}$ for $\alpha = .25$

r \ n	1000	5000	10,000	50,000	100,000	500,000	1,000,000
0	.99861	.99972	.99986	.99997	.99998	.999997	.999998
1	.99730	.99946	.99973	.99995	.99997	.999995	.999997
2	.99607	.99921	.99961	.99992	.99996	.999992	.999996
3	.99489	.99898	.99949	.99990	.99995	.999990	.999995
4	.99376	.99875	.99938	.99988	.99994	.999988	.999994
5	.99257	.99851	.99926	.99985	.99993	.999985	.999993
6	.99143	.99828	.99914	.99983	.99991	.999983	.999991
7	.99030	.99806	.99903	.99981	.99990	.999981	.999990
8	.98920	.99784	.99892	.99978	.99989	.999978	.999989
9	.98812	.99762	.99881	.99976	.99988	.999976	.999988
10	.98698	.99739	.99870	.99974	.99987	.999974	.999987
11	.98584	.99716	.99858	.99972	.99986	.999972	.999986
12	.98472	.99693	.99847	.99969	.99985	.999969	.999985
13	.98362	.99671	.99836	.99967	.99984	.999967	.999984
14	.98254	.99648	.99825	.99965	.99983	.999965	.999983
15	.98146	.99626	.99815	.99963	.99982	.999963	.999982
16	.98030	.99605	.99803	.99961	.99980	.999961	.999980
17	.97926	.99584	.99793	.99958	.99979	.999958	.999979
18	.97822	.99563	.99782	.99956	.99978	.999956	.999978
19	.97718	.99542	.99772	.99954	.99977	.999954	.999977
20	.97614	.99522	.99761	.99952	.99976	.999952	.999976
30	.96459	.99308	.99646	.99931	.99965	.999931	.999965
40	.95439	.99096	.99544	.99910	.99954	.999910	.999954
50	.94448	.98887	.99443	.99889	.99944	.999889	.999944
60	.93445	.98684	.99344	.99868	.99934	.999868	.999934
70	.92443	.98481	.99243	.99847	.99924	.999847	.999924
80	.91441	.98282	.99143	.99828	.99914	.999828	.999914
90	.90440	.98083	.99044	.99808	.99904	.999808	.999904
100	.89439	.97885	.98942	.99788	.99894	.999788	.999894
200	.79437	.95885	.97942	.99588	.99794	.999588	.999794
500	.49434	.89884	.94942	.98988	.99494	.998988	.999494

TABLE 5(e)

Values of $\dfrac{1}{1+\left(\frac{r+1}{n-r}\right)F_\alpha(2r+2,2n-2r)}$ for $\alpha = .50$

r \ n	1000	5000	10,000	50,000	100,000	500,000	1,000,000
0	.99931	.99986	.99993	.99999	.99999	.999999	.999999
1	.99832	.99966	.99983	.99997	.99998	.999997	.999998
2	.99733	.99947	.99973	.99995	.99997	.999995	.999997
3	.99633	.99927	.99963	.99993	.99996	.999993	.999996
4	.99533	.99907	.99953	.99991	.99995	.999991	.999995
5	.99435	.99887	.99943	.99989	.99994	.999989	.999994
6	.99334	.99867	.99933	.99987	.99993	.999987	.999993
7	.99234	.99847	.99923	.99985	.99992	.999985	.999992
8	.99134	.99827	.99913	.99983	.99991	.999983	.999991
9	.99033	.99807	.99903	.99981	.99990	.999981	.999990
10	.98934	.99787	.99893	.99979	.99989	.999979	.999989
11	.98832	.99767	.99883	.99977	.99988	.999977	.999988
12	.98730	.99747	.99873	.99975	.99987	.999975	.999987
13	.98630	.99728	.99863	.99973	.99986	.999973	.999986
14	.98531	.99708	.99853	.99971	.99985	.999971	.999985
15	.98432	.99687	.99843	.99969	.99984	.999969	.999984
16	.98333	.99666	.99833	.99967	.99983	.999967	.999983
17	.98234	.99646	.99823	.99965	.99982	.999965	.999982
18	.98135	.99627	.99814	.99963	.99981	.999963	.999981
19	.98036	.99607	.99804	.99961	.99980	.999961	.999980
20	.97936	.99587	.99793	.99959	.99979	.999959	.999979
30	.96937	.99386	.99693	.99939	.99969	.999939	.999969
40	.95936	.99186	.99593	.99919	.99959	.999919	.999959
50	.94936	.98987	.99494	.99899	.99949	.999899	.999949
60	.93935	.98788	.99394	.99879	.99939	.999879	.999939
70	.92934	.98588	.99293	.99859	.99929	.999859	.999929
80	.91935	.98387	.99193	.99839	.99919	.999839	.999919
90	.90936	.98186	.99094	.99819	.99909	.999819	.999909
100	.89935	.97986	.98993	.99799	.99899	.999799	.999899
200	.79942	.95987	.97993	.99599	.99799	.999599	.999799
500	.49956	.89986	.94993	.98999	.99499	.998999	.999499

Numerical Example for Tables 5(a) through 5(e)

1000 items are drawn at random from a lot and placed on life test. The test runs for 100 hours. Suppose that 20 failures occur, then we can be:

(a) 99% confident of the assertion that at least 96.72% of the items in the lot survive 100 hours;

(b) 95% confident of the assertion that at least 97.11% of the items in the lot survive 100 hours;

(c) 90% confident of the assertion that at least 97.30% of the items in the lot survive 100 hours;

(d) 75% confident of the assertion that at least 97.51% of the items in the lot survive 100 hours;

(e) 50% confident of the assertion that at least 97.94% of the items in the lot survive 100 hours.

Based on the data in Problem 1, Section 3.1 we can be 95% confident that the underlying distribution of life lies between the two lines drawn in the figure