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UNIVERSITY OF ILLINOIS

DEPARTMENT OF CIVIL ENGINEERING

URBANA, ILLINOIS

DEVELOPMENT OF PROCEDURES FOR RAPID COMPUTATION OF DYNAMIC STRUCTURAL RESPONSE

by

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Final Report
For the Period From
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This investigation is under the general guidance of Dr. N. M. Newmark, Head of the Department of Civil Engineering. The project was under the direction of Dr. A. S. Veletsos, Assistant Professor of Civil Engineering, until June 15, 1957 when Mr. John D. Haltiwanger, Assistant Professor of Civil Engineering replaced Dr. Veletsos as Director. The immediate supervisor is Mr. John W. Melin, Research Associate in Civil Engineering. The studies described in this report were carried out by Charles D. Bigelov, Samuel Sutcliffe, Research Assistants in Civil Engineering, and John Melin. Assistance in this work was also given by William H. Beutjer, an undergraduate in Civil Engineering.

The author wishes to acknowledge the fine cooperation of the Digital Computer Laboratory personnel in the performance of the many numerical studies on the ILLIAC, the digital computer at the University of Illinois.

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DEVELOPMENT OF PROCEDURES FOR RAPID COMPUTATION OF DYNAMIC STRUCTURAL RESPONSE

INTRODUCTION

Objectives of Program

This program is concerned primarily with the behavior of structures, structural elements, and complicated assemblages of elements subjected to the dynamic forces arising from air blast or ground shock. This behavior is measured by deflection or stress, permanent deformation, damage, or collapse. The general objectives are:

1. Development of charts or of rapid "rule-of-thumb" methods for the prediction of critical levels of blast intensity to produce severe damage or collapse of specific structural types.

2. Construction of charts or other procedures for the rapid determination of the magnitude of response of specific structural types to various levels of intensity of blast or shock.

3. Critical examination of the accuracy of predictions of critical levels of force, or of magnitude of response, in terms of the uncertainties in values of blast parameters, structural characteristics, and material parameters.

4. Survey of the present state of knowledge relative to those items in Objective 3 which cause the greatest uncertainty in the predictions, and preparation of plans for further study of these items.

5. Development of methods of calculation suitable for the accurate solution of problems of dynamic response in complicated structures using high speed automatic digital computers, preparation of codes for the determination of dynamic response of important classes of structures, and where

necessary, calculation of responses for a range of values of structural and blast parameters.

6. Coordination of research efforts and results with the activities of other research groups working in the field of weapons effects.

Scope

The determination of the over-pressure level required to give a specific amount of damage to a particular structure is complicated, due not only to the inadequate knowledge of loading and resistance but also the lack of sufficient time for a thorough, or complete, analysis. The loading depends not only on the over-pressure but also upon the individual elements of the structure which may or may not transmit load to the main frame depending on their individual properties. Thus, the actual loading curve is likely to vary erratically with time.

The resistance function is also subject to numerous uncertainties. The properties of the materials of which the structure is made vary considerably since the type of framing, method of connection, rate of loading, etc., each affects the over-all resistance of the structure.

These inherent uncertainties preclude the possibility of an "exact" analysis; thus, certain simplifying assumptions are necessary and justifiable. In general, the time loading curve has been replaced with a triangular load pulse with or without concentrated impulses. The true resistance curve has been approximated by a bilinear function having an initial elastic line followed by a second straight line for displacements greater than the yield displacement. The second line may have any slope depending upon the resistance characteristics of the structure. Using these simplifying

assumptions, several rapid methods have been developed which yield surprisingly accurate results considering the uncertainties involved. Extensive studies have been carried out to evaluate the errors which result from each of these simplifying assumptions and to modify these procedures so that they will yield the maximum feasible accuracy without unduly complicating the analysis.

In order to develop and verify the simplified methods of analysis, it has been necessary to solve by the most accurate methods available a multiplicity of problems, thus forming a basis for the evaluation of the various simplified procedures.

The previous studies of this project have been concerned primarily with the compilation of data using accurate analysis procedures on the ILLIAC, the University of Illinois automatic digital computer. These data have been presented in numerous charts of various forms in previous reports of this project. Some of these data have been replotted in other forms and a comparison study is being conducted to determine the optimum form, or forms, for the presentation of these data.

Further studies of the effects of rise time of the triangular force pulse have been made. These studies have a two-fold objective in that the best possible data presentation form is being sought and methods of replacing the delayed rise force pulse with an initial peak pulse which causes the equivalent amount of damage are also being studied.

The dynamic responses of multistory structures have been studied and preliminary results are given in this report. These data are presented in charts whereon are shown the relationships which exist among the numerous parameters involved.

A detailed study of the natural frequencies of multi-story, multi-bay frames was conducted using the ILLIAC to obtain numerical solutions. From these data an empirical formula for the calculation of natural frequencies has been developed. This study is described in the Appendix to this report. Using either the charts or formuli given there, the natural frequency of any of a multitude of multi-story structures may be easily obtained.

SIMPLE STRUCTURAL SYSTEMS

Single-Degree-of-Freedom System

If a structure is subjected to an impact or a force pulse which is a function of time, it will respond in various modes or configurations. These modes may vary not only for different structures but also for different types of force pulses acting on the same structure. In the simplest case the structure responds in one mode or predominately in one mode so that the configuration at any time is defined by the displacement of one point on the structure. This simple structure lends itself to computation of response due to dynamic loads quite readily and, thus, has been used for the effects-of-parameter-variation studies of this project. Although this analysis is exact only for a model having a rigid mass supported by a weightless spring, it has proven to be quite applicable in the case of more complicated structures. This is due primarily to the fact that all structures tend to respond in one mode or configuration. In general the accuracy depends upon the care taken in choosing the parameters for the replacement system. This model is shown in Fig. 1.

In making such a replacement, the deflection of the mass is usually proportional to the displacement of some point on the structure, for example the top of one of the columns of a one-story building. The mass, or an equivalent mass, is assumed to be concentrated at this point. The dynamic forces which act on this particular point are related to the net force on the actual structure. The period of the single-degree-of-freedom model is related to the first slope of the replacement resistance function and thus may or may not be equal to the period of the actual structure. Almost any structure may be replaced by this model; however, the accuracy of any analysis depends

on the complexity of the actual structure and the care used in choosing the corresponding model parameters.

When the actual structure is of one story and has an essentially rigid roof mass and light columns, then this method produces very reliable results.

The studies conducted on this project are primarily concerned with the dynamic characteristics of the model which, as stated above, are, to a large degree, directly related to the general behavior of actual structures.

Parameters

The parameters considered have been varied in such a manner that, within practical limits, the response of any structure to any force pulse could be predicted. Exact analyses of particular structures were not considered although programs were developed for use on the ILLIAC by which the response of a structure having any resistance function subjected to any loading pulse can be computed. However these same programs were used in the analyses of the idealized systems from which the data presented in the charts of this project were obtained. The general type of resistance used in the response studies is shown in Fig. 2a.

The resistance function used was composed of a linear portion for displacements less than the yield displacement and a second linear portion for displacements greater than the yield displacement as shown in Fig. 2a. The resistance function which decays beyond yield is sometimes called an unstable resistance function since the structure will collapse for any load above a certain critical load.

If, after yielding, the load is gradually removed, the unloading resistance curve is assumed to be parallel to the initial elastic portion.

For example, if a structure is loaded such that it reaches a maximum displacement of four times the yield displacement, and the load is then decreased until the deflection is reduced to the yield displacement before being increased again, the resistance of the model would be defined as shown by lines one through six of Fig. 2b. The resistance increases linearly until yielding (1); then continues to increase at a slower rate due to work hardening (2) until the load is reduced. As the load is decreased, the structure unloads elastically (3) until it yields in the opposite direction. "Negative" yielding, as defined by (4), continues until the reduction in load is stopped. If the load is then increased gradually, the resistance function increases linearly parallel to the initial elastic portion (5) until it intersects the positive work hardening curve, along which it continues (6) with increasing load.

The parameters used in all of the studies have been reduced to dimensionless form so that the application to any specific problem may be easily accomplished. The nomenclature is listed and defined on page 19 of this report.

Blast Parameters

The overpressure-time function is generally defined as an exponential curve which decays from an initial peak value. The force which such a pulse produces on an actual structure may vary erratically with time depending on the shape and strength characteristics of the components of the structure. However, a triangular force pulse was selected for the studies made on this project since it approximates quite well the loading for a great many structures and is also comparatively easy to use in analyses of the simple replacement system. To these triangular force pulses have been

added concentrated impulses, thereby making the studies more general. Two of the force functions which have been studied are shown in Fig. 3. An initially peaked triangular force pulse with a concentrated initial impulse is shown in Fig. 3a. The response to this type of load function of structures having a wide variety of bilinear resistances has been studied extensively on this project. A delayed rise triangular force pulse either with or without two equal concentrated impulses is depicted in Fig. 3b. The two impulses were applied initially and at the time of maximum pulse as shown. Force functions having more than two concentrated impulses were not considered even though a study of such functions would be of considerable academic interest. The use of a larger number of impulses would have increased the complexity of data presentation tremendously. Furthermore, from a practical point of view, the vast majority of actual load functions can be approximated adequately by the use of only one or two impulse concentrations in addition to a triangular force pulse of finite duration.

Equation of Motion

The equation of motion for the single-degree-of-freedom system is derived using Newton's law which states that the product of the mass of a body and its' acceleration is always equal to the unbalanced force acting on the body, i.e., the difference between the load and resistance. This law is expressed in Equation (1).

$$M\ddot{x} + q(x) = p(t) \qquad (1)$$

where M is the mass of the system, $\ddot{x}$ is the second derivative of the displacement with respect to time (acceleration), q(x) is the resistance of the structure as a function of displacement, and p(t) is the applied force as a function of time. The solution of the above equation for response (x)

as a function of time has been coded for the ILLIAC using a numerical integration procedure. The procedure has been discussed in detail in previous reports of this project.

The effect of impulse is considered by instantaneous changes in velocities at the time of the impulse, i.e., $\Delta\dot{x} = I_i/M$ at t_i.

Maximum Response Charts

For most of the studies of response of simple systems to the various force pulse types, the results have been presented in the form of charts. In every case the forms of the charts were chosen which would portray best the effects of the parameters being studied at that time. Thus, various forms have appeared in the previous reports of this project. These forms have been reviewed in an effort to find a more condensed, yet more useful form for the presentation of all of the data computed during these response studies. As a result, sample sets of other possible chart forms have been developed for the presentation of these data.

These new charts are presented in Figs. 4 and 5. The resistance considered is elasto-plastic, the second slope of the bilinear resistance function being zero. This model was subjected to an initial peak triangular force pulse with an initial impulse. The study was conducted using a dimensionless form for the parameters. The peak force is expressed in terms of the yield resistance, the response in terms of yield displacement, the pulse duration in terms of the natural period of the system, and the impulse in terms of the product of the yield resistance and the period.

The charts shown in Figs. 4a, b, c, and d are for constant values of response, $\mu = x_m/x_y = 2$, 5, 10, and 30 respectively. The peak force required to obtain the given response is plotted as a function of the pulse

duration for lines of constant impulse. The pressure-duration for any value of impulse may be obtained quite accurately by linear interpolation between impulse lines on a particular chart. Interpolation for values of maximum response between those for which charts are plotted is not satisfactory in many cases since the variations are not linear. For a given impulse and duration, the pressure necessary to cause a displacement of three times the yield will be less than the corresponding pressure required for $\mu = 5$ and will be greater than the value for $\mu = 2$, but, without computation, it is impossible to state its exact value. The load levels for $\mu = 2$ and $\mu = 5$ may differ by 200 per cent or more in some cases. These charts offer one of the most accurate and useful methods of depicting the effects of impulse.

The same data are plotted in a different form in Figs. 5a, b, c, d, e, and f. In this set, the relationship between peak pressure and maximum response is depicted for lines of constant impulse and charts of constant pulse duration. In Fig. 4 it can be seen that the curves are very smooth and almost linear over large ranges of pulse duration. Consequently, it was possible to present quite accurately these same data in a comparatively few charts of the form of Fig. 5 by a judicious choice of pulse durations. For any given values of response, impulse, and duration, the peak pressure may be obtained by linear interpolation and the result obtained should be correct within engineering accuracy although this accuracy is limited to the analysis and does not include errors incurred in computing the replacement force pulse or the replacement resistance function.

On the charts described above, the time of maximum response is not shown but could be added. The time of maximum response is not directly

applicable to response studies but in choosing replacement force pulses, this factor may be important, since any forces acting after maximum response has been attained do not affect the damage pressure level.

The above studies do not consider the effect of delayed rise of the force pulse on the response of a structure. In charts 6a-d and 7a-b the effects of rise time are shown. The resistance is elasto-plastic and the force pulse is shown in Fig. 3b. In the final report for July 1954, charts for the response of a delayed rise triangular force pulse with two impulses were shown for specific rise times of 0 and 0.5 times the period. The charts for response considering rise times of 1.0, 1.5, and 2.0 times the period of the system were presented in the interim report of 1 January 1956. Interpolation between specific rise times has proven to be unreliable using rise time as a principal variable. This was presented in the 1 July 1956 Final Report of this project. The study of rise time versus response demonstrated that for a given peak pressure and duration the response varied by as much as 2000 per cent as the rise time varied from zero to a value equal the period of the simple system. The study also showed that a slight change in peak pressure could cause similar changes in the response. In order to demonstrate clearly this latter phenomenon, charts for peak pressure versus rise time for specific values of response ($\mu = x_m/x_y = 2, 5, 10, 30$) were developed and these results are presented in Figs. 6 and 7 of this report.

The most important result of these studies, as shown in these charts, is that the peak pressure necessary to produce a specific response is almost independent of rise time. If the rise time is less than one third of the total duration and if no impulses are considered, then the peak pressure required to produce a specified response does not vary

appreciably with rise time. If the effects of impulses are considered, then the above is not valid as indicated in Charts 7a and 7b, in which two equal impulses were considered. These studies of delayed time force pulses thus indicate that slight variations in rise time or peak pressure have a considerable effect on response values when the pulse duration is long compared to the natural period. That is, the response parameter is very unstable and may change from yield to collapse for a variation in the peak pressure of only 10 per cent or for a variation in rise time from zero to one times the natural period.

MULTI-DEGREE-OF-FREEDOM SYSTEMS

General

The response of multi-story buildings to blast-type loading is very complex since the structures do not behave, in general, as single-degree-of-freedom systems. The replacement of such a structure by a simple system is possible but the resulting predictions of damage pressure level are not as accurate as predictions for a single-story structure. A more accurate replacement model for the multi-story structure has been developed and the response characteristics of this model subjected to dynamic loads are being studied. The objective of these studies is the development of simple rules and charts for predicting damage pressure levels.

Shear-Beam Replacement System

The initial studies of multi-story structural response were calculated assuming that the masses were concentrated at the floor levels. The structure was thus reduced to an "n" degree-of-freedom system where "n" represents the number of stories. The column shears were assumed to be directly proportional to the relative displacements between stories. All the columns in a given story were assumed to yield at the same relative displacement and thus the resistance of all the columns in a given story was replaced by a single shear spring. This assumes that the girders and floor together are infinitely stiff. Using this analogy, a code was prepared for computation of the dynamic response using the ILLIAC.

Charts for the response of a three-story building subjected to initially peaked triangular force pulses were prepared and these results

were presented in the 1 July 1955 final report. These analyses were made using the shear-beam replacement system just described.

Multi-Story Building with Flexible Girders

A study of the effect of flexible girders on the response configurations of a multi-story structure subjected to initially peaked triangular force pulses has been conducted and the preliminary results of this study are presented in this report.

Replacement System for Multi-Story Building with Flexible Girders

The replacement system shown in Fig. 8 was used for the analysis of a multi-story frame structure with flexible girders. The effects of joint-rotation and plastic hinges are accounted for in the analysis. The masses are concentrated at the floor levels. No consideration was given to the effect on lateral deflections of axial changes in length of the columns and girders under stress. Although the ILLIAC code used will permit account to be taken of the column and girder moment changes which result from the changes in dead load eccentricity during the response of the structure, these effects were also neglected.

The joints of the frame are allowed to rotate by means of a moment distribution procedure. Each member of the bent is assumed to deform elastically until the moment at one or both ends exceeds the yield moment. When the yield moment is reached, a constant moment hinge is considered to have formed at that end of the member. This hinge remains as long as the relative rotation at the end of the member increases. When the direction of relative rotation at the end is reversed then the hinge is removed and complete elastic continuity is restored except that a kink due to the previous plastic

action remains at the point of the hinge. The method of analysis was described in the Interim Report 1 January 1957 of the project and is similar to the procedure used for single-degree-of-freedom systems.

Parameters

The applied force was an initially peaked triangular force pulse as shown in Fig. 8 which was assumed to act uniformly over the structure. For simplicity of analysis this uniform load was replaced by equivalent concentrated loads at the floor levels. The load concentration for the top story was only half as large as those which were assumed to act at each of the other floors. The duration of each of these concentrated force pulses was assumed to be the same. The force functions were the same as those used in the shear-beam response studies.

The columns of the frame were chosen such that their combined stiffness (assuming no end rotation) would agree exactly with the stiffness used in the shear-beam study. The stiffnesses of the several stories were, from top to bottom, 44, 132, and 220 kips per inch, respectively. This is clearly a 1-3-5 variation. Since the girders were flexible, the actual stiffness at any story depended not only on the column stiffnesses but also on girder stiffnesses and the displacements of the other floors. Girder stiffnesses of 2, 4 and 10 times the first story column stiffnesses were studied; although, for each structure, all girders were identical. Since in these studies, none of the girders yielded, the investigation was continued using more flexible girders. These further studies are not yet complete.

The fundamental periods of the replacement systems for relative girder stiffnesses of 2, 4, and 10 were found to be 1.108, 1.027 and 0.973

sec. respectively. For the shear beam, i.e., infinite girder stiffness, the period was 0.934 sec. These periods were calculated using a mass of 880 lb sec^2/in. each for the lower two stories and a mass of 440 lb sec^2/in. for the top story. To obtain these masses, a typical structure was actually designed and its masses were computed.

Results

The results of this study are shown in Fig. 9a, 9b, etc. The charts are plotted for peak pressure versus relative response. The peak pressure is expressed in kips with the relative response of each story expressed in inches. The yield displacement is one inch for the shear-beam structure but cannot be defined easily for the structures with flexible girders because the moments depend on the configuration of the entire structure and are not defined by the relative displacement of one story. In general, yield at the ends of the members in the flexible girder studies did occur at a relative story displacement of approximately one inch. The displacements for the smaller force pulse values are shown on an enlarged scale in addition to the regular scale. The displacement versus peak force is plotted for constant values of girder stiffness of 2, 4, 10 and ∞ times the stiffness of the lower column. The pulse duration is constant for each chart, these durations being 0.1, 0.5, 1.0, 3.0 sec. and infinite for Figs. 9a through 9e, respectively.

These charts demonstrate quite clearly the relative unimportance of girder flexibilities on the response of the structures for the range of parameters shown in the charts.

Natural Frequency

The natural period of a single-degree-of-freedom system or the natural periods of a multi-degree-of-freedom system are important in the study of dynamic response since the duration of the force pulses acting on a structure are measured in terms of the natural period or periods. The calculation of the natural period or its reciprocal, the frequency, for a single-degree-of-freedom system is straightforward but the calculation of frequencies for a multi-story shear-beam structure involves an eigen value problem which is usually solved by a numerical trial and error process. The problem is further complicated when flexible girders are considered.

A code for calculation of natural frequencies has been developed for use on the ILLIAC and a study of the frequencies of multi-story frames was conducted. This study and the results obtained therein are presented in the Appendix to this report.

PERSONNEL

Mr. Peter Hoadley has been assigned to work on this project as a half-time research assistant beginning June 15, 1957. Also effective June 15, 1957, Mr. John D. Haltiwanger has replaced Dr. A. S. Veletsos as director of this project. The program will continue under the immediate supervision of John W. Melin and under the general guidance of Dr. N. M. Newmark as before.

NOMENCLATURE

The nomenclature used in the main text of this report is summarized below. All nomenclature used in the Appendix in the discussion of the shear-beam study is summarized at the end of the Appendix.

Dimensional Quantity	Analogous Dimensionless Quantity
x = displacement of mass of single-degree-of freedom system.	x/x_y
$q(x)$ = resistance of spring to displacement x.	$q(x)/q_y$
x_y = "yield" displacement of spring at the point where the resistance function changes slope.	---
q_y = "yield" resistance of the spring at the point where the resistance function changes slope.	---
k_1 = initial, elastic slope of resistance function. $q_y = k_1 x_y$.	---
k_2 = second, inelastic slope of resistance function.	$k_2/k_1 = k$
x_m = maximum displacement of mass.	$x_m/x_y = \mu$
t = time.	t/T
Δt = time interval used for numerical integration.	$\Delta t/T$
$p(t)$ = pressure at time t.	$p(t)/q_y$
t_d = total duration of applied force.	t_d/T
t_1 = rise time of applied force.	t_1/T
p_m = peak force; damage pressure level; critical pressure level.	p_m/q_y

Dimensional Quantity	Analogous Dimensionless Quantity
I, I_o, I_1 = impulses for applied forces.	$I_i/q_y T$
M = magnitude of mass.	---
x = velocity of mass.	$\Delta t \dot{x}/x_y$
Δx_i = change in velocity due to impulse I_i at time t_i. $\Delta \dot{x}_i = I_i/M$ for $t = t_i$.	$\Delta t \Delta \dot{x}_i/x_y$
$\ddot{x}$ = acceleration of mass.	$(\Delta t)^2 \ddot{x}/x_y$
T = period of vibration for small deflections. $T = 2\pi\,(M/k_1)^{1/2}$.	---
S_o = the stiffness $(\frac{kEI}{L})$ of the girders of the three-story frame.	
S_1 = the stiffness $(\frac{kEI}{L})$ of the lower story column of the three-story frame.	

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4a		0	2
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4d		0	30
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5a		0	0.1
5b		0	0.4
5c		0	0.8
5d		0	1.5
5e		0	3.0
5f		0	100
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	k_2/k_1	I/q_yT	$\mu = x_m/x_y$
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6d	0	0	30
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			t_d/T
9a			0.1
9b			0.5
9c			1.0
9d			3.0
9e			∞

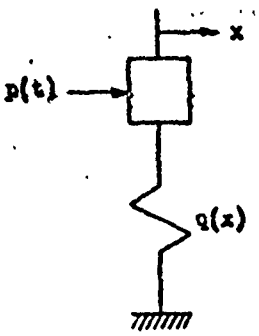
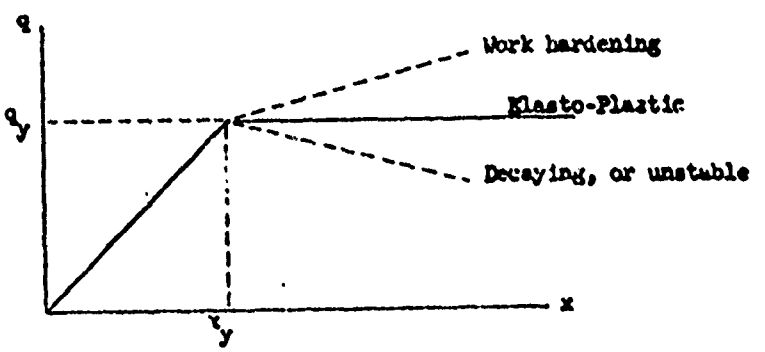
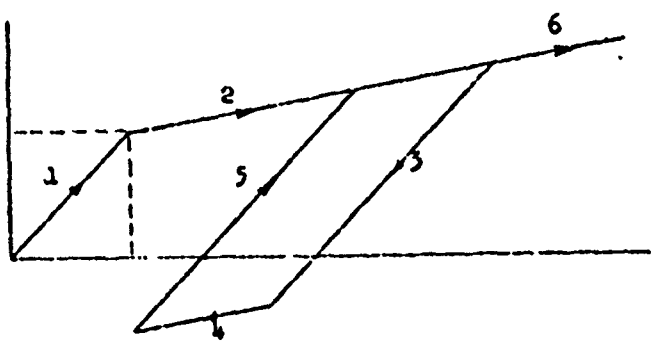


Fig. 1 Single-Degree-of-Freedom System

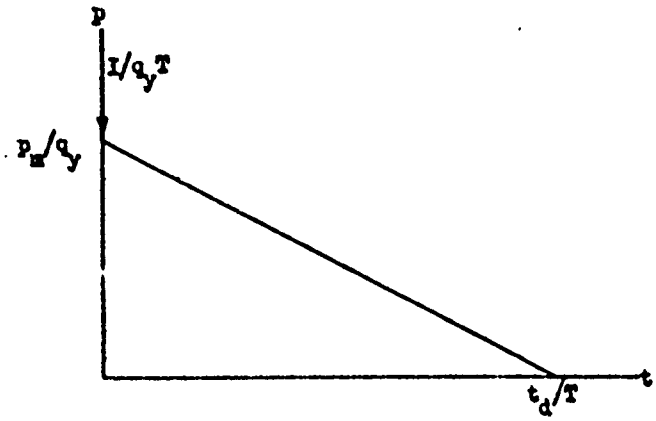


a.

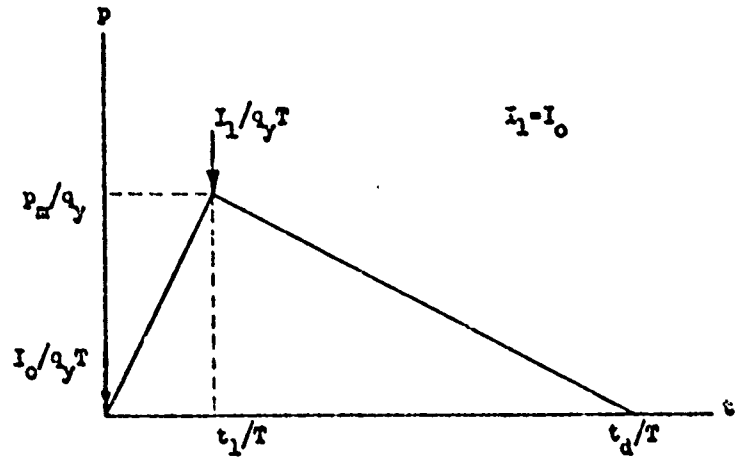


b.

Fig. 2 Resistance Function



a. Initial Peak Triangular Force Pulse



b. Delayed Rise Triangular Force Pulse

Fig. 3 Force Functions

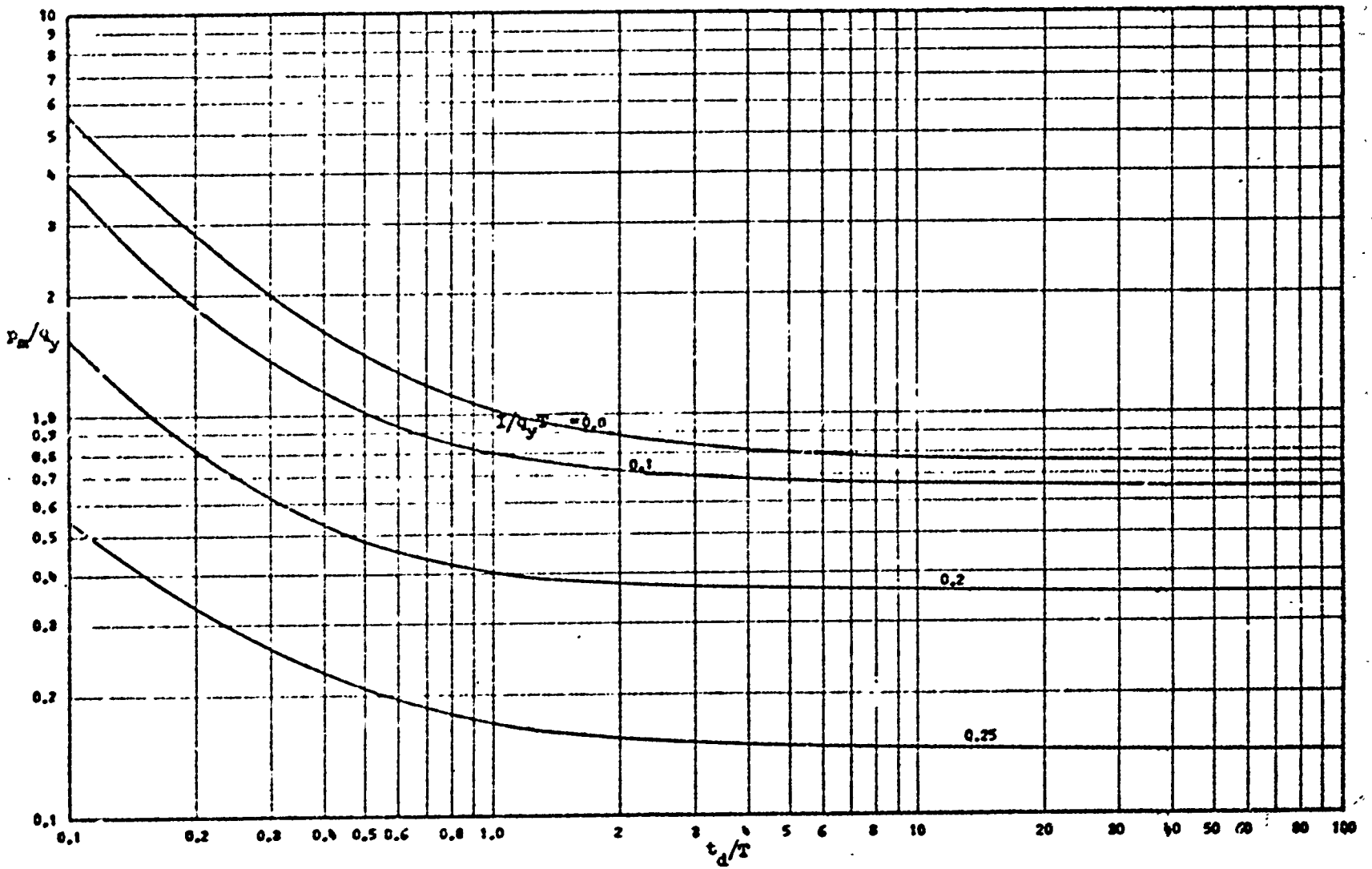


Fig. 4a Peak Pressure of Initial Peak Triangular Force Pulse

$K_2/K_1 = 0,\ \mu = x_m/x_y = 2$

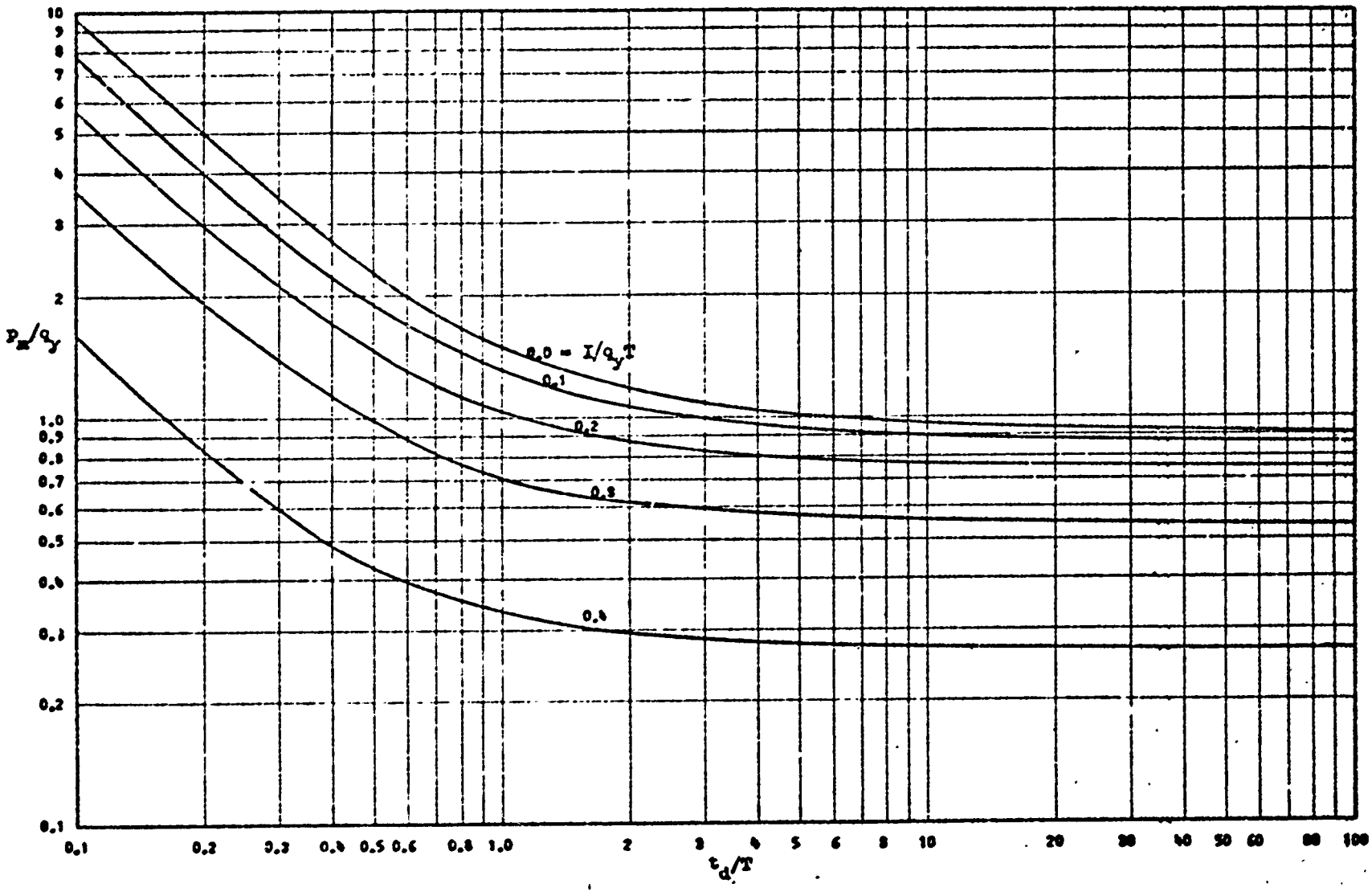


Fig. 4b Peak Pressure of Initial Peak Triangular Force Pulse

$K_2/K_1 = 0,\ \mu = x_m/x_y = 5$

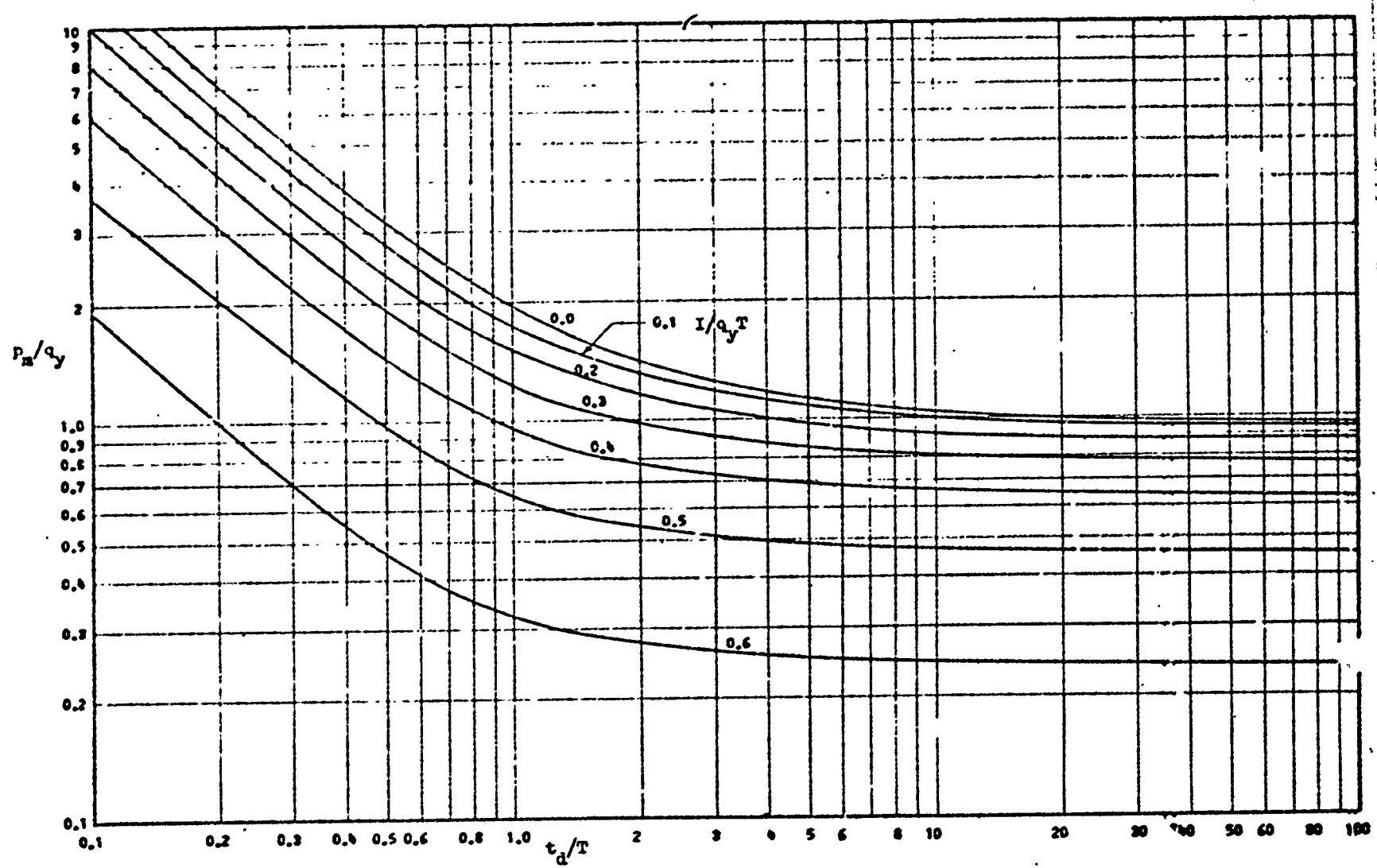


Fig. 4c Peak Pressure of Initial Peak Triangular Force Pulse

$K_2/K_1 = 0, \mu = x_m/x_y = 10$

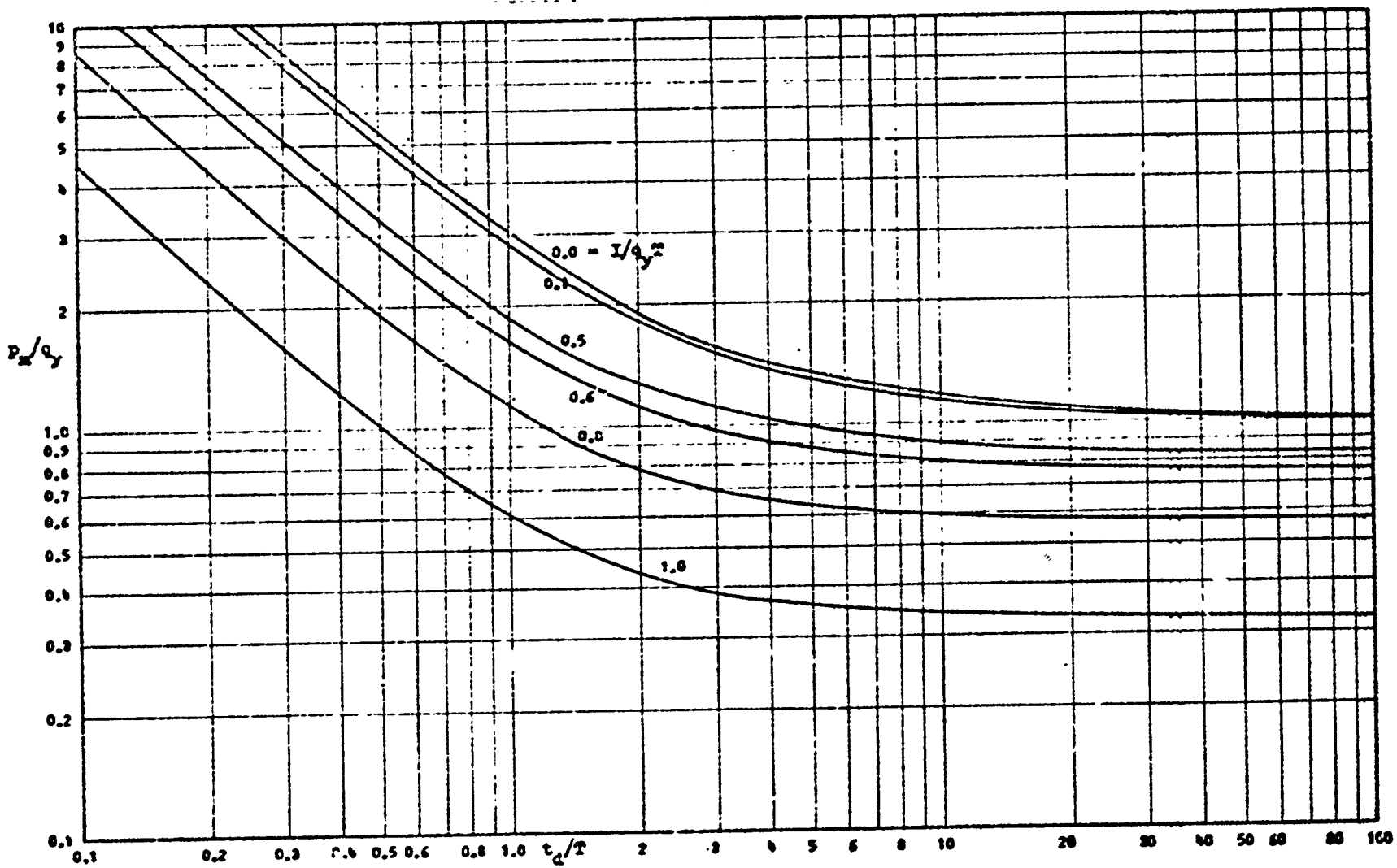


Fig. 4d Peak Pressure of Initial Peak Triangular Force Pulse

$K_2/K_1 = 0,\ \mu = x_m/x_y = 30$

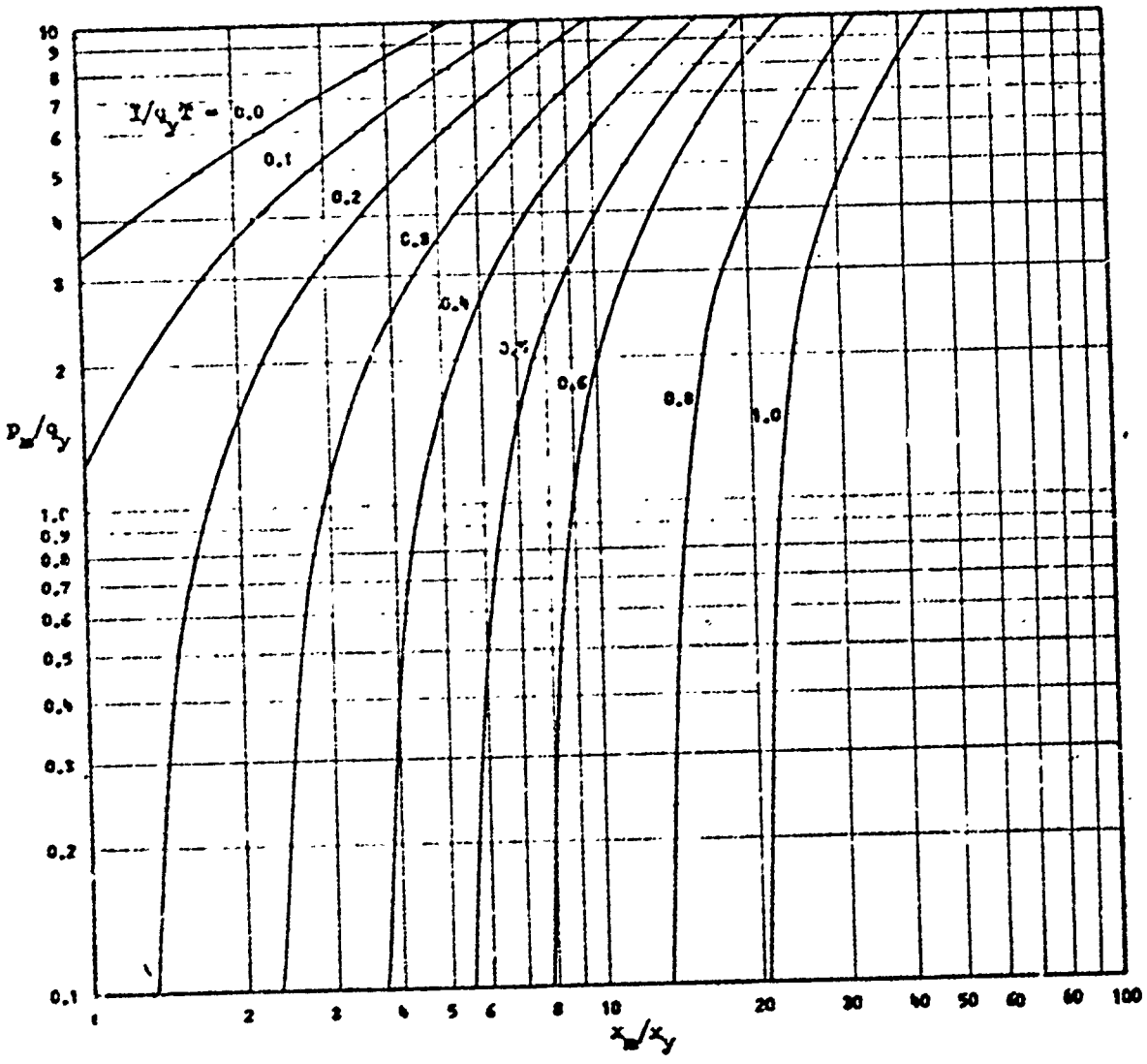


Fig. 5a Maximum Response to Initial Peak Triangular Force Pulse

$K_2/K_1 = 0$, $t_d/T = 0.1$

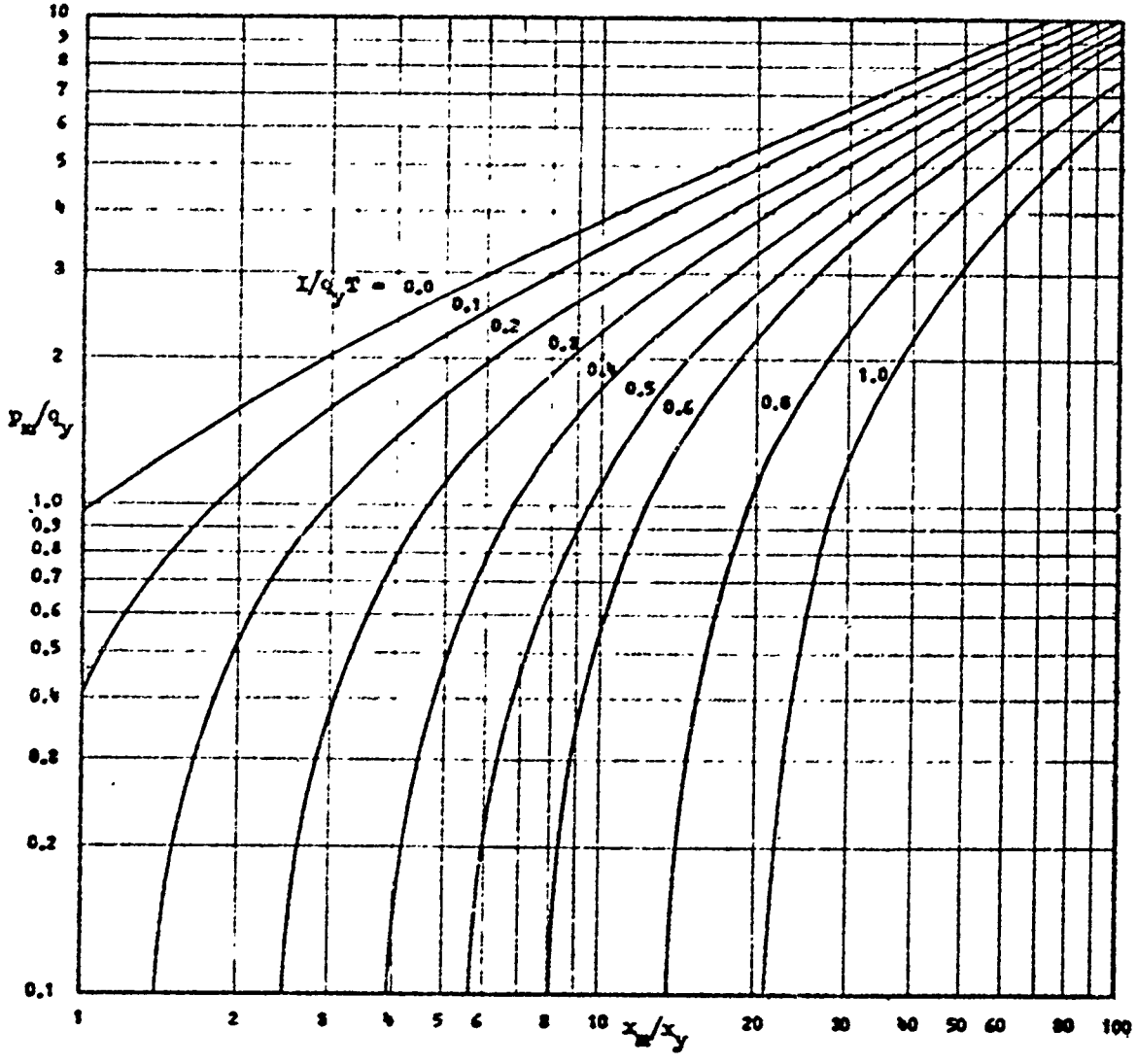


Fig. 5b Maximum Response to Initial Peak Triangular Force Pulse

$K_1/K_2 = 0.$ $t_1/T = 0.4$

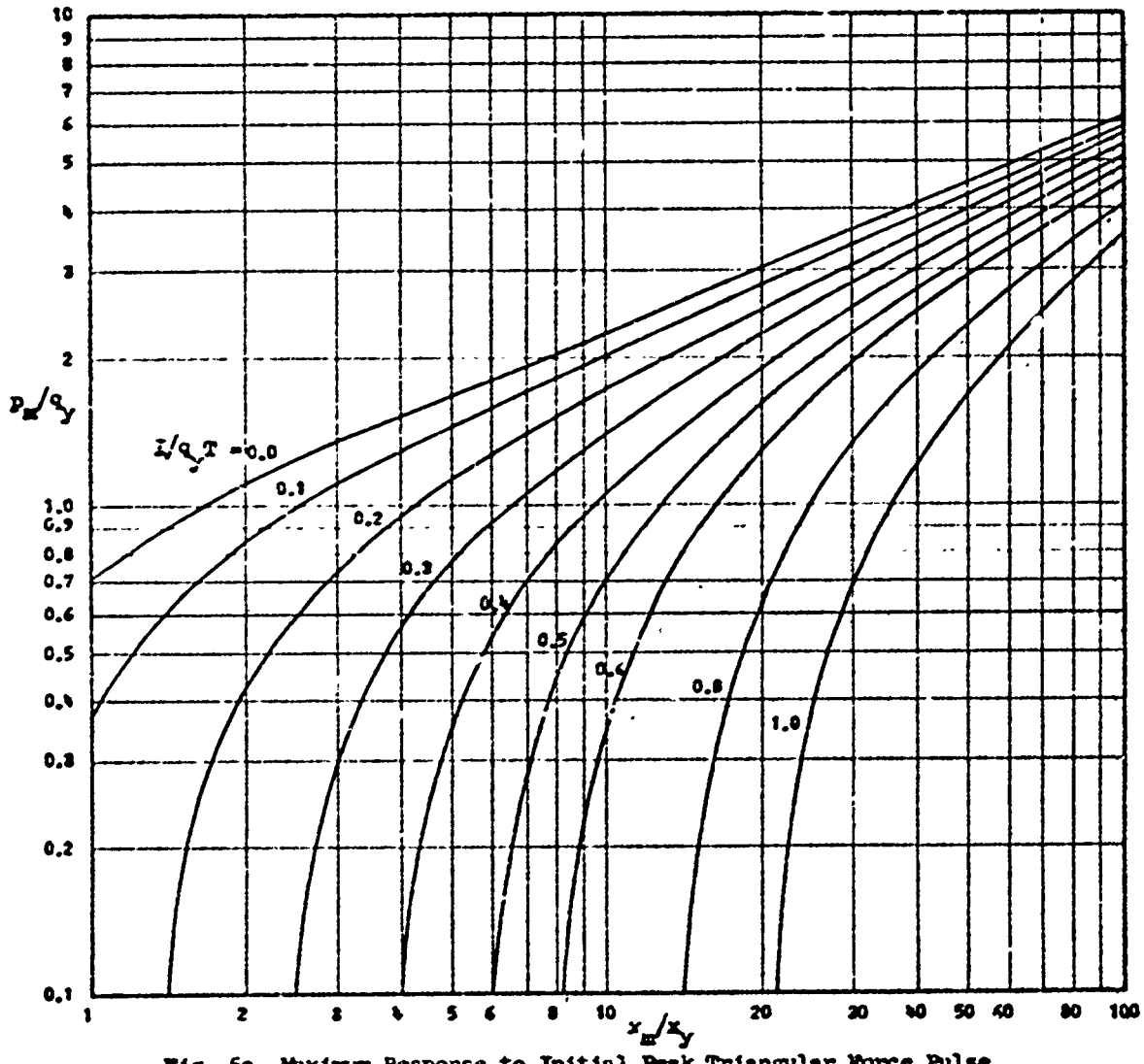


Fig. 5c Maximum Response to Initial Peak Triangular Force Pulse

$K_2/K_1 = 0$, $t_d/T = 0.8$

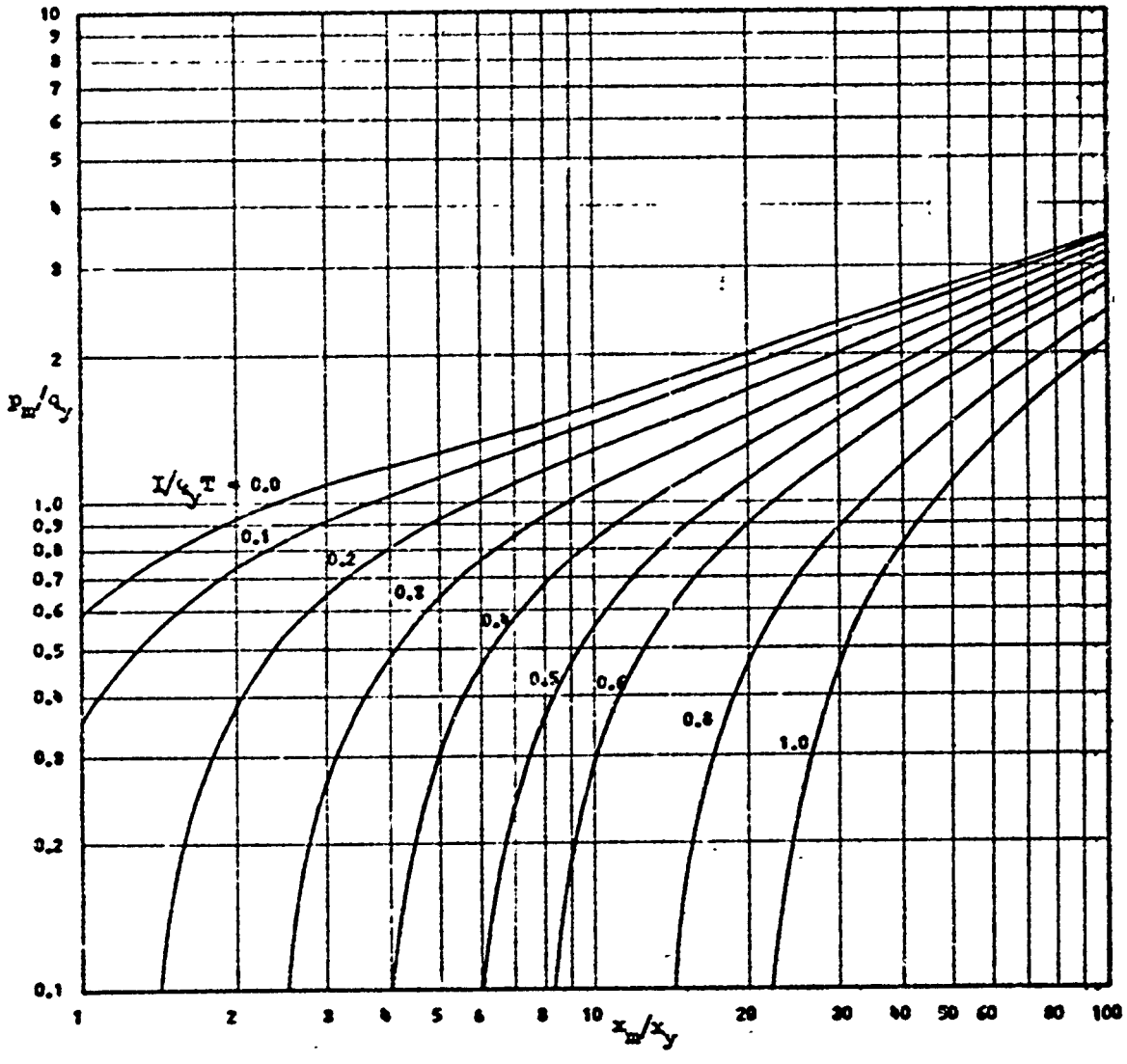


Fig. 5d Maximum Response to Initial Peak Triangular Force Pulse

$K_2/K_1 = 0$, $t_d/T = 1.5$

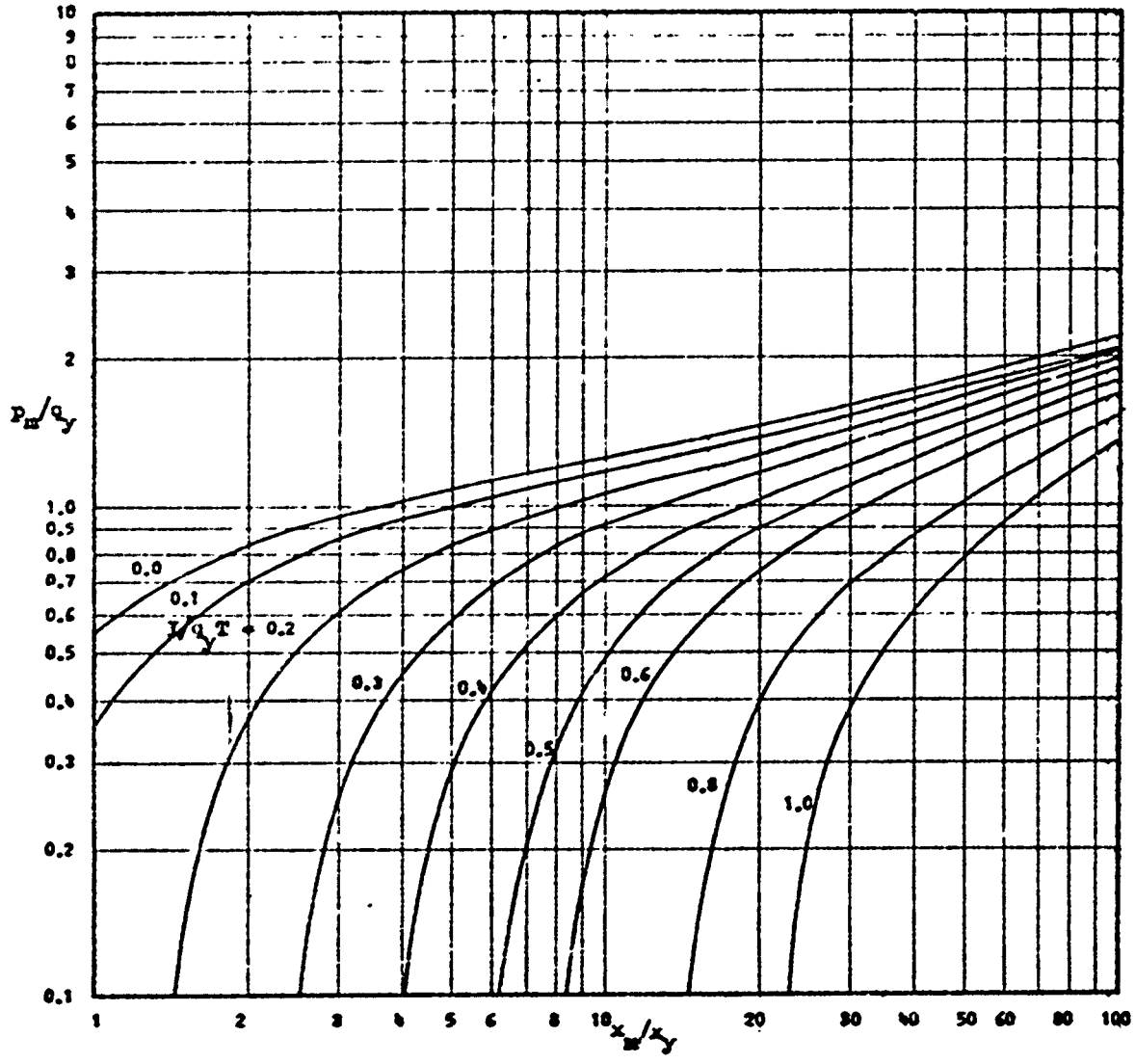


Fig. 5e Maximum Response to Initial Peak Triangular Force Pulse

$K_2/K_1 = 0$, $t_d/T = 3.0$

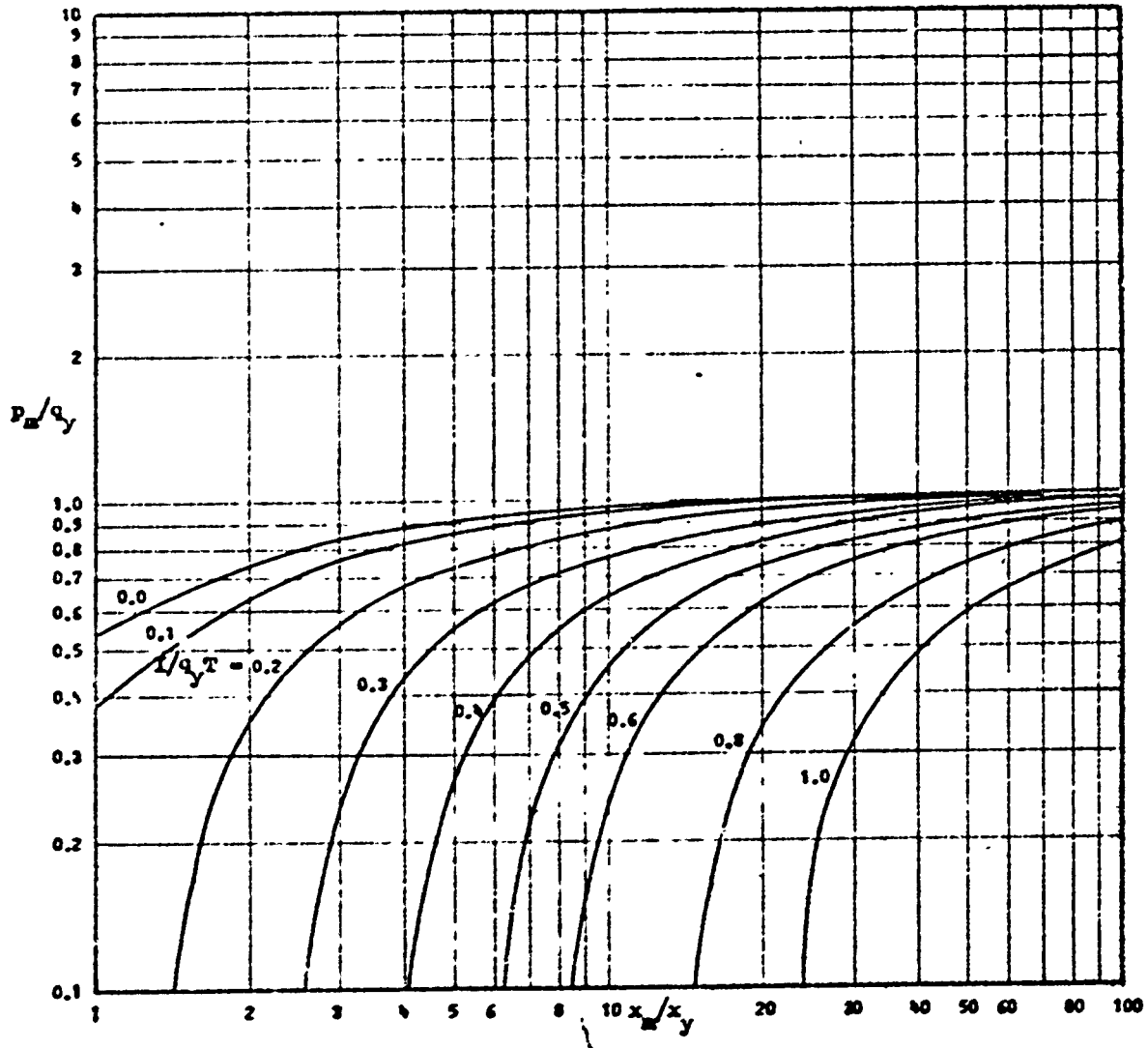


Fig. 5f Maximum Response to Initial Peak Triangular Force Pulse

$K_2/K_1 = 0$, $t_d/T = 100$

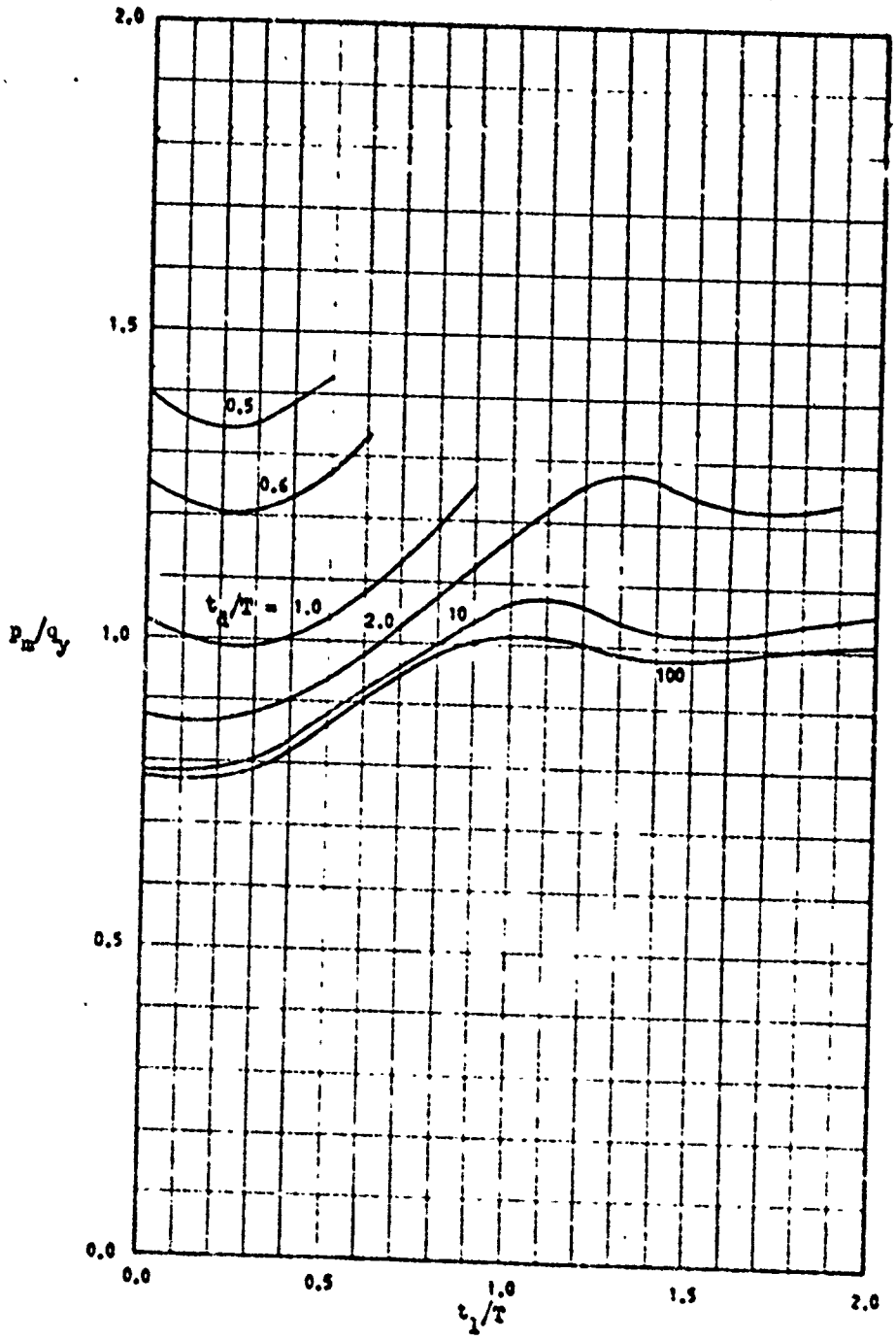


Fig. 6a Peak Pressure of Delayed Rise Triangular Force Pulse

$K_2/K_1 = 0$ $I/q_yT = 0$, $\mu = x_z/x_y = 2$

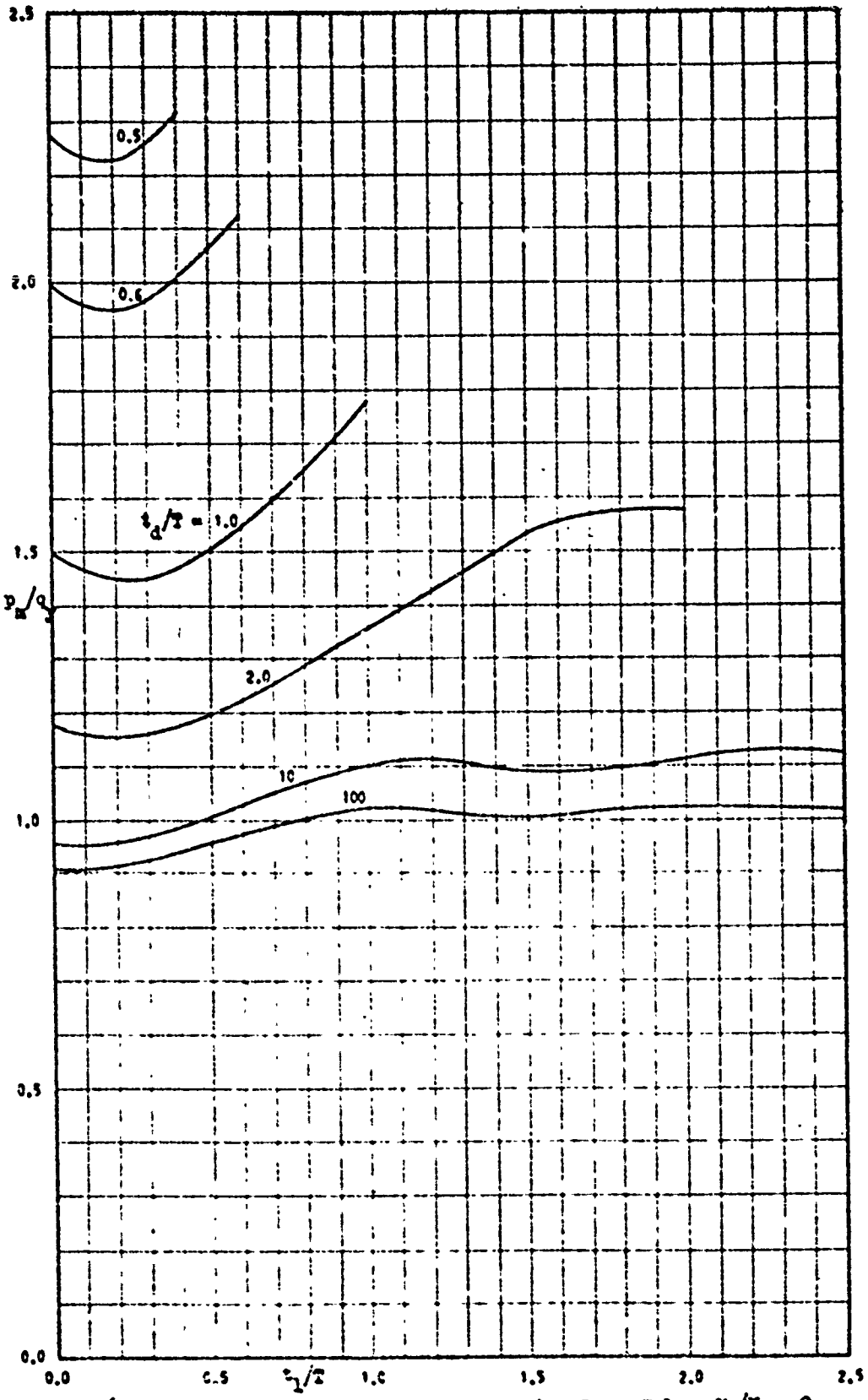


Fig. 6b Peak Pressure of Delayed Rise Triangular Force Pulse $K_2/K_1 = 0$
$1/q_y T = 0$, $\mu = 5$

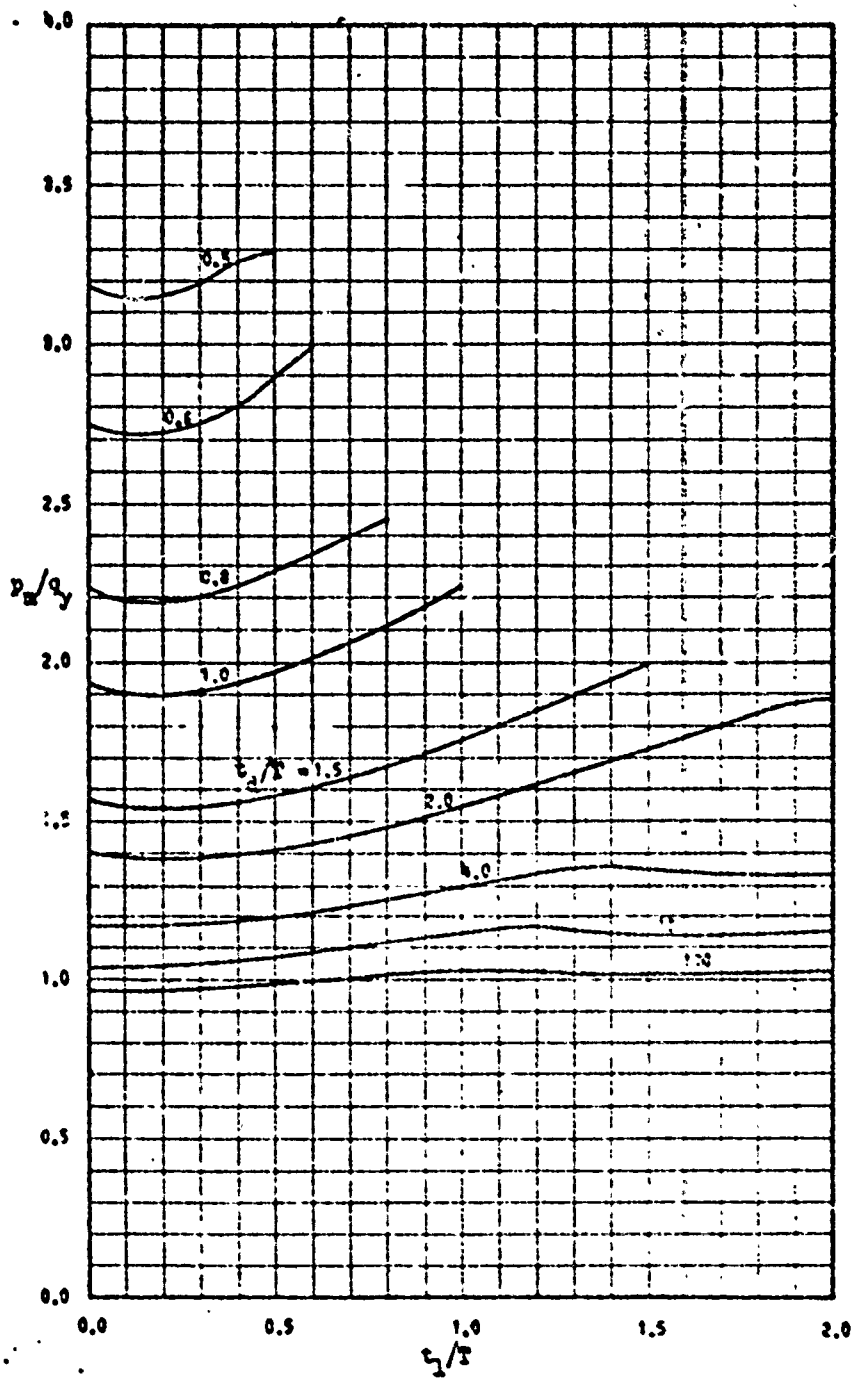


Fig. 6c Peak Pressure of Delayed Rise Triangular Force Pulse

$K_2/K_1 = 0$, $I/q_yT = 0$, $\mu = x_x/x_y = 10$

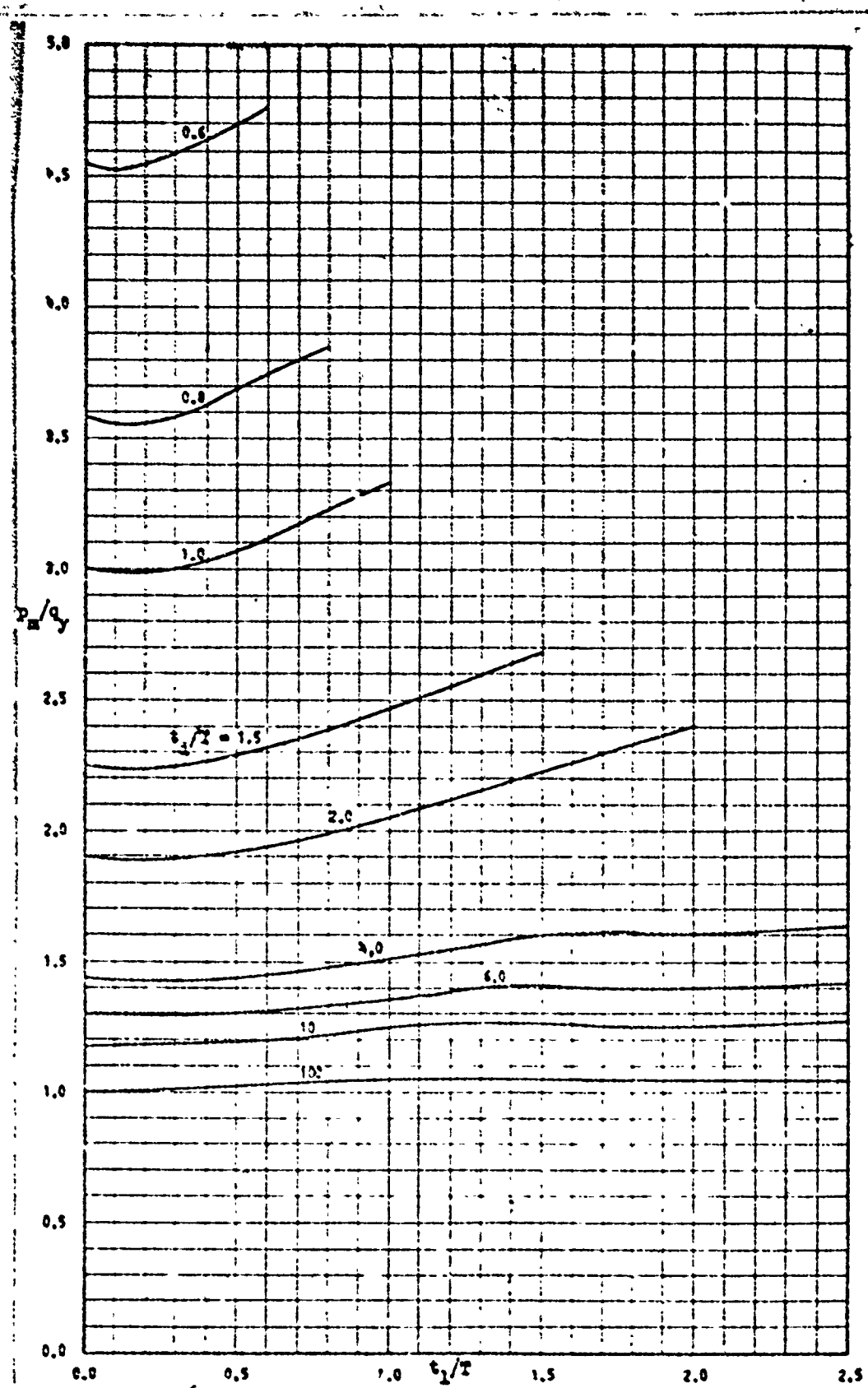


Fig. 6d Peak Pressure of Delayed Rise Triangular Force Pulse
$1/q_yT = 0$ $\mu = x_m/x_y = 30$

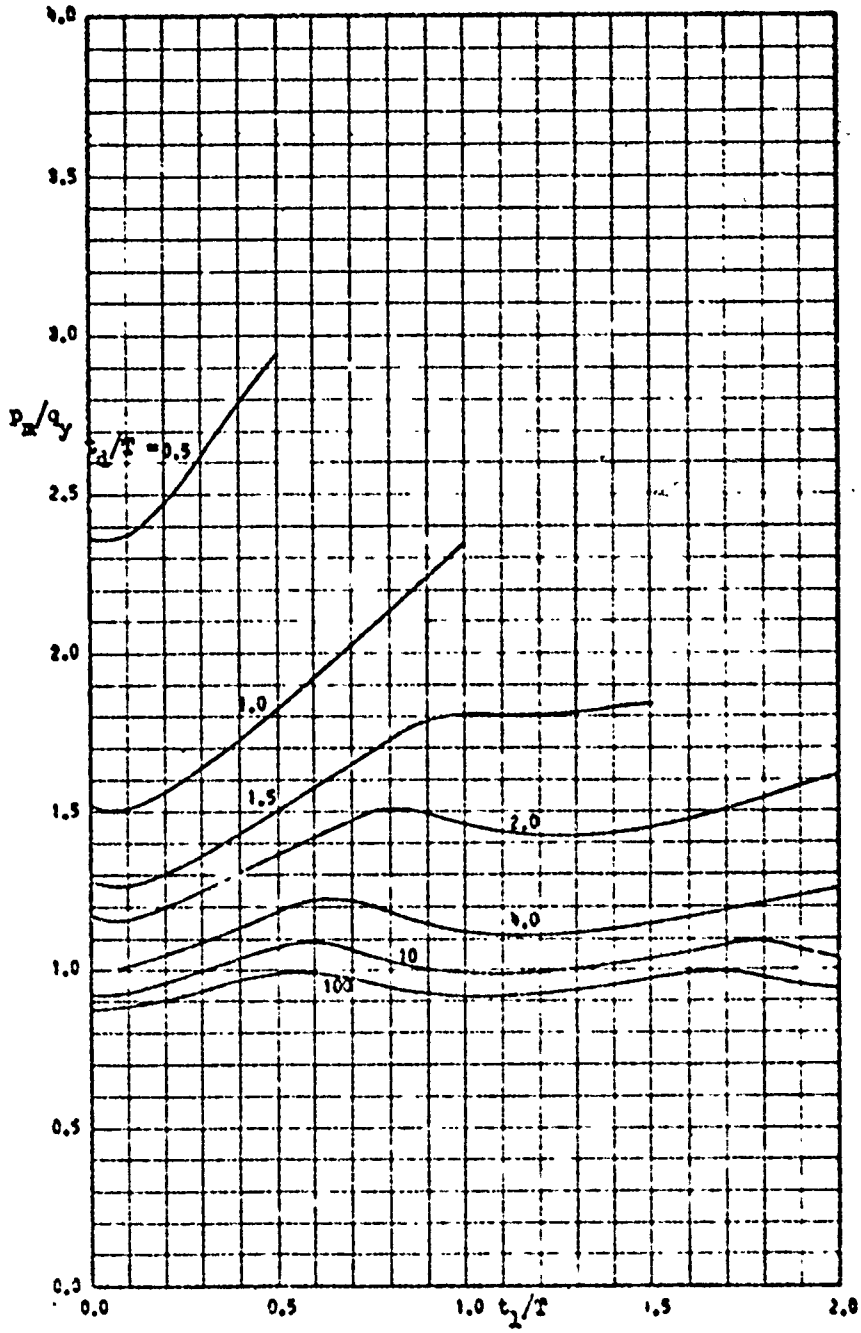


Fig. 7a Peak Pressure of Delayed Rise Triangular Force Pulse with Impulses

$I_o/q_yT = 0.1 \quad \mu = 10.0 \quad k = 0.0$

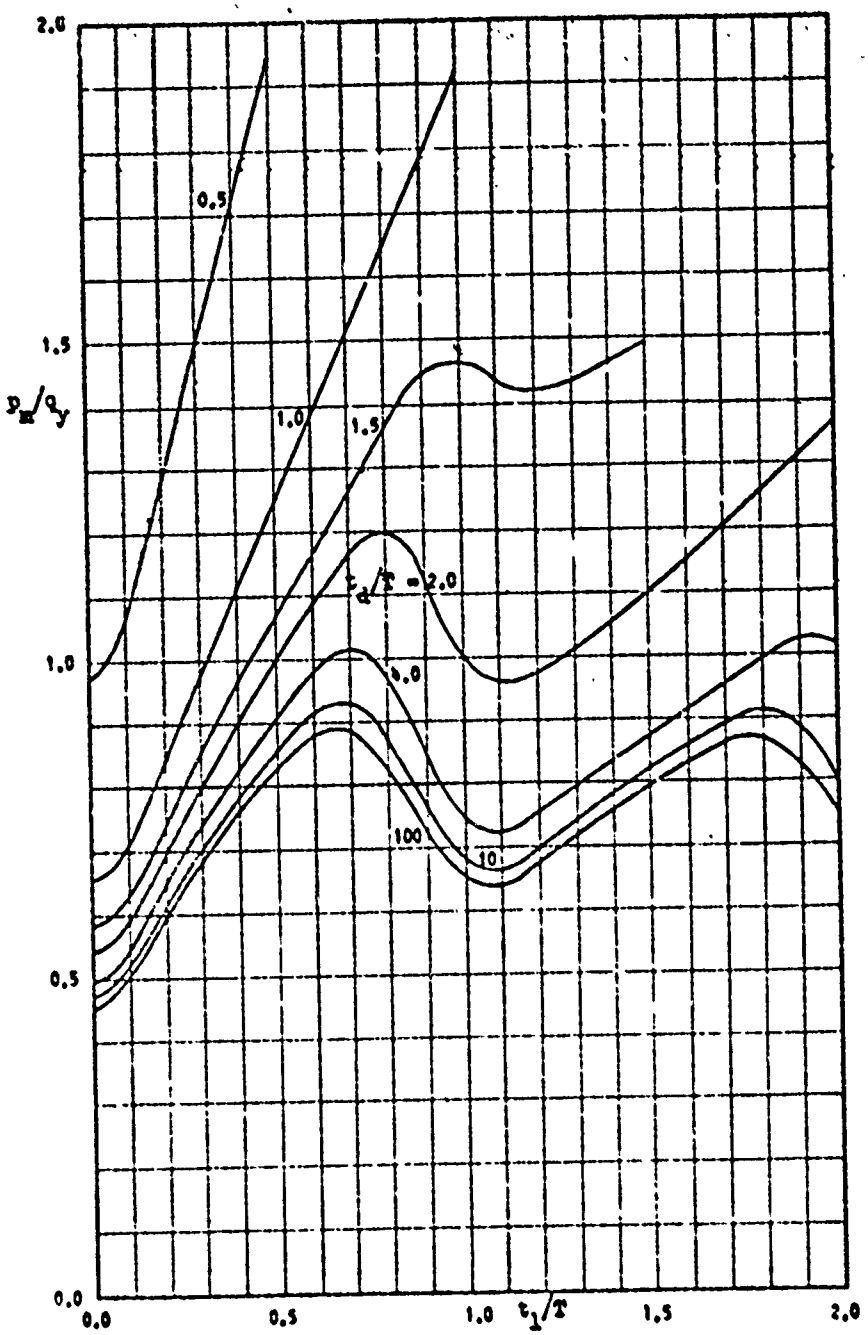
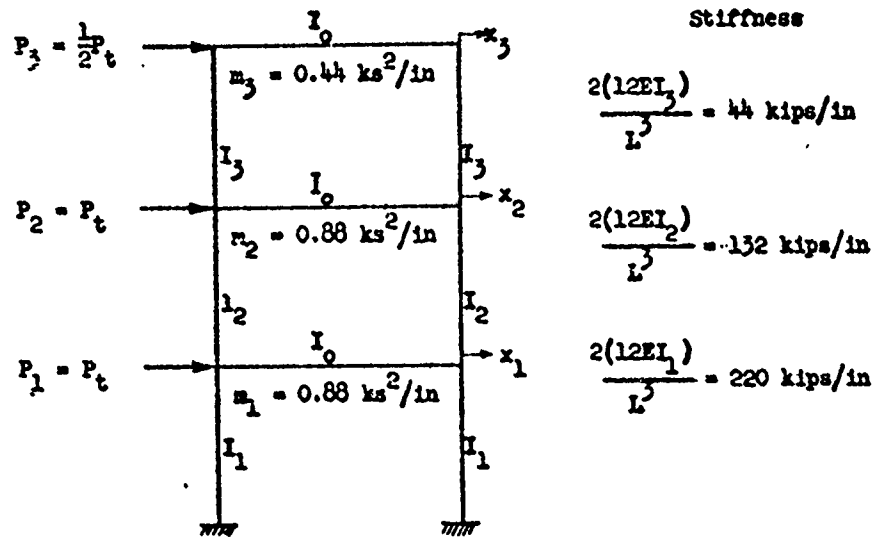
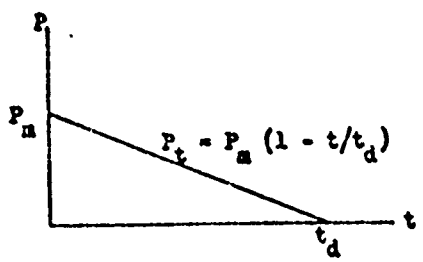


Fig. 7t Peak Pressure of Delayed Rise Triangular Force Pulse with Impulses

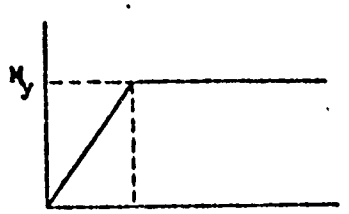
$I/q_yT = 0.25 \quad \mu = 10 \quad k = 0$



a. Three Mass Frame



b. Force Pulse on Each Story



c. Moment of End of Any Member Versus Rotation

Fig. 8 Three Story Frame

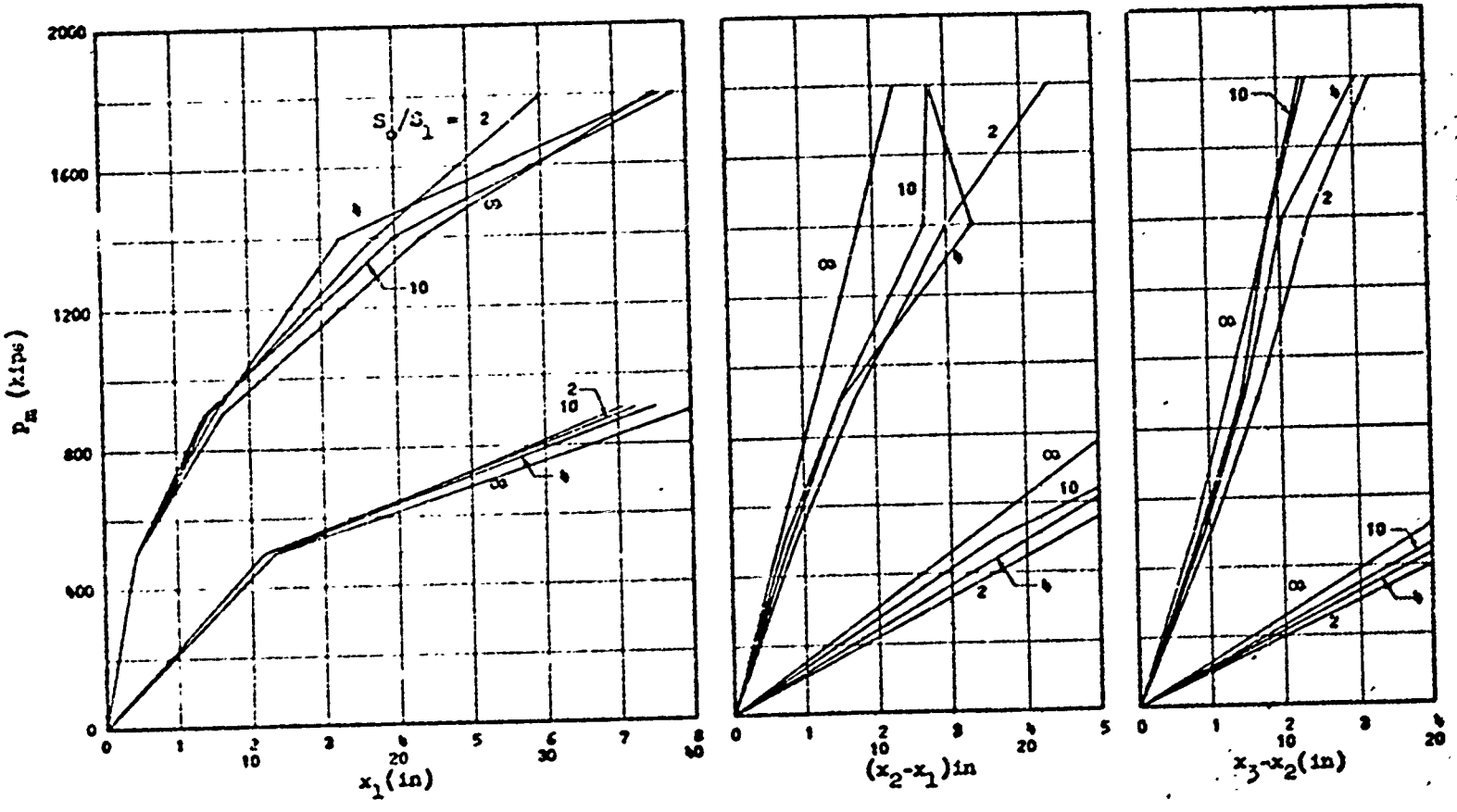


Fig. 9a Maximum Relative Response of Three-Story Frame Due to Initial Peak Triangular Force Pulse

$t_d = 0.1$ sec

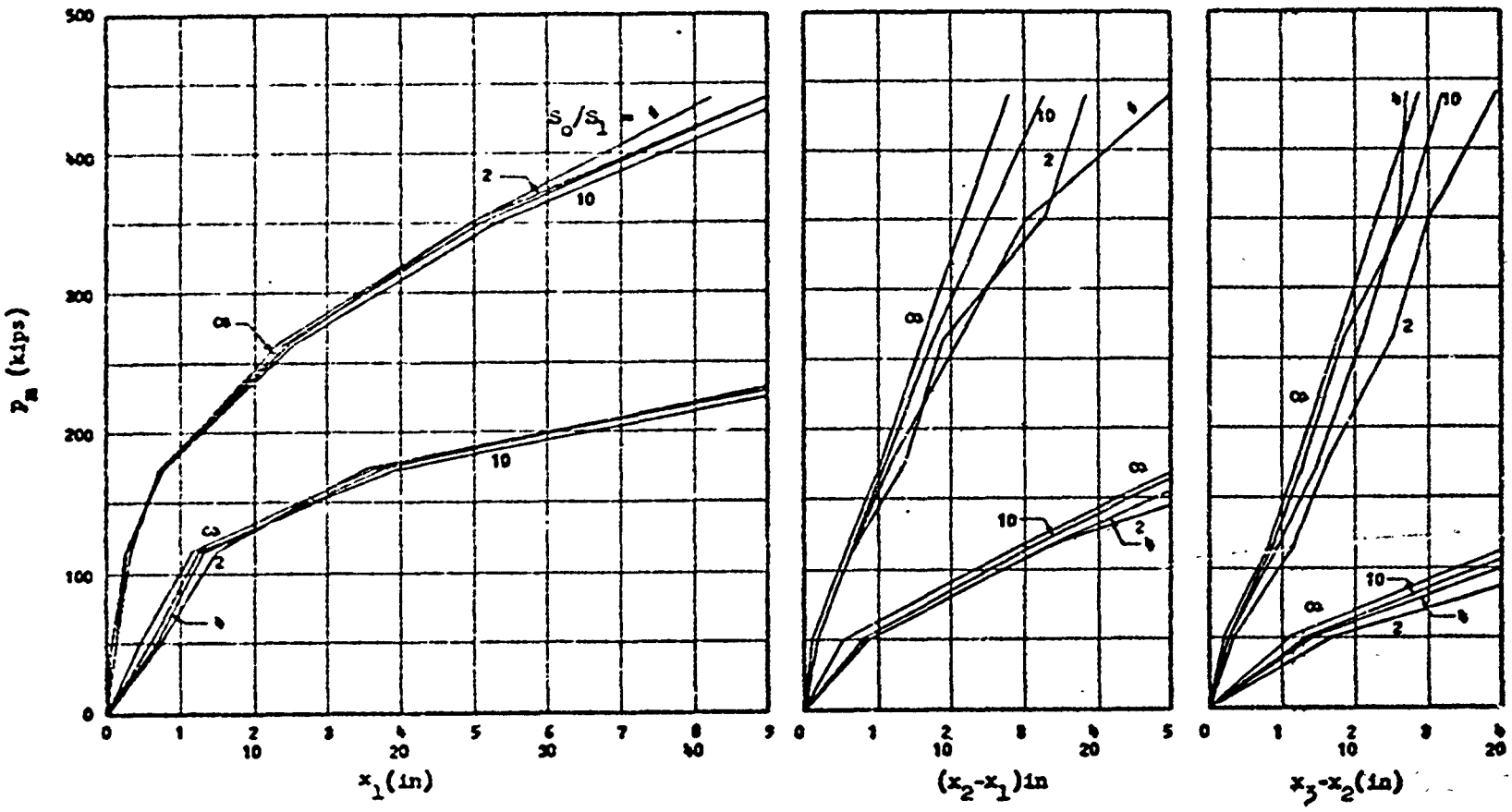


Fig. 9b Maximum Relative Response of Three-Story Frame Due to Initial Peak Triangular Force Pulse

t_d = 0.5 sec

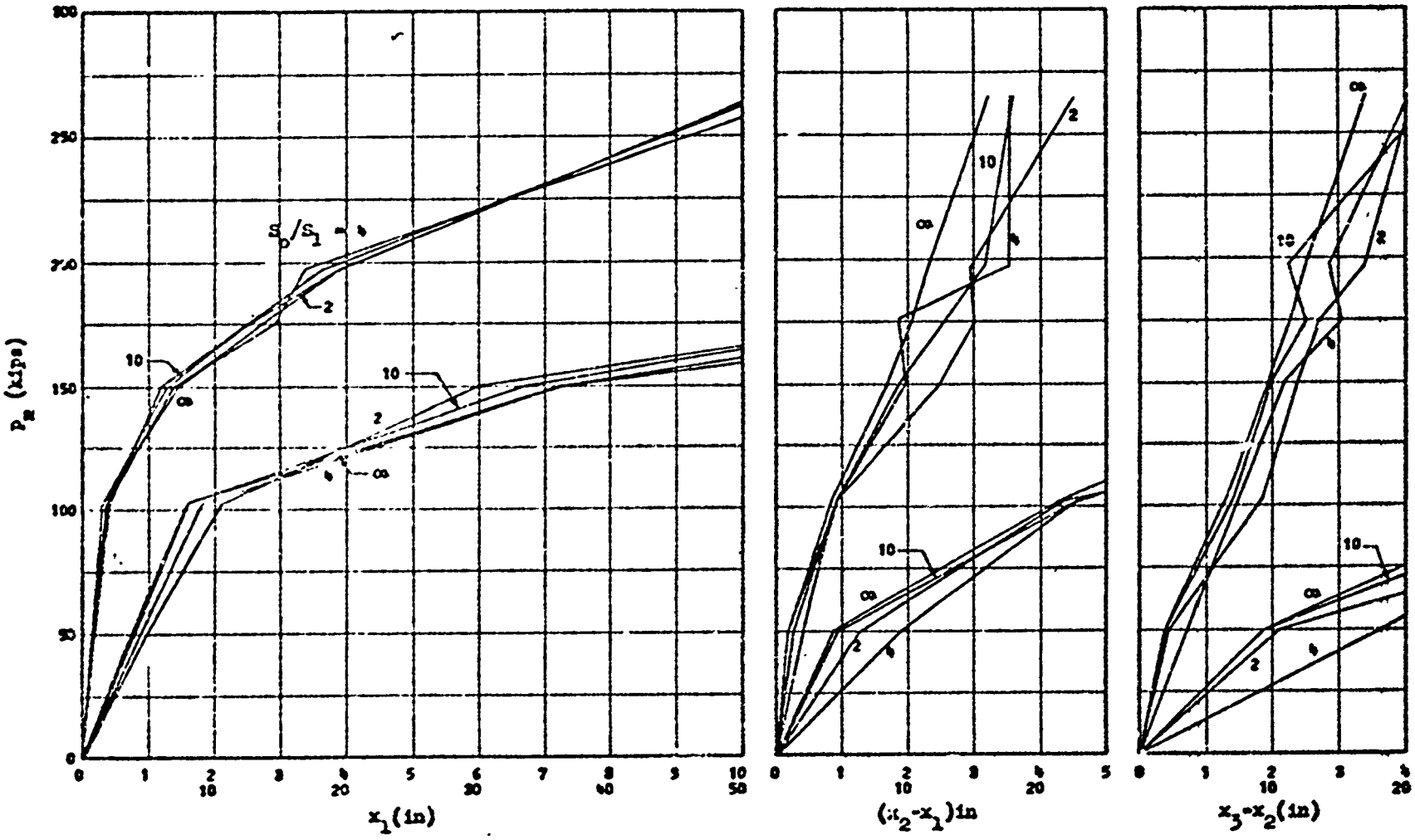
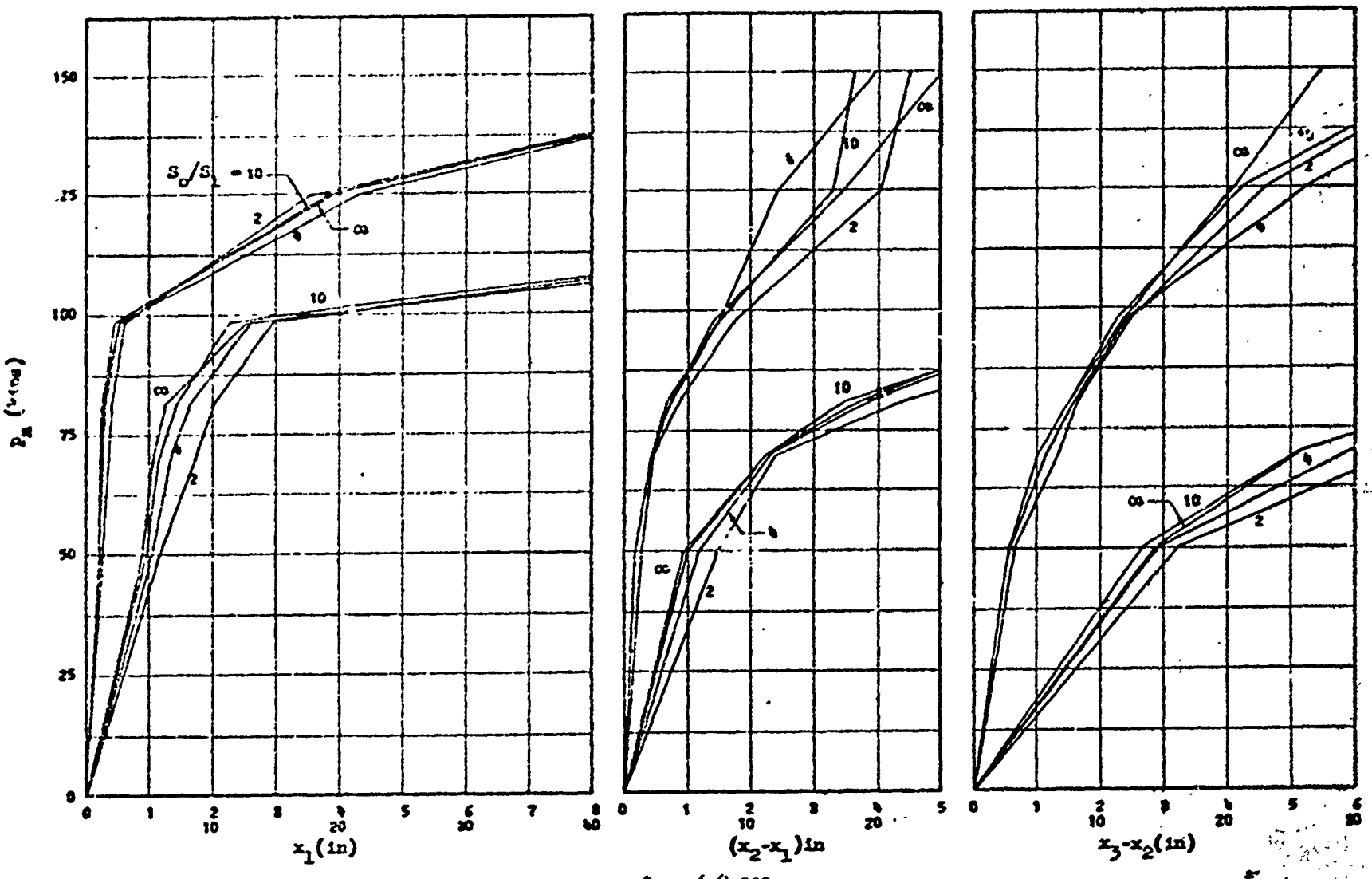


Fig. 9c Maximum Relative Response of Three-Story Frame Due to Initial Peak Triangular Force Pulse
$t_d = 1.0$ sec



t_d = 5.0 sec

Fig. 9d Maximum Relative Response of Three-Story Frame Due to Initial Peak Triangular Force Pulse

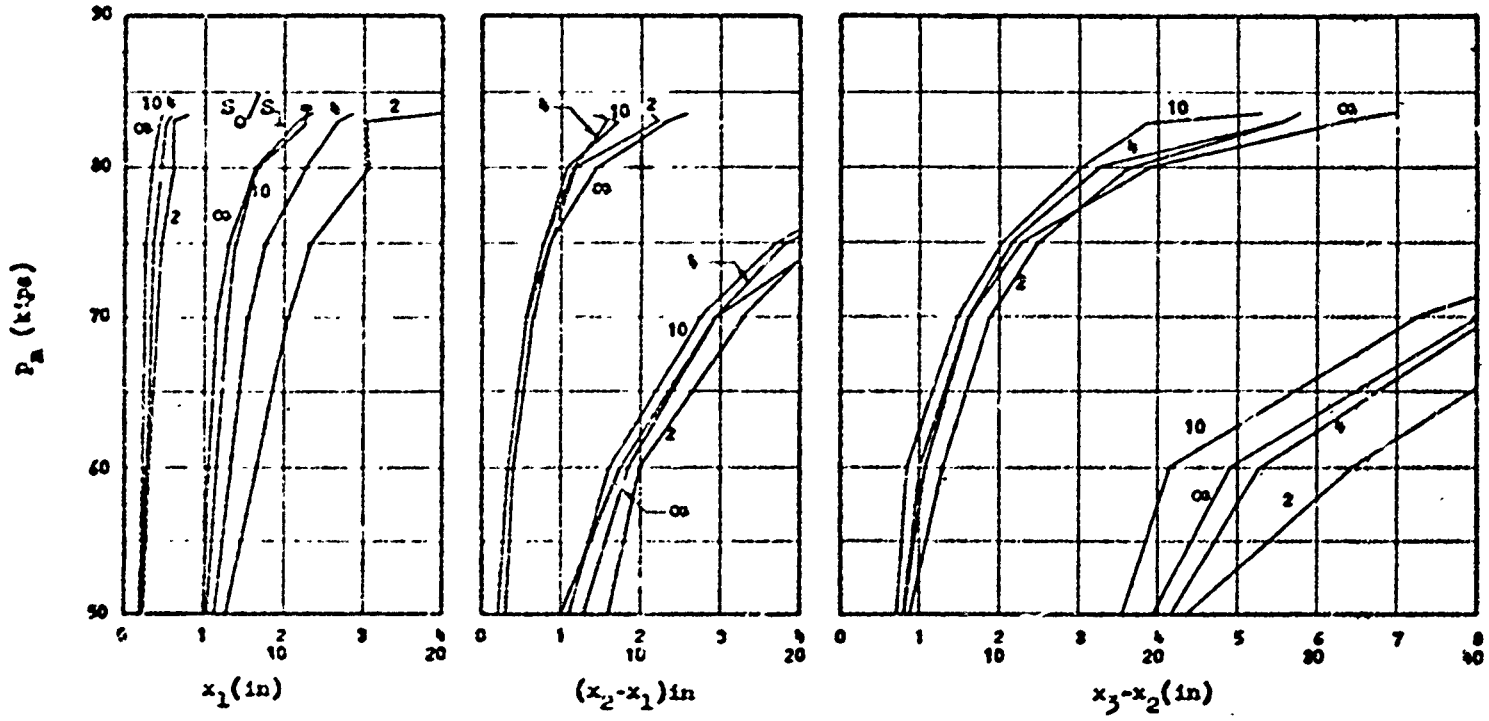


Fig. 9c Maximum Relative Response of Three-Story Frame Due to Initial Peak Triangular Force Pulse

$t_d = \infty$ sec

APPENDIX

Fundamental Natural Frequencies of Building Frames with Flexible Girders

INTRODUCTION

1. Object

One of the principal parameters defining the dynamic response of a multi-story, multi-bay building is the fundamental natural frequency of the structure. The purpose of this study was to develop simple, rapid methods for determination of this frequency.

2. Scope

The basic structures considered were one to ten-story frames of any number of bays. The stiffness of the girders was varied between zero and infinity, although for any value of girder stiffness all girders were considered to be identical. Variations in column stiffness were also considered, although the stiffnesses of all the columns in any one story were the same. The type of column stiffness variation considered was a linear variation; that is, the column stiffness was considered to vary from story to story as a straight line.

Structures with two extreme linear variations in column stiffness were studied first, then the results were extended to include structures with intermediate linear variations. The first type studied was a constant column structure, which has, of course, a linear variation of column stiffness of zero slope. The second type studied had a variation in relative column stiffness, starting from the top story, in the sequence 1, 2, 3.....N, where N is the number of stories.

The parameter, $s = \frac{k-1}{N-1}$, has been introduced to designate different linear variations in column stiffness, where N is the number of stories and k is defined as the ratio of $(\frac{EI}{L})$ for the lower column to $(\frac{EI}{L})$ for the top column, where E, I, and L are respectively the modulus of elasticity of the column material, the moment of inertia of the column section in a given story, and the length of the column per story.

The effect of different mass distributions was not considered. For all structures studied, the distributed column and girder masses were concentrated as point masses at the junctures of the columns and the girders. The total mass at each floor level was considered to be the same in each story.

The effect of rotation of column bases was not studied. All structures studied were considered to have fixed column bases.

METHOD OF ANALYSIS

3. Assumptions

The simplifications and assumptions used in this analysis are as follows:

1. The mass of the structure is concentrated as a series of point masses at the intersections of the girders and columns.
2. All joints are continuous, except when specified as pinned.
3. Deflections are small and in one plane only.
4. All members are linearly elastic.
5. All members deform by bending only, the effect of axial deformation being neglected.
6. The effect of vertical loads on the moments in the columns is neglected.

4. Method of Analysis

Consider a structural frame which has been idealized as described in the preceding section, and is vibrating in a natural mode of vibration such that:

$$y_j = A_j \sin(\omega t + \psi), \quad (j = 1, 2 \ldots N) \tag{1}$$

where y_j is the deflection of the jth story, A_j is the amplitude of vibration of the jth story, ω is the natural circular frequency in radians per second, t is the time in seconds, ψ is a phase angle, and N is the number of stories. The equations of motion for this system are:

$$m_j \ddot{y}_j = V_j, \quad (j = 1, 2 \ldots N) \tag{2}$$

where V_j are the total shearing forces from the adjacent columns and m_j is the total mass at the jth story level. Substituting (1) in (2) yields:

$$-A_j m_j \omega^2 \sin(\omega t + \psi) - V_j = 0, \quad (j = 1, 2 \ldots N) \tag{3}$$

Before these equations can be solved, the shearing forces V_j must be expressed in terms of the y_j.

If we consider the given structure to be deflected so that there is a unit translation of one of the stories, say the jth story, with zero translation of all other stories, then the holding force at any story, S_{ij}, is the negative of the total shear acting on the mass of the ith story due to a unit translation of the jth story. For any translation y_j of the jth story, the shearing force produced is $-S_{ij}y_j$. This applies for any combination of i and j, giving the total shear at any story for a given deflection configuration as:

$$V_j = -(S_{j1}y_1 + S_{j2}y_2 + \ldots + S_{jN}y_N) \tag{4}$$

Substitution of (1) into (4) yields:

$$V_j = -(S_{j1}A_1 + S_{j2}A_2 + \ldots + S_{jN}A_N)\sin(\omega t + \psi) \tag{5}$$

The equations of motion can now be written:

$$-A_j m_j \omega^2 \sin(\omega t + \psi) + (S_{j1}A_1 + S_{j2}A_2 + \ldots + S_{jN}A_N)\sin(\omega t + \psi) = 0,$$

$$(j = 1, 2 \ldots N) \tag{6}$$

Dividing out $\sin(\omega t + \psi)$ gives:

$$A_j m_j \omega^2 - S_{j1}A_1 - S_{j2}A_2 - \ldots - S_{jN}A_N = 0,$$

$$(j = 1, 2 \ldots N) \tag{7}$$

This set of equations in the N unknown $A_1, A_2, \ldots A_N$ will have solutions for the determinant of the equations set equal to zero. The values of ω_n satisfying this determinantal equation are the circular natural frequencies.

In matrix notation, this corresponds to the solution of the equation:

$$\left| [S] - \omega^2 [m] \right| = 0$$

where [S] is the symmetrical N x N matrix of the holding forces S_{ij} and [m] is the diagonal matrix of the masses, m_j. The matrix [S] has been called the "stiffness matrix" of the structure.

This problem was programed for solution on the ILLIAC, the digital computer of the University of Illinois. All the results reported in this Appendix were obtained with this program.

RESULTS

The results of this study are summarized in Figs. A1, A2, A3 and A4. In these graphs the variation of the relative natural frequency, C_1, versus the effective girder stiffness parameter, γ, has been plotted. The parameters C_1 and γ are defined as:

$$C_1 = \frac{\frac{f_1)_\gamma}{f_1)_o} - 1}{\frac{f_1)_\infty}{f_1)_o} - 1} \; ; \; \gamma = \frac{E_g I_g}{(EI)_{avg.}} \times \frac{L}{L_g} \cdot \frac{N+1}{2N} \cdot \frac{2M}{M+1} ,$$

where E, I, and L are respectively the modulus of elasticity of the column, the moment of inertia of the column section, and the column length per story; E_g, I_g, and L_g are respectively the modulus of elasticity of the girder, the moment of inertia of the girder, and the girder length per bay; N is the number of stories; and M is the number of bays. $f_1)_\gamma$ is the first natural frequency of the structure, in cycles per second, for any given value of the effective girder stiffness parameter γ.

The parameter C_1 was chosen because it serves to reduce the variation in the first natural frequency of any structure to the same scale, that is, to a variation between zero and one. The empirical factors $\frac{N+1}{2N}$ and $\frac{2M}{M+1}$ have been used in defining the girder stiffness parameter because they accurately define the effect of additional stories and/or bays on the natural frequencies as compared to a single-story, single-bay structure.

Figure A1 shows C_1 versus γ for various single-bay, uniform column structures ($s = 0$); Fig. A2 shows C_1 versus γ for various single-bay, linearly

varying column structures ($s = 1.0$); Fig. A3 shows C_1 versus γ for various two-bay structures, some with $s = 0$ and some with $s = 1.0$; Fig. A4 shows C_1 versus γ for various three-bay structures, some with $s = 0$ and some with $s = 1.0$. All of these curves have been approximated by the curve of C_1 versus γ for a one-bay, one-story structure. The equation of this curve is derived as follows: For the one-story, one-bay structure:

$$\frac{f_1)_\gamma}{f_1)_o} = \frac{f_1)_\infty}{f_1)_o} \sqrt{\frac{1 + 6\gamma}{4 + 6\gamma}} \tag{9}$$

Then since $\frac{f_1)_\infty}{f_1)_o} = 2$ in this case, we may write:

$$\frac{f_1)_\gamma}{f_1)_o} = \sqrt{\frac{1 + 6\gamma}{1 + 1.5\gamma}} \tag{10}$$

From this it follows (since $\frac{f_1)_\infty}{f_1)_o} - 1 = 1$) that

$$C_1 = \frac{\frac{f_1)_\gamma}{f_1)_o} - 1}{\frac{f_1)_\infty}{f_1)_o} - 1} = \sqrt{\frac{1 + 6\gamma}{1 + 1.5\gamma}} - 1 \tag{11}$$

If this equation is used to approximate the fundamental frequency variation of any structure, we may solve for $\frac{f_1)_\gamma}{f_1)_o}$ to yield:

$$\frac{f_1)_\gamma}{f_1)_o} = 1 + C_1 \left[\frac{f_1)_\infty}{f_1)_o} - 1\right] = 1 + \left[\sqrt{\frac{1 + 6\gamma}{1 + 1.5\gamma}} - 1\right] \left[\frac{f_1)_\infty}{f_1)_o} - 1\right] \tag{12}$$

It is necessary in the application of this formula to have both $f_1)_o$ and $f_1)_\infty$ for the given structure. This information is presented in figures A5 and A6, and tables A1 and A2. In Figure A5 is shown the variation of $\frac{f_1)_o^s}{f_1)_o^{s=0}}$ with the column stiffness parameter s. Figure A5 shows

the variation of $\frac{f_1)_\infty^s}{f_1)_\infty^{s=0}}$ with s. All values of $f_1)_0^s$ and $f_1)_\infty^s$ have been computed using the average stiffness of one column. Tables A1 and A2 contain the values of $f_1)_0^{s=0}$ and $f_1)_\infty^{s=0}$ for $1 \leq N \leq 10$. The values are expressed in terms of $(EI)_{avg.}$ of one column, L the length of the column in one story, and m the total mass at any one story.

These values apply to the one-bay structure. To find the value of $f_1)_0^{s=0}$ and $f_1)_\infty^{s=0}$ for multi-bay structures, one needs only to multiply the value from the table by $\sqrt{\frac{M+1}{2}}$. $(EI)_{avg.}$ still remains as the average stiffness of one column, L remains as the length of the column in one story, and m remains the total mass at one story level. With this information the fundamental frequency can be easily determined.

Example:

Consider a five-story, three-bay structure with $s = 1.0$, $\frac{E_g I_g}{L_g} = 2.78 \frac{(EI)_{avg.}}{L}$, and mass m at each story level.

$$\gamma = 2.78 \frac{6}{10} \cdot \frac{6}{4} = 2.5$$

$$C_1 = \sqrt{\frac{1+15}{1+3.75}} = 1.835$$

From Figure A5, $\frac{f_1)_0^{s=1.0}}{f_1)_0^{s=0}} = 1.176$

From Table A1, $f_1)_0^{s=0} = 0.026\sqrt{\frac{(EI)_{avg.}}{mL^3}}$ for a one-bay structure. For our

problem $f_1)_0^{s=1.0} = 0.026 \times 1.176 \sqrt{\frac{3+1}{2}} \sqrt{\frac{(EI)_{avg.}}{mL^3}} = 0.044 \sqrt{\frac{(EI)_{avg.}}{mL^3}}$

Similarly from figure A6 and table A2:

$$f_1)_\infty^{s=1.0} = 0.166 \times 1.394 \sqrt{\frac{3+1}{2}} \sqrt{\frac{(EI)_{avg.}}{mL^3}} = 0.327 \sqrt{\frac{(EI)_{avg.}}{mL^3}}$$

From these values:

$$\frac{f_1)_\gamma}{f_1)_0}^{=2.5} = 1 + 0.835 \left[\left|\frac{0.327}{0.044}\right| - 1\right] = 6.376$$

$$f_1)_{\gamma=2.5} = 6.376 \times 0.044 \sqrt{\frac{(EI)_{avg.}}{mL^3}} = 0.280 \sqrt{\frac{(EI)_{avg.}}{mL^3}}$$

The solution to this problem as obtained on the ILLIAC is $f_1)_{\gamma=2.5} = 0.271 \sqrt{\frac{(EI)_{avg.}}{mL^3}}$ for an error of 3.3%.

The errors incurred by the approximate formula were found in general to be less than 20% for $\gamma \geq 0.25$ for structures of $1 \leq N \leq 10$, $1 \leq M \leq \infty$, and $0 \leq s \leq 1$.

Nomenclature

The symbols used in this Appendix are defined where they first appear in the text. They are assembled here for convenient reference.

A_j = Amplitude of vibration of mass at j^{th} story

$$C_1 = \frac{\frac{f_1)_\gamma}{f_1)_\infty} - 1}{\frac{f_1)_\infty}{f_1)_0} - 1}$$

E = Modulus of elasticity of material of column

E_ℓ = Modulus of elasticity of material of girder

$f_n = \frac{1}{2\pi} \omega_n$ = n^{th} natural frequency, in cycles per second

I = Moment of inertia of column, in inches^4

I_g = Moment of inertia of girder, in inches^4

$$k = \frac{\frac{EI}{L}\text{ bottom}}{\frac{EI}{L}\text{ top}}$$

L = Length of column in each story, assumed to be constant throughout building

L_g = Length of girder in each bay

$m_j = m$ = Total mass at j^{th} story level

M = Number of bays

n = Order of frequency

N = Number of stories

$$s = \frac{k - 1}{N - 1}$$

S_{ij} = Holding force produced at i^{th} story due to a unit translation of the jth story

V_j = Total shear acting on mass of jth story

y_j = Displacement of jth story

t = Time in seconds

$\gamma = \frac{E_g I_g}{(EI)_{avg.}} \cdot \frac{L}{L_g} \cdot \frac{N + 1}{2N} \cdot \frac{2M}{M + 1}$ = Effective girder stiffness parameter. Effective ratio of girder stiffness to average column stiffness

ψ = Phase angle

ω_n = n^{th} natural circular frequency, in radians per second

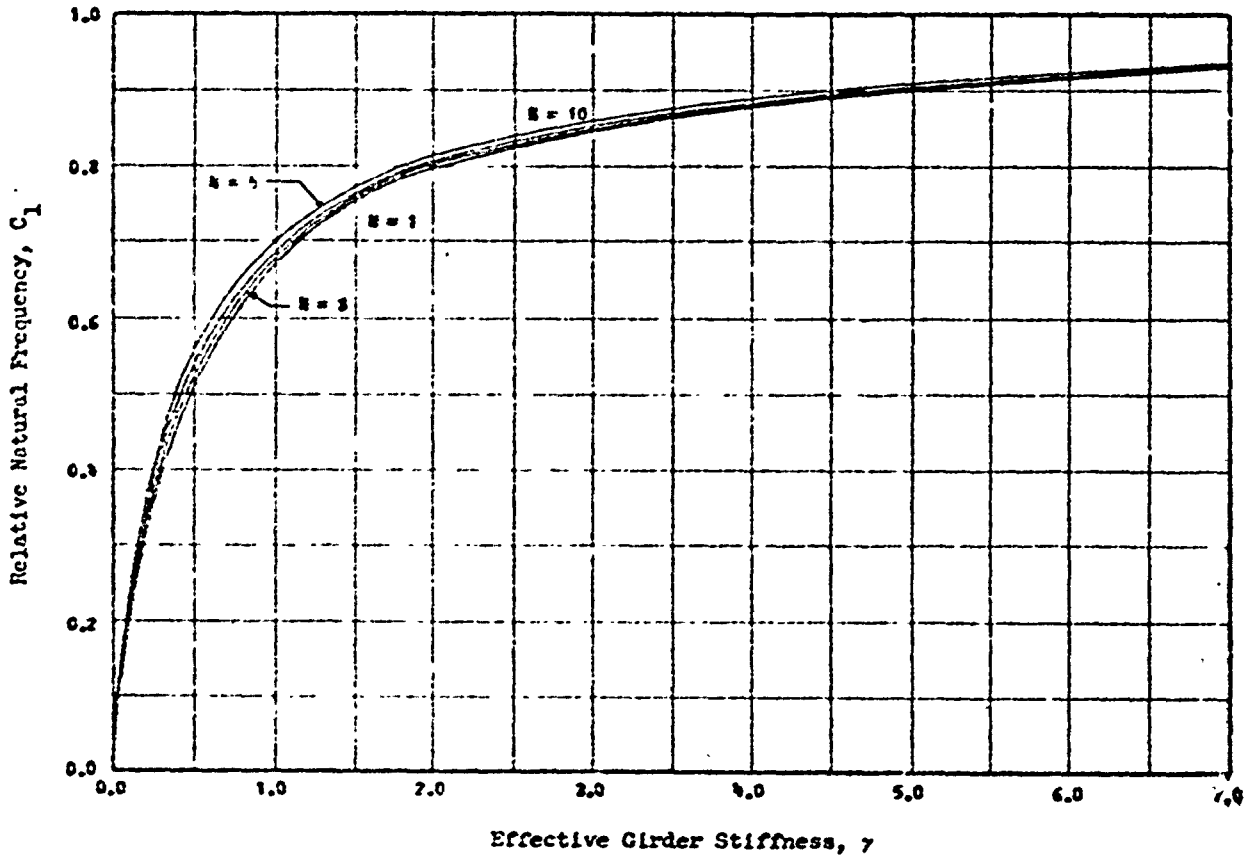


Fig. A1 Relative Natural Frequency for Single-Bay Structures s = o

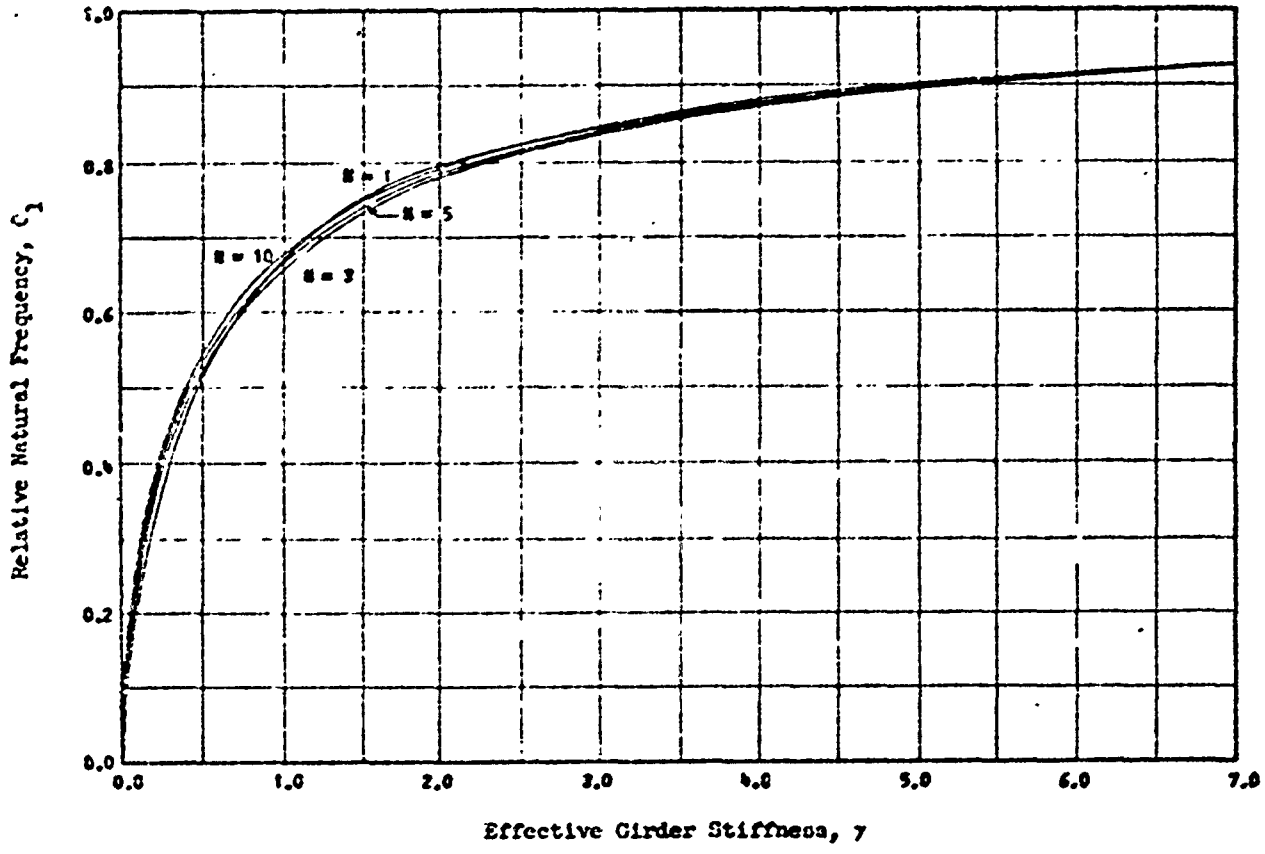


Fig. A2 Relative Natural Frequency for Single-Bay Structures s = 1.0

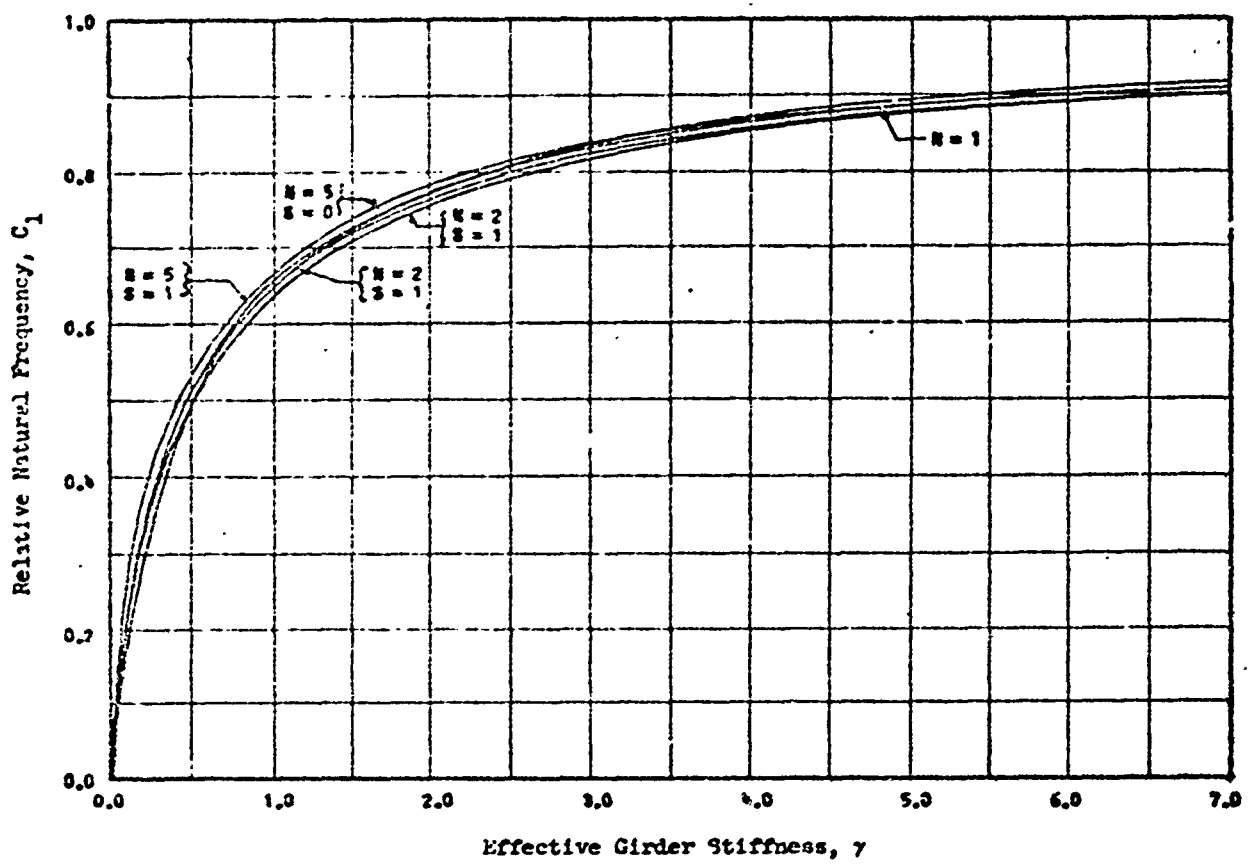


Fig. A3 Relative Natural Frequency for Two-Bay Structures

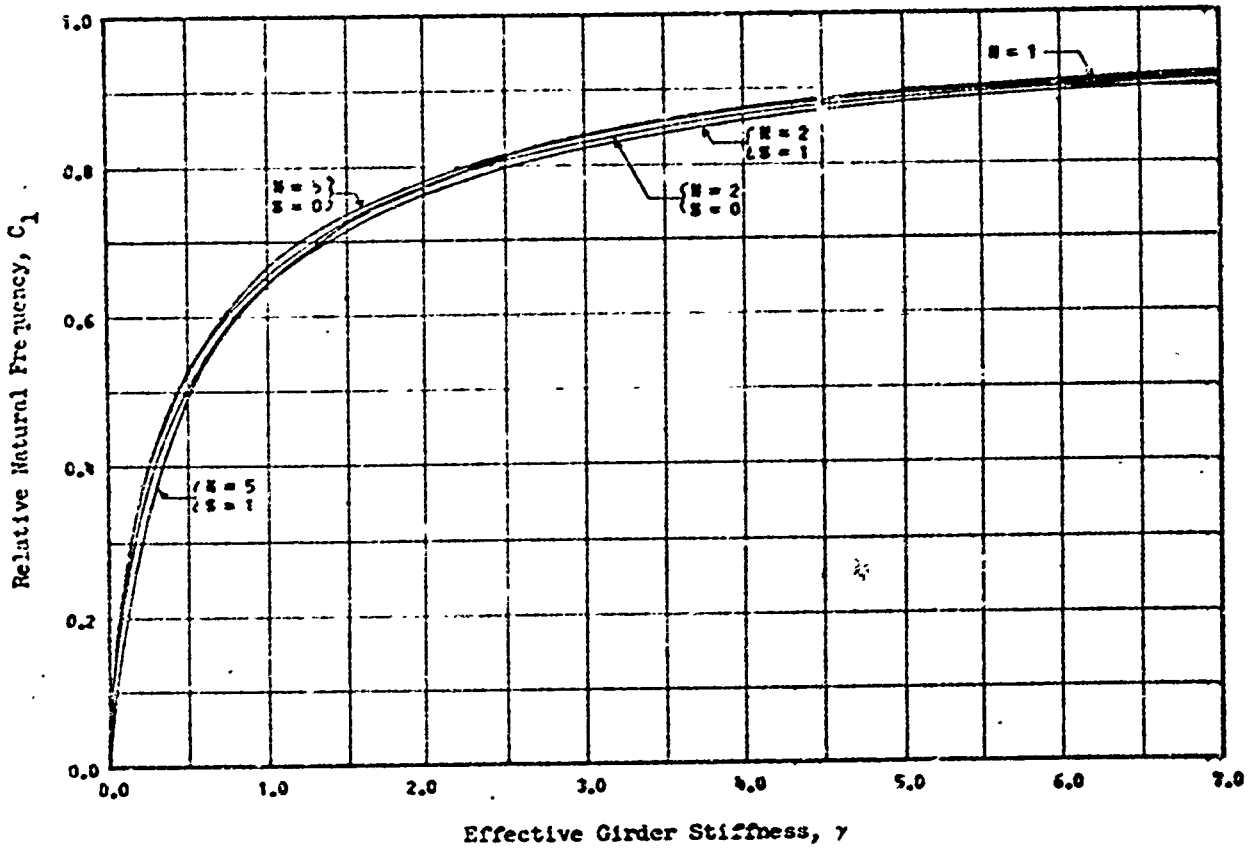


Fig. A4 Relative Natural Frequency for Three-Bay Structures

TABLE A1

First Natural Frequencies of Beam Type, Uniform Column Structures (s=o)

N	$f_1)^{s=o}_{\alpha=o} \left(\frac{1}{2\pi} \sqrt{\frac{(EI)_{avg.}}{mL^3}} \right)$
1	2.449
2	0.826
3	0.414
4	0.248
5	0.166
6	0.118
7	0.089
8	0.069
9	0.055
10	0.045

TABLE A2

First Natural Frequencies of Shear Beam, Uniform Column Structures (s=o)

N	$f_1)^{s=o}_{\gamma=\infty} \left(\frac{1}{2\pi} \sqrt{\frac{(EI)_{avg.}}{mL^3}} \right)$
1	4.899
2	3.028
3	2.180
4	1.701
5	1.394
6	1.181
7	1.024
8	0.904
9	0.809
10	0.732

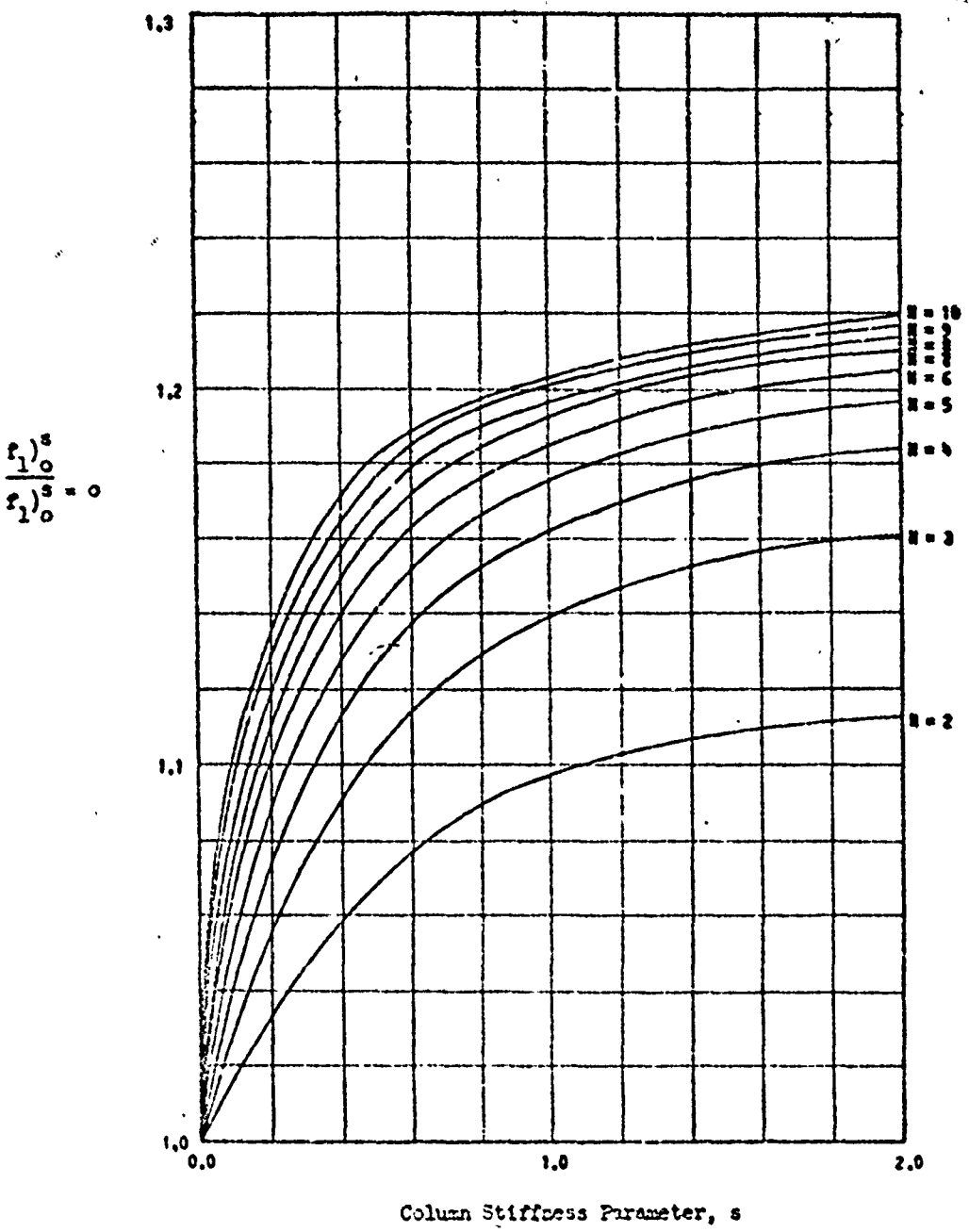


Fig. A5 Fundamental Frequencies for Beam Type Structures

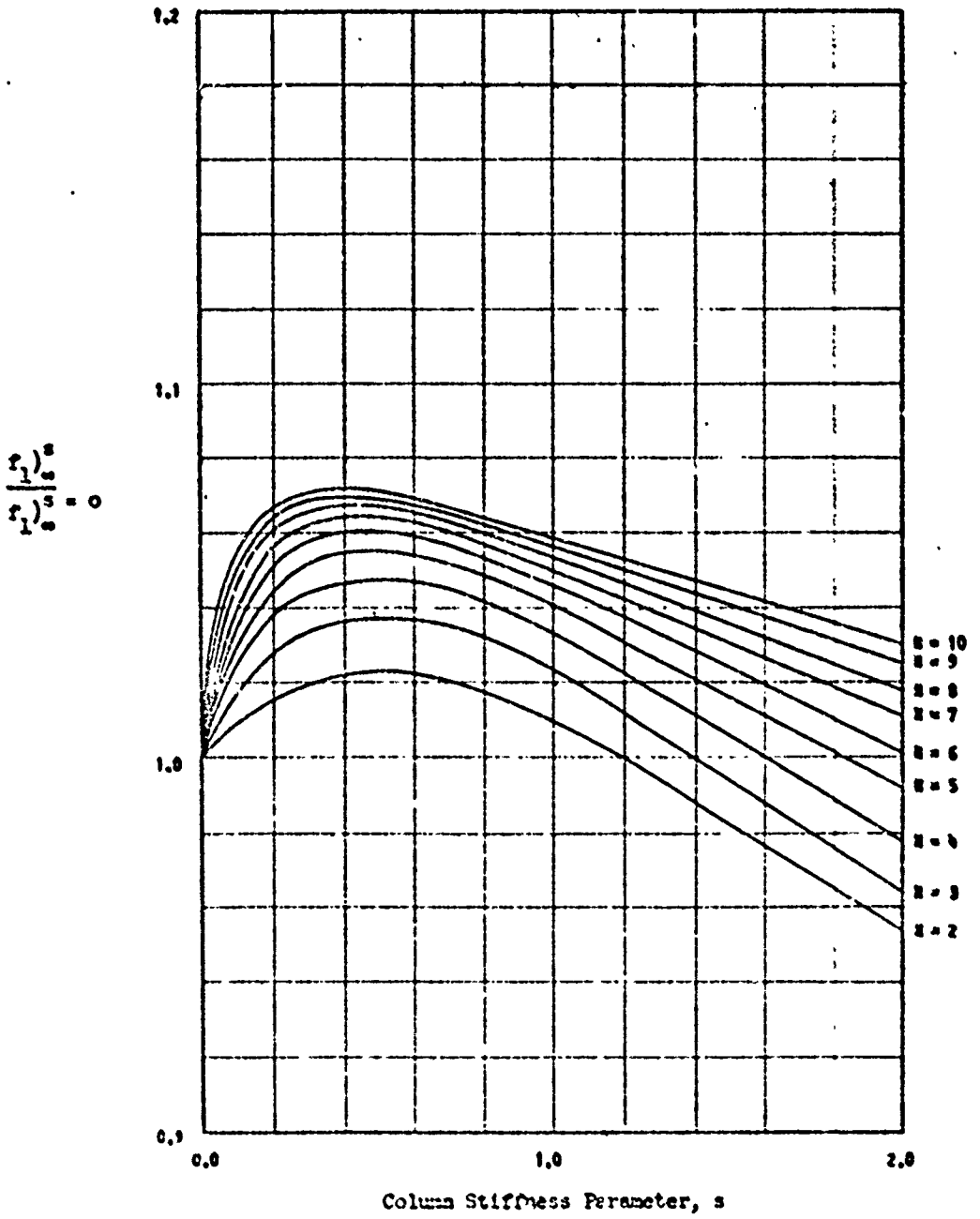


Fig. A6 Fundamental Frequencies of Shear-Beam Structures