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A DEFINITION OF STABLE INELASTIC MATERIAL*

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Abstract

The definitions given previously for work-hardening and perfect plasticity are broadened to cover viscous effects. As before, the system considered includes both the body and the time-dependent set of forces acting upon it. An external agency is supposed to apply an additional set of forces to the already loaded body. It is now postulated that the work done by the external agency on the additional displacements it produces is positive. Some of the restrictions thus imposed on permissible stress-strain relations are explored. Especial attention is paid to simple laws of creep. Uniqueness of solution also is studied.

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Introduction

In the description of material, terms such as elastic, viscous, and plastic are employed separately or in combination as in elastic-plastic, visco-elastic, and visco-plastic. Their meaning is quite clear in a qualitative sense. The difficulty lies in obtaining a broad and physically valid definition of terms which will lead to a useful mathematical formulation of general stress-strain relations. Any restrictions imposed limit the class of materials which are described and so are arbitrary to a considerable extent. In this paper and its predecessors (1) (2)* the restriction is to stability of material in the strict sense.

An elastic body is reversible and non-dissipative and time independent under isothermal conditions. All work done on the body is stored as strain energy and can be recovered on unloading. The stress-strain curve in simple tension may have a wide variety of shapes including the unstable portions shown in Figure 1. If stable elastic material is specified the drop in stress with increase in strain must be ruled out explicitly.

Plasticity denotes irreversibility and permanent or residual strain upon unloading. In the narrow but convenient terminology of the mathematical theory of plasticity, time independence of material properties is understood despite the frequent use of terms such as strain rate and velocity. Time always enters homogeneously in all equations of statics. A given loading path produces the same deformations regardless of the actual time history of the loading. Figure 2 gives possible simple tension curves for a material combining linear elasticity and work-hardening plasticity. The phenomenon of the upper yield point in mild steel is

* Numbers in parentheses refer to the Bibliography at the end of the paper.

a convincing demonstration of the fact that instability of material in the mechanical sense is encountered in the physical world. A gradually falling stress-strain curve likewise is a physical possibility although not found in structural metals when actual rather than nominal stress is plotted. If stable elastic-plastic material is specified, therefore, the drop in stress must be ruled out explicitly.

Time effects are present to some extent in all real materials. When the stress-strain curve is affected significantly by the rate of loading or more generally by the time history of the loading, where time enters in an essential manner, the viscosity or the modifier visco- is used. The idealization known as linear visco-elasticity may be visualized in terms of combinations of linearly elastic springs and Newtonian dashpots, Figure 3. Non-linearity of the springs and the dashpots poses no conceptual difficulty nor does the inclusion of elastic-plastic, "spring" elements. Falling stress-strain curves are easily produced by a simple time history in the most elementary Kelvin or Maxwell models. Instability now has a more obscure meaning and certainly must involve time. For example, a falling curve on a plot of stress versus strain-rate for a Maxwell type material would denote viscous instability, Figure 3.

Previous Postulate for Elastic and Plastic Materials.

A fundamental definition has been advanced previously for a stable elastic or work-hardening time-independent material (1) (2). Consider a body of such material at rest acted upon by a set of boundary tractions T_i and body forces F_i, Figure 4a. An external agency is supposed to add slowly a set of surface tractions ΔT_i and body forces ΔF_i. The displacements of the body will change as added force is applied and equilibrium is maintained. It was postulated that the work done by the external agency during the application of force must be positive and over a cycle of application and removal, positive or zero, zero only when purely elastic changes take place. In such a cycle, work cannot be extracted from the system of the body and the set of forces acting upon it. Materials following this postulated behavior are stable. A falling stress-strain curve in simple tension is ruled out as is far more complicated behavior which normally might not be considered as unstable. Despite the severity of the definition, the consequences do seem in accord with all existing intuitive concepts.

The implications or consequences of the fundamental definition are remarkably strong for the mathematical theory of plasticity. The yield or subsequent loading surface $f(\sigma_{ij}) = k^2$ which separates states of stress which can be joined by a purely elastic path from those requiring elastic-plastic action, Figure 5, must be convex. Furthermore, as shown, the plastic strain increment or strain "rate" must be normal to the surface at a smooth point $\dot{\varepsilon}_{ij}^{p} = \lambda \partial f/\partial\sigma_{ij}$ or lie between the outward normals to adjacent points at a corner.

Both convexity and normality had been postulated long before (3) (4) and it is certainly valid to question the value of replacing one set of postulates by another. One answer is that it is desirable to have the smallest number of postulates. A better statement, perhaps, is that fundamental definitions have many specific implications beyond those of immediate use. Direct experimental study and theoretical consideration of a basic postulate form the basis of unlimited future applications. On the other hand, it should be remembered that the definition of stable time-independent material is a definition and not a law of nature. Unstable materials of various kinds do exist. Also, frictional systems of the

Coulomb type do not follow the postulate (5). A sub-class of materials, therefore, is specified and the problem reduces to how well particular solids are described.

An Extended Postulate or Definition

The absence of time effects in the mathematical theory of elasticity and plasticity is not matched in the physical world (6). Bodies under constant load will creep, bodies under constant deformation will relax. For many structural metals at working temperatures the time effects are small enough to be ignored in practical applications and the previous postulate is adequate. However, all metals at elevated temperatures and most materials at room temperature display appreciable time effects. A modified and broadened postulate, therefore, is required.

When time or viscous effects are appreciable, it is no longer true that the work done by an external agency need be positive. Spring elements as in Figure 3 generally will contain energy which can be extracted from the system. A body under constant load usually does not maintain an unchanged configuration. Application of additional force to an already loaded body by an external agency is accompanied by a change in the displacement history, partly due to the existing force system and partly produced by the external agency. Figure 4 may now be reinterpreted. Figure 4a shows the surface tractions T_i and body forces F_i which are functions of time. The displacements u_i, the stresses σ_{ij}, and the strains ϵ_{ij} are likewise functions of time. Figure 4b shows the existing system plus the external agency and the combined effect. The added surface tractions ΔT_i and the added body forces ΔF_i also vary with time as do the changes in displacement, stress, and strain Δu_i, $\Delta\sigma_{ij}$, $\Delta\epsilon_{ij}$, which result from the action of the external agency superposed on the existing set of forces. Except for the very special fully linear case, it will not be true that ΔT_i and ΔF_i acting alone ($T_i = 0$, $F_i = 0$) would cause Δu_i, $\Delta\sigma_{ij}$, $\Delta\varepsilon_{ij}$. It is $T_i + \Delta T_i$ and $F_i + \Delta F_i$ which produce $u_i + \Delta u_i$, $\sigma_{ij} + \Delta\sigma_{ij}$, $\varepsilon_{ij} + \Delta\varepsilon_{ij}$, all of which are functions of time.

A fundamental definition of a stable inelastic (elastic-visco-plastic) material can now be stated:

<u>The work done by the external agency on the change in displacements it produces must be positive or zero.</u>

It is seen immediately that the previous postulate is included in this broader one because no change in displacement takes place in an elastic-workhardening material at constant load.

The definition may be put in mathematical form as

$$\int_{t=0}^{t_c} \left\{ \int_A \Delta T_i \Delta \dot{u}_i dA + \int_V \Delta F_i \Delta \dot{u}_i dV \right\} dt \geq 0 \qquad [1]$$

where the dot indicates derivative with respect to time, and $t = 0$ is the instant of time at which the external agency begins to apply force. The summation convention is followed for repeated subscripts in the same term, for example $\Delta T_i \, \Delta\dot{u}_i \equiv \Delta T_x \, \Delta\dot{u}_x + \Delta T_y \, \Delta\dot{u}_y + \Delta T_z \, \Delta\dot{u}_z$ and, for later use, $\sigma_{ij}\,\varepsilon_{ij} = \sigma_x\,\varepsilon_x + \sigma_y\,\varepsilon_y + \sigma_z\,\varepsilon_z + \tau_{xy}\,\gamma_{xy} + \tau_{yz}\,\gamma_{yz} + \tau_{zx}\,\gamma_{zx}$ in the usual notation.

It is sometimes more instructive to think of two alternative paths of loading $T_i^{(1)}$, $F_i^{(1)}$, and $T_i^{(2)}$, $F_i^{(2)}$, which diverge at $t = 0$, Figure 6. Each would produce its own u_i, σ_{ij}, ε_{ij}. The definition then has the mathematical form

$$\int_{t=0}^{t_c} \left\{ \int_A [T_i^{(2)} - T_i^{(1)}] [\dot{u}_i^{(2)} - \dot{u}_i^{(1)}] \, dA + \int_V [F_i^{(2)} - F_i^{(1)}] [\dot{u}_i^{(2)} - \dot{u}_i^{(1)}] \, dV \right\} dt \geq 0 \qquad [2]$$

which is really identical to [1]. It has the advantage of focussing attention on the meaning of the incremental quantities and the type of stability which is postulated.

Stability in the Large and Small - Permissibility of Path

Stability in the large and stability in the small correspond respectively to t_c unrestricted and to t_c limited to the immediate neighborhood of $t = 0$. As has been stated the requirement of stability is an arbitrary one in a sense. The requirement that the tensile stress-strain curve for an elastic-plastic body be a continuously rising one (complete stability) is not of real consequence when the stress point is on a portion of the curve which rises and small (infinitesimal) changes in stress are examined (stability in the small).

There is a further choice involved within each classification of stability. Any path labelled (1) in Figure 6 may be permitted or instead $T_i^{(1)}$ and $F_i^{(1)}$ may be restricted to constant values, their values at $t = 0$. It must be kept in mind always that inelastic materials are irreversible and not all paths are permissible paths in general. For example, at $t = t_c$ the system will be at A if it follows path (1) and at B if it follows path (2). With few exceptions, a straight line path from A to B, $T_i = T_i^{(1)} + \alpha(T_i^{(2)} - T_i^{(1)})$, $F_i = F_i^{(1)} + \alpha(F_i^{(2)} - F_i^{(1)})$, with α going from 0 to 1, will not change the state at (1) to the state at (2). Furthermore, no neighboring path to the straightline will exist which is capable of giving the same final state B. In fact for the most general materials there is no available path whatsoever from (1) to (2); path (2) is the only way of producing the state of the material at B.

In view of the irreversibility there is no clear decision to be made on how much stability to require. Stability in the small with path (1) restricted to the point corresponding to $t = 0$ would seem essential. The generalized definition without restriction, or stability of material in the complete sense, seems desirable. Some of its consequences will be explored in what follows.

Stress-strain Relations

The consequences of the basic postulate with respect to stress-strain relations are strongest and seen most easily for a homogeneous system whose mass or kinetic energy can be neglected. Surface tractions, body forces, and velocities in Inequality [2] then can be replaced by stresses σ_{ij} and strain rates $\dot{\epsilon}_{ij}$ through the theorem of virtual work. For any continuous u_i'

$$\int_A T_i^* u_i' \, dA + \int_V F_i^* u_i' \, dV = \int_V \sigma_{ij}^* \epsilon_{ij}' \, dV \qquad [3]$$

where T_i^*, F_i^*, σ_{ij}^* are any set of equilibrium values and u_i', ϵ_{ij}' are any compatible set (7). There need be no relation between the equilibrium and the compatible sets. Substitution of $T_i^{(2)} - T_i^{(1)}$ for T_i^*, $\dot{u}_i^{(2)} - \dot{u}_i^{(1)}$ for u_i', etc., in the virtual work expression [3] thus is permissible although in general $T_i^{(2)} - T_i^{(1)}$, $F_i^{(2)} - F_i^{(1)}$ alone will not produce $\dot{u}_i^{(2)} - \dot{u}_i^{(1)}$. Integration over the volume can be omitted for a homogeneous state of stress and strain with the result

$$\int_{t=0}^{t_c} [\sigma_{ij}^{(2)} - \sigma_{ij}^{(1)}][\dot{\varepsilon}_{ij}^{(2)} - \dot{\varepsilon}_{ij}^{(1)}]\, dt \geq 0 \qquad [4]$$

In the neighborhood of $t = 0$ the stresses and strain rates can be expanded in a power series for each of the paths (1) and (2), the subscript zero denoting the value at $t = 0$.

$$\begin{aligned} \sigma_{ij} &= \sigma_{ij}]_o + \dot{\sigma}_{ij}]_o t + \ddot{\sigma}_{ij}]_o\, t^2/2 + \ldots \\ \dot{\varepsilon}_{ij} &= \dot{\varepsilon}_{ij}]_o + \ddot{\varepsilon}_{ij}]_o\, t + \dddot{\varepsilon}_{ij}\, t^2/2 + \ldots \end{aligned} \qquad [5]$$

The two paths are identical up to $t = 0$ so that stress and strain are the same at $t = 0$, $\sigma_{ij}^{(1)}]_o = \sigma_{ij}^{(2)}]_o$ and $\varepsilon_{ij}^{(1)}]_o = \varepsilon_{ij}^{(2)}]_o$. Time derivatives of these quantities need not be the same so that strain rates and stress rates may differ. For t_c very close to $t = 0$, Inequality [4] gives

$$\begin{aligned} &[\dot{\sigma}_{ij}^{(2)} - \dot{\sigma}_{ij}^{(1)}]_o[\dot{\varepsilon}_{ij}^{(2)} - \dot{\varepsilon}_{ij}^{(1)}]_o\, t_c^2/2 \\ &+ \left\{[\dot{\sigma}_{ij}^{(2)} - \dot{\sigma}_{ij}^{(1)}]_o[\ddot{\varepsilon}_{ij}^{(2)} - \ddot{\varepsilon}_{ij}^{(1)}]_o + \tfrac{1}{2}[\ddot{\sigma}_{ij}^{(2)} - \ddot{\sigma}_{ij}^{(1)}]_o[\dot{\varepsilon}_{ij}^{(2)} - \dot{\varepsilon}_{ij}^{(1)}]_o\right\} t_c^3/3 \\ &+ \ldots \geq 0 \end{aligned} \qquad [6]$$

As t_c may be made vanishingly small, it is necessary that

$$[\dot{\sigma}_{ij}^{(2)} - \dot{\sigma}_{ij}^{(1)}]_o [\dot{\varepsilon}_{ij}^{(2)} - \dot{\varepsilon}_{ij}^{(1)}]_o \geq 0 \qquad [7]$$

If, however, $\dot{\varepsilon}_{ij}^{(2)} \equiv \dot{\varepsilon}_{ij}^{(1)}$ then it is necessary that

$$[\dot{\sigma}_{ij}^{(2)} - \dot{\sigma}_{ij}^{(1)}]_o [\ddot{\varepsilon}_{ij}^{(2)} - \ddot{\varepsilon}_{ij}^{(1)}]_o \geq 0 \qquad [8]$$

This process may be continued to higher and higher time derivatives of strain and of stress as well if $\dot{\sigma}_{ij}^{(2)} \equiv \dot{\sigma}_{ij}^{(1)}$, etc. The general requirement in the sense of [7] and [8] is

$$\left[\frac{\partial^m \sigma_{ij}^{(2)}}{\partial t^m} - \frac{\partial^m \sigma_{ij}^{(1)}}{\partial t^m} \right]_o \left[\frac{\partial^n \varepsilon_{ij}^{(2)}}{\partial t^n} - \frac{\partial^n \varepsilon_{ij}^{(1)}}{\partial t^n} \right]_o \geq 0 \qquad [9]$$

with m and n each ≥ 1. Inequality [9] insures stability in the small in the broad sense.

If $\sigma_{ij}^{(1)}$ is restricted to remain constant at $\sigma_{ij}^{(1)}]_o$, as time goes on, for the more restricted definition of stability in the small, [7] and [8] are replaced by

$$\dot{\sigma}_{ij}^{(2)}]_o [\dot{\varepsilon}_{ij}^{(2)} - \dot{\varepsilon}_{ij}^{(1)}]_o \geq 0 \qquad [10]$$

$$\dot{\sigma}_{ij}^{(2)}]_o [\ddot{\varepsilon}_{ij}^{(2)} - \ddot{\varepsilon}_{ij}^{(1)}]_o \geq 0 \qquad [11]$$

For time independent materials the (1) terms disappear completely leaving the earlier result $\dot{\sigma}_{ij}\dot{\varepsilon}_{ij} \geq 0$.

Return now to stability in the large and to time dependent materials. If the path OA of Figure 6 is traversed very rapidly and then the system remains at A, and a similar time history holds for path OB, the fundamental postulate in the form [4] requires

$$(\sigma_{ij}^B - \sigma_{ij}^A)(\dot{\varepsilon}_{ij}^B - \dot{\varepsilon}_{ij}^A) \geq 0 \qquad [12]$$

In general, the strain rates $\dot{\varepsilon}_{ij}^{B}$ and $\dot{\varepsilon}_{ij}^{A}$ will depend, of course, upon this specially selected time history. Here the distinction between the broad and narrow definition of stability does not arise because the path OAB traversed with extreme rapidity gives the same state point B as path OB, for materials which need time to deform.

Many special paths may be considered but their treatment will be deferred. Specific results for plasticity, linear visco-elasticity, and creep will be discussed next.

Consequences for Plasticity

As has been mentioned, the previous postulate for time-independent materials, with $\sigma_{ij}^{(1)}$ held constant, has strong implications in plasticity. All yield and subsequent loading surfaces $f(\sigma_{ij}, \varepsilon_{ij},$ history of loading$) = k^2$ must be convex. The plastic strain rate vector $\dot{\varepsilon}_{ij}^{p}$ must be normal to the surface in stress space, Figure 5, in the extended sense. At a smooth point of $f = k^2$, for $\frac{\partial f}{\partial \sigma_{mn}} \dot{\sigma}_{mn} > 0$ or an outward pointing $\dot{\sigma}$ which produces plastic deformation.

$$\dot{\varepsilon}_{ij}^{p} = G \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{mn}} \dot{\sigma}_{mn} \qquad [13]$$

where G may be a function of stress, strain, and the prior history of loading. It may also depend upon the stress rate $\dot{\sigma}_{ij}$. Time is not truly relevant so that G must be homogeneous of order zero in $\dot{\sigma}$, e.g., $G = f^n [1 + \frac{(\dot{\sigma}_{pq}\dot{\sigma}_{qr}\dot{\sigma}_{rp})^2}{(\dot{\sigma}_{mn}\dot{\sigma}_{mn})^3}]$.

The broader definition in this paper, of stability in the small, restricts G still more. It is permissible to deal with plastic strain rates only and substitution in Inequality [7] gives

$$\left[\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij}^{(2)} - \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij}^{(1)} \right] \left[G^{(2)} \frac{\partial f}{\partial \sigma_{mn}} \dot{\sigma}_{mn}^{(2)} - G^{(1)} \frac{\partial f}{\partial \sigma_{mn}} \dot{\sigma}_{mn}^{(1)} \right] \geq 0 \quad [14]$$

The magnitude of the stress rate is arbitrary and can be chosen to give a constant value of the component $\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij}$ normal to the yield or loading surface. Suppose this is done for each pair (1) and (2) so that the left bracket of [14] is zero. The bracketed expression containing $G^{(1)}$ and $G^{(2)}$ then must likewise be zero. If not, a very small increase or decrease in the magnitude of $\dot{\sigma}_{ij}^{(2)}$ or the opposite for $\dot{\sigma}_{ij}^{(1)}$ will violate Inequality [14]. Therefore, $G^{(2)} = G^{(1)}$ or G is independent of $\dot{\sigma}_{ij}$. The stress-strain relation [13] thus is restricted to linearity in the increments or rates of stress and strain. All incremental stress-strain relations now in use are linear or are combinations of linear forms as a matter of convenience. Although no pre-conceived notions are upset, it is of great interest and possibly of value to know that a single postulate or definition leads inevitably to all the main features of plastic stress-strain relations.

Stability in the small alone was employed in this application of the postulate, but stability in the large is needed for convexity. As stated before, there is always the possibility that stability in the large with the new postulate may be more restrictive than is reasonable on physical grounds for all paths of loading in all materials.

Consequences for Linear Visco-elasticity

A linear visco-elastic material may be defined in terms of models, Figure 3, with linear springs and dashpots in combination (8). An equivalent definition may be given by one or more differential equations in which time derivatives of any order may appear but stress and strain enter linearly. The difference between two solutions of such equations is a solution as well. Therefore, if stress state and history $\sigma_{ij}^{(1)}$ produce $\dot{\varepsilon}_{ij}^{(1)}$ and stress state and history $\sigma_{ij}^{(2)}$ produce $\dot{\varepsilon}_{ij}^{(2)}$, $\sigma_{ij}^{(2)} - \sigma_{ij}^{(1)}$ will give $\dot{\varepsilon}_{ij}^{(2)} - \dot{\varepsilon}_{ij}^{(1)}$. A stable linear viscoelastic material according to definition as expressed by the basic Inequality [4] must obey

$$\int_{t=0}^{t_c} \sigma_{ij} \dot{\varepsilon}_{ij} \, dt \geq 0 \qquad [15]$$

This is no surprise at all as it is simply a statement that work must be done on an unstressed and unstrained viscoelastic material in order to deform it. Any combination of linear springs and dashpots thus provides a stable material. Although the result is not unexpected, it is comforting to know that the implications of the broadened definition are in accord with physical intuition.

Consequences for Creep

Non-linearity is the rule rather than the exception in inelastic materials. The creep of metals in particular is markedly non-linear. Consider now a Maxwell type material, Figure 3, which deforms indefinitely under constant load. The elastic response can be ignored in discussing the viscous. Suppose that, for convenience and for reasonable agreement with actual behavior, the viscous strain rate $\dot{\varepsilon}_{ij}^{v}$ is a function of stress alone. As the existing state of stress determines the strain rate, the stress rate has no immediate influence; $[\dot{\sigma}_{ij}^{(2)} - \dot{\sigma}_{ij}^{(1)}][\dot{\varepsilon}_{ij}^{(2)} - \dot{\varepsilon}_{ij}^{(1)}]$ in Inequality [7] is automatically zero at any instant of time when two paths of loading diverge. Inequality [8] for stability in the small, however, does provide a non-trivial restriction on the creep strain relation.

Stability in the large gives far more spectacular and useful results. Referring to Inequality [12], the restriction on the path or time history of loading is lifted. $(\sigma_{ij}^{B} - \sigma_{ij}^{A})(\dot{\varepsilon}_{ij}^{B} - \dot{\varepsilon}_{ij}^{A}) \geq 0$ applies to any two stress states at any time because $\dot{\varepsilon}$ is supposed to depend upon σ alone and so is not path dependent. In particular, σ_{ij}^{A} may be chosen as the stress free state and the result is

$$\sigma_{ij}^{B} \; \dot{\varepsilon}_{ij}^{B} = \dot{W}(\sigma_{ij}^{B}) \geq 0 \qquad [16]$$

or the rate of doing work is always positive. As the strain rate is viscous, and, therefore, dissipative, this too is an obvious result whereas [12] is not.

The restrictions on stress-strain relations in creep imposed by [12] and [16] are strong and may be explored for each special case. In pictorial terms as shown in Figures 7, 8, the strain rate vector makes an acute angle with the stress vector, $\sigma_{ij}^{B} \dot{\varepsilon}_{ij}^{B} \geq 0$; the increment in strain rate for any stress path is in the direction of the change in stress $(\sigma_{ij}^{B} - \sigma_{ij}^{A})(\dot{\varepsilon}_{ij}^{B} - \dot{\varepsilon}_{ij}^{A}) \geq 0$.

Suppose, for example, that it is possible to write

$$\dot{\varepsilon}_{ij} = \frac{\partial \varphi}{\partial \sigma_{ij}} \qquad [17]$$

where φ is a function of the existing state of stress only. The surface φ = constant in stress space then bears some pictorial resemblance to the yield surface in plasticity. The strain rate vector is normal to the surface. As illustrated in Figure 8a, a stress path tangential to the surface at one point cannot intersect the surface at another point. If it did, Inequality [12] would be violated. Normality therefore leads to convexity of the surface φ = constant. Normality in the extended sense, as at a corner of such a surface likewise requires convexity if [12] is to be satisfied, Figure 8b.

A common assumption for creep in simple tension (not necessarily in accord with the actual physical behavior) is that the strain rate is proportional to some power n of the stress (9). A generalization of this hypothesis is to choose φ in [17] as a homogeneous function of stress of degree $n+1$. Substituting in [16], the rate of doing work or rate of dissipation is from Euler's theorem on homogeneous functions

$$\dot{W}(\sigma_{ij}) \equiv \sigma_{ij}\,\dot{\varepsilon}_{ij} = \sigma_{ij}\frac{\partial\varphi}{\partial\sigma_{ij}} = (n+1)\varphi \qquad [18]$$

Under these conditions φ = constant is a surface in stress space of constant rate of dissipation of energy. Therefore $\dot{W}$ = constant defines a convex surface in stress space.

The additional assumption of isotropy requires $\dot{W}$ to be a function of the stress invariants, for example, the sum of the principal stresses J_1 and the invariants $J_2 = \frac{1}{2} s_{ij}\, s_{ij}$, $J_3 = \frac{1}{3} s_{ij}\, s_{jk}\, s_{ki}$ of the stress deviation tensor s_{ij}, (10). Considerable discussion of convex homogeneous functions of stress with these invariants has been given in papers on plasticity (10)(11) and need not be repeated here. As an example, the requirement of convexity restricts the constant c in $(n+1)\varphi = \dot{W} = b(J_2^3 - cJ_3^2)^n$ to no more than 2.25.

Uniqueness of Solution to Equilibrium Broundary Value Problems (12)(13)

Consider a body at time $t = 0$ whose state at each point can be specified completely as, for example, the initial unstressed and unstrained condition. Suppose surface tractions T_i are specified on the portion of the surface A_T, displacements u_i on the remaining portion A_u, and body forces F_i throughout the volume V, each given as a function of time. The question of uniqueness concerns the possibility of two (or more) solutions (1) and (2) for the interior stresses, strains, and displacements or any of their time derivatives.

Assume that two equilibrium solutions do exist which satisfy the boundary conditions. Substitute their difference in the equation of virtual work [3] just as in the description following Equation [3]. The general result is that the left hand side is always zero and for any order of time derivative of stress and strain

$$0 = \int_V \left[\frac{\partial^m}{\partial t^m} \sigma_{ij}^{(2)} - \frac{\partial^m}{\partial t^m} \sigma_{ij}^{(1)} \right] \left[\frac{\partial^n}{\partial t^n} \varepsilon_{ij}^{(2)} - \frac{\partial^n}{\partial t^n} \varepsilon_{ij}^{(1)} \right] dV \qquad [19]$$

which is reminiscent of Inequality [9] except that here the notation is meant to include the zero order as well, i.e.

$$[\sigma_{ij}^{(2)} - \sigma_{ij}^{(1)}][\varepsilon_{ij}^{(2)} - \varepsilon_{ij}^{(1)}], \quad [\sigma_{ij}^{(2)} - \sigma_{ij}^{(1)}][\dot{\varepsilon}_{ij}^{(2)} - \dot{\varepsilon}_{ij}^{(1)}], \quad \text{etc.}$$

Uniqueness in the sense of $\frac{\partial^m}{\partial t^m} \sigma_{ij}^{(2)} = \frac{\partial^m}{\partial t^m} \sigma_{ij}^{(1)}$ or $\frac{\partial^n}{\partial t^n} \varepsilon_{ij}^{(2)} = \frac{\partial^n}{\partial t^n} \varepsilon_{ij}^{(1)}$ is assured if the integrand of [19] is positive. Just as in the special case of plasticity, a discussion of uniqueness in such great generality cannot be more specific (13). Any lack of uniqueness which may be present in the stress-strain relation appears here as well. With this understanding the term uniqueness will be used without qualification in the following.

Reference to Inequality [7] shows that stability in the small guarantees uniqueness of $\dot{\sigma}_{ij}$ or $\dot{\varepsilon}_{ij}$ at each instant of time. If $\dot{\varepsilon}_{ij}$ is dependent on σ_{ij} and not $\dot{\sigma}_{ij}$, Inequality [8] takes over and $\ddot{\varepsilon}_{ij}$ or $\dot{\sigma}_{ij}$ are shown directly to be unique. This type of consideration is of importance in the study of dynamic problems in which mass or inertia cannot be neglected. If needed, this process can be continued to any term of Inequality [9]. The initial state and the complete history of loading, therefore, determine the stress and the strain uniquely throughout any body composed of inelastic material obeying the definition of stability in the small.

It might be expected that stability in the large guarantees uniqueness of stress independent of the path of loading. This is true for any material undergoing the rapid loading type of path described for Inequality [12]. It also is true for those stable materials following any creep law in which the strain rate depends upon the stress only because they too most obey Inequality [12]. It is not true for most materials.

Certainly the fundamental definition does require that if a path exists from A to B, Figure 6, the work done by the external agency on the change in displacements it produces must be positive or zero, $\int_A^B (\sigma_{ij}-\sigma_{ij}^A)(\dot{\varepsilon}_{ij}-\dot{\varepsilon}_{ij}^A)\,dt \geq 0$. However, in general the integral bears little or no relevance to the virtual work term $(\sigma_{ij}^B-\sigma_{ij}^A)(\dot{\varepsilon}_{ij}^B-\dot{\varepsilon}_{ij}^A)$ whose positiveness assures uniqueness. Furthermore, no path AB is a permissible one in the general case.

Dynamics

All of the emphasis so far has been on equilibrium problems. Mass and kinetic energy have been ignored, although velocity and acceleration terms are finite. As will be seen, the inclusion of dynamic terms does not affect the conclusions to be drawn and, in fact, makes less demands on the stress-strain relations. Homogeneous states of stress and strain are not appropriate and the steps which lead to Inequality [4] give instead the more inclusive

$$\int_{t=0}^{t_c} \left\{ \int_v [\sigma_{ij}^{(2)} - \sigma_{ij}^{(1)}][\dot{\varepsilon}_{ij}^{(2)} - \dot{\varepsilon}_{ij}^{(1)}] \, dv \right\} dt + \left\{ \int_v \frac{1}{2} \rho [\dot{u}_i^{(2)} - \dot{u}_c^{(1)}]^2 \, dV \right\}_{t=0}^{t=t_c} \geq 0 \tag{20}$$

where the second term is the value of the volume integral at t_c minus the value at the time of divergence of paths, $t = 0$. If the external agency applies load slowly, the acceleration $\ddot{u}_i$ is the same for the two paths (1) and (2), Figure 6. Necessarily, in all cases the velocity $\dot{u}_i^{(2)}$ throughout the volume is the same as $\dot{u}^{(1)}$ at $t = 0$ and the second term of [20] is positive or zero at all times t_c. The Inequality [4] is much more significant, therefore, than [20].

Conclusions

A single postulate or definition is advanced for all stable inelastic materials. Its implications are far reaching. In plasticity theory, which is a particularly well-explored subject, the concepts of convexity, normality, and incremental linearity all are consequences of the definition. An interesting result also is obtained for a generalization of the simple tension creep law $\dot{\varepsilon} = K\sigma^n$. The surface of constant rate of dissipation for the special case is convex and $\dot{\varepsilon}$ is normal to it just as for the yield surface in plasticity. However, non-linear viscous behavior coupled with elastic and plastic action is an almost unknown field. Much more must be done before the value of the postulate can be determined. Stability in the large is an especially troublesome point. Nevertheless, the uniqueness of solution assured by the definition is a good indication of its acceptability as a starting point.

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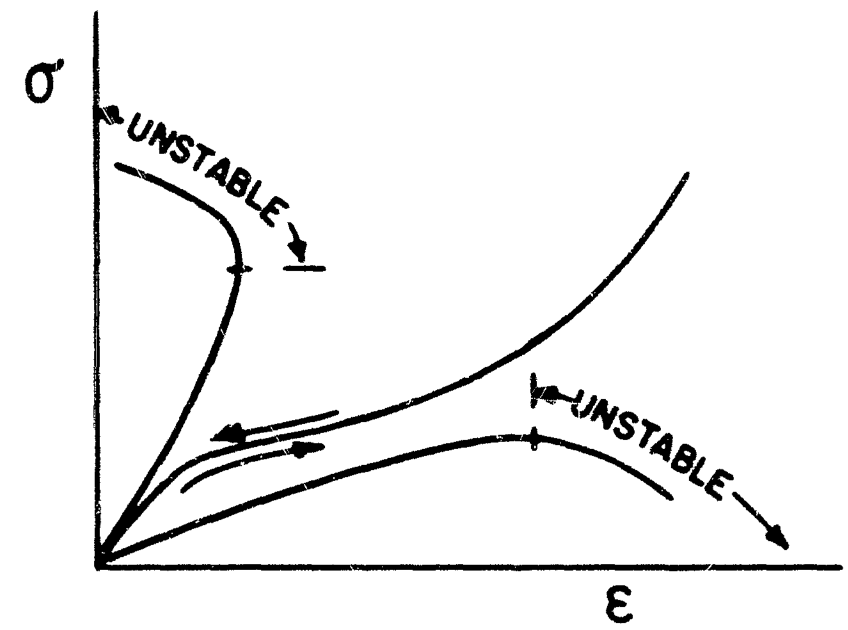

FIGURE 1. AN ELASTIC MATERIAL (REVERSIBLE): CURVE RISING TO THE RIGHT $\dot{\sigma}\dot{\varepsilon} > 0$

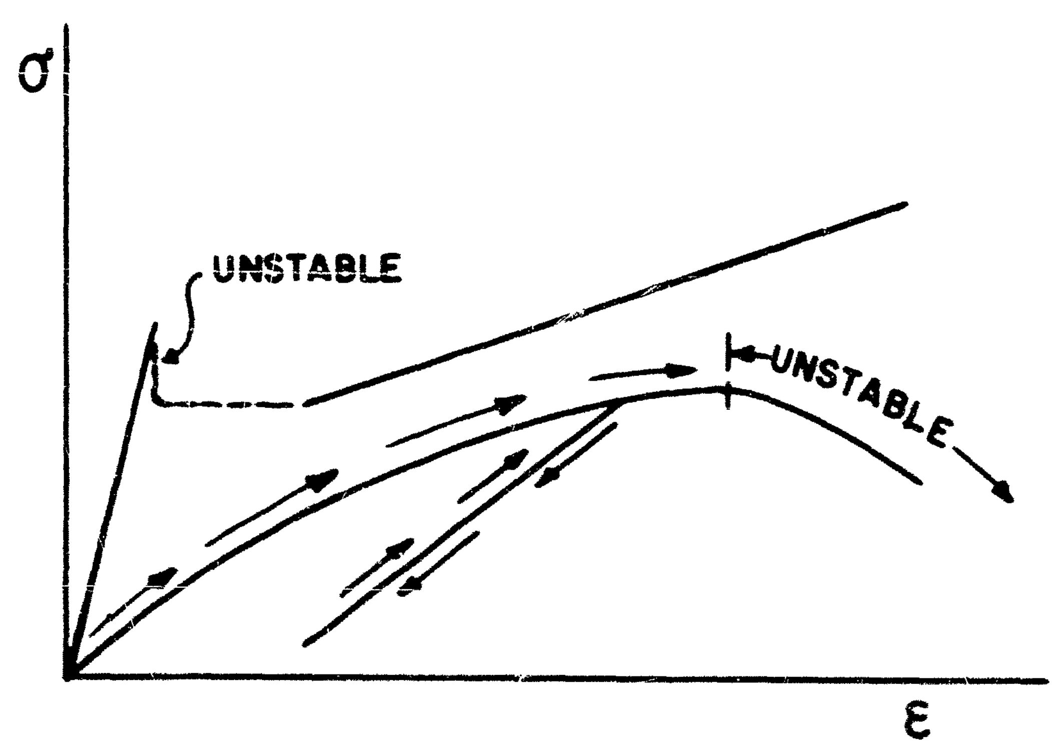

FIGURE 2. AN ELASTIC-PLASTIC MATERIAL (TIME-INDEPENDENT): RISING CURVE $\dot{\sigma}\dot{\varepsilon} > 0$

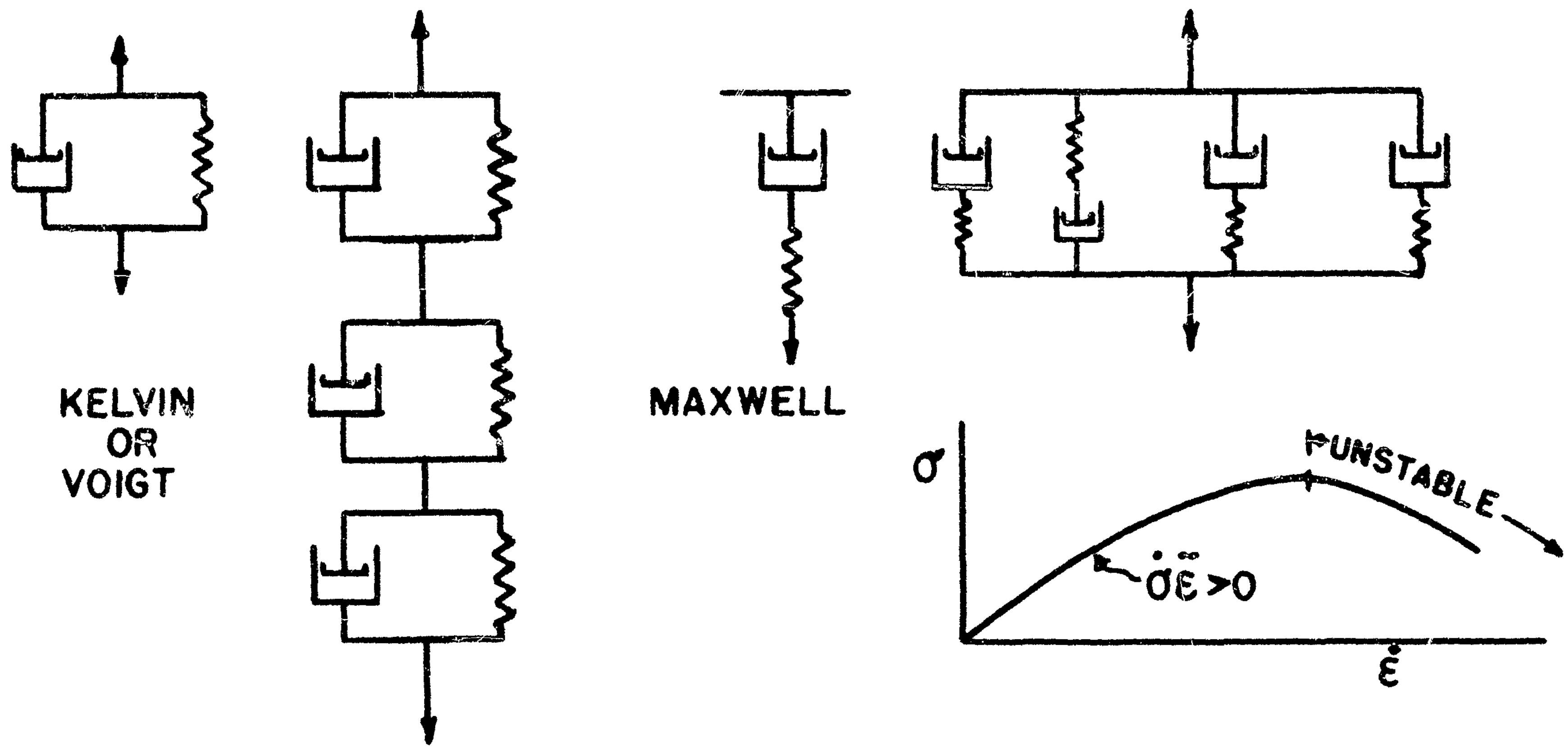

FIGURE 3. VISCOELASTIC MODELS

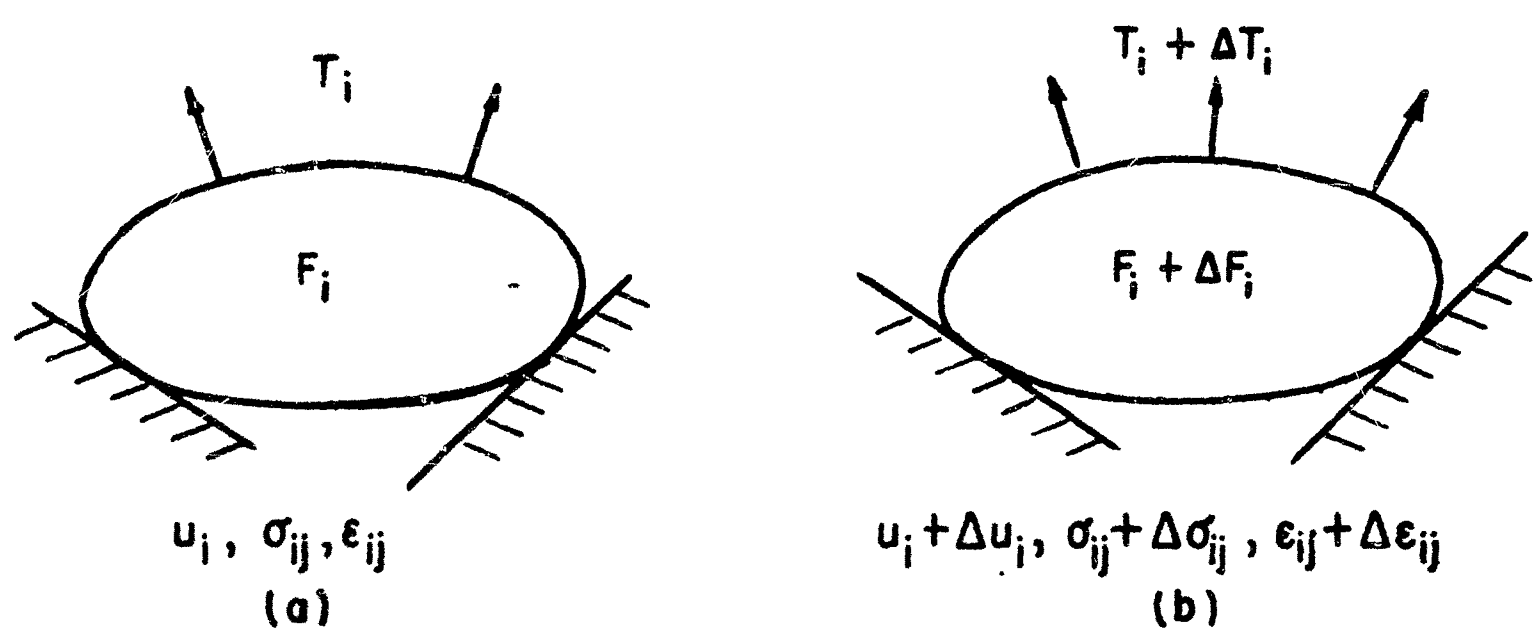

FIGURE 4. EXISTING SYSTEM (a) AND EXISTING SYSTEM PLUS EXTERNAL AGENCY (b)

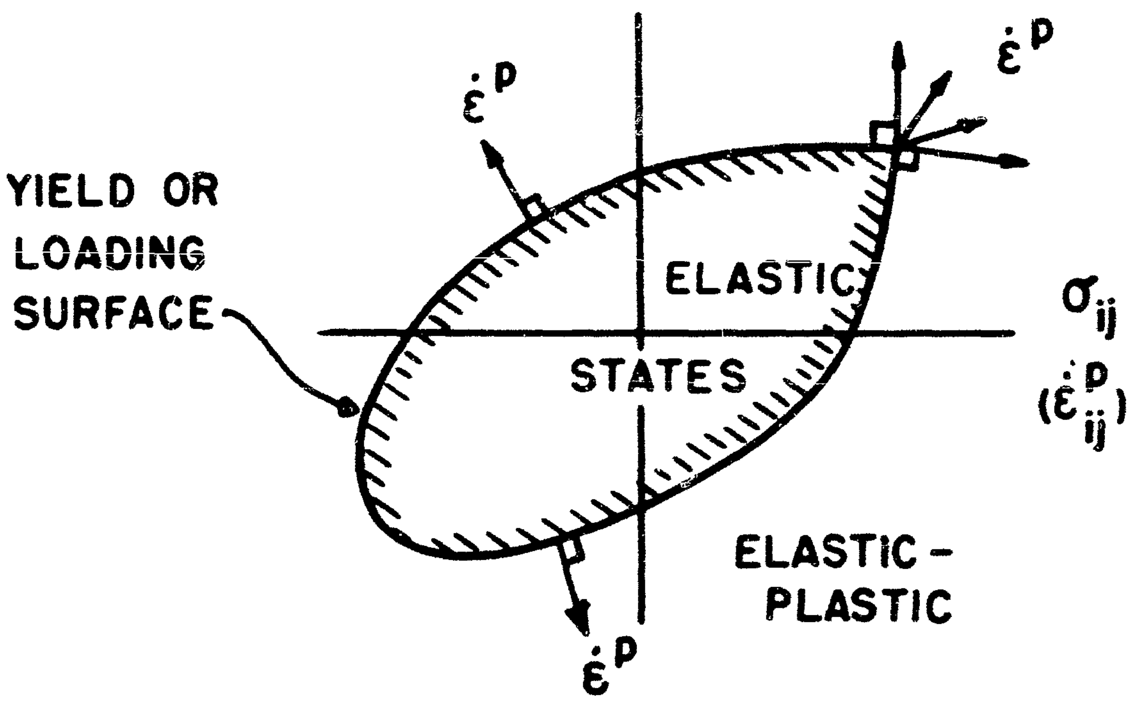

FIGURE 5. THE FUNDAMENTAL DEFINITION REQUIRES CONVEXITY AND NORMALITY IN PLASTICITY

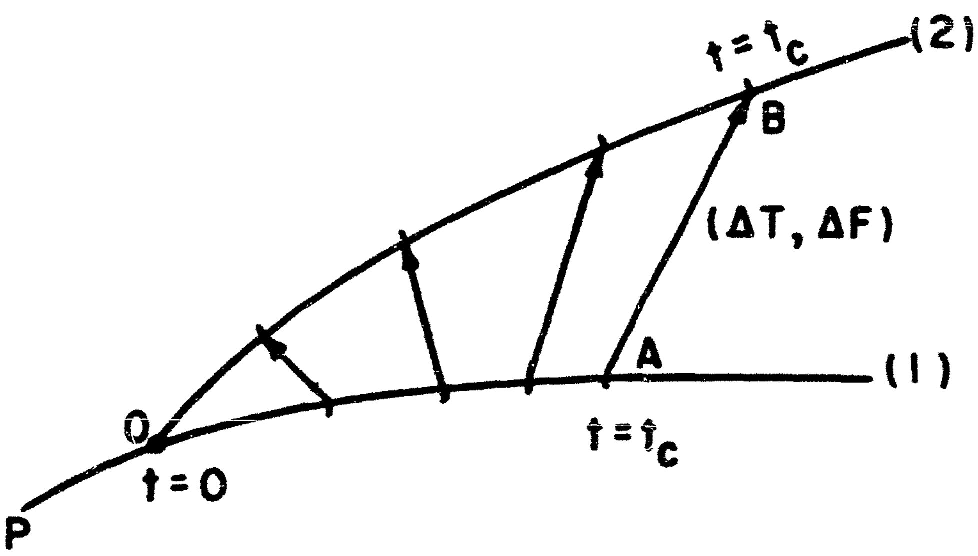

FIGURE 6. PATHS OF LOADING. (Arrows join points of the same time)

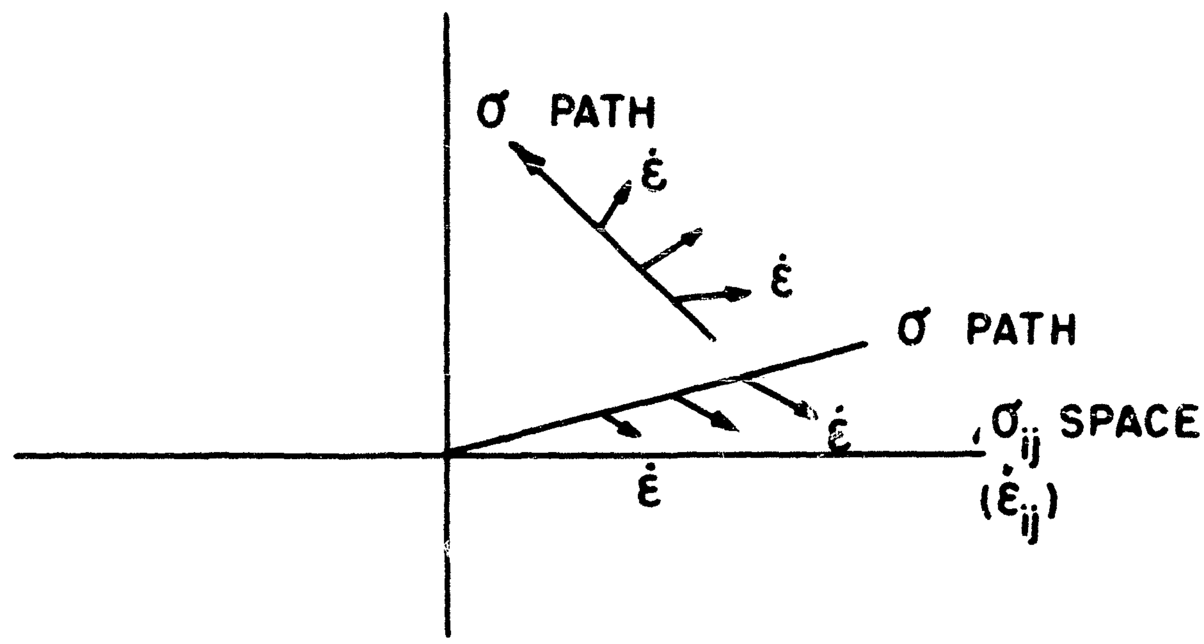

FIGURE 7. $[\sigma_{ij}^{B} - \sigma_{ij}^{A}][\dot{\varepsilon}_{ij}^{B} - \dot{\varepsilon}_{ij}^{A}] \geq 0$

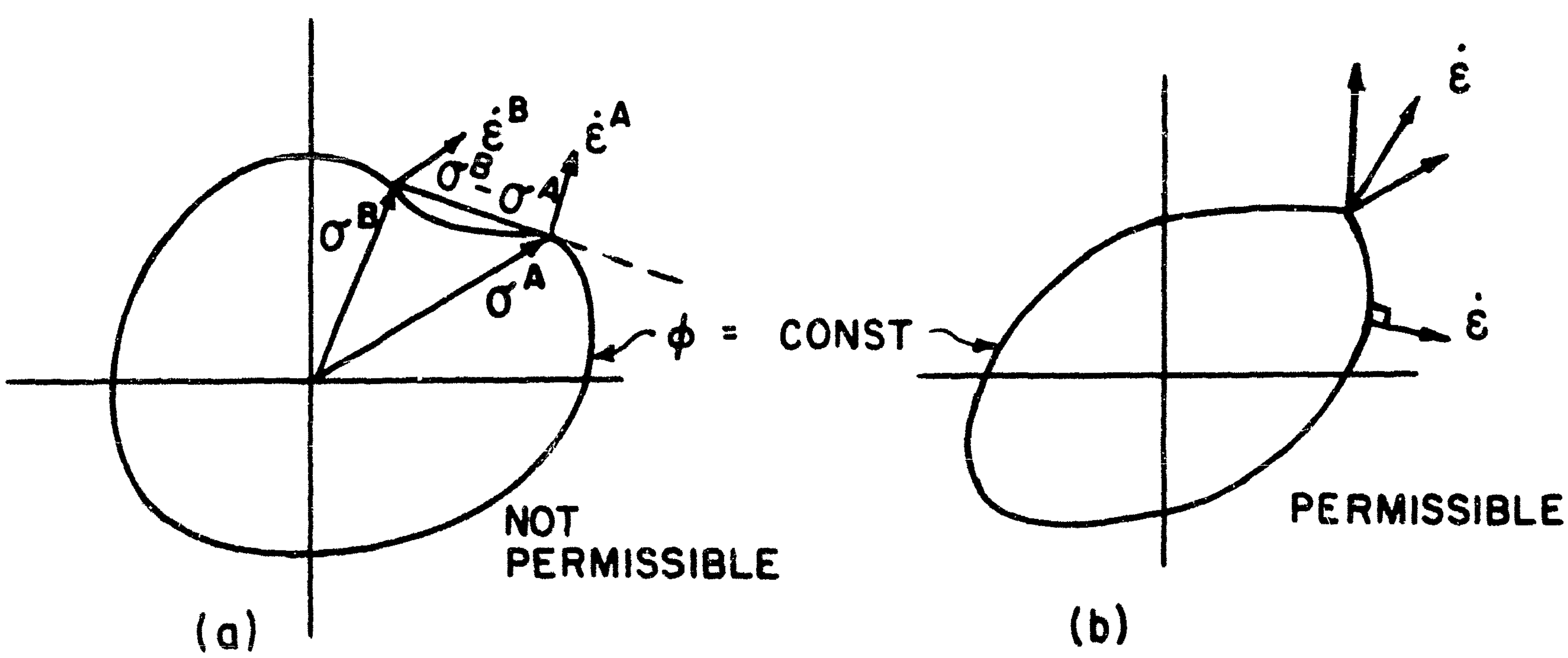

FIGURE 8. (a) NORMALITY, $\dot{\varepsilon}_{ij} = \partial\phi/\partial\sigma_{ij}$, AND INEQUALITY [12] REQUIRES CONVEXITY. (b) A CORNER ALSO IS PERMISSIBLE

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