INCOMPRESSIBLE POTENTIAL FLOW ABOUT
AXIALLY SYMMETRIC DUCTED BODY

PHILIP LEVINE

AERONAUTICAL RESEARCH LABORATORY

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ANALYSIS

The velocity potential and stream functions for a point source, sink, line source, ring vortex, and ring source are derived for the case of axially symmetric irrotational potential flow in an infinitely long sheet of constant density.

Methods for the development of axisymmetric body shapes in uniform sheeted flow are presented. Several practical applications are indicated.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

[Signature]

Colonel, USA
Chief, Aerodynamical Research Laboratory
Directorate of Laboratories
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a  radius of a ring or disk source, sink or doublet
b  natural logarithm base; e = 2.71828
C  stream function
D, \phi  modified Bessel functions of the first and second kind, respectively
\lambda  eigenvalue
\lambda, \nu  integral function of the first and second kind, respectively
R  strength of a source or sink form
S  strength of a doublet form
T  dimensionless radial coordinate; ratio of radial distance to duct radius
u  ratio of \nu/v

V  superimposed uniform flow velocity
W  dimensionless axial coordinate; ratio of axial distance to duct radius
Y  velocity potential
\xi, \eta  axial and radial velocity components respectively

\xi', \eta'  partial differentiation with respect to \( \xi \)
\eta', \xi'  partial differentiation with respect to \( \eta \)

1, 2, 3, 4, 5, ...  consecutive numbers
INTRODUCTION

The problem of determining the complete flow field about axially symmetric bodies occurs widely. Typical instances are the design and analysis of the performance of axisymmetric missiles and devices utilizing these bodies, and apparatus used to test these components, ring type flame holders, and turbine-machinery elements generally. In view of the numerous practical applications, it was decided to initiate a study of the problem for the case of incompressible flow.

Little theoretical work has been published concerning the flow about ducted axisymmetric bodies, especially for the cases where the body fills a portion of the cross-sectional area of the duct. The present theoretical study was well advanced before it was discovered that a solution for a point source in an infinitely long duct of constant cross-section had recently been published [Ref. 1]; however, no correction was made. However, in the expression for the velocity potential of a source at a given point, the expression is derived here, as it forms the basis for the development of a general expression of the study of the flow past ducted axisymmetric bodies.

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ANALYSIS

In a problem to be solved, the case of velocity, incompressible and potential flow about an inviscid body fixed on the axis of an infinitely large parallel plate of constant height, as

the wall is to be solved and the mathematical treatment are presented in Fig. 1.

The differential equation for the velocity potential of an

incompressible, incoherent flow is well known, and may be written as (Ref. 6):

\[ \nabla^2 \phi = 0 \]  

(1)

Fortunately, this equation is amenable to separation of variables, and consequently the solutions are obtained by superimposing the solutions by merely adding simple solutions together.

I. Potential Sources on the Right Side

An expression for the velocity potential of a point source will be developed first. For this case, the potential solutions for a sink and a doublet follow directly. These solutions may be superimposed on each other to construct asymmetric bodies of the desired shape.

Consider a source, of strength \( \mu \), as shown in Fig. 2, located at \( r = 0 \). The boundary conditions which must be satisfied are:

I. The source flow is uniform at \( r = \infty \). Thus,

\[ \phi = \frac{\mu}{2\pi} \log r \]  

II. At \( r = \infty \), the velocity of the source \( (2\mu) \), and at \( r = 0 \), the velocity of the source flow \( -\phi \). This follows from continuity as the total source flow exists in half about the (x) axis. The (1) sign accounts for the velocity direction as indicated in Fig. 1.

III. The radial component of the velocity is zero at the first wall. Thus,

\[ j_x = 0 \]  

IV. The source flow is symmetric about the (x) axis as well as the (z) axis. Hence, a singularity exists at \( r = 0 \). Hence,

\[ j_r = 0 \]  

for \( r \neq 0 \). Proceeding by the method of separation of variables, let
and substituting into Eq. 1, yields

\[ \frac{1}{2} \mathbf{R}_T + \frac{1}{2} \mathbf{R}_S = - \frac{\delta}{2} \mathbf{z}_S \times \mathbf{x} = \text{constant} \]  

(4)

There are three possibilities: \( x = 0, x = \pm \infty \), where \( \delta \) is a constant.

Examining the case of \( x = \infty \), \( x / \infty = 0 \), one finds

\[ \mathbf{z}_S = \mathbf{0} \]

\[ \mathbf{R}_{TT} \equiv \frac{1}{2} \mathbf{R}_T + \mathbf{J}^T \mathbf{R} = \mathbf{0} \]

The solutions to the above equations are well known as

\[ \mathbf{R} = \mathbf{Z}_1(\mathbf{r}) \mathbf{Z}_2(\mathbf{r}) \]

and

\[ \mathbf{F} = \mathbf{Z}_1(\mathbf{r}) \mathbf{F}_1(\mathbf{r}) + \mathbf{Z}_2(\mathbf{r}) \mathbf{F}_2(\mathbf{r}) \]

respectively, where \( \mathbf{Z}_1(\mathbf{r}) \) is the Bessel function of the first kind of zero order, and \( \mathbf{Z}_2(\mathbf{r}) \) is the Bessel function of the second kind of zero order. Therefore, a solution to Eq. 1 exists in the form

\[ \mathbf{S} = \left( D_1 e^{-j\mathbf{r}\cdot\mathbf{r}} + D_2 e^{j\mathbf{r}\cdot\mathbf{r}} \right) \left( F_1 r_0(\mathbf{r}) + F_2 r_0(\mathbf{r}) \right) \]

By satisfying conditions I and II, it is apparent that \( D_1 = 0 \) and either \( D_2 = 0 \) or \( r_0 > 0 \). Now, assuming \( r_0 > 0 \), in positive or negative modes, one can write

\[ \mathbf{S} = \left( D_2 e^{j\mathbf{r}\cdot\mathbf{r}} \right) \left( F_1 r_0(\mathbf{r}) + F_2 r_0(\mathbf{r}) \right) \]

Furthermore, to satisfy condition II, the solution must have the form,

\[ \mathbf{S} = \mathbf{S}_0 e^{j\mathbf{r}\cdot\mathbf{r}} \]
where the term \((\frac{a}{a_0})\) can be thought of as a coefficient \(a_0\) in the case of \(k = 0\).

The condition III requires that

\[ j_1(x) = 0 \]

Therefore, the values of \(j_1(x)\) are the nth-order roots of \((j_1)\), where \((j_1)\) is the Bessel function of the first kind of first order.

The form of the solution then becomes,

\[ A_n = 2m + A_0 e^{-in\alpha} \]

There is now a solution corresponding to each value of \(n\).

It should be noted that conditions I, II and III have been established to satisfy condition IV, a series solution can be built up, using Eq. 5, thereby.

\[ f = 2m + \sum_{n=1}^{\infty} A_n e^{-in\alpha} \]

Introducing condition IV into Eq. 6, one has,

\[ (\phi_n)_{2m} = \sum_{n=1}^{\infty} A_n e^{-in\alpha} \]

The orthogonality property of Bessel functions is such that (Ref. 3)

\[ \int_0^1 J_0(1_r) J_0(1_r) dr = \begin{cases} 0 & \text{if } n = m \\ 1 & \text{if } n = m \end{cases} \]

Using the above property, one obtains

\[ \int_0^1 r J_0(1_r) (\phi_n)_{2m} dr = 2m \int_0^1 r J_0(1_r) dr - A_n \int_0^1 r^2 J_0^2(1_r) dr \]

The second integral on the right can be readily evaluated (Ref. 3) as

\[ \int_0^1 r^2 J_0^2(1_r) dr = \frac{\pi}{2} e^{-2\alpha} \]

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The first integral on the right is zero, as
\[ \int_{0}^{1} (1-x^2) dx = \frac{1}{1^3} \left[ \frac{1}{1} \right]_{0}^{1} = 0 \]

Thus,
\[ A_n = \frac{2}{\pi 1^3} \int_{0}^{1} (1-x^2) J_n(1-x) dx \]

Taking the size of the source to be very small (close to a mathematical point), one can write, in accordance with Condition IV,

\[ A_n = \frac{2}{\pi 1^3} \lim_{r \to 0} \int_{0}^{1} (1-x^2) J_n(1-x) dx \]

But, \( \lim_{r \to 0} J_n(1-x) = 1 \), so that one has,

\[ A_n = \frac{2}{\pi 1^3} \lim_{r \to 0} \int_{0}^{1} (1-x^2) J_n(1-x) dx \]

The integral above represents the flow emitted by the source, which is \( 2\pi n \) in either direction by Gauss's theorem, so that

\[ \Lambda = \frac{2}{\pi 1^3} \]

A solution to Eq. 1, satisfying the four boundary conditions, can now be obtained by substituting the equation for the coefficients \( A_n \) into Eq. 6, yielding,

\[ f = \int_{0}^{1} \sum_{n=1}^{\infty} \frac{\sin \pi x}{\pi n} J_n(\omega' \rho) \]

(7)
Returning to Eq. 4, and examining the case of \( x = y^2 \), one finds that the boundary conditions cannot be satisfied with a series type of solution. Details of this case are presented in Appendix I.

The stream function corresponding to the above velocity potential is

\[
\psi = 2m^2 \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi}{L}}{n^2} J_1 (n \alpha r) \tag{8}
\]

**Sign convention:**
- \( (+) \) for \( x > 0 \)
- \( (-) \) for \( x < 0 \)

At \( x = 0 \), the stream function is discontinuous, so that approaching \( x = 0 \) from the left (see Fig. 2), \( \psi = -m \), while approaching \( x = 0 \) from the right, \( \psi = +m \).

That Eq. 8 represents the stream function may be seen by direct substitution into the following relationships for axisymmetric, incompressible, potential flow.

\[
\phi_x = -\frac{1}{r} \phi_r \quad \phi_y = \frac{1}{r} \phi_r \tag{9}
\]

The solution for a sink follows directly, as it influences the derived velocity potential and stream function only through the sign \( (-) \) associated with the strength \( (m) \).

The axial and radial velocity components can be determined by the appropriate partial differentiation of Eq. 7, so that,

\[
\phi_x = \pm \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi}{L}}{n^2} J_0 (n \alpha r) \tag{10}
\]

**Sign convention:**
- \( (+) \) for \( x > 0 \)
- \( (-) \) for \( x < 0 \)

\[
\phi_y = 0 \quad \text{at} \quad x = 0
\]

\[
\phi_r = \sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi}{L}}{n^2} J_1 (n \alpha r) \tag{11}
\]
In order to calculate the streamlines and velocity components, it is convenient to introduce:

\[ x' = \frac{vt}{
u}, \quad y' = \frac{vt}{2\nu}, \quad z' = \frac{vt}{4\nu} \]

Pursuantly, the values of \( J_0(x') \) and \( J_1(x') \) over the range \( r = 2(0.02)1 \) have been calculated in Reference 14 for the first ten roots of \( J_1 \). After the first ten roots have been used, the difference in the roots is close enough to \( x' \) so that additional terms can be rapidly evaluated by using the asymptotic approximation given in References 13 and 14.

\[
J_n(t) \longrightarrow (2/\pi t)^{1/2} \cos(t - n \pi /2) = \pi /4)
\]

For values of \( n > 15 \), the above relationship is in agreement with the exact value to within one percent (Ref. 17). For larger values of \( n \), the approximation becomes more accurate. Values of \( J_0, J_1, \) and \( J_2 \) were calculated on a Remington Rand 1103 computer at the Aeronomical Research Laboratory of the Wright Air Development Center. The results are presented in Tables I, II, III. Sufficient values are included in these tables to facilitate the analysis of flow about bodies of arbitrary shape.

The asymptotic relationship was used to calculate enough terms to assure convergence to five decimal places for the values of \( \alpha \) \# 4.1, \( z = 1.1 \), and \( n = 0.2 \). The remaining values are tabulated for the first ten roots of \( J_1 \) only. For \( n = 1.1 \), the first ten roots were sufficient to obtain convergence to five decimal places. For values of \( J_1 \), only one or two terms are necessary, hence the tables were restricted to values of \( n \leq 1 \). Scientific notation is used throughout the tables, so that the number and sign preceding the tabulated values locate the decimal place. For example, \( 2.4652 \times 10^{-2} \) may be written as .074652.

For a point source not located on the origin, one has only to consider that \( (x) \) (in Figs. 6, 10, 11) represents the distance along the \( (x) \) axis from the plane of the source to the point where the velocity or stream function is being evaluated. In this regard, the absolute value signs \( |x| \) were dropped throughout the tables for convenience.

2. Doublet or the Type Arts

To derive the velocity potential for a doublet, one can consider a sink at the origin and a source of the same strength at \( x = A \), \( r = 0 \), as shown in Fig. 3. The expressions for the source and sink follow from the development above.
The stresst doublet is defined as:

\[ \mathbf{N} = 2\pi \mathbf{m} \mathbf{a} \mathbf{a} \]

Thus, only the first term remains in the expansion above, so that the velocity potential for a doublet located at the origin is:

\[ \phi = -\sum_{n=1}^{\infty} \frac{A_n}{j^2 j_0(j_n)} J_0(j_n r) \quad \text{or} \quad \phi = -\frac{A_n}{j^2 j_0(j_n)} J_0(j_n r) \quad \text{(13)} \]

\[ F = 2\pi \sum_{n=1}^{\infty} \frac{A_n}{j^2 j_0(j_n)} J_1(j_n r) \quad \text{(14)} \]

One should note the sign convention adopted above for the doublet, for as derived, the axis of the doublet lies on the \( m \) axis and the doublet flow is counter clockwise above the \( m \) axis.

The streamlines for a doublet are shown in Figure 4. The calculations for the doublet were carried out in much the same manner as the case for the source streamlines. However, Eq. 14 converges much more slowly than does Eq. 5, requiring a greater number of terms.
In addition, the limit of Eq. 6 is known to be unity by symmetry and continuity, to be \( y = a \), but at \( x = 0 \), Eq. 14 is more difficult to evaluate. From the nature of the hodograph, the slope of the streamlines must be zero at \( x = a \), hence, Eq. 14 was evaluated for values of \( x \) close to zero, and then the streamlines were fitted in. To evaluate the value of Eq. 14 at \( x = a \), slopes of the curves corresponding to Eqs. 6 and 7 were found graphically at \( x = 0 \).

3. Ring Source

The velocity potential of a ring source can be found using the same line of attack as used for the single source. The boundary conditions put forth for the point source are valid in this case also, except for condition IV, which must be modified as follows:

IV(a) The source flow is symmetric about the \((r)\) axis. However, the ring \( r = a \) is a singularity. Thus,

\[
(\theta_a)_{x=0} = 0 \text{ for } r \neq a
\]

The total strength of the ring, which is taken to be uniformly distributed around the circumference, is \( (2\pi a) \), so that the total source flow emitted by the ring is \( (2\pi a) \). Therefore, Eqs. 6-10 valid for a ring source as well as a point source. To obtain the coefficients \((\theta_a)\), the condition must be satisfied that,

\[
\theta_a = \frac{\pi}{4a} \lim_{r \to 0} \int_0^{2\pi} (\theta_a)_{r=a} \, dr
\]

But this may be written as,

\[
\theta_a = \frac{\pi}{4a} \int_0^{2\pi} (\theta_a)_{r=a} \, dr = 0
\]

The integral now represents the flow emitted by the ring source, which is \((2\pi a)\) in either direction, so that,

\[
\theta_a = -\frac{\pi}{4a} \int_0^{2\pi} (\theta_a)_{r=a} \, dr
\]

The velocity potential for the ring source follows,

\[
\psi = \frac{2\pi a}{\pi} - \sum_{n=1}^{\infty} \frac{2^{1/2}}{n^2} \frac{\ln(\frac{4n^2}{\pi^2})}{(2n^2 - 1})(\psi)_{n} + \frac{2^{1/2}}{n^2} \pi \ln(\frac{4n^2}{\pi^2}) \psi(0, a)
\]

(15)
The corresponding stress function, satisfying Eq. 2, is

\[ \psi = \frac{k}{2\pi} \sum_{n=1}^{\infty} \frac{z_n^{1/2}}{j_n \pi^2 (j_n)} j_1(j_n x^2) j_0(j_n z) \]  

(16)

**Sign convention:**

- \( \psi \) for \( n > 0 \)
- \( \phi \) for \( n < 0 \)

At \( n = 0 \), \( \psi = \omega \), depending on whether the plane of the ring source is approached from the right or left.

A check on the validity of the solution can be made by taking the limit as \( (n \to 0) \) so that \( (j_0(j_n a) \to 1) \) and the expressions reduce to those found for a point source on the axis.

4. **Ring Doublet**

The velocity potential for a ring doublet and the corresponding stress function can be obtained by using the same technique as used to obtain the point doublet. The resultant equations are,

\[ \phi = -2\pi - 2\pi \sum_{n=1}^{\infty} \frac{z_n^{1/2}}{j_n \pi^2 (j_n)} j_1(j_n x^2) j_0(j_n z) \]  

(17)

\[ \psi = 2\pi \sum_{n=1}^{\infty} \frac{z_n^{1/2}}{j_n \pi^2 (j_n)} j_1(j_n x^2) j_0(j_n z) \]  

(18)

5. **Disk Source**

The velocity potential of a disk source can be derived using the same boundary conditions as for the case of the point source, except condition IV, which must be modified as,

**IV(b)** The source flow is symmetrical about the \((r)\) axis. However, the disk of radius \( r = a \) is a singularity. Thus,

\[ (\phi)_{r=a} = 0 \text{ for } r > a \]

The total strength of the doublet, which is taken to be uniformly distributed over the disk, is \( (a) \). Returning to Eq. 6, the coefficients \( (a_n) \) can be determined by utilizing condition IV(b) such that,

\[ a_n = -\frac{a^2}{2\pi} \int_{0}^{1} \left( (\psi)_{r=a} j_0(j_n r) \right) r \, dr \]

\]
Taking continuity, we have

\[ \phi = \int_0^a \frac{J_0(\alpha x)}{J_0^2(\alpha)} \, dx \]

or

\[ \phi = -\frac{1}{a} \int_0^a \frac{J_1(\alpha x)}{J_0^2(\alpha)} \, dx \]

The expression for the velocity potential follows,

\[ \phi = -2\pi - 4\pi \sum_{n=1}^{\infty} \frac{\alpha_{n-1} \alpha_n}{J_0^2(\alpha_n)} J_0(\alpha x) J_1(\alpha_n x) \]  \hspace{1cm} (19) 

\[ \psi = -2\pi \sum_{n=1}^{\infty} \frac{\alpha_{n-1} \alpha_n}{J_0^2(\alpha_n)} J_1(\alpha x) J_1(\alpha_n x) \]  \hspace{1cm} (20)

Sign convention: (+) for \( a > 0 \)
 (-) for \( a < 0 \)

At \( z = 0 \), \( \psi = \psi_0 \), depending on whether the plane of the disk source is approached from the right or left. The results can be checked by taking the limit as \( (a \to 0) \), where \( \psi_0(\alpha a) \to \psi_0(\alpha) \), so that the above expressions reduce to those for a point source.

Comparing Eqs. 20 and 6, it is apparent that the convergence of Eq. 20 will be faster, so that the use of disks for developing bodies of general shape rather than point sources should reduce the amount of computation.

6. Disk Doublet

The velocity potential for a disk doublet can be found by utilizing the techniques used for a point doublet. The velocity potential is found to be,

\[ \phi = -2\pi + 2\pi \sum_{n=1}^{\infty} \frac{\alpha_{n-1} \alpha_n}{J_0^2(\alpha_n)} J_0(\alpha x) J_1(\alpha_n x) \]  \hspace{1cm} (21)
The corresponding stream function is

\[
\psi = 2 \pi \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{i_n J_0'(i_n)} \, J_1(i_n r) \, J_1(i_n s) \tag{22}
\]

7. Ring Vortex

The velocity potential for a ring vortex is readily obtained by employing the property that a ring vortex is equivalent to a uniform distribution of dipoles over the surface bounded by it. (Ref. 2). The axes of the dipoles are taken normal to the surface everywhere, and the density of the distribution is taken to be equal to the strength of the vortex. Thus, by Eq. 21, one has,

\[
\phi = -2 \pi \alpha^2 - 4 \pi \alpha \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{i_n J_0'(i_n)} \, J_0(i_n r) \, J_1(i_n s) \tag{23}
\]

where \( \alpha \) is the strength of the vortex.

The corresponding stream function is

\[
\psi = 4 \pi \alpha r \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{i_n J_0'(i_n)} \, J_1(i_n r) \, J_1(i_n s) \tag{24}
\]

8. Line Source

Continuous distributions of sources, sinks and dipoles are often used to develop bodies of various shapes. Considering the case of a line source of constant strength per unit length \( (s_1) \), and letting \( (s_1) \) be the source coordinate, then,

\[
\phi = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{i_n J_0'(i_n)} \, J_0(i_n r) \, ds_1
\]

so that the velocity potential is

\[
\phi = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{i_n J_0'(i_n)} \, J_0(i_n r) \, (s_2 - s_1)
\]

where \( (s_1) \) and \( (s_2) \) denote the beginning and end of the source respectively, and \( (s_2 - s_1) \) is the length of the source. The corresponding stream function is

\[
\psi = -2 \pi \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{i_n J_0'(i_n)} \, J_1(i_n r) \, (s_2 - s_1)
\]
The results can be checked by taking the limit as $s_0 \to 0$ whereupon
the expressions reduce to those for a point source. The convergence
of the functions above does not appear to be any better than for the
case of a point source, so that the additional complexity of the
results may not warrant the use of continuous distributions. Linear
or exponential variations of the source strength can be treated in the
same manner as the above case, but the resultant expressions become
increasingly unwieldy.

**TYPICAL APPLICATIONS**

**A. SIMPLE BODIES**

The study of bodies of arbitrary and practical shape will be
initiated by first considering the simple shapes resulting when
single sources, matched, disk doublets and ring doublets of varying
strength are placed in a ducted uniform flow.

1. **Point Source in a Uniform Flow**

The velocity potential of a point source in a ducted uniform
flow is found by adding the velocity potential ($\Phi_0$) for a uniform
flow to Eq. 7. To obtain the corresponding stream function, one
has to add the term ($\Omega r^2/2$) to Eq. 6. In addition, a constant ($2\pi$)
must be subtracted from the stream function when the flow downstream
of the source is being considered.

The form of the stream function indicated above stems from
the mechanism by which a body streamline is developed when a point
source is placed in a uniform flow. The source flow going upstream
is turned back by the superimposed flow so that upstream of the source,
there is no net source flow (in the (a) direction) within the body
streamlines. However, downstream of the source, both the original
downstream source flow and the original upstream source flow which
have been turned back downstream now exist (i.e. the total source
flow). The usual convention for the stream function, is that a body
streamline is indicated when its value is zero, i.e., $\Phi = 0$. In
order to retain this convention, it is then necessary to adjust the
expression for the stream function downstream of the source by
subtracting a constant which accounts for the source flow within
the body streamlines. In this case, the constant is ($2\pi$) as the
stream function concerns only the flow in the upper half plane.
Radius were calculated for various source strengths using the results obtained from the analysis in the manner indicated above. To use the tables at the end of the text, the body streamlines are determined by the condition that,

\[ r^2 = \frac{U^2}{2m} \quad \text{for } s < 0 \]
\[ r^2 = \frac{n^2}{2m} + 1 \quad \text{for } s > 0 \]

The results for several values of \((U/m)\) are shown in Fig. 5.

The radial components of the velocities, made dimensionless with \((U)\), are simply,

\[ f_r = \frac{r^2}{U} f_r^0 \]

which can be evaluated directly from Table III.

The axial velocity components can be expressed similarly as,

\[ f_a = (1 + \frac{2m}{U} f_a^0) \]

\[ \text{sign convention: } (\Delta) \text{ for } s > 0 \]
\[ (\mu) \text{ for } s < 0 \]

At \( s = 0 \), \( f_a^0 = 0 \). The axial velocity components can be evaluated by using Table II.

There is a departure from the usual result that the velocity upstream at infinity is just that due to the uniform flow superimposed on the source flow. In this case, the velocity upstream at infinity is less than the superimposed value by \((2m)\). Similarly, the velocity downstream at infinity is greater by \((2m)\).

The radius of the body at the plane of the source can be quickly found by employing the continuity equation, thus,

\[ U = 2m = U(1 - r_o^2) \]

so that

\[ r_o^2 = \frac{2m}{U} \]

The radius of the body downstream at \( s = +\infty \) can also be determined by continuity as,
It follows that:

\[ \Delta \phi = \frac{2 \pi^2}{1 + \pi^2} \]

The formulas above indicate a criteria for the establishment of flow through the duct, namely, that \((U > 2a)\). In order to establish the stagnation point, one can use Table II to determine at what value of \(a\),

\[ (\Delta \phi)_{o.o} = \frac{\pi^2}{4a^2} \]

2. Doublet in a Uniform Flow

The velocity potential for a doublet in a uniform flow can be obtained by simply adding the velocity potential \((U)\) for a uniform flow to Eq. 13. The corresponding stream function can be obtained by adding \((U^2/2)\) to Eq. 14.

The body streamlines can be determined by using Table III, as the stream function can be written as,

\[ \psi = \frac{U^2}{2} + 2\pi \pi^2 \]

Bodies calculated for several values of \((U/D)\) are shown in Fig. 6. The calculation of the velocity was not attempted as the convergence close to the plane of the doublet is very poor.

3. Ring Doublet in a Uniform Flow

The velocity potential for a ring doublet in a uniform flow can be obtained by adding the velocity potential \((U)\) for a uniform flow to Eq. 17. The corresponding stream function can be obtained by adding \((U^2/2)\) to Eq. 18.

Care must be taken to observe the proper sign convention on \((H)\) for to obtain a body, the doublet flow on the \((x)\) axis and upstream of the doublet must be opposed to the direction of \((U)\). Hence, if \((U)\) is positive, \((P)\) is negative.

A body calculated for \(W/U = .5\) and \(a = .5\) is shown in Fig. 7.

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2. **Dipole Distributions in a Uniform Flow**

The velocity potential for a dipole distributed along a stagnation streamline can be obtained by adding the velocity potential \( \phi_d \) for a uniform flow to Eq. 21. The corresponding stream function can be obtained by adding \( \psi_d/2 \) to Eq. 22.

Dipole distributions calculated for \( N = 10 \) and several values of \( (a) \) are shown in Fig. 8.

3. **Reproduced Body Shapes**

The velocity and pressure distributions about bodies of special shapes are often of practical interest. Numerous methods for the determination of the source and sink distributions corresponding to a non-aerated body occur in the literature (e.g., Ref. 5). Applying similar techniques to the case at hand, the aim in this section is to outline a method for determining source and sink distributions which approximate a desired body shape. Having the source and sink distributions is all that is required to completely define the velocity and pressure distribution about the body.

4. **Discrete Distributions of Sources and Sinks**

From the analysis of a point source in a uniform flow, it is clear that specifying one coordinate of the body surface determines the strength of the source \( (s) \). Consequently, for each specified body coordinate, one must locate a source or sink in the flow. Since only axisymmetric bodies are being considered, the sources and sinks are located on the \( (a) \) axis. By judiciously choosing the location of these sources and sinks (i.e., close to the \( (a) \) coordinate specified or at least within the length of the desired body) one might expect that the body shape between the chosen coordinates will closely approximate the desired shape. To carry this approach out, it is convenient to let

\[
\gamma^+ = \frac{\gamma}{2} = \frac{\alpha^2}{2} + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2} J_m(\alpha^m) \quad (27)
\]

Then the stream function for the total \( (s) \) sources in a uniform flow is

\[
\psi^+ = \frac{2}{\alpha^2} - \frac{s}{\alpha^2} \quad (28)
\]

where

\[
\alpha^2 = \frac{s^2 - \alpha^2}{s^2}
\]

\( \alpha \) is the characteristic length of the body.
The sign associated with $\delta^2$ values may change abruptly, namely (+) when $2 \leq i \leq 6$ and (-) when $2 < i < 6$, and at $i = 7, \delta^2 = 1/2$, and the sign is negative. It was mentioned previously, but will be further emphasized here, that $\delta^1$ is the absolute value of the total distance from the source in question to the point where the stream function of velocity potential is being evaluated. The value of $\delta^1$ is then relative to the source and not dependent on the arbitrary location of the origin of the coordinate system.

A previous analysis of the source flow inside the body streamlines indicated that the condition $\delta^2 = 0$ defines a point on the body only if the net source flow upstream of this point (within the body) is zero. When a net source flow exists upstream of a particular point on the body, the condition

$$\delta^2 = 1/2 \sum_{u=1}^{u} a_u$$

on the body, where $(a_u)$ are the sources and sinks upstream of the body point.

As an example, consider that three body coordinates are specified, with sources and sinks located on the axis at the same values of $(z)$. Then, one has three equations,

$$\begin{align*}
1/2x_1^2 - s_1 \delta_{11} - a_2 \delta_{21} - s_3 \delta_{31} &= 0 \\
1/2x_2^2 - s_2 (1 - \delta_{12}) - a_3 \delta_{22} - s_3 \delta_{32} &= 0 \\
1/2x_3^2 - s_3 (1 - \delta_{13}) - a_1 (1 - \delta_{23}) - s_2 \delta_{33} &= 0
\end{align*}$$

The subscripts are chosen to correspond to the coordinates of the sources, $(x_1, x_2, x_3)$ and the body coordinates $(y_1, y_2, y_3)$. The first subscript on $\delta^2$ refers to the source location, while the second indicates the body coordinate. The above relationships can be readily put into matrix form:

$$\begin{pmatrix}
\delta_{11} & \delta_{12} & \delta_{13} \\
\delta_{21} & \delta_{22} & \delta_{23} \\
\delta_{31} & \delta_{32} & \delta_{33}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\delta_{11} \\
\delta_{22} \\
\delta_{33}
\end{pmatrix}
= \begin{pmatrix}
1/2x_1^2 \\
1/2x_2^2 \\
1/2x_3^2
\end{pmatrix}
$$

Several observations can be made on the above matrix, which permit one to write the matrix corresponding to any number of potential body coordinates in three-dimensional spaces. First, the values on the diagonal of the matrix must be equal to $(1/2)$. Second, the right of the diagonal are the sources and sinks upstream of the body points considered. Third, to the left of the diagonal are the sources and sinks upstream of the body points considered.
The axial velocity component can be found conveniently by letting

\[ (d_z^e)_z = \frac{d_z}{2\pi} = 1 + \sum_{n=1}^{\infty} \frac{-\ln n}{j_0^2(n) j_1(n)} J_0(1_n z) \] \hspace{1cm} (29)

The axial velocity at any point is then,

\[ (d_z^e)_z = 1 + \sum_{n=1}^{\infty} S_n (d_z^e)_n \] \hspace{1cm} (30)

where the sign is (+) for \( z > 0 \), and (-) for \( z < 0 \).

At \( z = 0 \), \( d_z^e = 0 \).

Solving Eq. 30 for \( (d_z^e)_z = 0 \) when \( r = 0 \), yields the location of the stagnation points.

The radial velocity component can be determined similarly, by letting,

\[ (d_r^e)_z = \sum_{n=1}^{\infty} \frac{-\ln n}{j_0^2(n) j_1(n)} j_1(1_n r) \] \hspace{1cm} (31)

Thus, the radial velocity at any point is,

\[ (d_r^e)_z = \sum_{n=1}^{\infty} S_n (d_r^e)_n \] \hspace{1cm} (32)

It may be desired to specify the total strength of the sources and sinks as an initial condition. This can be done conveniently by specifying the radius of the body at \( z = \pm \infty \). Then by continuity,

\[ 2 - \frac{\infty}{y} = (2 + \frac{\infty}{y})/2 - \frac{\infty}{y} \]

so that:

\[ \sum_{n=1}^{\infty} S_n = \frac{\infty}{y} = \frac{\infty}{y} 

\]
The examples were carried out with source coordinate points and the total source-sink strength specified. The details are presented in Appendix II, and the results in Figs. 9(a) and (b).

2. **Potential Distributions of Sources and Sink Disks**

Clearly, one could obtain generalised body shapes using only sets of source and sink distributions. However, the utilisation of source and sink disks appears to offer an advantage in that the source expressions converge more rapidly than for the case of point sources and sinks.

The use of this method is similar to the case where sources and sinks were used.

C. **Determination of Body Coordinates and Velocity Distribution**

**Discrete Distribution of Sources and Sinks**

The combined problem of specifying certain points on a body and certain flow conditions, and then solving for the complete body shape is often of interest. The technique for solving this problem is similar to that employed when determining the body shape alone.

A source or sink must be included for each point or flow condition specified. The appropriate equation relating these conditions is either 20, 30 or 32. Finally, one has a set of \( n \) linear equations in \( n \) unknowns to be solved. Upon their solution, the complete body shape can be determined using the methods described in the previous sections. An example is carried out in Appendix III. The results are shown in Fig. 10.

D. **Influence of Local Curvature**

Often in the design of ducted bodies, a question arises as to the influence of local body curvature on the adjacent velocity distribution. Two methods of varying the local body curvature were studied.

The first method studied, involved the calculation of bodies using eight specified body coordinates and point sources and sinks on the same. By maintaining some of the body points the same but varying others slightly, similar bodies with different (localized) curvatures were obtained. The results of this approach are shown in Fig. 9(a). The velocities were calculated at \( z = 0 \), and \( z = 0 \). The results are compared in Fig. 9(b). It can be seen that while the curvature effects were strong on the axial and radial components near the body, the effect on the magnitude of the resultant (perpendicular)
A second method tested was the use of a ring source in combination with a point source. The ring and point sources were placed in the same plane, with the radius of the ring being chosen so that desired for the body. While it would have been more desirable to superimpose a ring doublet on a point source, so that there would be no change in the mass flow, the computation of the ring doublet is excessively cumbersome due to the slow rate of convergence. Using a ring and point source, one can preserve the closeness of the ring to the body surface arbitrarily. This follows from the boundary conditions and continuity. Since it is known that in the plane of a point and ring source, there is no axial velocity component due to the source flow, the axial velocity in this plane is wholly that of the experienced uniform flow. By continuity, the radius of the body in the plane of the sources is

\[ \frac{c^2}{\varphi} = \frac{3}{2} \left( m_p + m_r \right) \]

where \( m_p \) and \( m_r \) are the strengths of the point and ring-sources respectively. Consequently, when the total source strengths are specified, the body radius is known in the plane of the sources and the radius of the ring can be placed as close to the body surface as desired.

Initial attempts were made to use examples using the above method. The initial calculation assumed a value of \( m_r = 0.5 m_p \) which turned out to be too small a value for the ring source as a negligible effect was had on the body shape. When the strength of the ring source was increased to \( m_r = 1.5 m_p \), significant changes in body contour occurred. The effect of the change in body contour on the velocity distribution along the duct wall was calculated, and the results are shown in Fig. 11.

**Velocity Distribution**

While the primary emphasis has been placed on developing the velocity potential and stream function for various sources and sink distributions inside ducts and associated methods for determining special body contours, considerable insight into the general nature of the flow about axially symmetric ducted bodies has been obtained. As is common from the nature of the solution, however, that a complete visualization is required for fully validated flow

...
The development of the velocity potential for a point source indicates that, if a source is a source like velocity equal to \( u \). Thus, the undisturbed velocity is not equal to the upper-pointed undisturbed velocity to the surface of a point source in an undisturbed velocity \( \psi = u \), but \( \psi = (u - \omega x) \) for upstream and \( (u + \omega x) \) for downstream.

An interesting finding, useful for comparing ducted and non-ducted flows, is that the body radius in the plane of the source (denoted here with bodies due to a single point source) is the same, i.e., \( R_0 = 2a/\omega \) in both cases, for the body radius at \( \omega = \infty \). However, the non-ducted case has \( R_0 = 4a/\omega \), as compared with the ducted case

\[
R_0^* = \frac{4a}{\omega(1 + 2a/\omega)}
\]

One additional point, the location of the stagnation point, will be compared for the two cases. In the case of the undisturbed body (for a single point source), it occurs at \( \theta = -\sqrt{nq/\omega} \), whereas in the ducted case, it may be located by writing \( \psi_0 = 0 \) in the form, \( 2a/\omega = 1 - n^2 \). Using Table II, for \( \omega = 0 \), one obtains the following comparisons:

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<table>
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</table>

In conclusion, it is seen that non-ducted and ducted point-source analyses are essentially the same for values of \( (a/\omega) \) of about 0.1 or smaller. The corresponding results for a ducted, indicates that the body shape is not essentially affected from a point model as it is greater than \( 0.3 \).
the series. Consequently, the use of doublets (ring and disk included) for the study of flow about bodies is particularly difficult. This difficulty, we see, arises because the polar expressions for velocity potential of a doublet is actually the result of differentiation of the corresponding velocity potential for a source. Thus, the series for the evaluation of the velocity components about a body due to a doublet in a uniform flow is difficult to compute. For the reasons indicated above, the study of doublet forms was abandoned.

For the ring doublet investigated (see Fig. 7), the shape of the inner and outer surfaces is quite different due to the nature of the ring doublet flow. It is likely that these shape differences would diminish if the strength of the ring doublet were decreased. Thus, it appears that the use of ring doublets as sources and sinks may be a practical way of studying the flow about ring type flame holders. Of course high speed computing devices would be required to conduct a detailed study on this problem.

The investigation of the disk doublet in a uniform flow was aided by the fact that the expressions for the velocity potential for a disk doublet converges better than the velocity potential of the other doublet forms studied. For the cases which were calculated, it appears that the resultant body, when the disk doublet is placed in a uniform flow, is very nearly a circle for body diameters up to one-half the duct diameter. For larger bodies, the shape becomes progressively blunter.

The computation of bodies of arbitrary shape is considerably reduced once the stream function and velocity components for a point source have been evaluated and tabulated. Thus E = E and D contain a sufficient range of these quantities to determine the complete flow fields about a wide variety of body shapes. For rapid and approximate studies, graphical methods may be used. For detailed and precise studies, the computation can be set up on an automatic computer in a number of ways.

The influence of local body curvature is of practical interest in the design of the hub of a turbomachine. The shape of the hub influences the blades leaving at least near the hub. An important design problem is how far away, starting from the hub, can the blade leading be affected by introducing a change in local curvature of the hub. Another problem is that of obtaining a particular velocity distribution at inlet to the machine which might be more desirable.

When a very important change is then introduced at the inlet, this change at the inlet influences the velocity distribution. The results would be important for design and performance, providing the changes in the contour are thoughtful in bringing about significant changes in the velocity distribution.
of wave forces a significant change in velocity distribution. Several
points were specified as body coordinates also, so that the basis
form of the hub would not be conspicuously different. The result,
shown in Fig. 10, indicates, in a preliminary way at least, that
the method of calculating the velocity distribution leads to large
modifications in body shape. Further, we conclude that the type
of velocity distribution, obtained in Fig. 9(b) are most likely to
exist under practical body shapes.
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### Table 1

Stream Function \( \Phi \) for a Point Source
### TABLE IX

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**Table III - continued**

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**Table III - continued**

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Solution of Equation: \( F_{xx} + 2F_{yy} = 0 \)

Resolving Eq. (1) and minimizing the value of \( F = F_0 \) one obtains:

\[
\begin{align*}
R_{xx} + 2R_{yy} &= 0 \\
R_{yy} + 2R_{xx} &= 0
\end{align*}
\]

The solutions to the above equations are well known, and can be written as,

\[
F = X_1 \cos \theta + Y_2 \sin \theta
\]

\[
F = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r)
\]

The third solution to Eq. (1) is,

\[
\hat{F} = \left( X_1 \cos \theta + Y_2 \sin \theta \right) \left( C_1 J_0(\lambda r) + C_2 Y_0(\lambda r) \right)
\]

where \( J_0 \) is the modified Bessel function of the first kind of zero order and \( Y_0 \) is the modified Bessel function of the second kind of zero order.

It is of interest to determine whether the boundary conditions for a point source can be satisfied with the solution above. Considering condition I, it is apparent that the above solution is periodic so that a series solution cannot represent this condition.

A series solution of theoretical interest can be found for the above solution that represents a flow within a duct of constant diameter. To satisfy the condition of zero radial velocity at the duct wall, one can write,

\[
R = I_0(\lambda r) X_1(\lambda) + R_y(\lambda) Y_1(\lambda)
\]

where \( I_0 \) is the modified Bessel function of the first kind of the first order, and \( Y_1 \) is the modified Bessel function of the second kind of the first order. The form of the solution is a result of the fact that neither \( I_0 \) nor \( Y_0 \) have real roots except at \( r = 0 \), (Ref. 6), so that a combination of the two functions is required to obtain a function that is zero for some value of \( x \), i.e., satisfies the condition of zero radial velocity at the duct wall.

A flow model, which can be satisfied by the solution obtained above, is that of a periodic distribution of sources and sinks along the duct axis. Considering that the sources and sinks are spaced by a period apart, and that flow symmetry exists at these points, one has,
\[ E_{n} = \sqrt{\left[ r_{0}(1, \lambda_{1}) \right]^2 \lambda_{1}(1, \lambda_{1}) \left( \lambda_{0}^{\prime} \cos \lambda_{0} - \lambda_{1}^{\prime} \cos \lambda_{1} \right) r_{0}(1, \lambda_{1}) \lambda_{1}(1, \lambda_{1})} \]

where \( a = 1, 2, 3, \ldots \).

Thus, \( Y_{2} = 0 \), and \( j = \pi / 2 \), so that one has,

\[ J = \sum_{n=2}^{2} A_{n} \left( \lambda_{0}(1, \lambda_{1}) \lambda_{1}(1, \lambda_{1}) = \lambda_{0}(1, \lambda_{1}) \lambda_{1}(1, \lambda_{1}) \right) \cos \lambda_{n} \quad (I-1) \]

To determine the coefficients, \( (A_{n}) \), Cauchy's theorem can be applied. With alternating sources and \( e^{i \omega t} \) of equal strength, one can use this theorem midway between them, i.e., when

\[ \theta = \frac{2a-1}{2} \theta \]

Differentiating Eq. I-1, with respect to \( (\theta) \), multiplying through by

\[ (r \sin \lambda_{p} \frac{2a-1}{2} \theta) \]

and integrating the result at

\[ \theta = \frac{2a-1}{2} \theta \]

yields for the left hand side,

\[ \frac{1}{2\pi} \int_{0}^{2\pi} \left( \lambda_{0}^{\prime} \lambda_{1} - \lambda_{0} \lambda_{1} \right) \sin \lambda_{p} \frac{2a-1}{2} \theta \pi \theta = 2a \]

The above integral is valid when \( (p) \) is odd, \( (j_{p} = \pi / 2b) \).

Thus, let \( p = 2a-1 \), where \( a = 1, 2, 3, \ldots \). The \((+ \) sign holds for odd values of \( (u) \), the \((- \) sign for even values. The value of the above integral is \( 2a \) as the flow of each source splits, \( 2 \pi a \) going in either direction.

for the right hand side (see above),

\[ E_{p} = \int_{0}^{2\pi} \left[ \lambda_{0}(1, \lambda_{1}) \lambda_{1}(1, \lambda_{1}) \right] \sin \lambda_{p} \frac{2a-1}{2} \theta \pi \theta = 2a \]

\[ \varphi_{0}(1, \lambda_{1}) \lambda_{1}(1, \lambda_{1}) \sin \lambda_{p} \frac{2a-1}{2} \theta \pi \theta = 2a \]

\[ \varphi_{p}(1, \lambda_{1}) \lambda_{1}(1, \lambda_{1}) \sin \lambda_{p} \frac{2a-1}{2} \theta \pi \theta = 2a \]
Performing the integration, noting that
\[ \lim_{n \to 0} r_{\lambda}^n L_1(\lambda r) = 1 \]
yields,
\[ n = \frac{A_p I_1(R_p)}{I_1}(p) \]
The resultant expression for the velocity potential is,
\[ \phi = \sum_{n=1}^{k} I_0 \left[ \frac{I_0(1 + \lambda r) - I_0(\lambda r) I_1(\lambda r)}{L_1(\lambda r)} \right] \text{ see } I_{10} \]
where \( I_{10} = (2n-1) \pi b \)
APPENDIX II

Calculation of Special Body Shapes

The body coordinates and source locations chosen are:

\[
\begin{array}{cccc}
 & a & x & b \\
a_1 & -59 & 30 & 85 & -165 \\
a_2 & -49 & 40 & 86 & 70 \\
a_3 & -365 & 59 & 87 & 39 \\
a_4 & -86 & 60 & 89 & 60 \\
\end{array}
\]

Instead of specifying eight body coordinates, seven are specified along with the condition that \( a = 100 \). Eight linear equations in eight unknowns can now be set up. To do this requires the determination of 64 coefficients. Taking the first body coordinate, and determining the values of the stream function there using Eq. 26,

\[
0 = 0.045 - 5.11 \cdot 4 - 2.21 - 8.23 - 3.41 - 4.21 - 3.41 - 0.71 - 8.61
\]

The values of \( f^0 \) are obtainable from Table I. For

\[
f^0_{11}, \ |M| = 0, \; \text{so that} \; \frac{f^0_{11}}{11} = 1/2. \; \text{For} \; f^0_{21}, \ |M| = 0.0, \; \text{so that} \; \frac{f^0_{21}}{21} = 0.565.
\]

Continuing in this manner, all the coefficients for the eight equations can be quickly found. Setting the result up in matrix form

\[
\begin{bmatrix}
0.500 & 0.346 & 0.266 & 0.177 & 0.126 & 0.066 & 0.026 & 0.016 \\
0.014 & 0.500 & 0.464 & 0.317 & 0.238 & 0.151 & 0.119 & 0.087 \\
0.022 & 0.022 & 0.500 & 0.419 & 0.334 & 0.252 & 0.200 & 0.154 \\
0.076 & 0.076 & 0.076 & 0.500 & 0.436 & 0.341 & 0.266 & 0.220 \\
0.099 & 0.099 & 0.099 & 0.099 & 0.500 & 0.459 & 0.389 & 0.327 \\
0.096 & 0.096 & 0.096 & 0.096 & 0.096 & 0.500 & 0.419 & 0.353 \\
0.091 & 0.091 & 0.091 & 0.091 & 0.091 & 0.091 & 0.500 & 0.489 \\
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
S_5 \\
S_6 \\
S_7 \\
S_8
\end{bmatrix}
\begin{bmatrix}
-0.45 \\
0.00 \\
0.00 \\
0.00 \\
0.00 \\
0.00 \\
0.00 \\
-0.50
\end{bmatrix}
\]

The above matrix was solved to hard, by diagonalizing it. The results are:

\[
\begin{align*}
S_1 &= 0.209 \quad S_2 &= -0.507 \\
S_3 &= 0.977 \quad S_4 &= -0.162 \\
S_5 &= -0.214 \quad S_6 &= 0.296 \quad S_7 &= -0.292 \quad S_8 &= 0.159
\end{align*}
\]
To obtain the coefficients, Tables II and III can be used. For the
axial velocity at z = 0, r = 1, one has by Eq. 30,

\[(A_v)_\text{ax} = 1 + s_1(A_v)_{12} + s_2(A_v)_{13} + s_3(A_v)_{14} + s_4(A_v)_{15} + s_5(A_v)_{16} + s_6(A_v)_{17} - s_7(A_v)_{18} - s_8(A_v)_{19}\]

To find \((A_v)^o\), the values \(z = -50, r = 1\) from Table II correspond
to \((A_v^o)_{12} = 0.720\). The values of \((A_v)_{12}\) are known from the solution of
matrix, and the remaining velocity terms can be found from the tables.
The radial velocities are calculated in the same manner, using
Table III.

**Ex. 2**

The body coordinates and source locations chosen are:

<table>
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<th>s</th>
<th>r</th>
<th>s</th>
<th>r</th>
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</thead>
<tbody>
<tr>
<td>s_1</td>
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<td>s_5</td>
<td>-.25</td>
</tr>
<tr>
<td>s_2</td>
<td>-.71</td>
<td>s_6</td>
<td>-.22</td>
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<tr>
<td>s_3</td>
<td>-.57</td>
<td>s_7</td>
<td>0</td>
</tr>
<tr>
<td>s_4</td>
<td>-.45</td>
<td>s_8</td>
<td>.70</td>
</tr>
</tbody>
</table>

The sum of the source strengths was fixed at .506. The results
of the calculation are:

\[s_1 = .009\]
\[s_2 = .016\]
\[s_3 = .073\]
\[s_4 = .057\]
\[s_5 = .116\]
\[s_6 = .234\]
\[s_7 = .208\]
\[s_8 = .180\]
The following conditions are prescribed:

Body coordinates:

<table>
<thead>
<tr>
<th>$x$</th>
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<td>.615</td>
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<tr>
<td>.52</td>
<td>.78</td>
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</table>

Velocity distribution at $z = 0$:

\[
(f_1)_{x=0} = 1.7(f_2)_{x=0} = 7 \\
(f_3)_{x=0} = 1.333(f_4)_{x=0} = 7 \\
(f_5)_{x=0} = 1.067(f_6)_{x=0} = 7
\]

The total source strength will be $\delta_0$. There are seven conditions specified, requiring seven sources and sinks. The location of these sources and sinks are:

<table>
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<tr>
<th>$z$</th>
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<tr>
<td>$s_1$</td>
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<tr>
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<tr>
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<td>$-0.25$</td>
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<tr>
<td>$s_5$</td>
<td>$0.36$</td>
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<tr>
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<td>$s_7$</td>
<td>$0.7$</td>
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Note that the arrangement of sources depends on the total used in Appendix II, where they were located at the same ($z$) coordinate as the body points.

The set up of these equations for the body coordinates follows the same approach used in Appendix II. The equation for the velocity conditions are set up as follows:
as \( r = 0 \), \( r = 1 \), and \( s = 0 \), \( s = 0.7 \), and equating the two expressions in the proper region, and equating the value of the necessary coefficients. This procedure is repeated three times, once for each velocity condition. Thereupon, the final system of equations may be written as,

\[
\begin{bmatrix}
0.756 & 0.634 & 0.502 & 0.366 & 0.267 & 0.156 & 0.114 \\
0.724 & 0.706 & 0.678 & 0.651 & 0.600 & 0.500 & 0.410 \\
0.679 & 0.671 & 0.666 & 0.656 & 0.634 & 0.599 & 0.520 \\
0.267 & 0.289 & 0.267 & 0.270 & 0.211 & 0 & 0.240 \\
0.192 & 0.215 & 0.224 & 0.216 & 0.169 & 0 & 0.158 \\
0.106 & 0.122 & 0.129 & 0.126 & 0.100 & 0 & 0.099 \\
1 & 1 & 1 & 1 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
3 \\
5.5 \\
3.5 \\
4 \\
3.5 \\
3.5 \\
3.5
\end{bmatrix}
\begin{bmatrix}
0.080 \\
0.245 \\
0.220 \\
0.200 \\
0.183 \\
0.067 \\
0.506
\end{bmatrix}
\]

The resultant values for the source and sink strengths are:

\[
S_1 = 5.691 \quad S_3 = 5.660 \quad S_5 = 25.175 \quad S_7 = 0.528
\]

\[
S_2 = 297 \quad S_4 = 17.385 \quad S_6 = 8.348
\]

Having found the source and sink strengths, the complete body shape can be found by utilising Eq. 26. The results are shown in Fig. 10.
REFERENCES

1. Lamb, H., "A Source In A Rotating Fluid", The Quarterly
   Journal of Applied Mathematics, Vol VIII, Part 1,
   Clarendon Press.

   New York.

3. Nabors, C. K., Jr., "ADVANCED ENGINEERING MATHEMATICS", First

4. Harvard Computation Laboratory, "Design and Operation of Digital
   Calculating Machinery - Progress Report #1", August 1955,
   Cambridge, Massachusetts.

   1966, Cambridge, at the University Press.

6. F. M., "Statics and Dynamics of Machinery", First Edition,

7. McLaughlin, E., "Bessel Functions For Engineers", First Edition,
   1934, Oxford University Press, London.

8. Von Karman, T., "Calculation of Pressure Distribution On Airship


    Parallel Walls", Journal of Applied Mechanics, Vol 22, No. 1,
    March 1955.

11. Sedov, L. I., and Sternberg, S., "Elliptic Integral Representa-
    tion of Axially Symmetric Flows", Research Project #5955,
    February 1, 1949, Illinois Institute of Technology.

    of Technology.

13. Streeter, V. L., "The Ring Doublet In Ideal Fluid Flow", Research
    Project #275, June 1959, Illinois Institute of Technology.

14. Wagner, F., and Brand, M., "Stream Functions and Velocity Fields
    of Spatial Source-Paths and Their Use in the Determination
    of the Motion of Fluids about Elliptical Bodies, with Examples",
    Translated from German Document, Air
    Documents Division, T-2, AF, Wright Field, Microfilm No. AO-1108,
    1940.


FIGURE 1. COORDINATE SYSTEM USED

FIGURE 2. STREAMLINES OF A POINT SOURCE
I.

FIGURE 3. DERIVATION OF THE VELOCITY POTENTIAL FOR A DOUBLET

FIGURE 4. STREAMLINES OF A DOUBLET
FIGURE 5. POINT SOURCES IN A UNIFORM FLOW

FIGURE 6. DOUBLETS IN A UNIFORM FLOW
Figure 7: Ring Doublet in a Uniform Flow

Figure 8: Disk Doublet in a Uniform Flow
(b) Velocity distributions for bodies (A, B, C)

Figure 3: Arbitrary body shapes investigated.
(7) PRESCRIBED VELOCITY DISTRIBUTION AT $z=0$

(8) RESULTANT BODY

FIGURE 10. RESULTANT BODY FOR A PRESCRIBED VELOCITY DISTRIBUTION