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DYNAMIC PROPERTIES OF LINEAR DECISION  
RULES IN PRODUCTION PLANNING

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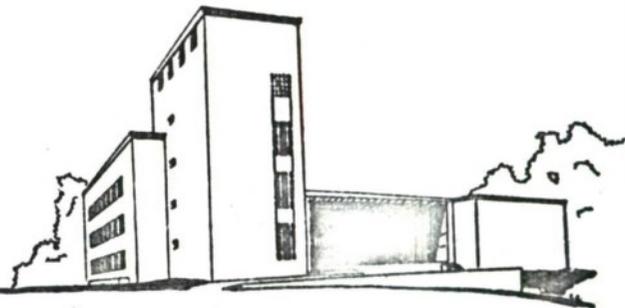
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IN PRODUCTION PLANNING

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GRADUATE SCHOOL of INDUSTRIAL ADMINISTRATION

William Lorimer Mellon, Founder

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DYNAMIC PROPERTIES OF LINEAR DECISION RULES  
IN PRODUCTION PLANNING

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## DYNAMIC PROPERTIES OF LINEAR DECISION RULES IN PRODUCTION PLANNING\*

### 1. Introduction

This paper is concerned with derivation of an optimal decision rule from an integral representing the costs associated with adjustments in production to changing sales forecasts. The cost formulation used here and the optimal rule associated with it are closely related to that examined by Holt, Modigliani, Simon, and the present writer [2,1]. However, the simple case to be examined will allow some of the dynamic performance characteristics of the scheduling rules and the effects of errors in estimates and expectations to be examined in a relatively convenient manner.

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\*In the preparation of this paper I have benefited from discussions with Charles C. Holt, Franco Modigliani, and Herbert A. Simon.

As a first approximation, the costs of inventory storage and depletion and of overtime pay to the work force of a factory might be represented by the following integral:

$$(1.1) \quad \bar{C} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[ \gamma^2 (H - H_c)^2 + (P - P_c)^2 \right] dt,$$

where  $\bar{C}$  is the cost per unit of time associated with inventory and overtime decisions,  $\gamma^2$  represents the relative costs of decreasing inventories to increasing the amount of overtime of the work force,  $H_c$  and  $P_c$  are the minimum-cost inventory level and production rate, respectively,  $H(t)$  represents the level of finished-goods inventories at time  $t$ ,  $P(t)$  represents the production rate at time  $t$ , and  $S(t)$  represents the rate of sales at time  $t$ . The last three variables are related by the condition:

$$(1.2) \quad \dot{H}(t) = P(t) - S(t),$$

that the rate of increase in inventories is equal to the rate by which production exceeds sales.

Since a rationale for a cost formulation of this sort has been given elsewhere [2], further explanation is probably not warranted here. Limited experience, however, indicates that estimation of the parameters is feasible and that the decision rules derived on the basis of such estimates are only moderately sensitive to errors.

In Section 2 we will be concerned with finding a rule for setting the production rate optimally, in the sense that the expected value of the cost function (1.1) is minimized, taking into account new information available to the decision-maker which is, in turn, reflected in revisions of forecasts of future sales. It will be shown that the optimal rule for this relatively

simple cost functional is as follows:

$$(1.3) \quad P(t) = \gamma \int_0^{\infty} e^{-\gamma \theta} S(t, \theta) d\theta - \gamma [H(t) - H_0]$$

where  $S(t, \theta)$  is the forecast of sales made at time  $t$  for time  $t + \theta$ . The production rate scheduled depends first upon a 'present value' of expected future sales as well as the initial state of inventories of finished goods. Through the latter term, corrections of errors in previous forecasts are gradually made. Subsequent sections will be concerned first with the response characteristics of the rule (1.3) for three simple kinds of forecasts. It will be shown that the rule is a 'low-pass' filter (cf. Simon [6]) and that inventory adjustments might lag behind sales changes for forecasts tending to extrapolate present sales conditions into the future. The sensitivity of the rules and costs associated with their performance to errors in estimating the coefficient  $\gamma^2$ , in the sales forecasts, and in forecasting over only a limited 'horizon' will then be examined.

Although the work stems from production scheduling problems in a specific firm, formulations of this sort appear to be applicable, with modifications, to other situations. It is possible, too, that explicit statements of a criterion as in (1.1) will provide some clues to better design of physical servomechanisms (e.g., control of temperature, pressure, and recycle rates in a cracking tower together with the design implications of better control). Furthermore, such techniques are potentially useful in the study of business fluctuations. In particular, the derivation of behavior hypotheses from cost criteria not only provides a bridge between the theories of the firm and business cycles, but also suggests the structure of such relations for statistical estimation (decision variables, data

relevant to these decisions, the role of expectations) and provides an independent source of estimates of the parameters of these relations.

## 2. Derivation of Optimal Rules

The problem posed before may be restated in a simpler form by extrapolating the 'certainty-equivalence' proof of Simon [7] to the situation in which decisions may change continuously in time. Let:

$P(t, \theta)$  be the rate of production planned at time  $t$  for time  $t+\theta$ ,

$H(t, \theta)$  be the level of inventories planned at time  $t$  for time  $t+\theta$ , and

$S(t, \theta)$  be the rate of sales forecasted at time  $t$  for time  $t+\theta$ .

The plan which minimizes the following cost functional is the optimal rule for the dynamic problem:

$$(2.1) \quad \bar{C} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [\lambda^2 (H - H_c)^2 + (P - P_c)^2] d\theta,$$

where  $\lambda^2$  is a positive number and, for simplicity,  $H_c$  and  $P_c$  are constants. We then wish to find a way of minimizing (2.1) subject to the following consistency relation:

$$(2.2) \quad \dot{H} = P - S,$$

the prime indicating differentiation with respect to  $\theta$ .

The problem is to find some derivative  $P^{(m)}(t, \theta)$  which, when set equal to  $P^{(m)}(t)$  the actual production rate at time  $t$ , will minimize the expected value of equation (1.1). The calculus of variations will yield the conditions for a minimum cost in the planning space. With the aid of the Laplace transform, we will then solve the resulting conditions for  $\theta = 0$ . These equations represent the optimal decision rule. Having gone through these manipulations, we will examine some of the qualitative properties of such rules, perhaps gaining insight as to possible applications and limitations of this approach.

2.1 Conditions for Minimum Cost. If we let  $P^*(0)$  be the optimal production plan over the future (we are dropping the argument  $t$  for the time being), any other plan may be written as a 'variation' from this optimum:

$$(2.3) \quad P = P^* + v'$$

Except that it be differentiable and satisfy certain end-point conditions to be specified below, the function  $v(\theta)$  is arbitrary. Since

$$H' = (P^* - S) + (P - P^*) = H^* + v',$$

it follows that

$$(2.4) \quad H = H^* + v$$

plus a constant which, as we shall see later, is equal to zero.

Substituting (2.3) and (2.4) into the functional (2.1), an equation of the following form is obtained:

$$(2.5) \quad \bar{C} = \bar{C}^* + 2L + Q.$$

the average cost using the policy  $P^*$  is given by:

$$(2.6) \quad \bar{C}^* = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [\delta^2 (H^* - H_c)^2 + (P^* - P_c)^2] d\theta.$$

$L$  represents the linear terms in  $v$  and its derivative:

$$(2.7) \quad L = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [\delta^2 (H^* - H_c) v + (P^* - P_c) v'] d\theta$$

and  $Q$  represents the quadratic terms:

$$(2.8) \quad Q = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [\delta^2 (v)^2 + (v')^2] d\theta.$$

Since  $P^*$  is the minimum-cost plan for production, for all the admissible functions (alternatives)  $v$ , we must have:

$$(2.9) \quad \bar{C} = \bar{C}^* + 2L + Q \geq \bar{C}^*,$$

and hence that

$$(2.10a) \quad L = 0, \text{ and}$$

$$(2.10b) \quad Q \geq 0$$

for all admissible functions  $v$ . Because the ratio of cost coefficients,  $\delta^2$ , is positive, we can see immediately from (2.8) that condition (2.10b) for a minimum is satisfied.

What, then, are the conditions under which  $L = 0$ ? Clearly, this holds for the trivial case  $H^* - H_c = P^* - P_c = 0$ , which implies a very special kind of sales forecast (without error).  $P^*$  and  $H^*$  may be found under more general conditions as follows. Equation (2.7) may be written in terms of  $v$  (and not its derivative) in the interval  $(0, T)$  and its values at the end-points as follows:

$$(2.11) \quad L = \lim_{T \rightarrow \infty} \frac{1}{T} \left\{ \int_0^T [\delta^2 (H^* - H_c) - (P^*)] v \, d\theta + (P^* - P_c) v \Big|_0^T \right\} = 0.$$

The integral will vanish if the following differential equation in planned future production rates and inventory levels is satisfied:

$$(2.12) \quad \delta^2 (H^* - H_c) - P^* = 0.$$

For the equation to be satisfied, it is also necessary to specify that  $v(0) = v(T) = 0$ . The condition on the 'variation' implies

$$(2.13a) \quad H^*(0) = H(0),$$

$$(2.13b) \quad H^*(T) = H(T).$$

That is, the initial and terminal inventory levels must be specified. The first condition means that  $H(0)$  is part of the data upon which the rate of production  $P^*(0)$  is set. We shall see below that if this condition were not imposed, it would be possible to have infinite average costs per unit time. The second condition, (2.13b), will not be taken seriously because in taking the limit as  $T \rightarrow \infty$ , the terminal inventory can be specified arbitrarily without affecting the decision for the immediate future.

2.2 Solution for Immediate Decision. Of course, the differential equation (2.12) is the classical Euler-Lagrange condition for an extremum

of a functional. We are not, however, so much interested in finding the time-path  $P^*(\theta)$  as in the initial decision  $P^*(0)$ . Subsequent plans will, of course, be made on the basis later information. To solve the differential equation for  $P^*(0)$  and later planned production rates we will use an integral transformation that has been very useful in the analysis of other linear systems.

The Laplace transform of a function  $f(\theta)$  over the interval  $(0, T)$  is defined as:

$$(2.14) \quad \bar{f}_T(p) = \int_0^T e^{-p\theta} f(\theta) d\theta.$$

The transformed function no longer depends, of course, upon  $\theta$ , but on the parameter  $p$ , as well as the upper limit of integration,  $T$ .  $\bar{f}_T(p)$  may be interpreted as the present-value of a time-series  $f(\theta)$ ; it is a function of the interest rate,  $p$ , and not time. It follows from integration by parts that:

$$(2.15) \quad \overline{f_T^{(k)}}(p) = p^k \bar{f}_T(p) - \sum_{j=1}^k p^{k-j} [f^{(j-1)}(0) - e^{-pT} f^{(j-1)}(T)],$$

defining

$$(2.16) \quad \bar{f}(p) = \lim_{T \rightarrow \infty} \bar{f}_T(p),$$

it follows that:

$$(2.17) \quad \overline{f^{(k)}}(p) = p^k \bar{f}(p) - \sum_{j=1}^k p^{k-j} f^{(j-1)}(0)$$

as long as  $f(\theta)$  and its derivatives do not approach infinity more quickly than a polynomial.

Denoting the transform of  $P^*(\theta)$  over the interval  $(0, T)$  by  $\bar{P}_T(p)$ , etc., we will transform the differential equation in  $\theta$  into an algebraic equation in  $p$ , obtaining:

$$(2.18) \quad \delta^2[\bar{H}_T - p^{-1} H_c(1 - e^{-pT})] - p \bar{P}_T + [P(0) - e^{-pT} P(T)] = 0.$$

From the definitional relation (2.2) we also have:

$$(2.18) \quad p \bar{H}_T - [H(0) - e^{-pT} H(T)] = \bar{P}_T - \bar{S}_T.$$

Substituting (2.19) into (2.18) and rearranging terms, we obtain:

$$(2.20) \quad (\delta^2 p^{-1} - p)\bar{P}_T - \delta^2 p^{-1} e^{-pT} [H(T) - H_0] - e^{-pT} P^*(T) + P^*(0) \\ = \delta^2 p^{-1} \left\{ \bar{S}_T - [H(0) - H_0] \right\} .$$

$\bar{S} = \lim_{T \rightarrow \infty} \bar{S}_T$  exists for all  $p$  having positive real parts if the sales forecast  $S(\theta)$  is bounded. In the limit as  $T \rightarrow \infty$ , the terminal inventory level,  $H(T)$ , becomes irrelevant and  $P^*(T)$  drops out, leaving:

$$(2.21) \quad (\delta^2 p^{-1} - p)\bar{P} + P^*(0) = \delta^2 p^{-1} \left\{ \bar{S} - [H(0) - H_0] \right\} .$$

The inventory level at the beginning of the period,  $H(0)$ , and the forecast of sales,  $S(\theta)$ , are pieces of information used in setting the production rate,  $P^*(0)$ , for the instant of time immediately ahead. That  $H(0)$  is not a decision variable comes from the end-point conditions from the variational analysis, viz.  $H^*(0) = H(0)$ . However,  $P^*(0)$  is not so specified. It may be chosen so that the differential equation (2.12), or equivalently the transform equation (2.21), will be satisfied.

$P^*(0)$  may be found as follows. The 'present value' of planned future production rates,  $\bar{P}$ , is finite for all  $p$  having positive real parts as long as  $P(\theta)$  does not approach infinity any faster than a polynomial (ordinarily it would be bounded). The coefficient of  $\bar{P}$  will vanish for values of  $p$  (say,  $p_k$ ) satisfying the equation:

$$(2.22) \quad \delta^2 p_k^{-1} - p_k = 0$$

as well as the condition  $\text{Re}(p) > 0$ . Consequently, the desired root is  $p_i$ , where:

$$(2.23) \quad p_k = \pm \delta, \quad k=1,2, \text{ respectively.}$$

Substituting into equation (2.21), we then obtain the expression for the optimal production rate in the 'next instant' of time as a function of initial inventory levels and a forecast of future sales:

$$(2.24) \quad P^*(0) = \delta \bar{S}(\sigma) - \delta [H(0) - H_0] .$$

Re-introducing the time argument,  $t$ , equation (2.24) may be written as:

$$(2.25) \quad P(t) = P^*(t, +0) = \gamma \int_0^{\infty} e^{-\delta \theta} S(t, \theta) d\theta - \delta [H(t) - H_0].$$

The rule states that the production rate to be set for the  $t$ 'th instant of time should be a weighted sum of future expected sales, less some partial correction of inventory levels (arising, in this case, from previous sales forecast errors). We can see from the equation a rationale for the stipulation  $H^*(t, +0) = H(t)$ . There has been no restriction upon the forecast function. Consequently, if  $H(t)$  were directly set by the rule and there were a discontinuous change in the forecast,  $P(t) = \dot{H}(t) + S(t)$  would become infinite, and so would the costs.

2.3 Solution for Distant Plans. It is also possible to calculate anticipated or planned rates of production in the future. We will start with equation (2.20), interpreting  $T$  not as the end of the planning horizon as before but as the running argument in the 'planning space'. Since  $H^*(t)$  is no longer the specified end-point, the asterisk is used. By choosing  $P_k$  as  $\pm \delta$ , respectively, the identity (2.20) gives the following two equations in  $H^*(T)$  and  $P^*(T)$ :

$$(2.26a) \quad -\delta [H^*(T) - H_0] - P^*(T) = \delta e^{-\delta T} [\bar{S}_T(\delta) - \bar{S}(\delta)]$$

$$(2.26b) \quad \delta [H^*(T) - H_0] - P^*(T) = -\delta e^{-\delta T} [\bar{S}_T(-\delta) + \bar{S}(\delta)] + 2\delta e^{-\delta T} [H(0) - H_0].$$

Eliminating the term involving inventories from this system, the following expression for  $P^*(T)$  may be found:

$$(2.27) \quad P^*(T) = \frac{\delta}{2} \left[ \int_T^{\infty} e^{-\delta(\theta-T)} \cdot \int_0^T \delta(\theta-T) \cdot \int_0^{\infty} e^{-\delta(\theta+T)} \right] S(\theta) d\theta - \delta e^{-\delta T} [H(0) - H_0].$$

For large values of  $T$ , equation (2.27) may be approximated by:

$$(2.28) \quad P^*(T) = \frac{\gamma}{2} \int_0^{\infty} e^{-\gamma \theta} [S(T+\theta) + S(T-\theta)] d\theta.$$

The inventory adjustment term has dropped out essentially because all the adjustment will be expected to take place over a relatively short period of time. Since the calculation is completely in the 'expectations space' there will be no new 'shocks' from the forecast errors to correct.

2.4 Relation to the Rules Discrete in Time. It is worth-while to examine at this point the relation of the 'rules' presented here with the work of Holt, Modigliani, and Simon [2]. If instead of varying continuously in time, the production rate were set for a period of length  $h$ , the cost function would take the form of a sum:

$$(2.29) \quad \bar{C} = \lim_{T \rightarrow \infty} \frac{h}{T} \sum_{\theta=h}^T [\gamma^2 (H_\theta - H_c)^2 + h^{-2} (P_\theta - P_0)^2]$$

where  $H_\theta$  is the inventory level at the end of the  $\theta$ 'th period of time in the future.  $P_\theta$  is the amount of production during the  $\theta$ 'th period of length  $h$ , and the summation is over  $\theta = h, 2h, \dots, \frac{T}{h} h$ .

The decision rule which leads to minimizing the expected value of the cost function above may be shown to be:

$$(2.30) \quad P_{t+h} = \frac{(1-\lambda_1)}{\lambda_1} \sum_{\theta=h}^{\infty} \lambda_1^{\theta/h} S_{t,\theta} - (1-\lambda_1)(H_t - H_0).$$

The parameter  $\lambda_1$  is related to the cost ratio  $\gamma^2$  as follows:

$$(2.31) \quad \lambda_1 = \frac{1}{2} [2 + h^2 \gamma^2 - \sqrt{h^2 \gamma^2 (4 + h^2 \gamma^2)}].$$

If the period is rather short, the parameter  $\lambda_1$  is approximately  $1 - h\gamma$ . Consequently the production rule may be written as

$$(2.32) \quad P_{t+h} = h\delta \sum_{\theta=h}^{\infty} (1-h\delta)^{(\theta-h)/h} S_{t,\theta} - h\delta (H_t - H_c)$$

As we let  $h$  approach zero, we have

$$(2.33) \quad P(t) = \frac{1}{h} P_{t+h} = \delta \sum_{\theta=h}^{\infty} (1-h\delta)^{(\theta-h)/h} \frac{1}{h} S_{t,\theta} - \delta (H_t - H_c) \\ = \delta \int_0^{\infty} e^{-\delta\theta} S(t, \theta) d\theta - \delta [H(t) - H_c].$$

This is, of course, identical with the continuous rule (2.25) which has been previously obtained.

If, on the other extreme, we let the time interval become very large, then  $\lambda_1$  is approximately equal to  $(h\delta)^{-2}$ . Here the rule would become

$$(2.24) \quad P_{t+h} = (1-h^{-2}\delta^{-2}) \sum_{\theta=h}^{\infty} (h\delta)^{-2(\theta-1)} S_{t,\theta} - (1-h^{-2}\delta^{-2})(H_t - H_c) \\ \sim S_{t,h} - (H_t - H_c).$$

This limit of the production adjustment which is due to Metzler [4] and analogous to that of Vassian [8], can, of course, be obtained from a cost function involving inventory costs alone.

### 3. Dynamic Performance Characteristics

3.1 Response to a Sinusoidal Input. Returning now to the decision rule, equation (2.25), it is evident that the dynamic performance will depend critically upon the type of forecast that is used. We will examine the performance of the rule for a sinusoidal sales pattern for three types of forecasts:

1. 'Perfect':  $S(t, \theta) = S(t+\theta),$
2. 'Null':  $S(t, \theta) = 0,$
3. 'Naive':  $S(t, \theta) = S(t).$

Other possibilities involve mistakes as to amplitude, frequency, and phase of the fluctuations, but we will not consider these here.

With the perfect forecast, the production performance may be obtained directly from the Euler-Lagrange equation (2.12) to be:

$$(3.1) \quad \gamma^2 P(t) - \ddot{P}(t) = \gamma^2 S(t).$$

The relative magnitude of production fluctuations to sales fluctuations is:

$$(3.2) \quad |Y_P| = \frac{1}{1 + \left(\frac{\omega}{\gamma}\right)^2}$$

where  $\omega$  is the angular frequency of fluctuations of sales. The lag in production adjustments behind sales,  $\angle Y_P$ , would be zero, as we would expect with a perfect forecast. The relative response  $|Y_P|$  is close to unity for  $\omega < \gamma$ , but drops off sharply for large  $\omega$ . In other words, production adjusts to slow changes in sales, but ignores those of a very short duration. The 'cut-off' frequency depends, of course, upon the relative costs of inventory storage and overtime, which are reflected in the coefficient  $\gamma^2$ . The relative magnitude of inventory adjustments to the fluctuations in sales is given by:

$$(3.3) \quad |Z_P| = \frac{1}{\gamma} \frac{\left(\frac{\omega}{\gamma}\right)}{1 + \left(\frac{\omega}{\gamma}\right)^2}$$

and  $\angle Z_P = \pi/2$  (i.e. the inventory reaches its maximum level when production equals sales and both are increasing). The function  $|Z_P|$  is small for very high and very low frequencies, reaching a maximum of  $1/2\gamma$  for  $\omega = \gamma$ .

The analysis with the perfect forecast would give the best performance. It is, however, dynamically unstable, so that any errors in the forecast would ultimately 'blow up'. It can be shown that the performance of the rule is, in general, dynamically stable if the forecast is what servo-mechanism engineers call 'physically realizable' (i.e. depends only upon past data).

Perhaps the most simple realizable forecast is the 'null' forecast, that future sales are expected to be equal to some long-term average. With  $S(t,0) = 0$ , the rule (2.25) becomes:

$$(3.4) \quad P(t) = -\gamma[H(t) - H_0].$$

Consequently the relative magnitude of production and sales fluctuations is:

$$(3.5) \quad |Y_0| = \frac{1}{\sqrt{1 + (\frac{\omega}{\gamma})^2}} = \sqrt{|Y_P|}.$$

Since all values of the function are less than unity, the rule with the null forecast responds more to sales fluctuations than that with the perfect forecast for all frequencies. The angle by which production lags behind sales is given by:

$$(3.6) \quad \angle Y_0 = \tan^{-1}\left(-\frac{\omega}{\gamma}\right);$$

increasing with the frequency. The magnitude of response of the inventory level is:

$$(3.7) \quad |Z_0| = \frac{i}{\gamma} \frac{1}{\sqrt{1 + (\frac{\omega}{\gamma})^2}}$$

with the same lag as production.

A 'naive' forecast is equivalent to the stipulation that  $S(t,0) = S(t)$ . This leads to the production rule:

$$(3.8) \quad P(t) = S(t) - \gamma[H(t) - H_0].$$

In a way, this case is rather uninteresting, because production will equal sales and consequently there will be no fluctuations in the level of inventories.

3.2 Sales Impulse Anticipated T Units of Time in Advance. Another way of examining the effects of the rule is the following. Suppose sales and production have been maintained for a long time at a constant rate

(which can be assumed equal to zero without loss of generality). At time  $t = T$ , an impulse of sales, which was anticipated at time  $t = 0$ , occurs. It is then possible to examine the cost of production as a function of  $T$ , the length of time in advance that such a sale is expected.

The sales function would then be

$$(3.9) \quad S(t) = \delta(t-T)$$

where  $\delta$  is the Dirac delta. The sales forecast function would be

$$(3.10) \quad S(t, \theta) = \begin{cases} \delta(t+\theta-T), & 0 \leq t \leq T, 0 \leq \theta, \\ 0, & \text{otherwise.} \end{cases}$$

If  $T = 0$ , there would be no advance notice of the sale; if  $T = \infty$ , there would be perfect advance knowledge. These two extremes have been examined by Holt and Simon [3] with different methods than those to be used here.

With some reasonably convenient operations involving the Laplace transform, equations (3.9)-(3.10) and the decision rule lead to the following function for the production 'response';

$$(3.11) \quad P(t) = \begin{cases} 0, & t \leq 0, \\ \gamma e^{-\delta T} \cosh \delta t, & 0 < t \leq T, \\ \gamma e^{-\delta t} \cosh \delta T, & T \leq t. \end{cases}$$

Similarly, the time-path of the inventory levels may be found by integration to be:

$$(3.12) \quad H(t) = \begin{cases} 0, & t \leq 0, \\ e^{-\delta T} \sinh \delta t, & 0 \leq t < T, \\ -e^{-\delta t} \sinh \delta T, & T < t. \end{cases}$$

Substitution of (3.11) and (3.12) into the cost function (1.1) leads to the total cost:

$$(3.13) \quad C(T) = \gamma (1 + e^{-2\delta T})/2.$$

Evidently  $C$  is a decreasing function of  $T$  and

$$C(0) = \gamma, \quad C(\infty) = \gamma/2.$$

The 'success' of the forecast is defined as:

$$(3.14) \quad s(T) = \frac{C(0) - C(T)}{C(0) - C(\infty)}.$$

This function measures the fraction of total cost savings in forecasting can be realized with a forecast having horizon  $T$ . From the cost function (3.13), the 'success' function becomes:

$$(3.15) \quad s(T) = 1 - e^{-2\delta^2 T}.$$

With  $\delta^2 = .09$ , a forecast horizon of 5 months would be 95% 'successful.'

Although the entire future is relevant to some extent, the cost effect of errors declines for this case quite rapidly. If the cost of inventory deviations is higher, only shorter horizons would be required; as the storage cost completely dominates the cost of deviations in the production rate, only the immediate future is relevant (as in the Metzler model [4]).

#### 4. Effects of Forecast and Estimation Errors

Since imperfections in the information available to the decision-maker is an important characteristic of planning problems, it is certainly relevant to examine the effects of various kinds of errors upon not only the decisions that are made but also the extra costs that are incurred as a result of these errors. Possible sources of error in the types of models considered here might be summarized as follows:

##### 1. Estimation of parameters:

- i. Cost coefficients ( $\delta^2$ ),
- ii. Forecasts ( $S(t, \theta) - S(t, 0)$ ),
- iii. Feedback information ( $H(t, 0) - H(t)$ );

2. Performance (e.g., differences between planned production and actual production,  $P(t,0) - P(t)$ );
3. Specification of the criterion function:
  - i. Time grid: planning horizon (T) and decision frequency ( $1/h$  for the rule discrete in time, infinite for the continuous rule),
  - ii. Aggregation of components: product of the firm and the organizational units and locations,
  - iii. Excluded components: decision variables (e.g., raw materials purchases, work-in-process inventories, etc.) and costs,
  - iv. More fundamental assumptions: the quadratic model, lack of unilateral restrictions, the expected value of the function as a criterion for choice, etc.

Although little can yet be said about the sources of error toward the bottom of the list, some statements can be made about the effects of cost estimation errors and information, control, and forecast errors. This we do below.

4.1 Errors in Cost Coefficients. Since the parameters of the decision rules depend upon the values of the cost coefficients, the behavior indicated will be costly to the extent that incorrect estimates of the coefficients were used. In the case involving inventory and production rate costs, an error in the ratio  $\sigma^2$  and the parameters  $H_c$  and  $P_c$  will result in different decisions (and hence higher costs).\*

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\* Costing errors in decision rules was first carried out by Roberson, Holt, and Modigliani [5] for the rule based on inventory, overtime, and labor turnover costs. H. Theil has made extensive investigations for a static quadratic welfare function in work that is as yet unpublished.

The cost functional (1.1) may be written as:

$$(4.1) \quad \bar{C} = \gamma^2 \overline{(H-H_0)^2} + \overline{(P-P_0)^2}.$$

The bars indicate the time average of the square of inventory and production deviations, respectively.

Assume that sales can be forecasted perfectly, that there is no error in estimating  $H_0$  ( $P_0$  is irrelevant), and that sales may be represented by the undamped sinusoid:

$$(4.2) \quad S(t) = \bar{S} + A \cos \omega t, \quad A = \text{const.}$$

The resulting production rates (if sales are perfectly known) will obey the following differential equation:

$$(4.3) \quad \ddot{P}(t) + c^2 P(t) = c^2 S(t)$$

if  $c^2$  is the estimate of  $\gamma^2$ . The production performance would therefore be given by:

$$(4.4) \quad P(t) = \frac{c^2}{c^2 + \omega^2} A \cos \omega t + \bar{S}$$

-- i.e., production would have the same frequency and phase as sales and an amplitude no greater than that of sales whatever the value of the estimate of the cost ratio.

The time average of production fluctuations would then be:

$$(4.5) \quad \overline{(P-P_0)^2} = \frac{A^2}{2} \frac{c^4}{(c^2 + \omega^2)^2} + (\bar{S} - P_0)^2.$$

Similarly, it can be shown that the average of the inventory fluctuations is:

$$(4.6) \quad \overline{(H-H_0)^2} = \frac{A^2}{2} \frac{\omega^2}{(c^2 + \omega^2)^2}.$$

If  $c^2$  were too large, then more of the fluctuations in sales would be 'taken up' in production fluctuations than would be desirable (and conversely), but

neither of these errors in the estimate of the cost coefficient would make much difference at low frequencies (i.e. if the sales rate changes slowly).

Assuming that  $\bar{S} = P_c$ , the time average of costs can be written as a function of the true coefficient  $c^2$ , its estimate  $\hat{c}^2$ , and the frequency of sales fluctuations from (4.1) as follows:

$$(4.7) \quad \bar{C} = \frac{A^2}{2} \frac{(\hat{c}^2 \omega^2 + c^4)}{(\omega^2 + c^2)^2}$$

Similar results would, of course, be obtained with a rule with which decisions are made at distinct points of time. The relative increase in costs due to an error of estimating the cost coefficient is:

$$(4.8) \quad \frac{\bar{C} - \bar{C}_{\min}}{\bar{C}_{\min}} = \left[ \frac{\hat{c}^2 - c^2}{\omega^2 + c^2} \right]^2$$

If, for example, the cost ratio  $c^2$  were 10% greater than  $\hat{c}^2$ ,  $\delta = .3$  per month (as is approximately the case for the factory studied in [2]), and  $\omega = .525$  radians/month (one cycle per year), then the error would increase costs by a factor of about 0.18%. It is a distinct advantage that the performance of the rule and the cost of the performance is not very sensitive to errors in the estimates of the coefficients. (Extreme insensitivity is, however, equivalent to stating that costs are not affected much by the scheduling decisions.) Furthermore, substantial deviations from predictions of inventory fluctuations made with rules of this sort might be expected because cost differences might be hard for firms to perceive.

Equation (4.8) is obviously a function of the frequency of the sales fluctuations. One can see that the costs 'peak' at the angular frequency  $\omega = c$ , the square-root of the estimated cost. The maximum value of the

relative cost difference is:

$$(4.9) \quad \frac{1}{4} \left( \frac{y}{c} - \frac{c}{x} \right)^2.$$

Thus, the maximum relative increase in cost for  $c^2 = 1.10 \gamma^2$  would be 0.23%.

4.2 Forecast, Control, and Information Errors. Letting  $h = 1$ , the discrete form of the rule, (2.30), becomes:

$$(4.10) \quad P_{t,1} = \frac{1-\lambda_1}{\lambda_1} \left[ \sum_{\theta=1}^{\infty} \lambda_1^{\theta} S_{t,\theta} - \lambda_1 (H_{t,0} - H_0) \right]$$

where  $P_{t,1}$  is the amount of production planned for the coming period,  $H_{t,0}$  if the (imperfect) estimate of inventories at the end of the preceding period, and  $S_{t,\theta}$  is the forecast made at time  $t$  for sales in the  $(t+\theta)$ 'th period. If there were no errors in these variables, they would be equal to  $P_{t+1}$ ,  $H_t$ , and  $S_{t+\theta}$ , respectively.

Suppose that the time-series  $S_t$  is stationary and independent:

$$(4.11) \quad S_t = \bar{S} + \varepsilon_{st}, \quad \bar{\varepsilon}_s = 0, \quad \overline{\varepsilon_s^2} > 0,$$

and that the forecast of future sales is:

$$(4.12) \quad S_{t,\theta} = \bar{S}.$$

We will further stipulate the 'information' error:

$$(4.13) \quad H_{t,0} = H_t + \varepsilon_{\mu t}, \quad \bar{\varepsilon}_H = 0, \quad \overline{\varepsilon_H^2} > 0,$$

and the 'control' error:

$$(4.14) \quad P_{t,1} = P_{t+1} + \varepsilon_{pt}, \quad \bar{\varepsilon}_P = 0, \quad \overline{\varepsilon_P^2} > 0.$$

We will assume that the errors are statistically independent.

Actual production in the  $(t+1)$ 'st period is then described by:

$$(4.15) \quad P_{t+1} = -\varepsilon_{p,t+1} - (1-\lambda_1)(H_t + \varepsilon_{\mu t} - H_0) + \bar{S}.$$

The difference equation above can readily be solved with the aid of the power-series transform:

$$(4.16) \quad P_{t+1} = (1-\lambda_1) \sum_{s=0}^{\infty} \lambda_1^s \epsilon_{P,t-s} - \epsilon_{P,t+1} + (1-\lambda_1) \sum_{s=0}^{\infty} \lambda_1^s \delta_{S,t-s} \\ + \frac{(1-\lambda_1)}{\lambda_1} \sum_{s=1}^{\infty} \lambda_1^s \epsilon_{H,t-s} - (1-\lambda_1) \epsilon_{Ht} + \bar{S}.$$

From the relation  $H_t - H_{t-1} = P_t - S_t$ , we can also find:

$$(4.17) \quad H_t = -\sum_{s=0}^{\infty} \lambda_1^s \epsilon_{P,t-s} - (1-\lambda_1) \lambda_1^{-1} \sum_{s=1}^{\infty} \lambda_1^s \epsilon_{H,t-s} + H_0 - \sum_{s=0}^{\infty} \lambda_1^s \epsilon_{S,t-s}.$$

Consequently, the mean-square deviation of production is:

$$(4.18) \quad \overline{(P-P_0)^2} = \frac{2}{1+\lambda_1} \overline{\epsilon_P^2} + \frac{2(1-\lambda_1)^2}{1+\lambda_1} \overline{\epsilon_H^2} + \frac{1-\lambda_1}{1+\lambda_1} \overline{\epsilon_S^2} + (\bar{S} - P_0)^2.$$

The deviation in inventories is:

$$(4.19) \quad \overline{(H-H_0)^2} = \frac{1}{1-\lambda_1^2} \overline{\epsilon_P^2} + \frac{1-\lambda_1}{1-\lambda_1^2} \overline{\epsilon_H^2} + \frac{1}{1-\lambda_1^2} \overline{\epsilon_S^2}.$$

The average cost of these errors will then be:

$$(4.20) \quad \bar{C} = \frac{(1-\lambda_1)^2}{\lambda_1} \overline{(H-H_0)^2} + \overline{(P-P_0)^2} \\ = \frac{1}{\lambda_1} \overline{\epsilon_P^2} + \frac{(1-\lambda_1)^2}{\lambda_1} \overline{\epsilon_H^2} + \frac{1-\lambda_1}{\lambda_1} \overline{\epsilon_S^2} + (\bar{S} - P_0)^2.$$

Since  $\lambda_1$  is always less than unity (and real), it is possible to draw certain conclusions about the effect of the three errors upon average costs. The rule can clearly do reasonably well with sloppy information about existing inventory levels, while deviations from planned production are rather costly. Although the model is not particularly sensitive to forecast errors if they are serially independent, the rule is quite sensitive to a bias in the forecast.

For example, if  $\gamma^2 = .09$ , as in the previous illustrations,  $\lambda_1$  would be equal to .74, and the cost function (4.20) would become:

$$(4.21) \quad \bar{C} = 1.35 \bar{\epsilon}_P^2 + .09 \bar{\epsilon}_H^2 + .35 \bar{\epsilon}_S^2 + (\bar{S} - P_0)^2.$$

Of course, the analysis of the effects of error is by no means complete. We have examined their effect for a rather simple cost function — and even here for special cases. To have done otherwise would have led us into too much analytical underbrush. However, such analyses do help answer the question of what constitutes a good forecast. Obviously, one cannot tell unless he has some idea of the effects of errors upon the criterion for choice (e.g. production costs). The costs as we have formulated them are quite insensitive to moderate errors. Again, interpreting the 'decision rules' as behavioral hypotheses, the insensitivity of costs to small errors suggests deviations of a fair magnitude about predicted output. If in addition a 'threshold level' in cost perceptions is postulated, it may, however, be possible to estimate the error of the prediction, at least at the microeconomic level.

References

1. Charles C. Holt, Franco Modigliani, and John F. Muth, 'Derivation of a Linear Decision Rule for Production and Employment,' Mgmt. Sci. 2: 159-177 (1956).
2. \_\_\_\_\_, \_\_\_\_\_, and Herbert A. Simon, 'A Linear Decision Rule for Production and Employment Scheduling,' Mgmt. Sci. 2: 1-30 (1955).
3. \_\_\_\_\_ and Herbert A. Simon, 'Optimal Decision Rules for Production and Inventory Control,' Proc. Conf. Prod. and Inv. Control (Case Inst. of Tech., 1954).
4. Lloyd A. Metzler, 'The Nature and Stability of Inventory Cycles,' Rev. Econ. Stat. 23: 113-129 (1941).
5. Clyde E. Roberson, Charles C. Holt, and Franco Modigliani, 'The Response of a Linear Decision Rule to Sinusoidal Sales, and Accuracy Requirements of the Cost Parameters,' O.N.R. Research Memorandum No. 32 (Carnegie Inst. of Tech., 1955).
6. Herbert A. Simon, 'On the Application of Servomechanism Theory in the Study of Production Control,' Econometrica 20: 247-268 (1952).
7. \_\_\_\_\_, 'Dynamic Programming under Uncertainty with a Quadratic Criterion Function,' Econometrica 24: 74-81 (1956).
8. Herbert J. Vassian, 'Application of Discrete Variable Servo Theory to Inventory Control,' Jour. Opns. Res. Soc. Amer. 3: 272-282 (1955).

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