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ON VALUES ASSOCIATED WITH A STOCHASTIC SEQUENCE

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1. Introduction

Let \( \{z_n\}_{n=1}^{\infty} \) be a sequence of random variables with a known joint distribution. We are allowed to observe the \( z_n \) sequentially, stopping anywhere we please; the decision to stop with \( z_n \) must be a function of \( z_1, \ldots, z_n \) only (and not of \( z_{n+1}, \ldots \)). If we decide to stop with \( z_n \), we are to receive a reward \( x_n = f_n(z_1, \ldots, z_n) \) where \( f_n \) is a known function for each \( n \). Let \( t \) denote any rule which tells us when to stop and for which \( E(x_t) \) exists, and let \( v \) denote the supremum of \( E(x_t) \) over all such \( t \). How can we find the value of \( v \), and what stopping rule will achieve \( v \) or come close to it?

2. Definition of the \( \gamma_n \) sequence

We proceed to give a more precise definition of \( v \) and associated concepts. We assume given always

(a) a probability space \((\Omega, \mathcal{F}, P)\) with points \( \omega \);
(b) a nondecreasing sequence \( \{\mathcal{F}_n\}_{n=1}^{\infty} \) of sub-Borel fields of \( \mathcal{F} \);
(c) a sequence \( \{x_n\}_{n=1}^{\infty} \) of random variables \( x_n = x_n(\omega) \) such that for each \( n \geq 1 \), \( x_n \) is measurable \( (\mathcal{F}_n) \) and \( E(x_n) < \infty \).

(In terms of the intuitive background of the first paragraph, \( \mathcal{F}_n \) is the Borel field \( \mathcal{B}(z_1, \ldots, z_n) \) generated by \( z_1, \ldots, z_n \). Having served the purpose of defining the \( \mathcal{F}_n \) and \( x_n \), the \( z_n \) disappear in the general theory which follows.) Any random variable \( (\text{r.v.}) \) \( t \) with values \( 1, 2, \ldots \) (not including \( \infty \)) such that the event \([t = n]\) (that is, the set of all \( \omega \) such that \( t(\omega) = n \)) belongs to \( \mathcal{F}_n \) for each \( n \geq 1 \), is called a stopping variable \( (\text{s.v.}) \); \( x_t = x_{t(\omega)}(\omega) \) is then a r.v. Let \( C \) denote the class of all \( t \) for which \( E(x_t) < \infty \). We define the value of the stochastic sequence \( \{x_n, \mathcal{F}_n\}_{n=1}^{\infty} \) to be

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Similarly, for each \( n \geq 1 \) we denote by \( C_n \) the class of all \( t \) in \( C \) such that \( P[t \geq n] = 1 \), and set
\[
(2) \quad v_n = \sup_{t \in C_n} E(x_t).
\]
Then
\[
(3) \quad C = C_1 \supset C_2 \supset \cdots \quad \text{and} \quad v = v_1 \geq v_2 \geq \cdots ;
\]
since \( t = n \in C_n \), we have \( v_n \geq E(x_n) \geq -\infty \).

For any family \((y_t, t \in T)\) of r.v.'s we define \( y = \operatorname{ess\, sup}_{t \in T} y_t \) if (a) \( y \) is a r.v. such that \( P[y \geq y_t] = 1 \) for each \( t \) in \( T \), and (b) if \( z \) is any r.v. such that \( P[z \geq y_t] = 1 \) for each \( t \) in \( T \), then \( P[z \geq y] = 1 \). It is known that there always exists a sequence \( \{t_k\}^* \) in \( T \) such that
\[
(4) \quad \sup_{t \in T} y_{t_k} = \operatorname{ess\, sup}_{t \in T} y_t.
\]
We may therefore define for each \( n \geq 1 \) a r.v. \( \gamma_n \) measurable \((\mathcal{F}_n)\) by
\[
(5) \quad \gamma_n = \operatorname{ess\, sup}_{t \in C_n} E(x_t | \mathcal{F}_n);
\]
then \( \gamma_n \geq x_n \) (equalities and inequalities are understood to hold up to sets of \( P \)-measure 0) and \( E(\gamma_n^-) \leq E(x_n^-) < \infty \).

It might seem more natural to consider, instead of \( C_n \), the larger class \( \bar{C}_n \) of all s.v.'s \( t \) such that \( P[t \geq n] = 1 \) and \( E(x_t) \) exists, that is \( E(x_t^-) \) and \( E(x_t^+) \) not both infinite. However, this would yield the same \( v_n \) and \( \gamma_n \). For if \( t \in \bar{C}_n \), define
\[
(6) \quad \gamma'_n = \left\{ \begin{array}{ll} E(x_t | \mathcal{F}_n) & \text{if } E(x_t | \mathcal{F}_n) \geq x_n, \\ n & \text{otherwise}. \end{array} \right.
\]
Then setting \( A = [E(x_t | \mathcal{F}_n) \geq x_n] \), we have
\[
(7) \quad E(x_t^-) \leq E(x_n^-) + \int_A x_t^-.
\]
But \(-\infty < \int_A x_t \leq \int_A x_t^- \), so \( \int_A x_t^- < \infty \). Hence, \( E(x_t^-) < \infty \) and \( t' \in C_n \). Now \( E(x_{t'} | \mathcal{F}_n) = \max (x_n, E(x_t | \mathcal{F}_n)) \geq E(x_t | \mathcal{F}_n) \), and hence \( E(x_t^-) \geq E(x_t) \). It follows that \( v_n \) and \( \gamma_n \) are unchanged if we replace \( C_n \) by \( \bar{C}_n \) in their definitions.

3. Some lemmas

**Lemma 1.** For each \( n \geq 1 \) there exists a sequence \( \{t_k\}^* \) in \( C_n \) such that
\[
(8) \quad x_n \leq E(x_{t_k} | \mathcal{F}_n) \uparrow \gamma_n \quad \text{as } k \to \infty.
\]

**Proof.** Choose \( \{t_k\}^* \) in \( C_n \) with \( t_1 = n \) such that \( \gamma_n = \sup_k E(x_{t_k} | \mathcal{F}_n) \). By lemmas 2 and 3 below, we can assume that (8) holds.

**Lemma 2.** For any \( t \in C_n \), define \( t' = \text{first } k \geq n \text{ such that } E(x_t | \mathcal{F}_n) \leq x_k \). Then
(a) $t' \leq t$, $t' \in C_n$,
(b) $E(x_t | \mathcal{F}_n) \geq E(x_t | \mathcal{F}_j)$,
(c) $t' > j \geq n \Rightarrow E(x_{t'} | \mathcal{F}_j) > x_j$.

**Proof.** If $t = j \geq n$, then $E(x_t | \mathcal{F}_j) = x_j$, so $t' \leq j$; hence, $t' \leq t$. Now

$$E(x_{t'}) = \sum_{k=n}^{\infty} \int_{[t'=k]} x_k \leq \sum_{k=n}^{\infty} \int_{[t'=k]} E^{-}(x_t | \mathcal{F}_k) \leq \sum_{k=n}^{\infty} \int_{[t'=k]} E(x_{t'} | \mathcal{F}_k)$$

$$= E(x_{t'}) < \infty,$$

so that $t' \in C_n$. Hence (a) holds. For any $A \in \mathcal{F}_j$ with $j \geq n$,

$$\int_{A[\tau' \geq j]} x_{t'} = \sum_{k=j}^{\infty} \int_{A[\tau'=k]} x_k \geq \sum_{k=j}^{\infty} \int_{A[\tau'=k]} E(x_t | \mathcal{F}_k) = \int_{A[\tau' \geq j]} x_t.$$ 

Putting $j = n$ gives (b). For $t' > j$ we obtain $E(x_{t'} | \mathcal{F}_j) \geq E(x_t | \mathcal{F}_j) > x_j$, which gives (c).

Any $t' \in C_n$ satisfying (c) of lemma 2 will be called $n$-regular.

**Lemma 3.** Let $\{t_1, \ldots, t_i\} \subseteq C_n$ be $n$-regular for some fixed $n \geq 1$, and define $\tau_i = \max\{t_1, \ldots, t_i\}$. Then $\tau_i \in C_n$ is $n$-regular and

$$\max_{1 \leq k \leq i} E(x_{t_k} | \mathcal{F}_n) \leq E(x_{\tau_i} | \mathcal{F}_n) \leq E(x_{\tau_i} | \mathcal{F}_j).$$

**Proof.** That $\tau_i \in C_n$ is clear. For $j \geq n$ and $A \in \mathcal{F}_j$,

$$\int_{A[\tau_i \geq j]} x_{\tau_i} = \sum_{k=j}^{\infty} \left( \int_{A[\tau_i=k \geq t_{i-1}]} x_{t_{i-1}} + \int_{A[\tau_i=k < t_{i-1}]} x_k \right)$$

$$\leq \sum_{k=j}^{\infty} \left( \int_{A[\tau_i=k \geq t_{i-1}]} x_{t_{i-1}} + \int_{A[\tau_i=k < t_{i-1}]} x_{t_{i-1}} \right)$$

$$= \int_{A[\tau_i \geq j]} x_{t_{i-1}}.$$ 

For $j = n$, this gives

$$E(x_{\tau_i} | \mathcal{F}_n) \geq E(x_{\tau_i} | \mathcal{F}_n) \geq \cdots \geq E(x_{\tau_i} | \mathcal{F}_n) = E(x_{\tau_i} | \mathcal{F}_n),$$

and hence, by symmetry,

$$E(x_{\tau_i} | \mathcal{F}_n) \geq \max_{1 \leq k \leq i} E(x_{t_k} | \mathcal{F}_n).$$

To prove that $\tau_i$ is $n$-regular, we observe by the above that

$$\tau_i \geq j \Rightarrow E(x_{\tau_i} | \mathcal{F}_j) \leq E(x_{\tau_i} | \mathcal{F}_j).$$

Since $t_1$ is $n$-regular,

$$t_1 < j \Rightarrow x_j < E(x_{t_1} | \mathcal{F}_j) = E(x_{t_1} | \mathcal{F}_j) \leq \cdots \leq E(x_{t_1} | \mathcal{F}_j),$$

and by symmetry,

$$\tau_i > j \Rightarrow x_j < E(x_{\tau_i} | \mathcal{F}_j).$$
4. The fundamental theorem

**Theorem 1.** The following relations hold:

(a) \( \gamma_n = \max (x_n, E(\gamma_{n+1}|\mathcal{F}_n)) \), 
\( (n \geq 1) \)

(b) \( E(\gamma_n) = v_n \).

**Proof.** (a). Given any \( t \in C_n \), let \( t' = \max (t, n + 1) \in C_{n+1} \) and set \( A = [t = n] \), and \( I_A \) is indicator function of \( A \). Then

\[
E(x_t|\mathcal{F}_n) = I_A \cdot x_n + I_{t-A} \cdot E(x_{t'}|\mathcal{F}_n) \\
= I_A \cdot x_n + I_{t-A} \cdot E(E(x_{t'}|\mathcal{F}_{n+1})|\mathcal{F}_n) \\
\leq I_A \cdot x_n + I_{t-A} \cdot E(\gamma_{n+1}|\mathcal{F}_n) \leq \max (x_n, E(\gamma_{n+1}|\mathcal{F}_n)).
\]

To prove the reverse inequality, choose, by lemma 1, \( \{t_k\}_1^\infty \in C_{n+1} \) such that

\[
x_{n+1} \leq E(x_{t_k}|\mathcal{F}_{n+1}) \uparrow \gamma_{n+1} \quad \text{as} \quad k \to \infty;
\]
then by the monotone convergence theorem for conditional expectations,

\[
(18) \quad E(\gamma_{n+1}|\mathcal{F}_n) = E(\lim_{k \to \infty} E(x_{t_k}|\mathcal{F}_{n+1})|\mathcal{F}_n) = \lim_{k \to \infty} E(x_{t_k}|\mathcal{F}_n) \leq \gamma_n.
\]

And since \( t = n \) is in \( C_n \), \( x_n = E(x_t|\mathcal{F}_n) \leq \gamma_n \). This completes the proof of (a).

(b). Since for each \( t \) in \( C_n \), \( E(x_t|\mathcal{F}_n) \leq \gamma_n \), \( E(x_t) \leq E(\gamma_n) \), so \( v_n \leq E(\gamma_n) \).

Now choose \( \{t_k\}_1^\infty \) in \( C_n \), according to lemma 1; then

\[
(20) \quad E(\gamma_n) = \lim_{k \to \infty} E(x_{t_k}) \leq v_n.
\]

**Lemma 4.** If \( t \in C \), then

\[
(22) \quad t \geq n \Rightarrow E(x_t|\mathcal{F}_n) \leq \gamma_n \quad \text{and} \quad E(x_t^-|\mathcal{F}_n) \geq \gamma_n^-.
\]

**Proof.** Set \( t' = \max (t, n) \in C_n \). By definition of \( \gamma_n \),

\[
(23) \quad t \geq n \Rightarrow E(x_t|\mathcal{F}_n) = E(x_{t'}|\mathcal{F}_n) \leq \gamma_n,
\]

and hence

\[
(24) \quad t \geq n \Rightarrow E(x_t^-|\mathcal{F}_n) \geq E^-(x_t|\mathcal{F}_n) \geq \gamma_n^-.
\]

5. The r.v. \( \sigma \)

We define the r.v. \( \sigma = \text{first } n \geq 1 \text{ such that } x_n = \gamma_n \quad (= \infty \text{ if no such } n \text{ exists}). \)

In general, \( P[\sigma < \infty] < 1 \), so that \( \sigma \) is not always a s.v.

**Lemma 5.** If \( t \in C \), then \( t' = \min (t, \sigma) \in C \) and \( E(x_{t'}) \geq E(x_t) \).

**Proof.** From lemma 4 we have

\[
(26) \quad E(x_t^-) = \int_{\{t' = t\}} x_{t'}^- + \sum_{n=1}^\infty \int_{\{t > n = e\}} x_{t'}^- \geq \int_{\{t' = t\}} x_{t'}^- + \sum_{n=1}^\infty \int_{\{t > n = e\}} \gamma_n^- \\
= \int_{\{t' = t\}} x_{t'}^- + \sum_{n=1}^\infty \int_{\{t > n = e\} \gamma_n^-} = E(x_{t'}),
\]
so that $t' \in C$. The same argument without the $-$ and with reversed inequality proves the inequality $E(x_t) \leq E(x_{t'})$.

A s.v. $t \in C$ is optimal if $v = E(x_t)$. A s.v. $t$ in $C$ is regular if it is 1-regular; that is, if for each $n \geq 1$, $t > n \implies E(x_t|\bar{\sigma}_n) > x_n$.

**Theorem 2.** (a) If $\sigma \in C$ and is regular, then it is optimal. (b) If $v < \infty$ and an optimal s.v. exists, then $\sigma \in C$ and is optimal and regular; moreover, $\sigma$ is the minimal optimal s.v. and

$$
\sigma \geq n \Rightarrow E(x_\sigma|\bar{\sigma}_n) = E(\gamma_\sigma|\bar{\sigma}_n) = \gamma_n \quad (n \geq 1).
$$

**Proof.** (a) If $\sigma \in C$ and is regular, then $\sigma > n \Rightarrow E(x_\sigma|\bar{\sigma}_n) > x_n$ for each $n \geq 1$. And for any $t \in C$, $\sigma = n$, $t \geq n \Rightarrow E(x_t|\bar{\sigma}_n) \leq \gamma_n = x_n$ by lemma 4. Hence by lemma 1 of [1], $\sigma$ is optimal.

(b) Since $v < \infty$, $v_n = E(\gamma_n) < \infty$ for each $n \geq 1$. Let $s$ in $C$ be any optimal s.v., set $A = \{s = n < \sigma\}$, and suppose $P(A) > 0$. Then

$$
\int_A \gamma_n > \int_A x_n + \epsilon \quad \text{for some } \epsilon > 0.
$$

Choose $\{t_k\}^\infty$ in $C_n$ by lemma 1; then $\int_A x_{t_k} \uparrow \int_A \gamma_n$, so that we can find $k$ so large that $\int_A x_{t_k} > \int_A \gamma_n - \epsilon$. Set

$$
s' = \begin{cases} s & \text{off } A, \\ t_k & \text{on } A; \end{cases}
$$

then it is easy to see that $s'$ is a s.v. in $C$. But

$$
E(x_{s'}) = \int_{u - A} x_s + \int_A x_{t_k} > \int_{u - A} x_s + \int_A x_n = E(x_s),
$$

a contradiction. Hence $P(A) = 0$, and thus $P[\sigma \leq s] = 1$, so $\sigma$ is a s.v. By lemma 5, $\sigma = \min(s, \sigma)$ is in $C$ and $\sigma$ is optimal and minimal.

For any $n \geq 1$, let $A = \{E(x_\sigma|\bar{\sigma}_n) < \gamma_n, \sigma > n\} \in \bar{\sigma}_n$. If $P(A) > 0$, then $\int_A \gamma_n > \int_A x_\sigma$, since $E(\gamma_n) \leq E(\gamma_\sigma) = v < \infty$. By lemma 1, there exists $t$ in $C_n$ such that $\int_A x_t > \int_A x_\sigma$. Define

$$
\tau = \begin{cases} t & \text{on } A, \\ \sigma & \text{off } A; \end{cases}
$$

then it is easy to see that $\tau$ is a s.v. in $C$ and $E(x_\tau) > E(x_\sigma) = v$, a contradiction. Hence $P(A) = 0$, and by lemma 4,

$$
\sigma > n \Rightarrow E(\gamma_\sigma|\bar{\sigma}_n) = E(x_\sigma|\bar{\sigma}_n) = \gamma_n > x_n,
$$

so $\sigma$ is regular and the last part of (b) holds.

6. Bounded stopping variables

The r.v.‘s $\gamma_n$ and the constants $v_n$ are in general impossible to compute directly. To this end we define for any $N \geq 1$ and $1 \leq n \leq N$ the expressions

$$
C_n^N = \text{all } t \in C_n \text{ such that } P[t \leq N] = 1; v_n^N = \sup_{t \in C_n^N} E(x_t);
$$

$$
\gamma_n^N = \esssup_{t \in C_n^N} E(x_t|\bar{\sigma}_n).
$$
Then

\[ -\infty < E(x_n) = v_n^* \leq v_{n+1}^* \leq \cdots \leq v_n \text{ and } x_n = \gamma_n^* \leq \gamma_{n+1}^* \leq \cdots \leq \gamma_n, \]

so that we can define

\[ v'_n = \lim_{N \to \infty} v'_N, \quad \gamma'_n = \lim_{N \to \infty} \gamma'_N, \]

and we have

\[ -\infty < E(x_n) \leq v'_n \leq v_n, \quad x_n \leq \gamma'_n \leq \gamma_n. \]

By the argument of theorem 1 applied to the finite sequence \( \{x_n\}_1^N \), we have

\[ \gamma_N^* = x_N, \]

\[ \gamma_n^* = \max (x_n, E(\gamma_{n+1}^* | \mathcal{F}_n)), \quad (n = 1, \ldots, N - 1), \]

and \( E(\gamma_n^*) = v_n^* \), so that \( \gamma_n^* \) and \( v_n^* \) are computable by recursion. By the monotone convergence theorem for expectations and conditional expectations, \( E(\gamma'_n) = v'_n \), and

\[ \gamma'_n = \max (x_n, E(\gamma'_{n+1} | \mathcal{F}_n)), \quad (n \geq 1). \]

Hence \( \{\gamma'_n\}_1^\infty \) satisfies the same recursion relation as does \( \{\gamma_n\}_1^\infty \). (In \[2\], \( \gamma_n^* = \beta_n^* \), \( \gamma'_n = \beta'_n \).)

**Theorem 3.** If the condition \( A^- \): \( E(\sup_n x_n^+) < \infty \) holds, then

\[ \gamma'_n = \gamma_n \quad \text{and} \quad v'_n = v_n, \quad (n \geq 1). \]

**Proof.** For any \( t \in C_n \) and \( A \in \mathcal{F}_n \),

\[ \int_{A_{[t \leq N]}^*} x_t \leq \int_{A_{[t \leq N]}^*} x_{\min(t,N)} + \int_{A_{[t > N]}^*} x_{N}. \]

Since \( E(x_{\min(t,N)} | \mathcal{F}_n) \leq \gamma_N^* \leq \gamma_n', \)

\[ \int_{A_{[t \leq N]}^*} x_t \leq \int_A \gamma_n' + \int_{A_{[t > N]}^*} (\sup_m x_m^*). \]

Letting \( N \to \infty \),

\[ \int_A x_t \leq \int_A \gamma_n', \quad E(x_t | \mathcal{F}_n) \leq \gamma_n', \quad \gamma_n \leq \gamma_n', \]

so \( \gamma_n = \gamma_n' \) and \( v_n = v_n' \).

**Corollary.** If \( A^- \) holds and \( \{x_n\}_1^\infty \) is Markovian, and \( \mathcal{F}_n = \sigma(x_1, \ldots, x_n) \), then \( \gamma_n = E(\gamma_n | x_n) \).

**Proof.** The Markovian property of \( \{x_n\}_1^\infty \) implies (by downward induction on \( n \)) \( \gamma_n^* = E(\gamma_n^* | x_n^* \) which entails \( \gamma_n' = E(\gamma_n' | x_n) \), and then \( \gamma_n = E(\gamma_n | x_n) \). (The assumption \( A^- \) will be dropped in the corollary to theorem 9.)

**7. Supermartingales**

A sequence \( \{y_n\}_1^\infty \) of r.v.'s is a supermartingale (or lower semimartingale) if for each \( n \geq 1 \), \( y_n \) is measurable \( (\mathcal{F}_n) \), \( E(y_n) \) exists, \( -\infty \leq E(y_n) \leq \infty \), and \( E(y_{n+1} | \mathcal{F}_n) \leq y_n \). We shall denote by \( D \) the class of all supermartingales \( \{y_n\}_1^\infty \) such that \( y_n \geq x_n \) for each \( n \geq 1 \). The sequences \( \{\gamma_n\}_1^\infty \) and \( \{\gamma_n'\}_1^\infty \) are in \( D \).
Theorem 4. The sequence \( \{y_n'\} \) is the minimal element of \( D \).

Proof. For any \( \{y_n\} \) in \( D \),

\[
y_n \geq x_n = y_n',
\]

so that

\[
y_i \geq \lim_{n \to \infty} y_n' = y_i', \quad (i \geq 1).
\]

We shall define various types of "regularity" for elements of \( D \), according to the class of s.v.'s \( t \) for which \( E(y_t) \) is assumed to exist and the relation

\[
t \geq n \Rightarrow E(y_t|\sigma_n) \leq y_n, \quad (n \geq 1)
\]
to hold. An element \( \{y_n\} \) of \( D \) is said to be

(a) regular if for every s.v. \( t \), \( E(y_t) \) exists and (46) holds;
(b) semiregular if for every s.v. \( t \) such that \( E(y_t) \) exists, (46) holds;
(c) C-regular if for every s.v. \( t \in C \) (for which \( E(y_t) \) necessarily exists),

(46) holds.

Clearly, for elements of \( D \), regular \( \Rightarrow \) semiregular \( \Rightarrow \) C-regular.

We shall use the notation \( A^+ : E(\sup_n x_n^+) < \infty, A^* : E(x_t) \) exists for every s.v. \( t \). Clearly, \( A^+ \Rightarrow A^* \Leftrightarrow A^- \).

Lemma 6. If \( A^* \) holds, then for any \( \epsilon > 0 \) and \( n \geq 1 \), there exists \( s \in C_n \) such that

\[
E(x_s|\sigma_n) > \gamma_n - \epsilon \quad \text{on } [\gamma_n < \infty].
\]

Proof. Choose \( \{t_k\} \in C_n \) by lemma 1. On \( [\gamma_n < \infty] \) define \( \alpha = \) first \( k \geq 1 \) such that \( E(x_t|\sigma_n) > \gamma_n - \epsilon \), and set

\[
s = \begin{cases} t_\alpha \text{ on } [\gamma_n < \infty] \\ n \text{ elsewhere.} \end{cases}
\]

Then \( E(x_s) \) exists, and on \( [\gamma_n < \infty] \), \( E(x_s|\sigma_n) > \gamma_n - \epsilon \). Hence,

\[
E(x_s) \geq \int_{[\gamma_n < \infty]} (\gamma_n - \epsilon) + \int_{[\gamma_n = \infty]} x_n > -\infty,
\]

so that \( s \in C_n \).

Lemma 7. (a) Condition \( A^- \) implies \( E(\gamma'_t) = E((\gamma'_t)') < \infty \) for every s.v. \( t \),
and (b) condition \( A^+ \) implies \( E((\gamma'_t)^+) \leq E(\gamma'_t +) < \infty \) for every s.v. \( t \).

Proof. (a) Since by theorem 3 \( x_n \leq \gamma'_n \), \( \gamma'_n = (\gamma'_t)' \leq \sup x_n' \).
(b) Since

\[
\gamma^+_n = \text{ess sup}_{t \in C_n} E^+(x_t|\sigma_n) \leq E(\sup x_t^+|\sigma_n),
\]

then

\[
E((\gamma'_t)^+) \leq E(\gamma'_t) = \sum_{n=1}^{\infty} \int_{[t=n]} \gamma^+_n \leq \sum_{n=1}^{\infty} \int_{[t=n]} E(\sup x_t^+|\sigma_n) = E(\sup x_t^+).
\]
Theorem 5. (a) If \( \{y_n\}_n^* \in D \) and is C-regular, then \( y_n \geq \gamma_n \) for each \( n \geq 1 \);
(b) \( A^* \Rightarrow \{\gamma_n\}_n^* \) is semiregular;
(c) \( A^- \) or \( A^+ \Rightarrow \{\gamma_n\}_n^* \) is regular;
(d) \( \{\gamma_n\}_n^* \) is C-regular.

Proof. (a) If \( \{y_n\}_n^* \in D \) and is C-regular, then

\[
\gamma_n = \text{ess sup}_{t \in \mathcal{C}_n} E(x_t|\mathcal{F}_n) \leq \text{ess sup}_{t \in \mathcal{C}_n} E(y_t|\mathcal{F}_n) \leq y_n.
\]

(b) Let \( \tau \) be any s.v. such that \( P[\tau \geq n] = 1 \) and \( E(\gamma) \) exists. For arbitrary \( \epsilon > 0, k \geq n, \) and \( m \geq 1, \) setting \( A_m = [\gamma_n < m], \) we have

\[
m \geq \int_{A_m} \gamma_n \geq \int_{A_m} \gamma_{n+1} \geq \cdots \geq \int_{A_m} \gamma_k \geq \cdots,
\]

so that \( \gamma_k < \infty \) on \( A_m. \) Hence, \( \gamma_k < \infty \) on \( A = [\gamma_n < \infty]. \) By lemma 6, we can choose \( t_k \in \mathcal{C}_k \) such that

\[
E(x_{t_k}|\mathcal{F}_n) > \gamma_k - \epsilon
\]
on \( A. \)

Define

\[
t = \begin{cases} t_k & \text{on } A[\tau = k], \\ \tau & \text{off } A.
\end{cases}
\]

Then \( E(x_t) \) exists, and on \( A, \)

\[
E(x_t|\mathcal{F}_n) = E\left(\sum_{k=n}^{n} I_{[\tau=k]} E(x_t|\mathcal{F}_t)\right) \geq E\left(\sum_{k=n}^{n} I_{[\tau=k]} (\gamma_k - \epsilon)|\mathcal{F}_n\right) = E(\gamma|\mathcal{F}_n) - \epsilon;
\]

and therefore on \( A, \) by the remark preceding lemma 1,

\[
\gamma_n = \text{ess sup}_{t \in \mathcal{C}_n} E(x_t|\mathcal{F}_n) \geq E(\gamma|\mathcal{F}_n) - \epsilon
\]
(recall that \( \mathcal{C}_n = \) all s.v.'s \( t \geq n \) such that \( E(x_t) \) exists). Hence,

\[
\gamma_n \geq E(\gamma|\mathcal{F}_n)
\]
on \( \Omega. \)

Now let \( t \) be any s.v. such that \( E(\gamma_t) \) exists. Set \( \tau = \max(t, n). \) Then if \( E(\gamma^+_t) = \infty, E(\gamma^-_t) < \infty, \) and hence

\[
E(\gamma^-_t) = \int_{[t>n]} \gamma^-_t + \int_{[t \leq n]} \gamma^-_n < \infty,
\]
while if \( E(\gamma^+_t) < \infty, \) then

\[
E(\gamma^+_t) = \int_{[t>n]} \gamma^+_t + \int_{[t \leq n]} \gamma^+_n < \infty,
\]
since

\[
\int_{[t \leq n]} \gamma_t = \sum_{k=1}^{n} \int_{[t=k]} \gamma_k \geq \sum_{k=1}^{n} \int_{[t=k]} \gamma_n = \int_{[t \leq n]} \gamma_n.
\]

Hence \( E(\gamma_t) \) exists. By the previous result, \( \gamma_n \geq E(\gamma_t|\mathcal{F}_n), \) and hence,

\[
t \geq n \Rightarrow \gamma_n \geq E(\gamma_t|\mathcal{F}_n) = E(\gamma_t|\mathcal{F}_n).
\]

(c) This statement follows from (b) and lemma 7.
For $0 \leq b < \infty$, let $x_n(b) = \min(x_n, b)$, and let $\gamma_n^b (\leq \gamma_n)$ denote $\gamma_n$ for the sequence $\{x_n(b)\}_{\infty}^{\infty}$. As $b \to \infty$, $-x_n^b \leq \gamma_n^b \leq \gamma_n$, say, where $\gamma_n^b \leq \gamma_n$, and for any $t$ in $C_n$, $x_t(b) \geq -x_t^b$, so that $E(x_t(b) | \mathcal{F}_n) \uparrow E(x_t | \mathcal{F}_n)$. Since $\gamma_n \geq \gamma_n^b \geq E(x_t(b) | \mathcal{F}_n)$, $\gamma_n \geq E(x_t | \mathcal{F}_n)$, and hence $\gamma_n \geq \gamma_n^b$, $\gamma_n = \gamma_n$. Now if $t \in C$, then by (c), $t \geq n \Rightarrow E(\gamma^b_n | \mathcal{F}_n) \leq \gamma_n^b \leq \gamma_n$. As $b \to \infty$, since $\gamma_t^b \geq -x_t^b$ and $E(x_t^b) < \infty$, $t \geq n \Rightarrow E(\gamma_n | \mathcal{F}_n) \leq \gamma_n$, so $\{\gamma_n\}^b$ is $\mathcal{C}$-regular.

**Corollary 1.** (a) The sequence $\{\gamma_n\}^b$ is the minimal $\mathcal{C}$-regular element of $D$.

(b) Condition $A^*$ implies that $\{\gamma_n\}^b$ is the minimal semiregular element of $D$.

(c) Either $A^-$ or $A^+$ implies that $\{\gamma_n\}^b$ is the minimal regular element of $D$.

We remark that under $A^-$, $E(\sup_n \gamma_n^b) \leq E(\sup_n x_n^b) < \infty$. Hence, by a well-known theorem, $\{\gamma_n\}^b$ is regular, and similarly for $\{\gamma_n\}^b$. By theorems 4 and 5(a), $\{\gamma_n\}^b = \{\gamma_n\}^b$, which gives an alternative proof of theorem 3.

**Corollary 2.** If $\gamma_n^b = \text{ess sup}_{t \in \mathcal{C}_n} E(\min (x_t, b) | \mathcal{F}_n)$, then

$$\gamma_n = \lim_{b \to \infty} \gamma_n^b.$$  

8. Almost optimal stopping variables

**Lemma 8.** If $v < \infty$, then for any $\epsilon > 0$, $P[x_n \geq \gamma_n - \epsilon, \text{i.o.}] = 1$.

**Proof.** Since $\infty > v = E(\gamma_n) \geq E(\gamma_n) \geq \cdots$, we have $P[\gamma_n < \infty] = 1$ for each $n \geq 1$. Choose any $\epsilon > 0$ and $r > 0$, and define for $n \geq 1$,

$$B_n = \left[ E(x_n | \mathcal{F}_n) > \gamma_n - \frac{\epsilon}{r} \right],$$

where $\{t_n\}^\infty_{\infty}$ is chosen by lemma 1 for each $n \geq 1$ so that $t_n \in C_n$ and $P(B_n) > 1 - 1/r$ (convergence a.e. $\Rightarrow$ convergence in probability). Define

$$B = [x_n < \gamma_n - \epsilon \text{ for all } n \geq m]$$

where $m$ is any fixed positive integer. Then

$$x_n \leq \gamma_n - \epsilon I_B$$

for $n \geq m$,

so on $B_n$ for any $n \geq m$,

$$\gamma_n - \frac{\epsilon}{r} < E(x_n | \mathcal{F}_n) \leq E(\gamma_n | \mathcal{F}_n) - \epsilon P(B | \mathcal{F}_n)$$

$$\leq \gamma_n - \epsilon P(B | \mathcal{F}_n)$$

by theorem 5(d).

Hence on $B_n$, $P(B | \mathcal{F}_n) \leq 1/r$, and therefore $P(BB_n) \leq 1/r$. It follows that $P(B) \leq P(BB_n) + P(B \cap B_n) \leq (1/r) + (1/r) = (2/r)$. Since $r$ can be arbitrarily large, $P(B) = 0$, and therefore,

$$P[x_n \geq \gamma_n - \epsilon \text{ for some } n \geq m] = 1$$

and

$$P[x_n \geq \gamma_n - \epsilon, \text{i.o.}] = \lim_{m \to \infty} = 1.$$

**Theorem 6.** For any $\epsilon > 0$, define

$$s = \text{first } n \geq 1 \text{ such that } x_n \geq \gamma_n - \epsilon \text{ (s = \infty if no such } n \text{ exists)}.$$
Assume the following:

(a) \( P[s < \infty] = 1 \),
(b) \( E(x_n) \) exists,
(c) \( \lim \inf_{n \to \infty} \int_{[s > n]} E^+(\gamma_{n+1}|\mathcal{F}_n) = 0 \).

Then \( E(x_n) \geq v - \epsilon \).

**Proof.** We can assume \( E(x_n) < \infty \). Since \( \gamma_n \leq x_n + \epsilon \), \( E(\gamma_n) < \infty \). Now

\[
\begin{align*}
v &= E(\gamma_1) = \int_{[s = 1]} \gamma_s + \int_{[s > 1]} E(\gamma_2|\mathcal{F}_1) \\
    &= \int_{[s = 1]} \gamma_s + \int_{[s > 2]} \gamma_s + \int_{[s > 2]} E(\gamma_3|\mathcal{F}_2) = \cdots \\
    &= \int_{[1 \leq s \leq n]} \gamma_s + \int_{[s > n]} E(\gamma_{n+1}|\mathcal{F}_n) \leq \int_{[1 \leq s \leq n]} \gamma_s + \int_{[s > n]} E^+(\gamma_{n+1}|\mathcal{F}_n).
\end{align*}
\]

Letting \( n \to \infty \), \( v \leq E(\gamma_1) \leq E(x_n) + \epsilon \).

**Corollary.** For any \( \epsilon \geq 0 \), define \( s \) by (70). Then

(i) for \( \epsilon > 0 \), \( A^+ \Rightarrow P[s < \infty] = 1 \) and \( E(x_n) \geq v - \epsilon \);
(ii) for \( \epsilon = 0 \), \( \{A^+, P[s < \infty] = 1\} \Rightarrow E(x_n) = v \).

**Proof.** Condition \( A^+ \) implies \( v < \infty \), and by lemma 8, this implies that \( P[s < \infty] = 1 \). Condition \( A^+ \) also implies (b) and (c).

**Theorem 7.** Let \( \{\alpha_n\}_n \) be any sequence of r.v.'s such that \( \alpha_n \) is \( (\mathcal{F}_n) \) measurable and \( E(\alpha_n) \) exists for each \( n \geq 1 \), and such that

(a) \( \alpha_n = \max(x_n, E(\alpha_{n+1}|\mathcal{F}_n)) \),
(b) \( P[x_n \geq \alpha_n - \epsilon \text{ i.o.}] = 1 \) for every \( \epsilon > 0 \),
(c) \( \{E^+(\alpha_{n+1}|\mathcal{F}_n)\}_n \) is uniformly integrable,
(d) either \( E(\sup_{n} \alpha_n) < \infty \), or \( A^+ \) holds.

Then for each \( n \geq 1 \), \( \alpha_n \leq \gamma_n \).

**Proof.** For \( m \geq 1 \), \( A \in \mathcal{F}_m \), and \( \epsilon > 0 \), define \( t = \text{first } n \geq m \) such that \( x_n \geq \alpha_n - \epsilon \). Then \( P[m \leq t < \infty] = 1 \). If the first part of (d) holds, then \( E(\alpha_t^n) < \infty \), and since \( x_t \geq \alpha_t - \epsilon \), it follows that \( E(x_t^n) < \infty \), and hence, by theorem 5(d),

\[
\int_A \alpha_t \leq \int_A x_t + \epsilon \leq \int_A \gamma_t + \epsilon \leq \int_A \gamma_m + \epsilon.
\]

If \( A^+ \) holds, then \( E(\alpha_t^n) \leq E(x_t^n) + \epsilon < \infty \), and the same result follows from theorem 5(e). Now

\[
\int_A \alpha_m = \int_{A[t=m]} \alpha_t + \int_{A[t>m]} \alpha_{m+1} = \cdots = \int_{A[m \leq t \leq m+k]} \alpha_t + \int_{A[t \geq m+k]} E^+(\alpha_{m+k+1}|\mathcal{F}_{m+k}).
\]

Letting \( k \to \infty \), it follows from (c) that

\[
\int_A \alpha_m \leq \int_A \alpha_t \leq \int_A \gamma_m + \epsilon,
\]

so since \( \epsilon \) was arbitrarily small, \( \int_A \alpha_m \leq \int_A \gamma_m \), and therefore, \( \alpha_m \leq \gamma_m \).
Corollary. Assume that $A^-$ holds. If $\{\alpha_n\}^\infty_1$ is any sequence such that $\alpha_n$ is measurable $(\mathcal{F}_n)$, $E(\alpha_n)$ exists for each $n \geq 1$, and (a), (b), and (c) hold, then

$$\alpha'_n = \gamma'_n.$$ (75)

Proof. By theorems 7, 3, and 4, since $A^-$ implies (d),

$$\gamma'_n \leq \alpha'_n \leq \gamma'_n = \gamma'_n.$$ (76)

9. A theorem of Dynkin

We next prove a slight generalization of a theorem of Dynkin [3]. Let $\{z_n\}^\infty_1$ be a homogeneous discrete time Markov process with arbitrary state space $Z$. For any nonnegative measurable function $g(\cdot)$ on $Z$, define the function $P_g(\cdot)$ by

$$P_g(z) = E(g(z_{n+1}) | z_n = z),$$ (77)

and set

$$Q_g = \max (g, P_g), \quad Q_g^{k+1} = Q(Q_g^k), \quad (k \geq 0), \quad Q_g^0 = g.$$ (78)

Then $g \leq Q_g \leq Q_g^2 \leq \cdots$, so

$$h = \lim_{N \to \infty} Q_g^N$$ (79)

exists. Let $\mathcal{F}_n = \mathcal{B}(z_1, \ldots, z_n)$ and consider the sequence $\{x_n\}^\infty_1$ with $x_n = g(z_n)$.

Theorem 8. For the process defined above, $\sup_t E(g(z_t)) = E(h(z_1))$.

Proof. By theorem 3,

$$\gamma_1 = \gamma'_1 = \lim_{N \to \infty} \gamma_N^N,$$ (80)

where

$$\gamma_N^N = g(z_N),$$
$$\gamma_{N-1}^N = \max (g(z_{N-1}), E(g(z_N) | z_{N-1})) = Q_g(z_{N-1}),$$
$$\gamma_{N-2}^N = \max (g(z_{N-2}), E(Q_g(z_{N-1}) | z_{N-2})) = \max (g(z_{N-2}), P Q_g(z_{N-2}))$$

(81)

$$= \max (g(z_{N-2}), P Q_g(z_{N-2}), P Q_g(z_{N-2})) = Q_g^2(z_{N-2}),$$

Hence $\gamma_1 = h(z_1)$ and $v = E(\gamma_1) = E(h(z_1))$.

10. The triple limit theorem

Lemma 9. Assume $A^+$ holds, and define

$$x_n(a) = \max (x_n, -a), \quad (0 \leq a < \infty),$$ (82)

$$\gamma_{a}^\infty = \text{ess sup}_{n \geq 1} E(x_n(a) | \mathcal{F}_n).$$

Then

$$\gamma_a = \lim_{a \to \infty} \gamma_{a}^\infty.$$ (83)
Proof. Since $\gamma_n^* = \max (x_n(a), E(\gamma_{n+1}^*|\mathcal{F}_n))$ and $\gamma_n(a) \downarrow \gamma_n^*$, say, as $a \to \infty$, where $\gamma_n^* \geq \gamma_n$, it follows from $A^+$ that $\gamma_n^* = \max (x_n, E(\gamma_{n+1}^*|\mathcal{F}_n))$. For any $\epsilon > 0$ and $m \geq 1$, define $s = \text{first } n \geq m$ such that $x_n \geq \gamma_n^* - \epsilon$ ($= \infty$ if no such $n$ exists). Then $\{\gamma_{\text{min}}(s, n) \}^{s-m}$ is a martingale, since

$$E(\gamma^*_{\text{min}}(s,n+1)) = I_{[s > n]} E(\gamma^*_{n+1}|\mathcal{F}_n) + I_{[s \leq n]} E(\gamma_n^*|\mathcal{F}_n)$$

$$= I_{[s > n]} \cdot \gamma^*_n + I_{[s = m]} \cdot \gamma^*_m + \cdots + I_{[s = n]} \cdot \gamma^*_n = \gamma^*_{\text{min}}(s,n).$$

Since $E((\gamma^*_{\text{min}}(s,n))^+) \leq E(\sup_n x_n^+) < \infty$, and since $E((\gamma_n^* -)^-) < \infty$, we have by a martingale convergence theorem,

$$\gamma^*_{\text{min}}(s,n) \to \text{a finite limit as } n \to \infty,$$

and hence,

$$\gamma_n^* \to \text{a finite limit on } [s = \infty] \quad \text{as } n \to \infty.$$

But on $[s = \infty]$, $\gamma_n^* > x_n + \epsilon$ for $n \geq m$, so

$$\limsup_n x_n \leq \limsup_n \gamma_n^* - \epsilon \quad \text{on } [s = \infty].$$

Since $\gamma_n^* \leq E(\sup_{j \geq m} x_j(a)|\mathcal{F}_n)$ for $n \geq m$,

$$\limsup_n \gamma_n^* \leq \limsup_n \gamma_n^* \leq \sup_{j \geq m} x_j(a),$$

and hence,

$$\limsup_n \gamma_n^* \leq \limsup \sup_n x_n(a) = \max (\limsup_n x_n, -a),$$

and

$$\limsup_n \gamma_n^* \leq \limsup x_n,$$

but $\gamma_n^* \geq x_n$. Hence,

$$\limsup_n \gamma_n^* = \limsup_n x_n,$$

contradicting (87) unless $P[s = \infty] = 0$. Hence,

$$P[x_n \geq \gamma_n^* - \epsilon, \text{i.o.}] = 1,$$

and by theorem 7, $\gamma_n^* \leq \gamma_n$. Therefore, $\gamma_n^* = \gamma_n$.

Theorem 9. The random variables $\gamma_n$ are equal to

$$\gamma_n = \lim_{b \to \infty} \lim_{a \to -\infty} \lim_{N \to \infty} \gamma^N_n(a, b),$$

where

$$\gamma^N_n(a, b) = \text{ess sup}_{P[n \leq i \leq N]=1} E(x_i(a, b)|\mathcal{F}_n)$$

and

$$x(a, b) = \begin{cases} a & \text{if } x < a, \\ x & \text{if } a \leq x \leq b, \\ b & \text{if } x > b. \end{cases}$$

Proof. This follows from lemma 9, theorem 3, and corollary 2 of theorem 5.

Corollary 1. The values $v_n$ are equal to
(96) \[ \lim_{b \to a} \lim_{N \to \infty} v_n^*(a, b). \]

**Corollary 2.** If \( \{x_n\} \) is Markovian and \( \mathfrak{F}_n = \mathfrak{B}(x_1, \ldots, x_n) \), then

\[ \gamma_n = E(\gamma_n|x_n). \]

If the \( x_n \) are independent, then

\[ E(\gamma_{n+1}|\mathfrak{F}_n) = E(\gamma_{n+1}) = v_{n+1}, \]

and the \( v_n \) satisfy the recursion relation

\[ v_n = E(\max(x_n, v_{n+1})), \quad (n \geq 1). \]

**PROOF.** By induction \( \gamma^N(a, b) = E(\gamma^N(a, b)|x_n) \) from \( n = N \) down, as in the proof of the corollary of theorem 3. Letting \( N, a, b \) become infinite yields (97). Under independence,

\[ E(\gamma_{n+1}|\mathfrak{F}_n) = E(E(\gamma_{n+1}|x_{n+1})|\mathfrak{F}_n) = E(\gamma_{n+1}) = v_{n+1}. \]

And from \( \gamma_n = \max(x_n, E(\gamma_{n+1}|\mathfrak{F}_n)) = \max(x_n, v_{n+1}) \), we obtain (99) on taking expectations.

11. Remarks on the independent case

**THEOREM 10.** Let the \( \{x_n\} \) be independent with \( \mathfrak{F}_n = \mathfrak{B}(x_1, \ldots, x_n) \). Set \( s = \text{first } n \geq 1 \text{ such that } x_n \geq \gamma_n - \epsilon \) for \( \epsilon > 0 \) (\( = \infty \) if no such \( n \) exists). Then

\[ v < \infty \Rightarrow P[s < \infty] = 1, \]

and if in addition \( E(x_n) \) exists, then

\[ E(x_n) \geq v - \epsilon. \]

**PROOF.** By lemma 8 and theorem 6, since by (87)

\[ \int_{[s > n]} E^+(\gamma_{n+1}|\mathfrak{F}_n) = \int_{[s > n]} v_{n+1}^+ = v_{n+1}^+ P[s > n] \leq v^+ P[s > n] \to 0. \]

We remark that when \( \epsilon = 0 \) the conditions \( v < \infty, P[s < \infty] = 1, E(x_n) \) exists, imply \( E(x_n) = v \).

**THEOREM 11.** Let the \( \{x_n\} \) be independent with \( \mathfrak{F}_n = \mathfrak{B}(x_1, \ldots, x_n) \), and let \( \{\alpha_n\} \) be any sequence of r.v.'s such that \( \alpha_n \) is measurable \( (\mathfrak{F}_n) \) and \( E(\alpha_n) \) exists, \( n \geq 1 \). If

(a) \[ \alpha_n = \max(x_n, E(\alpha_{n+1}|\mathfrak{F}_n)), \quad (n \geq 1), \]

(b) \[ P(x_n \geq \alpha_n - \epsilon \text{ i.o.}) = 1 \text{ for every } \epsilon > 0, \]

(c) \[ E(\alpha_{n+1}|\mathfrak{F}_n) = c_n = \text{constant, with } E(\alpha_1) = c_1 < \infty, \]

(d) \( A^+ \) holds, or \( \liminf_n E(x_n) > -\infty \),

then

\[ \alpha_n \leq \gamma_n, \quad (n \geq 1). \]

**PROOF.** Define \( A \) and \( t \) as in theorem 7. Since
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(105) \[ c_n = E \{ \max (x_{n+1}, c_{n+1}) \mid \mathcal{F}_n \} \geq c_{n+1}, \]
we have

(106) \[
\int_A \alpha_m = \int_{A[m \leq t \leq m+k]} \alpha_t + \int_{A[t > m+k]} \alpha_{m+k+1} \\
= \int_{A[m \leq t \leq m+k]} \alpha_t + \int_{A[t > m+k]} c_{m+k} \\
\leq \int_{A[m \leq t \leq m+k]} \alpha_t + c_t P[t > m + k].
\]

Hence under \( A^+ \) (or \( A^- \)),

(107) \[
\int_A \alpha_m \leq \liminf_{k \to \infty} \int_{A[m \leq t \leq m+k]} \alpha_t \leq \liminf_{k \to \infty} \int_{A[m \leq t \leq m+k]} x_t + \epsilon \\
\leq \liminf_{k \to \infty} \int_{A[m \leq t \leq m+k]} \gamma_t + \epsilon = \int_A \gamma_t + \epsilon \leq \int_A \gamma_m + \epsilon
\]
by theorem 5(c), so \( \alpha_m \leq \gamma_m \). If the second part of (d) holds, then since \( c_n \downarrow c \), say, where \( c \geq \liminf_n E(x_n) > -\infty \), and \( x_t \geq c_t - \epsilon \geq c - \epsilon \), it follows that \( E(x_t^-) < \infty \), so theorem 5(d) yields the same conclusion.

REMARKS. 1. Lemmas 2 and 3 are slight extensions of lemmas 1 and 2 of [2].

2. Theorem 1 has been proved independently by G. Haggstrom [4] when \( E[x_n] < \infty \) and \( E(\sup_n x_n^+) < \infty \), as have theorem 4, corollary 1(c) of theorem 5 under \( A^+ \), and the corollary of theorem 6. The latter was also proved by J. L. Snell [5].

3. We are greatly indebted to Mr. D. Siegmund for improvements in the statement and proof of many of our results. In particular, theorem 9 is largely due to him.

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