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STATISTICS

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A CLASS OF OPTIMAL STOPPING PROBLEMS

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1. Introduction and summary

Let \( x_1, x_2, \ldots \), be independent random variables uniformly distributed on the interval \([0, 1]\). We observe them sequentially, and must stop with some \( x_i \), \( 1 \leq i < \infty \); the decision whether to stop with any \( x_i \) must be a function of the values \( x_1, \ldots, x_i \) only. (For a general discussion of optimal stopping problems we refer to [1], [3].) If we stop with \( x_i \) we lose the amount \( \alpha x_i \), where \( \alpha \geq 0 \) is a given constant. What is the minimal expected loss we can achieve by the proper choice of a stopping rule?

Let \( C \) denote the class of all possible stopping rules \( t \); then we wish to evaluate the function

\[
v(\alpha) = \inf_{t \in C} E(t^* x_i).
\]

If there exists a \( t \) in \( C \) such that \( E(t^* x_i) = v(\alpha) \), we say that \( t \) is optimal for that value of \( \alpha \). Let \( C^N \) for \( N \geq 1 \) denote the class of all \( t \) in \( C \) such that \( P[t \leq N] = 1 \); then \( C^1 \subset C^2 \subset \cdots \subset C \), and hence, defining

\[
v^N(\alpha) = \inf_{t \in C^N} E(t^* x_i),
\]

we have

\[
\frac{1}{2} = v^1(\alpha) \geq v^2(\alpha) \geq \cdots \geq v(\alpha) \geq 0.
\]

We shall show that as \( N \to \infty \),

\[
v^N(\alpha) \approx \begin{cases} 
2(1 - \alpha)/N^{1-\alpha} & \text{for } 0 \leq \alpha < 1, \\
2/\log N & \text{for } \alpha = 1,
\end{cases}
\]

from which it follows that

\[
v(\alpha) = 0, \quad \text{for } 0 \leq \alpha \leq 1.
\]

(For \( \alpha = 0 \), J. P. Gilbert and F. Mosteller [4] give the expression \( v^N(0) \approx 2/(N + \log (N + 1) + 1.767) \); this case is closely related to a problem of optimal selection considered in [2]. It can be shown that \( Nv^N(0) \uparrow 2 \) as \( N \to \infty \).)

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We shall show, moreover, that
\begin{equation}
0 < v(\alpha) < \frac{1}{2}, \quad \text{for } 1 < \alpha \leq 1.4,
\end{equation}
and that the relation
\begin{equation}
\lim_{N \to \infty} v_N(\alpha) = v(\alpha)
\end{equation}
holds for all \( \alpha \geq 0 \). No optimal rule exists for \( 0 < \alpha \leq 1 \) by (5), since \( E(t^n x_i) > 0 \) for every \( t \) in \( C \). We shall show that an optimal rule does exist for every \( \alpha > 1 \); when \( v(\alpha) = \frac{1}{2} \) the optimal rule is \( t = 1 \), but for any \( \alpha \) such that \( 0 < v(\alpha) < \frac{1}{2} \) the optimal rule \( t \) is such that \( Et = \infty \). The function \( v(\alpha) \) is continuous for all \( \alpha \geq 0 \).

2. Proof of (4)

For any fixed \( \alpha \geq 0 \) and \( N \geq 1 \), set \( v_{N+1}^N = \infty \) and define
\begin{equation}
v_i^N = E\{\min (i^n x_i, v_{i+1}^N)\} = \int_0^1 \min (i^n x, v_{i+1}^N) \, dx \quad (i = N, \cdots, 1).
\end{equation}
The constants \( v_i^N \) can be computed recursively from (8), and by a familiar argument it follows that
\begin{equation}
v_N^N(\alpha) = v_N^N = E(t^n x_i),
\end{equation}
where
\begin{equation}
t = \text{first } i \geq 1 \text{ such that } i^n x_i \leq v_{i+1}^N.
\end{equation}

For the remainder of this section we shall regard \( N \) as a fixed positive integer and \( \alpha \) as a fixed constant such that \( 0 \leq \alpha \leq 1 \); for brevity we shall write \( v_i \) for \( v_i^N \). Then from (8),
\begin{equation}
v_i \leq E(i^n x_i) = i^n/2, \quad (i = 1, \cdots, N),
\end{equation}
so that
\begin{equation}
v_{i+1} i^{-\alpha} \leq \frac{1}{2} \left( \frac{i+1}{i} \right)^\alpha \leq \frac{1}{2} \cdot 2^\alpha \leq 1, \quad (i = 1, \cdots, N - 1).
\end{equation}
Hence from (8),
\begin{equation}
v_i = \int_0^{v_i i^{-\alpha}} i^n x \, dx + (1 - v_i i^{-\alpha})v_{i+1}
= v_{i+1} \left( 1 - \frac{v_{i+1}}{2i^\alpha} \right), \quad (i = 1, \cdots, N - 1).
\end{equation}

Noting that \( v_i > 0 \) for \( i = 1, \cdots, N \), we can rewrite (13) as
\begin{equation}
\frac{1}{v_i} = \frac{1}{v_{i+1}} + \frac{1}{2 i^\alpha - v_{i+1}} = \frac{1}{v_{i+1}} + \frac{1}{2 i^\alpha} + \frac{v_{i+1}}{2 i^\alpha (2 i^\alpha - v_{i+1})},
\end{equation}
\( (i = 1, \cdots, N - 1) \).

Summing (14) for \( i = 1, \cdots, N - 1 \) and noting that from (8)
we obtain the formula

$$v_N = \frac{N^\alpha}{2},$$

(15)

We shall show at the end of this section that, setting

$$I_N = \frac{1}{2} \sum_{i=1}^{N-1} \frac{1}{i^\alpha}, \quad J_N = \frac{1}{2} \sum_{i=1}^{N-1} \frac{v_{i+1}}{i^\alpha(2i^\alpha - v_{i+1})},$$

(16)

we have as $N \to \infty$

$$J_N = o(I_N), \quad I_N \sim \frac{N^{1-\alpha}/2(1 - \alpha)}{\log N/2}, \quad \alpha < 1.$$  \hspace{1em} \text{(18)}

Relations (4) follow from (9), (16), and (18).

**Proof of (18).** The second part of (18) follows from the relation

$$I_N \sim \frac{1}{2} \int_1^N \frac{dt}{t^\alpha},$$

(19)

The first part of (18) follows from two lemmas.

**Lemma 1.** The following inequality holds:

$$v_i \leq \frac{2N^\alpha}{N - i + 1}, \quad (i = 1, \ldots, N).$$

(20)

**Proof.** Equation (20) holds for $i = N$ by (15). Suppose it holds for some $i + 1 = 2, \ldots, N$; we shall show that it holds for $i$ also.

(a). If $2N^\alpha/(N - i) > \bar{i}^\alpha$, then by (11),

$$v_i \leq \frac{\bar{i}^\alpha}{2} \leq \frac{N^\alpha}{N - i} \leq \frac{2N^\alpha}{N - i + 1}$$

(21)

(b). If $2N^\alpha/(N - i) \leq i^\alpha$, then setting

$$f(x) = x \left(1 - \frac{x}{2\bar{i}^\alpha}\right), \quad f'(x) = 1 - \frac{x}{\bar{i}^\alpha} \geq 0, \quad \text{for} \quad x \leq \bar{i}^\alpha,$$

so by (13)

$$v_i = f(v_{i+1}) \leq f\left(\frac{2N^\alpha}{N - i}\right) = \frac{2N^\alpha}{N - i} \left(1 - \frac{N^\alpha}{i^\alpha(N - i)}\right) \leq \frac{2N^\alpha}{N - i + 1},$$

(22)

which completes the proof.

From (12) and (20) we have

$$J_N = \frac{1}{2} \sum_{i=1}^{N-1} \frac{v_{i+1}}{i^\alpha(2i^\alpha - v_{i+1})} \leq \frac{N^\alpha}{N - i} \sum_{i=1}^{N-1} \frac{1}{i^\alpha(2i^\alpha - v_{i+1})}.$$

(24)

To prove the first part of (18), in view of the second part, it will suffice to show the following.

**Lemma 2.** As $N \to \infty$,

$$N^\alpha \sum_{i=1}^{N-1} \frac{1}{(N - i)^{2\alpha}} = \begin{cases} o(N^{1-\alpha}), & (0 \leq \alpha < 1), \\ 0(1), & (\alpha = 1). \end{cases}$$

(25)
Proof. (a). Assume $0 < \alpha < 1$. For any $0 < \delta < 1$, the left side of (25) can be written as

\[ N^\alpha \left( \sum_{1}^{[N]} + \sum_{[N]+1}^{N-1} \right) \frac{1}{(N-i)\delta^{2\alpha}} \leq N^\alpha \left( \frac{1}{N(1-\delta)} \sum_{i=1}^{N-1} \frac{1}{i^\alpha} + N(1-\delta)(\delta N)^{-2\alpha} \right) \]

\[ \sim N^\alpha \left( \frac{1}{N(1-\delta)} \frac{1}{1-\alpha} + N(1-\delta)(\delta N)^{-2\alpha} \right) \sim \frac{(1-\delta)N^{1-\alpha}}{\delta^{2\alpha}}. \]

Hence,

\[ \lim_{N \to \infty} \frac{J_N}{N^{1-\alpha}} \leq \frac{1-\delta}{\delta^{2\alpha}}. \]

Since $\delta$ can be arbitrarily near 1, the left-hand side of (27) must be 0.

(b). Assume $\alpha = 1$. We have for the left-hand side of (25), setting $M = \lfloor N/2 \rfloor$,

\[ N \sum_{1}^{N-1} \frac{1}{(N-i)\delta^{2\alpha}} = N \left( \sum_{1}^{M} + \sum_{M+1}^{N-1} \right) \frac{1}{(N-i)\delta^{2\alpha}} \leq 2 \sum_{i=1}^{M} \frac{i^{-2}}{\delta^{2\alpha}} + N \sum_{M+1}^{N-1} i^{-2} \]

\[ \leq 2 \int_{1/2}^{\infty} \frac{dt}{t^{2\alpha}} + N \left( \frac{N}{2} \right)^{2} = 0(1). \]

3. An optimal rule exists for $\alpha > 1$ and $v(\alpha) > 0$

Define $z_n = \inf_{i \geq n} (i^\alpha x_i)$. Then for any constant $0 \leq A \leq n^\alpha$, we have

\[ P[z_n \geq A] = P[i^\alpha x_i \geq A ; i \geq n] = \prod_{i=n}^{\infty} \left( 1 - \frac{A}{i^\alpha} \right). \]

Hence,

\[ P \left[ z_1 \geq \frac{1}{2} \right] = \prod_{i=1}^{\infty} \left( 1 - \frac{1}{2^\alpha} \right) > 0, \]

and therefore,

\[ v(\alpha) \geq E z_1 > 0. \]

Next, for any $A > 0$,

\[ \sum_{1}^{\infty} P[n^\alpha x_n \leq A] \leq \sum_{1}^{\infty} \frac{A}{n^\alpha} < \infty. \]

Hence, by the Borel-Cantelli lemma,

\[ P[\lim_{n \to \infty} n^\alpha x_n = \infty] = 1. \]

The existence of an optimal $t$ for $\alpha > 1$ now follows from lemma 4 of [1].

4. For $\alpha \geq \frac{3}{2}, v(\alpha) = \frac{1}{2}$

We define for $i = 1, 2, \cdots$, and any fixed $\alpha \geq 0$,

\[ v_i = \inf_{t \in C_i} E(t^\alpha x_i), \]

\[ P \left[ z_1 \geq \frac{1}{2} \right] = \prod_{i=1}^{\infty} \left( 1 - \frac{1}{2^\alpha} \right) > 0, \]

and therefore,

\[ v(\alpha) \geq E z_1 > 0. \]

Next, for any $A > 0$,

\[ \sum_{1}^{\infty} P[n^\alpha x_n \leq A] \leq \sum_{1}^{\infty} \frac{A}{n^\alpha} < \infty. \]

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where \( C_i \) denotes the class of all \( t \in C \) such that \( P[t \geq i] = 1 \). Then \( v(\alpha) = v_1 \leq v_2 \leq \cdots \). It can be shown [3], although it is not trivial to prove, that in analogy with (8),

\[
v_i = E(\min (i^\alpha x, v_{i+1})) = \int_0^1 \min (i^\alpha x, v_{i+1}) \, dx, \quad (i \geq 1).
\]

It follows that

\[
v_i \leq \frac{i^\alpha}{2^i} \quad (i \geq 1).
\]

From now on in this section we shall assume that \( 1 < \alpha \leq \frac{3}{2} \). Then

\[
v_{i+1}i^{-\alpha} \leq \frac{1}{2} \left( \frac{i + 1}{i} \right)^\alpha \leq \frac{1}{2} \left( \frac{3}{2} \right)^\alpha \leq 1, \quad (i \geq 2).
\]

Hence, as in (13),

\[
v_i = v_{i+1} \left( 1 - \frac{v_{i+1}}{2i^\alpha} \right), \quad (i \geq 2),
\]

and since \( v_i = v(\alpha) > 0 \) for \( \alpha > 1 \) by (31), we have as in (14),

\[
\frac{1}{v_i} = \frac{1}{v_{i+1}} + \frac{1}{2i^\alpha - v_{i+1}}, \quad (i \geq 2).
\]

Summing (39) for \( i = n, \cdots, m - 1 \), we obtain

\[
\frac{1}{v_n} = \frac{1}{v_m} + \sum_{n}^{m-1} \frac{1}{2i^\alpha - v_{i+1}}, \quad (2 \leq n \leq m).
\]

From (29), for any \( A > 0 \), we have as \( m \to \infty \),

\[
P[\varepsilon_m \geq A] = \prod_{m} \left( 1 - \frac{A}{v^\alpha} \right) \to 1,
\]

thus \( E\varepsilon_m \to \infty \), and since \( v_m \geq E\varepsilon_m \), it follows that \( v_m \to \infty \). Hence from (40),

\[
\frac{1}{v_n} = \sum_{n}^{m} \frac{1}{2i^\alpha - v_{i+1}}, \quad (n \geq 2).
\]

From (42) and (37) we have for \( n \geq 1 \),

\[
\frac{1}{(\alpha - 1)n^{\alpha - 1}} \geq \sum_{n+1}^{\infty} \frac{1}{i^\alpha} \geq \frac{1}{v_{n+1}} \geq \frac{1}{2} \sum_{n+1}^{\infty} \frac{1}{i^\alpha} \geq \frac{1}{2} \int_{n+1}^{\infty} \frac{dt}{t^\alpha} = \frac{1}{2(\alpha - 1)(n + 1)^{\alpha - 1}}
\]

and hence,

\[
\frac{\alpha - 1}{n} \leq \frac{v_{n+1}}{n^\alpha} \leq \frac{2(\alpha - 1)}{n + 1} \left( \frac{n + 1}{n} \right)^\alpha, \quad (n \geq 1).
\]

We shall now show that \( v_1 > 1 \) for \( \alpha = \frac{3}{2} \). It will follow from (35) that \( v_1 = \frac{1}{2} \) and that \( t = 1 \) is optimal for \( \frac{3}{2} \); the same is true a fortiori for any \( \alpha \geq \frac{3}{2} \).

From (38) we obtain

\[
v_{i+1} = i^\alpha - \sqrt{i^{2\alpha} - 2i^\alpha v_i}, \quad (i \geq 2);
\]
the + sign being excluded because of (37). Suppose now that \( v_2 \leq 1 \) for \( \alpha = \frac{3}{4} \).

Then by (45),

\[
\begin{align*}
v_2 &\leq 2^{3/2} - \sqrt{8 - 2.2^{3/2}} = 1.3, \\
v_4 &\leq 3^{3/2} - \sqrt{27 - 2\sqrt{27}(1.3)} = 1.52, \\
v_6 &\leq 4^{3/2} - \sqrt{64 - 16(1.52)} = 1.7.
\end{align*}
\]

(46)

On the other hand, by (44) we have for \( \alpha = \frac{3}{4} \),

\[
\frac{v_{n+1}}{n^{3/2}} \leq \frac{1}{n+1} \left( \frac{n+1}{n} \right)^{3/2} \leq \frac{1}{6} \left( \frac{6}{5} \right)^{3/2} \leq \frac{11}{50}, \quad (n \geq 5).
\]

(47)

Hence, from (42) for \( \alpha = \frac{3}{4} \),

\[
\frac{1}{v_n} = \sum_{i=1}^{n} \frac{1}{2^{i^2} - v_{i+1}} \leq \sum_{i=5}^{n} \frac{1}{2^{i^2} \left( 1 - \frac{v_{i+1}}{2^{i^2}} \right)} \leq \sum_{i=5}^{n} \frac{1}{2^{i^2} \left( 1 - \frac{11}{100} \right)} \leq 50 \int_{0/2}^{\sqrt{2}} dt/\alpha \leq \frac{50}{89} \frac{1}{\alpha - 1} \frac{\sqrt{2}}{9} \frac{100}{89} \cdot \frac{\sqrt{2}}{3} < \frac{1}{1.7},
\]

contradicting (46). Hence \( v_2 > 1 \) for \( \alpha = \frac{3}{4} \).

5. If \( 1 < \alpha \leq 1.4 \), then \( v(\alpha) < \frac{1}{2} \)

By (44) we have for \( \alpha = \frac{3}{4} \),

\[
v_2 \leq \frac{1}{4} \cdot 3^{3/2} < \frac{5}{8},
\]

and hence by (38), \( v_2 < \frac{5}{8}(1 - (5/4.2^{3/2})) < 1 \). Hence by (35), \( v_1 = v(\frac{3}{4}) < \frac{1}{2} \).

For \( \alpha > 1 \), an optimal \( t \) exists by section 3, and from ([3], theorem 2), a minimal optimal \( t \) is defined by

\[
t = \text{first } n \geq 1 \text{ such that } x_n \leq \frac{v_{n+1}}{n^\alpha}.
\]

(50)

Let \( \alpha \) be any constant \( > 1 \) such that \( v(\alpha) < \frac{1}{2} \). Then \( P[t > 1] > 0 \) by (50), and for \( \alpha < \frac{3}{4} \) we have from (44) that

\[
\frac{v_{n+1}}{n^\alpha} \leq \frac{1}{n + 1} \left( \frac{n+1}{n} \right)^2 < \frac{n+1}{n^\alpha} < 1, \quad \text{for } n \geq 2.
\]

(51)

Hence, \( P[t > N] > 0 \) for every \( N \geq 1 \), so \( t \) is not bounded. In fact, if \( 1 < \alpha = (3 - \epsilon)/2 \) for some \( \epsilon > 0 \), then from (44)

\[
\frac{v_{n+1}}{n^\alpha} \leq (1 - \epsilon) \left( \frac{n+1}{n^2} \right) \leq \frac{1}{n} \quad \text{for } n \geq \frac{1 - \epsilon}{\epsilon}.
\]

(52)

Hence, if \( v(\alpha) < \frac{1}{2} \), so that \( P[t > N] > 0 \) for every \( N \geq 1 \), it follows that for \( n > N \geq \frac{1 - \epsilon}{\epsilon} \) and some \( K > 0 \),
(53) \[ P[t > n] \geq K \left(1 - \frac{1}{N}\right) \left(1 - \frac{1}{N + 1}\right) \cdots \left(1 - \frac{1}{n}\right) = K \cdot \frac{N - 1}{n}, \]

so that \( Et = \sum_{i=0}^{\infty} P[t > n] = \infty. \)

We thus have for \( \alpha > 1 \): either \( 0 < v(\alpha) < \frac{1}{2} \) and \( Et = \infty \), or \( v(\alpha) = \frac{1}{2} \) and \( t = 1 \), where \( t \) is optimal for that \( \alpha \). The least value \( \alpha^* \) such that \( v(\alpha^*) = \frac{1}{2} \) is not known to us, but by the results of this and the previous section, it lies between 1.4 and 1.5.

6. The identification of optimal rules for \( 1 < \alpha \)

For \( N = 1, 2, \ldots \), define \( t_N \) by (10). Then \( t_N \leq t_{N+1} \leq \cdots \). Let \( b_i = \lim_{N \to \infty} v_i^N \). Then from (8),

\[
(54) \quad b_i = \int_0^1 \min (i^{x_i}, b_{i+1}) \, dx, \quad (i = 1, 2, \ldots).
\]

Define

\[
(55) \quad s = \text{first } i \geq 1 \text{ such that } i^{x_i} \leq b_{i+1} \text{ if such an } i \text{ exists},
\]

\[ = \infty \text{ otherwise.} \]

Then \([1] s = \lim_{N \to \infty} t_N. \) Since \( v_i^N \geq v_i \) for each \( N \), \( b_i \geq v_i \). Therefore \( s \leq t \), where \( t \) is an optimal rule defined by (50). We shall now show that \( s = t \) by showing that \( b_i = v_i \) for all \( i \geq 1 \).

From (54) we have

\[
(56) \quad b_i \leq i^{x/2}, \quad (i \geq 1),
\]

and hence as in (37) and (39), for some \( i_0 = i_0(d), \)

\[
(57) \quad b_{i+1} i^{x-i} \leq 1, \quad (i \geq i_0),
\]

\[
\frac{1}{b_i} = \frac{1}{b_{i+1}} + \frac{1}{2i^{x} - b_{i+1}}, \quad (i \geq i_0).
\]

Since \( b_i \geq v_i \to \infty \) as \( i \to \infty \), we have, as in (42),

\[
(58) \quad \frac{1}{b_n} = \sum_{i=1}^{N} \frac{1}{2i^{x} - b_{i+1}}, \quad (n \geq i_0).
\]

Assume that for some \( j \geq 1, \) \( b_j > v_j \). Then by (35) and (54) this inequality must hold for some \( i_1 \geq i_0 \) (since if \( j < i_0 \) and \( b_i \leq v_i \), then \( b_j \leq v_j \)), and hence for every \( i \geq i_1 \). Hence by (42) and (54),

\[
(59) \quad \frac{1}{v_i} = \sum_{i=i_1}^{\infty} \frac{1}{2i^{x} - v_{i+1}} < \sum_{i=i_1}^{\infty} \frac{1}{2i^{x} - b_{i+1}} = \frac{1}{b_i},
\]

a contradiction. Hence \( b_j = v_j \) for all \( j \geq 1 \).

It follows from the above that for \( 1 < \alpha \),

\[
(60) \quad v(\alpha) = v_1 = b_1 = \lim_{N \to \infty} v_i^N = \lim_{N \to \infty} v_\alpha(N).
\]

That this relation holds also for \( 0 \leq \alpha \leq 1 \) has been shown already.
7. Continuity of $v(\alpha)$

From (60), which holds for any $\alpha \geq 0$, given $\epsilon > 0$ we can find $N = N(\alpha, \epsilon)$ so large that

$$v(\alpha) + \frac{\epsilon}{2} \geq v^N(\alpha) = E(t^ax_i)$$

for some $t$ in $CN$. Hence for $\alpha' > \alpha$,

$$v(\alpha) \leq v(\alpha') \leq E(t^{\alpha'}x_i) \leq N^{\alpha'-\alpha}E(t^ax_i) \leq N^{\alpha'-\alpha} \left(v(\alpha) + \frac{\epsilon}{2}\right) \leq v(\alpha) + \epsilon,$$

provided that $\alpha' - \alpha$ is sufficiently small. Hence $v(\alpha)$ is continuous on the right for each $\alpha \geq 0$.

Since $v(\alpha)$ is nondecreasing in $\alpha$ for each fixed $i \geq 1$, we have by the bounded or monotone convergence theorem for integrals from (35)

$$v_i(\alpha - 0) = \lim_{\epsilon \to 0} v_i(\alpha - \epsilon) = \lim_{\epsilon \to 0} \int_0^1 \min (t^{\alpha-\epsilon}, v_{i+1}(\alpha - \epsilon)) \, dx$$

$$= \int_0^1 \min (t^\alpha, v_{i+1}(\alpha - 0)) \, dx \quad (i \geq 1),$$

and by the remark preceding (42), $\lim_{n \to \infty} v_n(\alpha - 0) = \infty$ for $\alpha > 1$. Hence, as in the preceding section, (58) holds with $b_n$ replaced by $v_n(\alpha - 0)$, and the argument shows that $v_n(\alpha - 0) = v_n(\alpha)$. In particular, $v_n(\alpha - 0) = v(\alpha)$, which shows that $v(\alpha)$ is continuous on the left for $\alpha > 1$. Since $v(\alpha) = 0$ for $0 \leq \alpha < 1$, it follows that $v(\alpha)$ is continuous on the left for each $\alpha \geq 0$.

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