Elementary Closures for Integer Programs *

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Abstract

In integer programming, the elementary closure associated with a family of cuts is the convex set defined by the intersection of all the cuts in the family. In this paper, we compare the elementary closures arising from several classical families of cuts: three versions of Gomory’s fractional cuts, three versions of Gomory’s mixed integer cuts, two versions of intersection cuts and their strengthened forms, Chvátal cuts, MIR cuts, lift-and-project cuts without and with strengthening, two versions of disjunctive cuts, Sherali-Adams cuts and Lovász-Schrijver cuts with positive semi-definiteness constraints.

Key Words: Integer programming, cutting plane, elementary closure.

1 Introduction

Recently, the integer programming community has emphasized that many of the cuts found in the literature are essentially the same. Chvátal cuts [12] are equivalent to Gomory fractional cuts [20, 21, 23]. Lift-and-project cuts [4] are disjunctive cuts [3]. Gomory mixed integer cuts [22], disjunctive cuts [2, 9, 24] and mixed integer rounding cuts [28] are equivalent [26].

It is natural to ask which of these cuts are intrinsically different. This is the purpose of

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this paper. Our approach is to compare the corresponding elementary closures. Given an integer program $P_I \equiv \{(x, y) \in \mathbb{Z}_+^{n+p} \mid Ax + Gy \leq b\}$ and a family $F$ of inequalities $\alpha x + \gamma y \leq \beta$ generated from $P \equiv \{(x, y) \in \mathbb{R}_+^{n+p} \mid Ax + Gy \leq b\}$ and valid for $P_I$, the elementary closure $P_F$ is a convex set obtained as the intersection of all the inequalities in $F$. This concept was introduced by Chvátal [12] for a specific family of cuts. In this paper, we consider elementary closures associated with other families of cuts. Some of the cuts that we consider are designed for mixed integer programs, others just for mixed 0,1 programs or pure integer programs. They can all be applied to pure 0,1 programs. We compare eighteen families of cuts that have appeared in the literature by comparing the corresponding eighteen elementary closures. We show that ten of these sets are distinct and we establish all inclusion relationships between them.

2 Several Classical Families of Cuts

Let $A$ and $G$ be given rational matrices (dimensions $m \times n$ and $m \times p$ respectively) and $b$ a given rational column vector (dimension $m$). Consider the polyhedron $P \equiv \{(x, y) \in \mathbb{R}_+^{n+p} \mid Ax + Gy \leq b\}$. A cut is a valid inequality for the set $P_I \equiv \{(x, y) \in \mathbb{Z}_+^{n+p} \mid Ax + Gy \leq b\}$. We will sometimes restrict our attention to the pure case ($p=0$) or to the mixed 0,1 case (the constraints $Ax + Gy \leq b$ contain the inequalities $0 \leq x \leq 1$). In all cases, we assume without loss of generality that $x \geq 0$ and $y \geq 0$ are part of the constraints for $P$.

2.1 Chvátal cuts and $P_C$

In the pure case, the inequality $\lfloor uA \rfloor x \leq \lfloor ub \rfloor$ is valid for $P_I$ for any $u \in \mathbb{R}_+^m$. Here, $\lfloor uA \rfloor$ denotes the vector obtained from the vector $uA$ by rounding down every component to an integer. These cuts are known as Chvátal cuts [12]. Let $P_C$ denote the corresponding elementary closure.

2.2 Gomory fractional cuts and $P_F$, $P_{FB}$, $P_{FBF}$

In the pure case, Gomory [20, 21, 23] introduced fractional cuts when the constraints are in equality form. Assume, without loss of generality, that $A$ and $b$ are integral. Note that $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ can be equivalently expressed as $P' = \{(x, s) \in \mathbb{R}_+^{n+m} \mid Ax + s = b, \ s \geq 0\}$. Let $P'_I = \{(x, s) \in \mathbb{Z}_+^{n+m} \mid Ax + s = b, \ s \geq 0\}$. For any $u \in \mathbb{R}^m$, the inequality $f_u^I(\vec{x}) \geq f_0$ is
valid for $P'_I$, where $f = u(A, I) - [u(A, I)]$ and $f_0 = ub - [ub]$. Plugging $s = b - Ax$ into it, we get the fractional cut $|uA| x - |u| Ax \leq |ub| - |u| b$ in the space of $x$. Let $P_{F}$ denote the elementary closure of the fractional cuts.

Originally, Gomory proposed to generate fractional cuts from basic feasible solutions of $P'$. Let $z = (\bar{z}) \in \mathbb{R}^{n+m}_+$. Then $P' = \{ z \in \mathbb{R}^{n+m}_+ ((A, I)z = b) \}$. Consider a basic feasible solution $\bar{z}$ of $P'$. Let $B$ and $N$ be the index sets of its basic and nonbasic variables respectively. Then, for $i \in B$,

$$z_i - \sum_{j \in N} \bar{a}_{ij} z_j = \bar{a}_{i0}.$$

is obtained as a linear combination of the equations in the original system $Ax + s = b$ with some multipliers $u \in \mathbb{R}^m$. Clearly, $u$ depends on $\bar{z}$ and $i$, and there is only a finite number of such choices. Any Gomory fractional cut obtained this way is called a fractional cut from a basic feasible solution. Let $P_{FB}$ be the corresponding elementary closure. Similarly, let $P_{FB}$ be the elementary closure of the fractional cuts generated from all the basic solutions.

### 2.3 Gomory mixed integer cuts and $P_{M1}$, $P_{M1B}$, $P_{M1BF}$

In [22], Gomory introduced cuts that are at least as strong as his fractional cuts in the pure case and, in addition, can be applied to the mixed case. Let $P' = \{ (x, y, s) \in \mathbb{R}^{n+p+m}_+ | Ax + Gy + s = b, s \geq 0 \}$ and $P'_f = \{ x \in \mathbb{Z}^n, (y, s) \in \mathbb{R}^{p+m}_+ | Ax + (G, I)g(y) = b, s \geq 0 \}$. Introduce $z = (\bar{z})$. For any $u \in \mathbb{R}^m$, let $\bar{a} = uA$, $\bar{g} = u(G, I)$ and $\bar{b} = ub$. Let $\bar{a}_i = |\bar{a}_i| + f_i$ and $\bar{b} = \bar{b} + f_0$. The following inequality, known as Gomory mixed integer cuts [22], is valid for $P'_f$:

$$\sum_{(i, j) : f_i \leq f_0} f_i x_i + \frac{f_0}{1 - f_0} \sum_{(i, j) : f_i > f_0} (1 - f_i) x_i + \sum_{(j, z) : g_j \geq 0} \bar{g}_j z_j - \frac{f_0}{1 - f_0} \sum_{(j, z) : g_j < 0} \bar{g}_j z_j \geq f_0.$$

Plugging $s = b - Ax - Gy$ into it, we get a valid cut for $P_f$. Let $P_{M1}$ denote the corresponding elementary closure.

Just as we defined fractional cuts from basic feasible solutions, we can also define mixed integer cuts from basic feasible solutions.

Let $P_{M1BF}$ be the elementary closure of the mixed integer cuts generated from basic feasible solutions and let $P_{M1B}$ be the elementary closure of the mixed integer cuts generated from all the basic solutions.
2.4 Mixed integer rounding cuts and $P_{MIR}$

Let $c^1 x + hy \leq c_0^1$ and $c^2 x + hy \leq c_0^2$ be two valid inequalities for $P$, and $\pi = c^2 - c^1 \in \mathbb{Z}^n$, $\pi_0 = [c_0^2 - c_0^1]$ and $\gamma = c_0^2 - c_0^1 - \pi_0$. Then the inequality

$$\pi x + (c^1 x + hy - c_0^1)/(1 - \gamma) \leq \pi_0$$

is valid for $P_I$ [28]. These cuts were introduced by Nemhauser and Wolsey [28] and are called mixed integer rounding cuts. Let $P_{MIR}$ be the corresponding elementary closure.

2.5 Disjunctive cuts and $P_D$

For any $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$, define $P(\pi, \pi_0)$ to be the convex hull of the union of the two polyhedra $P \cap \{\pi x \leq \pi_0\}$ and $P \cap \{\pi x \geq \pi_0 + 1\}$. Inequalities that are valid for some $P(\pi, \pi_0)$ are called disjunctive cuts or D-cuts. Disjunctive cuts were introduced by Balas [3]. They are also called split cuts [15]. D-cuts can be obtained as follows [2, 9, 24]: if $cx + hy - \alpha(\pi x - \pi_0) \leq c_0$ and $cx + hy + \beta(\pi x - \pi_0 - 1) \leq c_0$ are two valid inequalities for $P$, where $\alpha \geq 0$, $\beta \geq 0$, $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$, then $cx + hy \leq c_0$ is a D-cut and, conversely, all D-cuts can be obtained this way.

$$P_D \equiv \cap_{(\pi, \pi_0) \in \mathbb{Z}^{n+1}} P(\pi, \pi_0)$$  is the elementary closure of the D-cuts.

2.6 Simple disjunctive cuts and $P_{SD}$

In the mixed 0,1 case, the D-cuts can be specialized to those arising from the disjunction $(x_j \leq 0)$ or $(x_j \geq 1)$, i.e., valid inequalities for the union of the polyhedra $P \cap \{x_j \leq 0\}$ and $P \cap \{x_j \geq 1\}$. Let $P_{SD}$ denote the elementary closure of these simple D-cuts.

2.7 Lift-and-project cuts and $P_{L&P}$, $P_{L&P+S}$

These cuts can be obtained in the mixed 0,1 case. For a given $j \in \{1, \ldots, n\}$, form the nonlinear system

$$(1 - x_j)(b - Ax - Gy) \geq 0$$

$$x_j(b - Ax - Gy) \geq 0$$
and linearize it by substituting \( x_j \) for \( x_j^2 \), \( z_i \) for \( x_i x_j, i \neq j \); and \( w_i \) for \( x_j y_i \). Then project this higher dimensional polyhedron back into the \((x, y)\) space. The resulting polyhedron is denoted by \( P_j \). It is easy to see that \( P_I \subseteq P_j \). Therefore any valid inequality for \( P_{L \& P} \equiv \cap_j P_j \) is valid for \( P_I \). Valid inequalities for \( P_{L \& P} \) are called lift-and-project cuts [4]. See Sherali and Adams [30] and Lovász and Schrijver [25] for earlier related work.

It can be shown [3, 4] that any valid inequality \( \alpha^T \beta \geq 0 \) for \( P_j \) corresponds to a feasible solution \((\alpha, \beta) \in \mathbb{R}^{n+p+1}\) of the polyhedral cone

\[
\begin{array}{ccc}
\alpha + u(A, G) & -u_0 e_j & \geq 0, \\
\alpha + v(A, G) & -v_0 e_j & \geq 0, \\
-u b & = \beta, \\
-v b & + v_0 & = \beta, \\
u, v & \geq 0,
\end{array}
\]

where \( e_j \) denotes the \( j \)-th unit vector. Furthermore the extreme rays of the cone \( Q \) correspond to the facets of \( P_j \) when \( P_j \neq \emptyset \). Notice that if \((\alpha, \beta)\) is an extreme ray of \( Q \), then \( \alpha = \max\{\alpha^1, \alpha^2\} \), where

\[
\begin{align*}
\alpha^1 &= u_0 e_j - u(A, G), \\
\alpha^2 &= v_0 e_j - v(A, G).
\end{align*}
\]

The lift-and-project cut \( \alpha^T \beta \geq 0 \) is derived from the 0-1 condition on a single variable \( x_j \), even though we allow the choice of \( x_j \) to be arbitrary. These cuts can be strengthened by using integrality conditions on the other variables \( x_i \) for \( i \neq j \) as shown by Balas and Jeroslow [5, 7]. The strengthened lift-and-project cut \( \gamma^T \beta \geq 0 \) is defined as follows:

\[
\gamma_k = \min\{\alpha^1_k + u_0 [\tilde{m}_k], \alpha^2_k - v_0 [\tilde{m}_k]\} \text{ for } k = 1, \ldots, n, \\
\gamma_k = \max\{\alpha^1_k, \alpha^2_k\} \text{ for } k = n + 1, \ldots, n + p,
\]

where \( \alpha^1 \) and \( \alpha^2 \) are defined above and \( \tilde{m}_k = \frac{\alpha^2_k - \alpha^1_k}{u_0 + v_0} \).

The strengthened lift-and-project cuts can be viewed as valid inequalities for the union of the polyhedra \( P \cap \{\pi x \leq 0\} \) and \( P \cap \{\pi x \geq 1\} \) for a special choice of \( \pi \) (see proof of Theorem 2.2 in [5]).

The elementary closure of the strengthened lift-and-project cuts is denoted by \( P_{L \& P + S} \).
2.8 Intersection cuts and $P_{IBF}$, $P_{IB}$, $P_{IBF+S}$, $P_{IB+S}$

Let $P' = \{(x, y, s) \in R^{n+p+m} | Ax + Gy + s = b, \ s \geq 0\}$ and let $(\bar{x}, \bar{y}, \bar{z})$ be a basic feasible solution. Let $(\bar{x}, \bar{y})$ be the corresponding point in $(n+p)$-space. Note that $(\bar{x}, \bar{y})$ is an extreme point of $P$ and the $n + p$ nonbasic variables in $(\bar{x}, \bar{y}, \bar{z})$ uniquely define $n + p$ extreme rays originating at $(\bar{x}, \bar{y})$. Let $C$ denote the cone defined by these $n + p$ rays. Valid inequalities for $C \cap \{x_k \leq \pi_0\}$ and $C \cap \{x_k \geq \pi_0 + 1\}$ where $\pi_0 \in Z$, are called intersection cuts from basic feasible solutions. This notion was introduced by Balas in a more general context [1]. Let $P_{IBF}$ be the corresponding elementary closure.

If we consider the cone $C$ defined by a basic solution $(\bar{x}, \bar{y}, \bar{z})$ which may not be feasible, then a valid inequality for $C \cap \{x_k \leq \pi_0\}$ and $C \cap \{x_k \geq \pi_0 + 1\}$, is called an intersection cuts from a basic solution. Denote the corresponding elementary closure by $P_{IB}$.

Just as lift-and-project cuts, intersection cuts can be strengthened by using integrality conditions on the other variables $x_i$ for $i \neq k$. The elementary closures of the strengthened intersection cuts from basic feasible solutions and the strengthened intersection cuts from basic solutions are respectively denoted by $P_{IBF+S}$ and $P_{IB+S}$.

2.9 Sherali-Adams cuts and $P_{SA}$

The Sherali-Adams cuts can be obtained in the mixed 0,1 case as follows. Form the nonlinear system

$$
(1 - x_1)(b - Ax - Gy) \geq 0
$$

$$
x_1(b - Ax - Gy) \geq 0
$$

$$
...
$$

$$
(1 - x_n)(b - Ax - Gy) \geq 0
$$

$$
x_n(b - Ax - Gy) \geq 0
$$

and linearize it by substituting $x_j$ for $x_j^2$; $z_{ij}$ for $x_i x_j$, $i \neq j$; and $w_{ik}$ for $x_i y_k$. Then project this higher dimensional polyhedron $M$ back into the $(x, y)$ space. The resulting polyhedron is denoted by $P_{SA}$. Any valid inequality for $P_{SA}$ is valid for $P_I$ and is called a Sherali-Adams cut [30]. Sherali and Adams also introduced other families, obtained with more multiplication of terms, but we do not consider them here.
2.10 Lovász-Schrijver cuts and $P_{LS}$

The Lovász-Schrijver cuts are obtained in the mixed 0,1 case by imposing an additional constraint on the polyhedron $M$ defined in Section 2.9. Let $Z$ be the square matrix with $n+1$ rows and columns defined as follows. Row 0 is the vector $(1, x)$, column 0 is its transpose, and for $i, j \geq 1$, $Z$ has entry $z_{ij}$ if $i \neq j$ and $z_{ii} = x_i$ otherwise. Lovász and Schrijver [25] proposed to tighten the set $M$ by imposing that the matrix $Z$ be positive semi-definite. Let $M = M \cap \{Z | Z$ is a positive semi-definite matrix}$. The projection of $M$ into the $(x, y)$ space is denoted by $P_{LS}$. One can verify that $P_I \subseteq P_{LS} \subseteq P$. Any valid inequality for $P_{LS}$ is a Lovász-Schrijver cut.

3 Comparison of the Elementary Closures

All the cuts defined in Section 2 are meaningful in the case of pure 0,1 programs. The purpose of this section is to compare the corresponding eighteen elementary closures in this case. When appropriate, we state the results in more general cases (for example, pure integer programs, mixed 0,1 programs).

The comparison between the eighteen elementary closures is illustrated in Figure 1. If $A$ and $B$ denote two kinds of cuts, we use the following convention in Figure 1: $A = B$ means that the elementary closures $P_A$ and $P_B$ are identical, $A \rightarrow B$ means that $P_A \supseteq P_B$ and the inclusion is strict for some instances, and $A$ not related to $B$ in the figure means that for some instances $P_A \nsubseteq P_B$ and for other instances $P_B \nsubseteq P_A$.

(1) $P_C = P_F$:  

Now we are in the pure case $Ax \leq b$ with $x \in Z^+_n$. Without loss of generality, $A$ and $b$ are assumed to be integral. The proof of Theorem 6.34 in [13] implies that any Chvátal cut has the form $[uA]x \leq [ub]$ for some multiplier vector $0 \leq u < 1$. Since $[u] = 0$, this is the Gomory fractional cut $[uA]x - [u]Ax \leq [ub] - [u]b$. Conversely, any fractional cut $[uA] - [u]Ax \leq [ub] - [u]b$ can be written as $[uA - [u]A]x \leq [ub - [u]b]$, since $A$ and $b$ are integral. But this is a Chvátal cut $[\lambda A] \leq [\lambda b]$, where $\lambda = u - [u]$.

(2) $P_{SD} = P_{LS}$:

In the mixed 0,1 case, $(P \cap \{x_j \leq 0\}) \cup (P \cap \{x_j \geq 1\}) = P \cap \{(x, y) \in R^{n+p} : x_j \in \{0, 1\}\}$
Figure 1: Comparison of Elementary Closures
Therefore $P_{SD} = \cap_{j=1}^{n} \text{conv}(P \cap \{(x, y) \in R^{n+P}: x_j \in \{0, 1\}\})$, which is $P_{L&P}$ by Theorem 2.1 in [4].

(3) $P_{MIR} = P_D = P_{M1}$:

Nemhauser and Wolsey [28] showed these two equalities.

To show that $P_{MIR} \subseteq P_D$, we show that any $D$-cut can be obtained by the MIR procedure. Let $cx + hy \leq c_0$ be a $D$-cut. Then there exist $(\pi, \pi_0) \in Z^{n+1}$ and $\alpha, \beta > 0$ such that $cx + hy - \alpha(\pi x - \pi_0) \leq c_0$ and $cx + hy + \beta(\pi x - \pi_0 - 1) \leq c_0$ are both valid inequalities for $P$. Multiplying both inequalities by $\frac{1}{\alpha + \beta}$ and applying the MIR procedure with $\gamma = \frac{\beta}{\alpha + \beta}$ gives, after simplifications, $cx + hy \leq c_0$.

$P_D \subseteq P_{M1}$ since it is well known that Gomory mixed integer cuts can be derived from a disjunction expressing integrality conditions on the $x$ variables.

Now we show that $P_{M1} \subseteq P_{MIR}$. We show that any MIR cut can be obtained using Gomory’s mixed integer procedure. Consider two valid inequalities for $P$, say $c^1 x + hy \leq c^1_0$ and $c^2 x + hy \leq c^2_0$ where $\pi = c^2 - c^1 \in Z^n$. After introducing two nonnegative slack variables $s_1$ and $s_2$, we get $c^1 x + hy + s_1 = c^1_0$ and $c^2 x + hy + s_2 = c^2_0$. Subtracting the first equation from the second, we get $(c^2 - c^1)x - s_1 + s_2 = c^2_0 - c^1_0$. The Gomory mixed integer cut generated from this equation is $\frac{c^2}{c^2 - c^1} s_1 + s_2 \geq \gamma$, where $\gamma = c^2 - c^1 - \frac{c^2_0 - c^1_0}{c^2 - c^1}$, or equivalently $(c^2 - c^1)x - \frac{1}{c^2 - c^1} s_1 \leq \frac{c^2_0 - c^1_0}{c^2 - c^1}$. Plugging $s_1 = -c^1 x - hy + c^1_0$ into it, we obtain the mixed integer rounding cut $\pi x + (c^1 x + hy - c^1_0)/(1 - \gamma) \leq \pi_0$, where $\pi_0 = c^2 - c^1_0$.

(4) $P_{IB} = P_{L&P}$, $P_{IB+S} = P_{L&P+S} = P_{M1B}$, $P_{IBF+S} = P_{M1BF}$:

These are new results of Balas and Perregaard [8]. They are obtained by establishing the correspondence between the basic feasible solutions of (*) in Section 2.7 and Gomory mixed integer cuts from basic solutions of $P'$.

(5) $P_{L&P+S} \subseteq P_{L&P}$ and the inclusion is strict for some $P$:

Obviously $P_{L&P+S} \subseteq P_{L&P}$ from their definitions. To see that the inclusion can be strict, consider the polytope $P = \{x \in R^2_+: -1/2 \leq x_1 - x_2 \leq 0, x_1 \leq 1, x_2 \leq 1\}$. Then $P_{L&P} = \{x \in R^2 | x_1 - x_2 \leq 0, -2x_1 + x_2 \leq 0, -x_1 + 2x_2 \leq 1\}$, while $P_{L&P+S} = \text{conv}(P_1) = \{x \in R^2_+ | x_1 - x_2 = 0, x_1 \leq 1, x_2 \leq 1\}$.

(6) $P_F \subseteq P_{FB} \subseteq P_{FBF}$, $P_{M1} \subseteq P_{M1B} \subseteq P_{M1BF} \subseteq P_{FBF}$, $P_{M1B} \subseteq P_{FB}$, $P_{IB+S} \subseteq P_{IB} \subseteq$
Theorem 3.1. \( P_{IBF}, P_{IBF+S} \subseteq P_{IBF} \) and each of these inclusions is strict for some \( P \):

These inclusions follow from the definitions. The fact that they can be strict follows from the following results, established later: The results in (15), (4) and (12) imply the strict inclusions in \( P_F \subseteq P_{FB} \subseteq P_{FBF} \), and the results in (7), (15), (4), (12) and (11) imply the strict inclusions in \( P_{MI} \subseteq P_{MIB} \subseteq P_{MIBF} \subseteq P_{FBF} \). The strict inclusion in \( P_{MIB} \subseteq P_{FB} \) follows from (11), and the strict inclusions in \( P_{IB+S} \subseteq P_{IB} \subseteq P_{IBF} \) follow from (5), (4) and (13). The strict inclusion in \( P_{IBF+S} \subseteq P_{IBF} \) holds because of (10).

(7) \( P_{MI} \subseteq P_C \) and the inclusion is strict for some \( P \):

In the pure case, the Gomory mixed integer cut obtained from the multiplier vector \( u \) is a strengthening of the Gomory fractional cut obtained from the same \( u \). The fact that the inclusion can be strict will be established in (11).

(8) \( P_{SA} \subseteq P_{L&P} \) and the inclusion is strict for some \( P \):

The inclusion \( P_{SA} \subseteq P_{L&P} \) follows from the definitions. The fact that the inclusion can be strict follows from (14).

(9) \( P_{LS} \subseteq P_{SA} \) and the inclusion is strict for some \( P \):

The inclusion follows from the definitions. Next we show that it can be strict. For a graph \( G = (V, E) \) and \( P = \{ x \in \mathbb{R}^n_+ \mid x_i + x_j \leq 1, \forall (i, j) \in E \} \), the set \( P_I \) is the stable set polytope \( \text{conv}(\{ x \in \{0,1\}^n \mid x_i + x_j \leq 1, \forall (i, j) \in E \}) \). Lovász and Schrijver [25] characterized \( P_{SA} \) as the polytope defined exactly by the constraints of \( P \) and the odd hole constraints. They showed that there are other constraints, like clique, odd wheel and odd antihole constraints, which are valid for \( P_{LS} \) but not for \( P_{SA} \).

(10) \( P_{LS} \nsubseteq P_{FBF} \) for some \( P \):

Let \( P = \{ x \in \mathbb{R}^2 \mid -x_1 + x_2 \leq 1/2, 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 \} \). Then \( P_{FBF} = \{ x \in \mathbb{R}^2 \mid x_1 - x_2 \geq 0, 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 \} = \text{conv}(P_I) \), since the Gomory fractional cut generated from the basic feasible solution \((0,1/2)\) is \( x_1 - x_2 \geq 0 \). However, the vector \((x_1, x_2)^T = (1/2 - \epsilon, 1/2 + \epsilon)^T \notin \text{conv}(P_I) \) is in \( P_{LS} \), where \( \epsilon \) is a sufficiently small positive number, because \((x_1, x_2, y_1) = (1/2 - \epsilon, 1/2 + \epsilon, 1/2 - \epsilon)\) satisfies both the inequalities in the higher dimensional polyhedron \( M \) and the positive semi-definiteness constraint. See also [14].
(11) \( P_C \not\subseteq P_{1BF} \) for some \( P \): 

For \( P = \{(x_1, x_2) \mid -2x_1 + x_2 \leq 0, 2x_1 + x_2 \leq 2, 0 \leq x_1, 0 \leq x_2 \leq 1\} \), we have \( P_{1BF} = \text{conv}(P_1) = \{(x_1, x_2) \mid 0 \leq x_1 \leq 1, x_2 = 0\} \).

In order to show \( \text{conv}(P_1) \subseteq P_C \) for this \( P \), let us consider an arbitrary valid inequality for \( P \) and justify that \((1/2, 1/2)\) satisfies the corresponding Chvátal cut. Any valid inequality for \( P \) must either be valid for both \((0, 0)\) and \((1, 1)\) or be valid for both \((1, 0)\) and \((0, 1)\). Hence the Chvátal rounding procedure will keep either both \((0, 0)\) and \((1, 1)\) or both \((1, 0)\) and \((0, 1)\) valid for the corresponding Chvátal cut. Therefore, the center point \((1/2, 1/2)\) is valid for the cut.

(12) \( P_{MIBF} \not\subseteq P_{FB} \) for some \( P \):

Let \( P = \{x \in \mathbb{R}_+^2 \mid 10x_1 + 2x_2 \leq 11, 2x_1 + 10x_2 \leq 11, 10x_1 + 10x_2 \leq 19, x_1 \leq 1, x_2 \leq 1\} \). The only 6 basic feasible solutions are \((11/12, 11/12)\), \((1/2, 1)\), \((1, 1/2)\), \((1, 0)\), \((0, 1)\), \((0, 0)\). The Gomory mixed integer cuts generated from basic feasible solutions do not yield \( x_1 + x_2 \leq 1 \), whereas \( P_{FB} = \text{conv}(P_1) = \{x \in \mathbb{R}_+^2 \mid x_1 + x_2 \leq 1\} \), since the Gomory fractional cut generated from the basic infeasible solution \((9/10, 1)\) is \( x_1 + x_2 \leq 1 \).

(13) \( P_{MIBF} \not\subseteq P_{L&P} \) for some \( P \):

Consider the polytope \( P = \{x \in \mathbb{R}_+^2 \mid 4x_1 + x_2 \leq 4, -4x_1 + x_2 \leq 0, x_2 \leq 1\} \). The basic feasible solutions are \((1/4, 1)\), \((3/4, 1)\), \((0, 0)\) and \((1, 0)\). We obtain \( P_{MIBF} = \{x \in \mathbb{R}_+^2 \mid -x_1 + x_2 \leq 0, x_1 + x_2 \leq 1\} \), while \( P_{L&P} = \text{conv}(P_1) = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, x_2 = 0\} \).

(14) \( P_D \not\subseteq P_{SA} \) for some \( P \):

Let \( P \) be the polytope of the Steiner triple system \( A_9 \) [11], [18]. Theorem 4.4 of [11] tells us that the inequality \( 1x \geq 27/7 \) is a Sherali-Adams cut.

The point \( x^* = (x_1^*, x_2^*, \ldots, x_9^*) = (3/7, 3/7, \ldots, 3/7) \) satisfies \( 1x \geq 27/7 \) as equality and satisfies all the constraints of \( P \) as strict inequalities. If we could show that \( x^* \) satisfies any \( D \)-cut as a strict inequality, then \( x^* \) is in the interior of \( P_D \), which means that the Sherali-Adams cut \( 1x \geq 27/7 \) is not valid for \( P_D \). Hence we have \( P_D \not\subseteq P_{SA} \) for this \( P \).

It is shown in [3] that an inequality \( ax \leq a_0 \) is a \( D \)-cut if and only if
\[
\begin{aligned}
\alpha &\leq \gamma_1 \pi + \theta_1 A, \\
\alpha &\leq -\gamma_2 \pi + \theta_2 A, \\
o_0 &\geq \gamma_1 \pi_0 + \theta_1 b, \\
o_0 &\geq -\gamma_2 (\pi_0 + 1) + \theta_2 b, \\
(a, o_0) &\in \mathbb{R}^{p+1}, \\
(\gamma_1, \theta_1), (\gamma_2, \theta_2) &\in \mathbb{R}^{m+1}, \\
(\pi, \pi_0) &\in \mathbb{Z}^{p+1}.
\end{aligned}
\]

Since we assume that \(ax \leq o_0\) is not valid for \(P\), we have \(\gamma_1 \neq 0\) and \(\gamma_2 \neq 0\). After dividing \(\gamma_1\) in the both sides of all the above inequalities and resetting the variables, we see that \(ax \leq o_0\) is a \(D\)-cut if and only if

\[
\begin{aligned}
\alpha &\leq \pi + \theta_1 A, \\
\alpha &\leq -\pi + \theta_2 A, \\
o_0 &\geq \pi_0 + \theta_1 b, \\
o_0 &\geq -\pi (\pi_0 + 1) + \theta_2 b, \\
\gamma > 0, \theta_1, \theta_2 &\in \mathbb{R}^{m}, \\
(a, o_0) &\in \mathbb{R}^{p+1}, \\
(\pi, \pi_0) &\in \mathbb{Z}^{p+1}.
\end{aligned}
\]

GAMS was used to verify the infeasibility of the nonlinear mixed integer system consisting of (\(\ast\ast\ast\)) for \(A_0\) plus the inequality constraint \(ax^* \geq o_0\) for \(x^* = (3/7, 3/7, \ldots, 3/7)\). We conclude that \(x^*\) is in the interior of \(P_D\), which leads to \(P_D \not\subseteq P_{SA}\) for this polytope \(P\).

(15) \(P_{L&P+S} \not\subseteq P_C\) for some \(P\):

Consider the polytope \(P = \{x \in \mathbb{R}^1 \mid \sum_{j \neq i} x_{ij} \leq 1, \text{ for } i = 1, 2, \ldots, 5\}\) for the matching problem on the complete graph with five nodes. The facet-defining inequality \(\sum_{ij \in E} x_{ij} \leq 2\) of conv\((P_j)\) is a Chvátal cut of rank one. We will show that this facet-defining inequality cannot be generated from any disjunction of \(P \cap \{\pi x \leq 0\}\) and \(P \cap \{\pi x \geq 1\}\), where \(\pi\) is integral. By the property of strengthened lift-and-project cuts shown in 2.7, \(\sum_{ij \in E} x_{ij} \leq 2\) is not valid for \(P_{L&P+S}\).

By the argument and (\(\ast\ast\ast\)) in (14), an inequality \(ax \leq o_0\) can be generated from a disjunction of \(P \cap \{\pi x \leq 0\}\) and \(P \cap \{\pi x \geq 1\}\), where \(P = \{x \in \mathbb{R}^n | Ax \leq b\}\), if and only if the following linear integer system is feasible:
\[
\begin{align*}
\alpha & \leq \pi + \theta_1 A, \\
\alpha \gamma' & \leq -\pi + \theta_2 A, \\
\alpha_0 & \geq \theta_1 b, \\
\alpha_0 \gamma' & \geq \theta'_2 b - 1, \\
\gamma' & > 0, \quad \theta_1, \theta_2' \in \mathbb{R}^+ \\
(\pi, \pi_0) & \in \mathbb{Z}^{n+1}.
\end{align*}
\]

For \(\sum_{i,j\in E} x_{ij} \leq 2\) and \(P = \{x \in \mathbb{R}^{|E|}_+ | \sum_{j \neq i} x_{ij} \leq 1, \text{ for } i = 1, 2, \ldots, 5\}\), we used LINDO to check numerically that the corresponding linear integer system is infeasible.

\section{Properties of Elementary Closures}

In this section we address two questions about the elementary closures introduced above: Are they polyhedra? Can linear functions be optimized in polynomial time over these elementary closures?

\(P_{IBF}, P_{FBF}, P_{FB}, P_{MIBF}\) (or \(P_{IIBF+S}\)) and \(P_{MIB}\) (or \(P_{IIB+S}, P_{L&P+S}\)) are polyhedra, since there are only finitely many constraints for them. But it is unknown whether one can optimize linear functions over them in polynomial time.

\(P_C\) (or \(P_F\)) is a polyhedron [12, 29], but it is NP-hard to optimize linear functions over it [16]. Recently, Bockmayr and Eisenbrand [10] showed that, in fixed dimension, \(P_C\) has a description with polynomially many inequalities.

\(P_{L&P}\) (or \(P_{SD}, P_{IB}\)) is obtained by projecting a polyhedron and therefore is a polyhedron. Furthermore, optimizing a linear function over it can be done by solving a polynomial size linear program in the higher dimensional space. Hence this optimization problem is solvable in polynomial time. So is \(P_{SA}\) for the same reason.

\(P_{LS}\) is a convex set. It is not a polyhedron in general, but one can optimize linear functions over \(P_{LS}\) in polynomial time [25]. Recently, Goemans and Tuncel [19] gave some conditions under which \(P_{LS} = P_{SA}\). Cook and Dash [14] considered combining Lovász-Schrijver cuts with Chvátal cuts.

\(P_{MI}\) (or \(P_{MIB}, P_D\)) is a polyhedron [15]. However, it is still an open problem whether the optimization of linear functions over it can be done in polynomial time.
References


[8] E. Balas and M. Perregaard, A precise correspondence between lift-and-project cuts, simple disjunctive cuts, and mixed integer Gomory cuts for 0 – 1 programming, 17th International Symposium on Mathematical Programming, Atlanta, Georgia (2000).


