On a divisibility relation for Lucas sequences

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\textbf{ABSTRACT}

In this note, we study the divisibility relation $U_m \mid U_{n+k}^s - U_n^s$, where $U := \{U_n\}_{n \geq 0}$ is the Lucas sequence of characteristic polynomial $x^2 - ax \pm 1$ and $k, m, n, s$ are positive integers.

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1. Introduction

Let $U := U(a, b) = \{U_n\}_{n \geq 0}$ be the Lucas sequence given by $U_0 = 0$, $U_1 = 1$ and

$$U_{n+2} = aU_{n+1} + bU_n \quad \text{for all} \quad n \geq 0, \quad \text{where} \quad b \in \{\pm 1\}.$$  \hspace{1cm} (1)

Its characteristic equation is $x^2 - ax - b = 0$ with roots

$$(\alpha, \beta) = \left(\frac{a + \sqrt{a^2 + 4b}}{2}, \frac{a - \sqrt{a^2 + 4b}}{2}\right).$$  \hspace{1cm} (2)

When $a \geq 1$, we have that $\alpha > 1 > |\beta|$. We assume that $\Delta = a^2 + 4b > 0$ and that $\alpha/\beta$ is not a root of unity. This only excludes the pairs $(a, b) \in \{(0, \pm 1), (\pm 1, -1), (2, -1)\}$ from the subsequent considerations. Here, we look at the relation

$$U_m \mid U_{n+k}^s - U_n^s,$$ \hspace{1cm} (3)

with positive integers $k, m, n, s$. Note that when $(a, b) = (1, 1)$, then $U_n = F_n$ is the $n$th Fibonacci number. Taking $k = 1$ and using the relations

$$F_{n+1} - F_n = F_{n-1},$$

$$F_{n+1} + F_n = F_{n+2},$$

$$F_{n+1}^2 + F_n^2 = F_{2n+1},$$

it follows that relation (3) holds with $s = 1, 2, 4$, and $m = n-1$, $n+1$, $2n+1$, respectively. Further, in [2], the authors assumed that $m$ and $n$ are coprime positive integers. In this case, $F_n$ and $F_m$ are coprime, so the rational number $F_{n+1}/F_n$ is defined modulo $F_m$. Then it was shown in [2] that if this last congruence class above has multiplicative order $s$ modulo $F_m$ and $s \notin \{1, 2, 4\}$, then

$$m < 500s^2.$$ \hspace{1cm} (4)

In this paper, we study the general divisibility relation (3) and prove the following result.

**Theorem 1.** Let $a$ be a non-zero integer, $b \in \{\pm 1\}$, and $k$ a positive integer. Assume that $(a, b) \notin \{(\pm 1, -1), (\pm 2, -1)\}$. Given a positive integer $m$, let $s$ be the smallest positive integer such that divisibility (3) holds. Then either $s \in \{1, 2, 4\}$, or

$$m < 20000(sk)^2.$$ \hspace{1cm} (5)
2. Preliminary results

We put $V := V(a, b) = \{V_n\}_{n \geq 0}$ for the Lucas companion of $U$ which has initial values $V_0 = 2$, $V_1 = a$ and satisfies the same recurrence relation $V_{n+2} = aV_{n+1} + bV_n$ for all $n \geq 0$. The Binet formulas for $U_n$ and $V_n$ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n$$

for all $n \geq 0$. (6)

The next result addresses the period of $\{U_n\}_{n \geq 0}$ modulo $U_m$, where $m \geq 1$ is fixed.

**Lemma 2.** The congruence

$$U_{n+4m} \equiv U_n \pmod{U_m}$$

holds for all $n \geq 0$, $m \geq 2$.

**Proof.** This follows because of the identity

$$U_{n+4m} - U_n = U_m V_m V_{n+2m},$$

which can be easily checked using the Binet formulas (6). □

The following is Lemma 1 in [2]. It has also appeared in other places.

**Lemma 3.** Let $X \geq 3$ be a real number. Let $a$ and $b$ be positive integers with $\max\{a, b\} \leq X$. Then there exist integers $u, v$ not both zero with $\max\{|u|, |v|\} \leq \sqrt{X}$ such that $|au + bv| \leq 3\sqrt{X}$.

The following lemma is well-known, but we include the proof for the reader’s convenience. In what follows, a unit means Dirichlet unit, that is an algebraic integer $\eta$ such that $\eta^{-1}$ is also an algebraic integer.

**Lemma 4.** Let $v > 1$ be an integer and $\zeta$ be a primitive $v$th root of unity. Then

$$\prod_{\gcd(k, v) = 1} (1 - \zeta^k) = \begin{cases} p, & \text{if } v = p^f \text{ is a prime power}, \\ 1, & \text{if } v \text{ has at least two distinct prime divisors}, \end{cases}$$

the product being over the residue classes mod $v$ coprime with $v$. In particular, in the second case, $1 - \zeta$ is a unit.
Proof. The product on the left of (8) is \( \Phi_v(1) \), where \( \Phi_v(X) \) denotes the \( v \)th cyclotomic polynomial. For \( v = p^k \) we have

\[
\Phi_{p^k}(X) = \frac{X^{p^k} - 1}{X^{p^{k-1}} - 1} = X^{p^{k-1}(p-2)} + X^{p^{k-1}(p-1)} + \cdots + X^{p^{k-1}} + 1,
\]

and \( \Phi_{p^k}(1) = p \), proving the prime power case. In particular, \( (1 - \zeta) \mid p \) in this case.

Now assume that \( v \) is divisible by two distinct primes \( p \) and \( p' \). Then \( \zeta^{v/p} \) is a primitive root of unity of order \( p \), which implies that in the ring \( \mathbb{Z}[\zeta] \) we have \( (1 - \zeta) \mid (1 - \zeta^{v/p}) \mid p \). Similarly, \( (1 - \zeta) \mid p' \). The divisibility relations \( (1 - \zeta) \mid p \) and \( (1 - \zeta) \mid p' \) imply that \( (1 - \zeta) \mid 1 \), that is, \( 1 - \zeta \) is a unit. Hence its \( \mathbb{Q}(\zeta)/\mathbb{Q} \)-norm is \( \pm1 \). Since it is obviously positive, it must be 1. But this norm is exactly the left-hand side of (8). \( \Box \)

This lemma has the following consequence, which is again well-known, but we did not find a suitable reference.

Corollary 5.

1. Assume that \( \zeta \) and \( \xi \) are roots of unity of coprime orders, and both distinct from 1. The \( \zeta - \xi \) is a unit.

   From now on \( m \) and \( n \) are positive integers and \( d = \gcd(m,n) \).

2. In \( \mathbb{Z}[x] \) we have the equality of ideals \( (x^m - 1, x^n - 1) = (x^d - 1) \).

3. Let \( \gamma \) be an algebraic integer in some number field \( K \). Then we have the equality of \( \mathcal{O}_K \)-ideals \( (\gamma^m - 1, \gamma^n - 1) = (\gamma^d - 1) \).

Proof. Item 1 follows from the second assertion of Lemma 4.

In item 2 it suffices to show that \( x^d - 1 \in (x^m - 1, x^n - 1) \). In the case \( d = 1 \) this reduces to showing that

\[
1 \in \left( \frac{x^m - 1}{x - 1}, \frac{x^n - 1}{x - 1} \right).
\]

The resultant of the polynomials \( \frac{x^m - 1}{x - 1} \) and \( \frac{x^n - 1}{x - 1} \) is the product of the factors of the form \( \zeta - \xi \), where \( \zeta \) and \( \xi \) are roots of unity of orders dividing \( m \) and \( n \), respectively, and none of \( \zeta, \xi \) is 1. If \( d = \gcd(m,n) = 1 \), then each factor is a unit by item 1. Hence, the resultant is a unit of \( \mathbb{Z} \), that is, \( \pm1 \), proving (9) in the case \( d = 1 \).

The case of arbitrary \( d \) reduces to the case \( d = 1 \). Indeed, by the latter, \( x^d - 1 \) belongs to the ideal \( (x^m - 1, x^n - 1) \) in the ring \( \mathbb{Z}[x^d] \). Hence, the same is true in the ring \( \mathbb{Z}[x] \).

Finally, item 3 is an immediate consequence of the previous item. \( \Box \)

We will use one simple property of cyclotomic polynomials. Recall that for a positive integer \( v \) we denote by \( \Phi_v(X) \) the \( v \)th cyclotomic polynomial. Then for \( \alpha > 1 \) we have the trivial estimate \( \Phi_v(\alpha) > (\alpha - 1)^{\varphi(v)} \) (where \( \varphi(v) \) is, of course, the Euler totient). We will need a slightly sharper estimate.
Lemma 6. Let \( v \) be a positive integer and \( \alpha > 1 \) a real number. Then for \( v > 1 \) we have

\[
\Phi_v(\alpha) > (\alpha(\alpha - 1))^{\varphi(v)/2}.
\] (10)

Proof. We use the identity

\[
\Phi_v(X) = \prod_{d|v} (X^d - 1)^{\mu(v/d)},
\]

where \( \mu(\cdot) \) is the Möbius function. We have clearly

\[
(\alpha^d - 1)^{\mu(v/d)} \geq \begin{cases} 
\alpha^d, & \mu(v/d) = -1, \\
\alpha^d \alpha - \alpha, & \mu(v/d) = 1.
\end{cases}
\] (11)

Moreover:

- denoting by \( \tau^*(v) \) the number of square-free divisors of \( v \), we have, for \( v > 1 \), exactly \( \tau^*(v)/2 \) divisors with \( \mu(v/d) = 1 \) and exactly \( \tau^*(v)/2 \) divisors with \( \mu(v/d) = -1 \);
- inequality (11) is strict for all \( d \mid v \) satisfying \( \mu(v/d) \neq 0 \), with at most one exception.

Hence, multiplying (11) for all \( d \mid v \) with \( \mu(v/d) \neq 0 \), and using the identity \( \varphi(v) = \sum_{d|v} d \mu(v/d) \), we obtain, for \( v > 1 \), the lower estimate

\[
\Phi_v(\alpha) > \alpha^{\varphi(v)} \left( \frac{\alpha - 1}{\alpha} \right)^{\tau^*(v)/2} = (\alpha(\alpha - 1))^{\varphi(v)/2}.
\] (12)

For \( v \notin \{1, 2, 6\} \), we have \( \tau^*(v) \leq \varphi(v) \), which implies

\[
|\Phi_v(\alpha)| > \alpha^{\varphi(v)} \left( \frac{\alpha - 1}{\alpha} \right)^{\varphi(v)/2} = (\alpha(\alpha - 1))^{\varphi(v)/2},
\]

proving (10) for \( v \notin \{1, 2, 6\} \). And for \( v \in \{2, 6\} \), this is obviously true. \( \Box \)

The following lemma is the workhorse of our argument.

Lemma 7. Let \( a, b \) and \( k \) be as in the statement of Theorem 1, and assume in addition that \( a \geq 1 \). Let \( v \geq 1 \) be an integer and \( \zeta \) a primitive \( v \)-th root of unity. Define \( \alpha \) as in (2) and assume that the numbers

\[
\alpha \quad \text{and} \quad \frac{\alpha^k - (-b)^k \zeta}{\alpha^k - \zeta}
\] (13)

are multiplicatively dependent. Then we have one of the following options:
(i) \((-b)^k = -1, \, v = 4;\)
(ii) \((a, b, k) \in \{(1, 1, 1), (2, 1, 1)\} \text{ and } v \in \{1, 2\};\)
(iii) \((-b)^k = 1, \, v \in \{1, 2\};\)
(iv) \((a, b, k) = (4, -1, 1) \text{ and } v \in \{4, 6\}.\)

Proof. We use the notation
\[ K = \mathbb{Q}(\alpha), \quad L = \mathbb{Q}(\zeta), \quad M = \mathbb{Q}(\alpha, \zeta), \quad \alpha_1 = \alpha^k, \quad \delta = (-b)^k. \]

Note that \(\delta\alpha_1^{-1} = \beta^k\) is the Galois conjugate of \(\alpha_1\).

Recall that we disregard the cases \((a, b) \in \{(1, -1), (2, -1)\}\). In addition to this, we will disregard the case \((a, b, k) = (1, 1, 1)\), because it is settled in Lemma 2 of [2]. This implies that
\[ \alpha_1 \geq 1 + \sqrt{2}. \quad (14) \]

When \(\delta = 1\) we can say more:
\[ \alpha_1 \in \left\{ \frac{3 + \sqrt{5}}{2}, 2 + \sqrt{3} \right\} \quad \text{or} \quad \alpha_1 \geq \frac{5 + \sqrt{21}}{2}. \quad (15) \]

We will also assume that we are not in one of the instances (i), (iii) above; this is equivalent to saying that
\[ \zeta^2 \neq \delta. \quad (16) \]

Since the numbers (13) are multiplicatively dependent, then the second of these numbers must be a unit (because the first is). In particular,
\[ (\alpha_1 - \zeta) \mid (\alpha_1 - \delta \bar{\zeta}) \]
in the ring \(O_M\), which implies that
\[ (\alpha_1 - \zeta) \mid (\zeta - \delta \bar{\zeta}). \quad (17) \]

This divisibility relation is very restrictive: we will see that is satisfied in very few cases, which can be verified by inspection.

We first show the following identity for the norm of \(\alpha_1 - \zeta:\)
\[ |N_{M/\mathbb{Q}}(\alpha_1 - \zeta)| = (\alpha_1^{-\varphi(v)} \Phi_v(\alpha_1) \Phi_v^*(\alpha_1))^{|M:L|/2}, \quad (18) \]
where \(\Phi_v(X)\) is the \(v\)th cyclotomic polynomial and
\[ v^* = \begin{cases} 
  v & \text{if } 4 \mid v \text{ or } \delta = 1, \\
  v/2 & \text{if } 2 \parallel v \text{ and } \delta = -1, \\
  2v & \text{if } 2 \nmid v \text{ and } \delta = -1. 
\end{cases} \] (19)

Note that
\[ \varphi(v^*) = \varphi(v), \quad \Phi_v(X) = \pm \Phi_v(\delta X), \quad \Phi_v(X^{-1}) = \pm X^{-\varphi(v)} \Phi_v(X), \]
the sign in last identity being “+” for \( v > 1 \) and the sign in the middle identity being “+” if \( \delta = 1 \) or \( \min\{v, v^*\} > 1 \).

Let us prove (18). When \( \alpha \not\in L \), the conjugates of \( \alpha_1 - \zeta \) are the \( 2\varphi(v) \) numbers \( \alpha_1 - \zeta' \) and \( \delta\alpha_1^{-1} - \zeta'' \), where both \( \zeta' \) and \( \zeta'' \) run through the set of primitive \( v \)th roots of unity. Hence, in this case
\[ |N_{M/Q}(\alpha_1 - \zeta)| = |\Phi_v(\alpha_1)\Phi_v(\delta\alpha_1^{-1})| = \alpha_1^{-\varphi(v)} \Phi_v(\alpha_1)\Phi_v^*(\alpha_1), \]
which is (18) in the case \( \alpha \not\in L \).

Now assume that \( \alpha \in L \), and set
\[ G = \text{Gal}(L/Q), \quad H = \text{Gal}(L/K), \]
for the Galois groups of the various extensions. The group \( H \) is a subgroup of \( G \) of index 2, and we have
\[ \alpha_1^\sigma = \begin{cases} 
  \alpha_1, & \sigma \in H, \\
  \delta\alpha_1^{-1}, & \sigma \in G \setminus H. 
\end{cases} \]
Hence,
\[ |N_{M/Q}(\alpha_1 - \zeta)| = |N_{L/Q}(\alpha_1 - \zeta)| \]
\[ = \prod_{\sigma \in H} |\alpha_1 - \zeta^\sigma| \prod_{\sigma \in G \setminus H} |\delta\alpha_1^{-1} - \zeta^\sigma| \]
\[ = \alpha_1^{-\varphi(v)/2} \prod_{\sigma \in H} |\alpha_1 - \zeta^\sigma| \prod_{\sigma \in G \setminus H} |\delta\alpha_1 - \zeta^\sigma|, \]
where in the second equality we used \( \alpha_1 \in \mathbb{R} \). In a similar fashion,
\[ |N_{M/Q}(\alpha_1 - \delta\bar{\zeta})| = \prod_{\sigma \in H} |\alpha_1 - \delta\bar{\zeta}^\sigma| \prod_{\sigma' \in G \setminus H} |\delta\alpha_1^{-1} - \delta\bar{\zeta}^\sigma| \]
\[ = \alpha_1^{-\varphi(v)/2} \prod_{\sigma \in H} |\delta\alpha_1 - \zeta^\sigma| \prod_{\sigma \in G \setminus H} |\alpha_1 - \zeta^\sigma|. \]
Since \( \frac{\alpha_1 - \delta\bar{\zeta}}{\alpha_1 - \zeta} \) is a unit, the two norms computed above are equal. Hence,
\[ |N_{M/Q}(\alpha_1 - \zeta)|^2 = |N_{M/Q}(\alpha_1 - \zeta)N_{M/Q}(\alpha_1 - \delta \bar{\zeta})| \]
\[ = \alpha_1^{-\varphi(v)} \prod_{\sigma \in G} |\alpha_1 - \zeta^\sigma| \prod_{\sigma \in G} |\delta \alpha_1 - \zeta^\sigma| \]
\[ = \alpha_1^{-\varphi(v)} \Phi_v(\alpha_1)\Phi_v^*(\alpha_1), \]

which proves (18) in the case \( \alpha \in L \) as well.

Combining (17) and (18) and recalling (16), we obtain the inequality
\[ 0 < \alpha_1^{-\varphi(v)/2} |\Phi_v(\alpha_1)\Phi_v^*(\alpha_1)|^{1/2} \leq |N_{L/Q}(1 - \delta\zeta^2)|. \tag{20} \]

This will be our basic tool.

Our next observation is that \( 1 - \delta\zeta^2 \) cannot be a unit. Indeed, if it is a unit, then the right-hand side of (20) is 1 and \( \min\{v, v^*\} > 1 \). Hence, applying Lemma 6, we obtain
\[ \alpha_1^{-\varphi(v)/2} (\alpha_1(\alpha_1 - 1))^{\varphi(v)/2} < 1, \]

which implies \( \alpha_1 < 2 \), contradicting (14).

Thus, \( 1 - \delta\zeta^2 \) is non-zero, but not a unit. Applying Lemma 4, we find that this is possible only in the following cases:
\[ v = p^\ell, \quad \delta = 1, \tag{21} \]
\[ v = 2p^\ell, \quad \delta = 1, \tag{22} \]
\[ v = 2^\ell, \quad \ell \geq 3, \tag{23} \]
\[ v \in \{1, 2, 4\}, \quad \delta \neq \zeta^2, \tag{24} \]

where (here and below) \( \ell \) is a positive integer and \( p \) is an odd prime number. We study these cases separately.

In the case (21), we have
\[ \Phi_v(X) = \Phi_v^*(X) = \frac{X^{p^\ell} - 1}{X^{p^\ell-1} - 1} \quad \text{and} \quad N_{L/Q}(1 - \zeta^2) = p \]
by Lemma 4. We obtain
\[ \frac{1}{\alpha_1^{p^{\ell-1}(p-1)/2}} \frac{\alpha_1^{p^\ell} - 1}{\alpha_1^{p^\ell-1} - 1} \leq p. \]

The left-hand side is strictly bounded from below by \( \alpha_1^{p^{\ell-1}(p-1)/2} \), which gives \( \alpha_1^{p^{\ell-1}} < p^{\frac{p-1}{2}} \). Checking with (15) leaves the only option
\[ \alpha_1 = \frac{3 + \sqrt{5}}{2}, \quad p^\ell = 3, \]

which is eliminated by direct verification.
In the case (22), we have
\[ \Phi_v(X) = \Phi_{v^\ast}(X) = \frac{X^{p^\ell} + 1}{X^{p^\ell-1} + 1} \quad \text{and} \quad N_{L/Q}(1 - \zeta^2) = p. \]

We obtain
\[ \frac{1}{\alpha_1^{p^\ell-1}(p-1)/2} \frac{\alpha_1^{p^\ell} + 1}{\alpha_1^{p^\ell-1} + 1} \leq p. \]

From (15), we deduce \( \alpha_1^{p^\ell-1} + 1 \leq 1.4 \alpha_1^{p^\ell-1} \), which implies the inequality \( \alpha_1^{p^\ell-1} < (1.4 p)^{\frac{2}{p^\ell}} \). Invoking again (15), we are left with the three options
\[ \alpha_1 = \frac{3 + \sqrt{5}}{2}, \quad p^\ell \in \{3, 5\}; \quad (25) \]
\[ \alpha_1 = 2 + \sqrt{3}, \quad p^\ell = 3. \quad (26) \]

The two cases in (25) are eliminated by verification, while (26) leads to \((a, b, k, v) = (4, -1, 1, 6)\), one of the two instances in (iv).

In the case (23), we have
\[ \Phi_v(X) = \Phi_{v^\ast}(X) = \frac{X^{2^\ell} - 1}{X^{2^\ell-1} - 1} \quad \text{and} \quad N_{L/Q}(1 - \delta \zeta^2) = 4. \]

We obtain
\[ \frac{1}{\alpha_1^{2^\ell-2}} \frac{\alpha_1^{2^\ell} - 1}{\alpha_1^{2^\ell-1} + 1} \leq 4, \]

which implies \( \alpha_1^{2^\ell-2} \leq 4 \). Since \( \ell \geq 3 \), this contradicts (14).

In the final case (24), it more convenient to use the divisibility relation (17) directly.

If \( v \in \{1, 2\} \), then \( \zeta^2 = 1 \) and \( \delta = -1 \). Taking the norm in both sides of (17), we obtain
\[ \alpha_1 - \alpha_1^{-1} = \text{Tr}_{K/Q}(\alpha_1) \mid 4. \]

Together with \( N_{K/Q}(\alpha_1) = \delta = -1 \) and inequality (14), this implies two possibilities:
\[ \alpha_1 = 1 + \sqrt{2}, \quad \alpha_1 = 2 + \sqrt{3}. \quad (27) \]

The latter is disqualified by inspection. The former yields \((a, b, k) = (2, 1, 1)\), which is (ii).

In a similar fashion one treats \( v = 4 \). In this case \( \zeta^2 = -1 \) and \( \delta = 1 \), and, taking the norm in (17), we obtain
\[ (\alpha_1 + \alpha_1^{-1})^2 = (\text{Tr}_{K/Q}(\alpha_1))^2 \mid 16. \]
We again have one of the options (27), but this time the former is eliminated by inspection, and the latter leads to \((a, b, k) = (4, -1, 1)\), the missing instance in (iv). This completes the proof of the lemma. \(\square\)

The following is a generalization of Lemma 4 from [2].

For a prime number \(p\) and a nonzero integer \(m\), we put \(\nu_p(m)\) for the exponent of the prime \(p\) in the factorization of \(m\). For a finite set of primes \(S\) and a positive integer \(m\), we put

\[
m_S = \prod_{p \in S} p^{\nu_p(m)}
\]

for the largest divisor of \(m\) whose prime factors are in \(S\). For any prime number \(p\) we put \(f_p\) for the index of appearance in the Lucas sequence \(\{U_n\}_{n \geq 0}\), which is the minimal positive integer \(k\) such that \(p \mid U_k\).

**Lemma 8.** Let \(a \geq 1\). If \(S\) is any finite set of primes and \(m\) is a positive integer, then

\[
(U_m)_S \leq \alpha^2 m \text{lcm}[U_{f_p} : p \in S].
\]

**Proof.** It is known that

\[
\nu_p(U_m) = \begin{cases}
0 & \text{if } m \not\equiv 0 \pmod{f_p}; \\
\nu_p(U_{f_p}) + \nu_p(m/f_p) & \text{if } m \equiv 0 \pmod{f_p}, \quad p \text{ odd}; \\
\nu_2(U_2) + \nu_2(m/2) & \text{if } m \equiv 0 \pmod{2}, \quad p = 2, \ a \text{ even}; \\
\nu_2(U_3) & \text{if } m \equiv 3 \pmod{6}, \quad p = 2, \ a \text{ odd}; \\
\nu_2(U_6) + \nu_2(m/2) & \text{if } m \equiv 0 \pmod{6}, \quad p = 2, \ a \text{ odd}.
\end{cases}
\]

The above relations follow easily from Proposition 2.1 in [1]. In particular, the inequality

\[
\nu_p(U_m) \leq \nu_p(U_{f_p}) + \nu_p(m) + \delta_{p,2}
\]

always holds with \(\delta_{p,2}\) being 0 if \(p\) is odd or \(p = 2\) and \(a\) is even and \(\nu_2((a^2 + 3b)/2)\) if \(p = 2\) and \(a\) is odd. We get that

\[
(U_m)_S \leq \left( \prod_{p \in S} p^{\nu_p(U_{f_p})} \right) \left( \prod_{p \mid m, p > 2} p^{\nu_p(m)} \right) 2^{\nu_2(m) + \nu_2((a^2 + 3b)/2)}
\]

\[
< \alpha^2 m \text{lcm}[U_{f_p} : p \in S],
\]

which is what we wanted to prove. For the last inequality above, we used the fact that

\[
2^{\nu_2((a^2 + 3b)/2)} \leq (a^2 + 3b)/2 = (\alpha^2 - \alpha\beta + \beta^2)/2 < \alpha^2.
\]

\(\square\)
3. Proof of Theorem 1

We assume that $m \geq 10000k$. Since $U_n$ is periodic modulo $U_m$ with period $4m$ (Lemma 2), we may assume that $n \leq 4m$. We split $U_m$ into various factors, as follows. Write

$$U_{n+k}^s - U_n^s = \prod_{d|s} \Phi_d(U_{n+k}, U_n),$$

where $\Phi_d(X,Y)$ is the homogenization of the cyclotomic polynomial $\Phi_d(X)$. We put

$s_1 := \text{lcm}[2, s]$, $S := \{p : p \mid 6s\}$ and

$$D := (U_m)_S;$$

$$A := \gcd(U_m/D, \prod_{d \leq 6, d \neq 5} \Phi_d(U_{n+k}, U_n));$$

$$E := \gcd(U_m/D, \prod_{d | s_1 \atop d = 5 \text{ or } d > 6} \Phi_d(U_{n+k}, U_n)).$$

Clearly,

$$U_m \mid ADE.$$

Before bounding $A$, $D$, $E$, let us comment on the sign of $a$. If $a < 0$, then we change the pair $(a, b)$ to $(-a, b)$. This has as effect replacing $(\alpha, \beta)$ by $(-\alpha, -\beta)$ and so $U_n(\alpha, \beta) = (-1)^{n-1}U_n(\alpha, \beta)$ for all $n \geq 0$. In particular, $U_m$ remains the same or changes sign. Further, if $k$ is even then

$$\Phi_d(U_{n+k}(-\alpha, -\beta), U_n(-\alpha, -\beta)) = \pm \Phi_d(U_{n+k}(\alpha, \beta), U_n(\alpha, \beta)),$$

while if $k$ is odd, then

$$\Phi_d(U_{n+k}(-\alpha, -\beta), U_n(-\alpha, -\beta)) = \pm \Phi_d(U_{n+k}(\alpha, \beta), -U_n(\alpha, \beta))$$

$$= \pm \Phi_{d^*}(U_{n+k}(\alpha, \beta), U_n(\alpha, \beta)),$$

where the $d^*$ has been defined at (19). Note that the sets $\{d \leq 6, d \neq 5\}$ and $\{d \mid s_1, d = 5 \text{ or } d > 6\}$ are closed under the operation $d \mapsto d^*$. Hence, $D$, $A$, $E$ do not change if we replace $a$ by $-a$, so we assume that $a > 0$. By the Binet formula (6), we get easily that the inequality

$$\alpha^{n-2} \leq U_n \leq \alpha^n \quad \text{is valid for all } n \geq 1.$$  \hfill (28)

We are now ready to bound $A$, $D$, $E$. 
The easiest to bound is $D$. Namely, by Lemma 8 and the fact that $f_p \leq p + 1$ for all $p | 6s$, we get

$$D \leq \alpha^2 m \prod_{p | 6s} U_{p+1} < ma^{2+\sum_{p | 6s}(p+1)} < \alpha^{6s+3+\log m/\log \alpha},$$

where we used the fact that $\sum_{p | t}(p+1) \leq t$, which is easily proved by induction on the number of distinct prime factors of $t$.

We next bound $E$.

Note that

$$E \mid \prod_{\zeta; \zeta^4 = 1 \atop \zeta \notin \{\pm 1, \pm i, \pm \omega, \pm \omega^2\}} (U_{n+k} - \zeta U_n),$$

where $\omega := e^{2\pi i/3}$ is a primitive root of unity of order 3.

Let $K = \mathbb{Q}(e^{2\pi i/3}, \alpha)$, which is a number field of degree $d \leq 2\phi(s_1) = 2\phi(s)$. Assume that there are $\ell$ roots of unity $\zeta$ participating in the product appearing in the right-hand side of (30) and label them $\zeta_1, \ldots, \zeta_\ell$. Write

$$E_i = \gcd(E, U_{n+k} - \zeta_i U_n) \text{ for all } i = 1, \ldots, \ell,$$

where $E_i$ are ideals in $\mathcal{O}_K$. Then relations (30) and (31) tell us that

$$EO_K \mid \prod_{i=1}^\ell E_i.$$ (32)

Our next goal is to bound the norm $|N_{K/Q}(E_i)|$ of $E_i$ for $i = 1, \ldots, \ell$. First of all, $U_m \in E_i$. Thus, with formula (6) and the fact that $\beta = (-b)\alpha^{-1}$, we get

$$\alpha^m \equiv (-b)^m \alpha^{-m} \pmod{E_i}.$$ (33)

Multiplying the above congruence by $\alpha^m$, we get

$$\alpha^{2m} \equiv (-b)^m \pmod{E_i}.$$ (34)

We next use formulae (6) and (31) to deduce that

$$(\alpha^{n+k} - (-b)^{n+k}\alpha^{-n-k}) - \zeta(\alpha^n - (-b)^n\alpha^{-n}) \equiv 0 \pmod{E_i}, \quad (\zeta := \zeta_i).$$

Multiplying both sides above by $\alpha^n$, we get

$$\alpha^{2n}(\alpha^k - \zeta) - (-b)^{n+k}(\alpha^{-k} - (-b)^k\zeta) \equiv 0 \pmod{E_i}.$$ (34)
Let us show that \( \alpha^k - \zeta \) and \( \mathcal{E}_i \) are coprime. Assume this is not so and let \( \pi \) be some prime ideal of \( \mathcal{O}_X \) dividing both \( \alpha^k - \zeta \) and \( \mathcal{E}_i \). Then we get \( \alpha^k \equiv \zeta \pmod{\pi} \) and so \( \alpha^{-k} \equiv (-b)^k \zeta \pmod{\pi} \) by (34). Multiplying these two congruences we get \( 1 \equiv (-b)^k \zeta^2 \pmod{\pi} \). Hence, \( \pi \mid 1 - (-b)^k \zeta^2 \). If this number is not zero, then, \( (-b)^k \zeta^2 \) is a root of unity whose order divides 6, so, by Lemma 6, we get that \( \pi \mid 6 \), which is impossible because \( \pi \mid \mathcal{E}_i \mid E \), and \( E \) is an integer coprime to 6. If the above number is zero, we get that \( \zeta^2 = \pm 1 \), so \( \zeta \in \{ \pm 1, \pm i \} \), but these values are excluded at this step. Thus, indeed \( \alpha^k - \zeta \) and \( \mathcal{E}_i \) are coprime, so \( \alpha^k - \zeta \) is invertible modulo \( \mathcal{E}_i \). Now congruence (34) shows that

\[
\alpha^{2n+k} \equiv (-b)^n \zeta \left( \frac{\alpha^k - (-b)k\zeta}{\alpha^k - \zeta} \right) \pmod{\mathcal{E}_i}.
\] (35)

We now apply Lemma 3 to \( a = 2m \) and \( b = 2n + k \leq 8m + k < 9m \) with the choice \( X = 9m \) to deduce that there exist integers \( u, v \) not both zero with \( \max\{|u|, |v|\} \leq \sqrt{X} \) such that \( |2mu + (2n + k)v| \leq 3\sqrt{X} \). We raise congruence (33) to \( u \) and congruence (35) to \( v \) and multiply the resulting congruences getting

\[
\alpha^{2mu + (2n+k)v} = (-b)^{mu + n}v \zeta^u \left( \frac{\alpha^k - (-b)k\zeta}{\alpha^k - \zeta} \right)^v \pmod{\mathcal{E}_i}.
\]

We record this as

\[
\alpha^R \equiv \eta \left( \frac{\alpha^k - (-b)k\zeta}{\alpha^k - \zeta} \right)^S \pmod{\mathcal{E}_i}
\] (36)

for suitable roots of unity \( \eta \) and \( \zeta \) of order dividing \( s_1 \) with \( \zeta \) not of order 1, 2, 3, 4, or 6, where \( R := 2mu + (2n + k)v \) and \( S := v \). We may assume that \( R \geq 0 \), for if not, we replace the pair \((u, v)\) by the pair \((-u, -v)\), thus replacing \((R, S)\) by \((-R, -S)\) and \( \eta \) by \( \eta^{-1} \) and leaving \( \zeta \) unaffected. We may additionally assume that \( S \geq 0 \), for if not, we replace \( S \) by \(-S\) and \( \zeta \) by \((b)^k \zeta\), again a root of unity of order dividing \( s_1 \) but not of order 1, 2, 3, 4, or 6 and leave \( R \) and \( \eta \) unaffected. Thus, \( \mathcal{E}_i \) divides the algebraic integer

\[
E_i = \alpha^R (\alpha^k - \zeta_i)^S - \eta_i (\alpha^k - (-b)k\zeta_i)^S.
\] (37)

Let us show that \( E_i \neq 0 \). If \( E_i = 0 \), we then get

\[
\alpha^R = \eta_i \left( \frac{\alpha - (-b)k\zeta_i}{\alpha - \zeta_i} \right)^S,
\]

and after raising both sides of the above equality to the power \( s_1 \), we get, since \( \eta_i^{s_1} = 1 \), that

\[
\alpha^{s_1 R} = \left( \frac{\alpha^k - (-b)k\zeta_i}{\alpha - \zeta_i} \right)^{S s_1}.
\]
Lemma 7 gives us a certain number of conditions all of which have \( \zeta_i \) or a root of unity of order 1, 2, 4, or 6, which is not our case. Thus, \( E_i \) is not equal to zero. We now bound the absolute values of the conjugates of \( E_i \). We find it more convenient to work with the associate of \( E_i \) given by

\[
G_i = \alpha^{-\lfloor R/2 \rfloor} E_i = \alpha^{R-\lfloor R/2 \rfloor} (\alpha^k - \zeta_i)S - \alpha^{-\lfloor R/2 \rfloor} \eta_i (\alpha^k - (-b)^{k/\zeta_i})S.
\]

Note that

\[
R \leq |2m + (2n + k)v| \leq 3\sqrt{X} = 9\sqrt{m}, \quad \text{and} \quad S = |v| \leq \sqrt{X} = 3\sqrt{m}.
\]

Let \( \sigma \) be an arbitrary element of \( G = \text{Gal}(K/\mathbb{Q}) \). We then have that \( \eta_i^\sigma = \eta_i', \zeta_i^\sigma = \zeta_i' \), where \( \eta_i' \) and \( \zeta_i' \) are roots of unity of order dividing \( s_1 \). Furthermore, \( \alpha^\sigma \in \{ \alpha, \beta \} \). If \( \alpha^\sigma = \alpha \), we then get

\[
|G_i^\sigma| = \left| \alpha^{R-\lfloor R/2 \rfloor} (\alpha^k - \zeta_i)S - \eta_i' (\alpha^{-\lfloor R/2 \rfloor} (\alpha - (-b)^{k/\zeta_i})S) \right|
\leq \alpha^{(R+1)/2} (\alpha^k + 1)^S + (\alpha^k + 1)^S
\leq 2\alpha^{(R+1)/2}(\alpha + 1)^{S+k} \leq \alpha^{2+(9\sqrt{m}+1)/2+6\sqrt{mk}}
\leq \alpha^{11\sqrt{mk}}, \quad (38)
\]

while if \( \alpha^\sigma = \beta \), we also get

\[
|G_i^\sigma| = \left| \beta^{R-\lfloor R/2 \rfloor} (\beta^k - \zeta_i')S - \beta^{-\lfloor R/2 \rfloor} \eta_i' (\beta^k - (-b)^{k/\zeta_i'})S \right|
\leq (\alpha^{-k} + 1)^S + \alpha^{R/2}(\alpha^{-k} + 1)^S
\leq \alpha^S + \alpha^{R/2+S} \leq 2\alpha^{R/2+S} \leq \alpha^{2+4.5\sqrt{m}+6\sqrt{m}}
\leq \alpha^{11\sqrt{mk}}.
\]

In the above, we used the fact that \( \alpha^{-k} + 1 \leq \alpha^{-1} + 1 \leq \alpha \). In conclusion, inequality (38) holds for all \( \sigma \in G \). Thus, if we write \( G_i^{(1)}, \ldots, G_i^{(d)} \) for the \( d \) conjugates of \( G_i \) in \( K \), we then get that

\[
|N_{K/\mathbb{Q}}(E_i)| \leq |N_{K/\mathbb{Q}}(E_i)| = |N_{K/\mathbb{Q}}(G_i)| \leq \alpha^{11dk\sqrt{m}},
\]

where the first inequality above follows because \( E_i \) divides \( E_i \); hence \( G_i \), and \( E_i \neq 0 \). Multiplying the above inequalities for \( i = 1, \ldots, \ell \), we get that

\[
E^\ell = |N_{K/\mathbb{Q}}(E)| = |N_{K/\mathbb{Q}}(EO_K)| \leq \left| N_{K/\mathbb{Q}} \left( \prod_{i=1}^\ell E_i \right) \right|
\leq \left| \prod_{i=1}^\ell N_{K/\mathbb{Q}}(G_i) \right| \leq \alpha^{11dk\sqrt{m}},
\]
therefore

\[ E \leq \alpha^{11kd\sqrt{m}} \leq \alpha^{22k\phi(s)\sqrt{m}} < \alpha^{22ks\sqrt{m}}. \]  

(39)

In the above, we used that \( d \leq 2\phi(s) \leq 2s. \)

We are now ready to estimate \( A. \) We write

\[ A_1 := \gcd(U_m, U_{n+k}^2 - U_n^2); \]
\[ A_2 := \gcd(U_m, U_{n+k}^2 + U_n^2); \]
\[ A_3 := \gcd(U_m, U_{n+k}^6 - U_n^6). \]

Clearly, \( A \leq A_1A_2A_3. \) We bound each of \( A_1, A_2, A_3. \) We first estimate \( A_1 \) and \( A_2 \) and deal with \( A_3 \) later. Write

\[ U_n^2 = \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right)^2 = \frac{\alpha^{2n} + 2(-b)^n + \alpha^{-2n}}{(\alpha + b\alpha^{-1})^2}; \]
\[ U_{n+k}^2 = \frac{\alpha^{2n+2k} + 2(-b)^n(-b)^k + \alpha^{-2n-2k}}{(\alpha + b\alpha^{-1})^2}. \]

Assume that \((-b)^k = 1. \) If \( \zeta \in \{ \pm i \}, \) then \((\alpha^k - (-b)^k\zeta)/(\alpha^k - \zeta) = (\alpha^k + \zeta)/(\alpha^k - \zeta) \) is multiplicatively independent with \( \alpha \) by Lemma 7. The argument which lead to inequality (39) shows that

\[ A_2 \leq \alpha^{11kd_1\sqrt{m}} \leq \alpha^{44k\sqrt{m}}, \]

(40)

where \( d_1 = 4 \) is the degree of the field \( \mathbb{Q}(\alpha, i). \) To estimate \( A_1, \) we set \( \gamma = -b\alpha^2 \) and, using that \((-b)^k = 1, \) we find

\[ U_{n+k}^2 - U_n^2 = \frac{\alpha^{2n+2k} + \alpha^{-2n-2k} - \alpha^{2n} - \alpha^{-2n}}{(\alpha + b\alpha^{-1})^2} \]
\[ = \alpha^{-2n-k} \frac{(\gamma^{2n+k} - 1)(\gamma^k - 1)}{(\gamma - 1)^2}, \]
\[ U_m = (-b\alpha)^{1-m} \frac{\gamma^m - 1}{\gamma - 1}. \]

In the ring of integers \( \mathcal{O} = \mathcal{O}_K \) of the quadratic field \( K = \mathbb{Q}(\alpha) \) consider the ideals

\[ a = \left( \frac{\gamma^m - 1}{\gamma - 1}, \frac{\gamma^{2n+k} - 1}{\gamma - 1} \right), \quad b = \left( \frac{\gamma^m - 1}{\gamma - 1}, \frac{\gamma^k - 1}{\gamma - 1} \right). \]

Clearly, \( A_1 | ab, \) whence
\[ A_1^2 = \mathcal{N}_{K/Q}(A_1) \leq |\mathcal{N}_{K/Q}(a)||\mathcal{N}_{K/Q}(b)|. \]

Clearly,
\[ |\mathcal{N}_{K/Q}(b)| \leq |\mathcal{N}_{K/Q} \left( \frac{(-b)^k \alpha^{2k} - 1}{\alpha + b\alpha - 1} \right)| = |\mathcal{N}_{K/Q}(U_k)| < \alpha^{2k}. \]

To estimate \(|\mathcal{N}_{K/Q}(a)|\), observe that \(a = (\gamma^d - 1)/(\gamma - 1)\) by item 3 of Corollary 5, where \(d = \gcd(m, 2n + k)\). Using the obvious inequality \(|\gamma^{-1}| \leq 1/2\), we get that
\[ |\mathcal{N}_{K/Q}(a)| = \left| \frac{\gamma^d - 1}{\gamma - 1} \frac{\gamma^{-d} - 1}{\gamma^{-1} - 1} \right| \leq 6|\gamma|^d = 6\alpha^{2d} < \alpha^{2d+4}. \]

Hence, \(A_1 \leq \alpha^{d+k+2}\). It is important to note that \(d \neq m\): otherwise we would have had \(U_m \mid U_{n+k}^2 - U_n^2\), contradicting our hypothesis about the minimality of \(s\). Therefore \(d\) is a proper divisor of \(m\), showing that
\[ A_1 \leq \alpha^{m/2+k+2}. \quad (41) \]

Thus, we have bounded \(A_1\) and \(A_2\) in the case \((-b)^k = 1\).

The case \((-b)^k = -1\) can be treated analogously, but \(A_1\) and \(A_2\) swap roles. This time for \(\zeta \in \{ \pm 1 \}\) the number \(\frac{\alpha^k - (-b)^k \zeta}{\alpha^k - \zeta}\) is multiplicatively independent of \(\alpha\) by Lemma 7, which implies the estimate
\[ A_1 \leq \alpha^{22k \sqrt{m}}. \quad (42) \]

Next, using that \((-b)^k = -1\), we find
\[ U_{n+k}^2 + U_n^2 = \alpha^{2-n-k} \frac{(\gamma^{2n+k} - 1)(\gamma^k - 1)}{(\gamma - 1)^2}, \]
and arguing exactly as in the case \((-b)^k = 1\), we get
\[ A_2 \leq \alpha^{m/2+k+2}. \quad (43) \]

Hence, we get that both in case \((-b)^k = 1\) and in case \((-b)^k = -1\), we have
\[ A_1A_2 \leq \alpha^{m/2+k+2+44k \sqrt{m}}. \quad (44) \]

Finally, for \(A_3\), we note that by Lemma 7, unless \(\alpha = 2 + \sqrt{3}\), we have that \(\frac{\alpha^k - (-b)^k \zeta}{\alpha^k - \zeta}\) is multiplicatively independent of \(\alpha\) for \(\zeta \in \{ \pm \omega, \pm \omega^2 \}\). Thus, writing
\[ A_{3, \pm} = \gcd(A_3, U_{n+k}^2 \pm U_{n+k} U_n + U_n^2), \]
we get, by arguing in the field $\mathbb{Q}(\alpha, e^{2\pi i/3})$ of degree 4 as we did in order to prove inequality (39), that

$$A_{3,\pm} \leq \alpha^{44k\sqrt{m}}, \quad (45)$$

which leads to

$$A_3 \leq A_{3,+} + A_{3,-} \leq \alpha^{88k\sqrt{m}}. \quad (46)$$

So, let us assume that $(a, b, k) = (4, 1, 1)$, so $\alpha = 2 + \sqrt{3}$. Note that since $U_t \equiv t \pmod{2}$, it follows that $U_{n+k}^s - U_n^s = U_{n+1}^s - U_n^s$ is odd and a multiple of $U_m$, therefore $m$ is odd.

For $\zeta \in \{\omega, \omega^2\}$, we have that $\frac{\alpha^k - (-b)^k\zeta}{\alpha^k - \zeta} = \frac{\alpha - \zeta}{\alpha - \zeta}$ are multiplicatively independent of $\alpha$, which leads, by the previous argument, to

$$A_{3,+} \leq \alpha^{44k\sqrt{m}}. \quad (47)$$

As for $A_{3,-}$, since

$$U_{n+1}^2 - U_n^2 = V_{2n+1}/4,$$

we have that

$$A_{3,-} \mid \gcd(U_m, V_{2n+1}) = 1,$$

where the last equality follows easily from the fact that $m$ is and $2n+1$ are both odd (see (iii) of the Main Theorem in [3]). Together with (47), we infer that inequality (46) holds in this last case as well. Together with (44), we get that the inequality

$$A \leq A_1 A_2 A_3 \leq \alpha^{m/2+k+2+132k\sqrt{m}} \quad (48)$$

holds in all instances.

Inequality (28) together with estimates (29), (48) and (39), give

$$\alpha^{m-2} \leq U_m = DAE \leq \alpha^{6s+3+\log m/\log \alpha + m/2+k+2+(132k+22ks)\sqrt{m}}.$$

Since $s \geq 3$, we have $132 + 22s \leq 66s$. Since also $1/\log \alpha < 3$, we get

$$m/2 \leq (6s + 7 + 3 \log m + k) + 66sk\sqrt{m}.$$

Since $m \geq 10000$, one checks that $6s + 7 + 3 \log m + k < ks\sqrt{m}$. Hence,

$$m \leq 134ks\sqrt{m}, \quad (49)$$

which leads to the desired inequality (5).
4. Comment

One may wonder if one can strengthen our main result Theorem 1 in such a way as to include also the instances \( s \in \{1, 2, 4\} \) maybe at the cost of eliminating finitely many exceptions in the pairs \((a, k)\). The fact that this is not so follows from the formulae:

\[
\begin{align*}
(i) & \quad U_{n+k} - U_n = U_{n+k/2}V_{k/2} \text{ for all } n \geq 0 \text{ when } b = 1 \text{ and } 2 \| k; \\
(ii) & \quad U_{n+k} + U_n = U_{n+k/2}V_{k/2} \text{ for all } n \geq 0 \text{ when } b = 1 \text{ and } 4 \mid k \text{ or when } b = -1 \text{ and } k \text{ is even;} \\
(iii) & \quad U_{2n+k}^2 + U_n^2 = U_{2n+k}U_k \text{ for all } n \geq 0 \text{ when } b = 1 \text{ and } k \text{ is odd},
\end{align*}
\]

which can be easily proved using the Binet formulas (6). Thus, taking \( m = n + k/2 \) (for \( k \) even) and \( m = 2n + k \) for \( k \) odd and \( b = 1 \), we get that divisibility (3) always holds with some \( s \in \{1, 2, 4\} \). We also note the “near-miss” \( U_{4n+2} | 4(U_{n+1}^6 - U_n^6) \) for all \( n \geq 0 \) if \((a, b, k) = (4, -1, 1)\).

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