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The Blended Finite Element Method for Multi-fluid Plasma Modeling

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ICOSAHOM16, June 27th - July 1st, 2016
Rio de Janeiro, Brazil
1 The Multi-fluid Plasma Model

2 Blended Finite Element Method
   - Blended Finite Element Method
   - Nodal Continuous Galerkin
   - Modal Discontinuous Galerkin
   - Model Verification

3 Results
   - 1D Soliton problem
   - Electromagnetic Plasma Shock Problem

4 Application
There are multiple plasma models.

- 3-Dimensions + 3-Velocities
- Evolve the particles position and velocity
- e.g. Particle-In-Cell models

- Ensemble average of particles distribution, \( f_s(x,v,t) \)
- Evolve the distribution function
- e.g. Vlasov-Maxwell models
Modeling each particle velocity and position is not practical.
Instead an average is performed to give a statistical description.
Calculate the number of particles per unit volume having approximately the velocity $v$ near the position $x$ and at time $t$, distribution function $f(v, x, t)$

$$
\rho_s = m_s \int f_s(v) \, dv
$$

$$
\rho_s u_s = m_s \int v f_s(v) \, dv
$$

$$
P_s = P_s = m_s \int w f_s(v) \, dv, \quad p_s = \frac{1}{3} m_s \int w^2 f_s(v) \, dv
$$

$$
H_s = m_s \int w^2 f_s(v) \, dv, \quad h_s = \frac{1}{2} m_s \int w^2 f_s(v) \, dv
$$

$$
w = v - u_s$$
BOLTZMANN EQUATION EVOLES $f_s$.

- The Boltzmann eqn:

\[
\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \frac{\partial f_s}{\partial t} \bigg|_c
\]

- Take the $0^{th}$, $1^{st}$, $2^{nd}$ moments of the Boltzmann Eqn.

\[
m_s \int v^n \frac{\partial f_s}{\partial t} dv + m_s \int v^{n+1} \cdot \frac{\partial f_s}{\partial \mathbf{x}} dv + q_s \int v^n (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} dv = m_s \int v^n \frac{\partial f_s}{\partial t} \bigg|_c dv
\]

- Each moment of the Boltzmann eqn gives an equation for the moment variable, and introduces the next higher moment variable.

- This process can go on indefinitely.
BOLTZMANN EQUATION Evolves $f_s$.

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s u_s) = \frac{\partial \rho_s}{\partial t} \bigg|_{\Gamma}$$

$$\frac{\partial \rho_s u_s}{\partial t} + \nabla \cdot (\rho_s u_s u_s + p_s I + \Pi_s) = \frac{\rho_s q_s}{m_s} (E + u_s \times B) - \sum_{s^*} R_{s,s^*} + \frac{\partial \rho_s u_s}{\partial t} \bigg|_{\Gamma}$$

$$\frac{\partial \varepsilon_s}{\partial t} + \nabla \cdot ((\varepsilon_s + p_s) I + \Pi_s) \cdot u_s + h_s) = \frac{\rho_s q_s}{m_s} u_s \cdot E + \sum_{s^*} Q_{s,s^*} + \frac{\partial \varepsilon_s}{\partial t} \bigg|_{\Gamma}$$

- System is truncated by relating higher moment variables to the lower ones
- The fluids are coupled to each other and to the electromagnetic fields through Maxwell’s equations and interaction source terms.
Advantages of the Model

<table>
<thead>
<tr>
<th>Kinetic</th>
<th>LTE, velocity moments</th>
<th>MFPM</th>
<th>ε₀→0, mₑ→0</th>
<th>MHD</th>
</tr>
</thead>
</table>

Ideal MHD Model is valid when:

- High collisionality, \( \tau_{ii}/\tau \ll 1 \)
- Small Larmor radius, \( r_{Li}/L \ll 1 \)
- Low Resistivity, \( \left( \frac{m_e}{m_i} \right)^{1/2} \left( \frac{r_{Li}}{L} \right)^2 \frac{\tau}{\tau_{ii}} \ll 1 \)

Multi-Fluid Plasma Model

- Less computationally expensive than kinetic models
- Multi-fluid effects become relevant at small spacial and temporal scales
- Finite electron mass and speed-of-light effects are included
- There is charge separation is modeled
- Displacement current effects are resolved in the MFPM
# OUTLINE

## 1 THE MULTI-FLUID PLASMA MODEL

## 2 BLENDED FINITE ELEMENT METHOD
- Blended Finite Element Method
- Nodal Continuous Galerkin
- Modal Discontinuous Galerkin
- Model Verification

## 3 RESULTS
- 1D Soliton problem
- Electromagnetic Plasma Shock Problem

## 4 APPLICATION
The MFPM has dispersive sources.

\[
\frac{\partial Q}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = S
\]

- The source Jacobian \( \frac{\partial S}{\partial Q} \) has imaginary eigenvalues.
- The equation system has dispersive sources.
- The dispersion is physical (may be difficult to distinguish from numerical dispersion).
- This dispersion is due to plasma waves that result from ion and electron plasma interactions with electromagnetic fields.
- An ideal numerical method for the MFPM should:
  - be high-order accurate
  - capture shocks
  - couple the flux and the sources
  - not impose strict time-step

Srinivasan et al, CCP 10 (2011)
BFEM simultaneously uses CG and DG.

- Solution to the electron and EM fields is smooth and does not shock

- Continuous Galerkin
  - Electron fluid and EM fields
    \[ Q = \sum_i q_i v_i \]

- Discontinuous Galerkin
  - Multiple ion and neutral fluids
    \[ Q = \sum_i c_i v_i \]
**Implicit Continuous Galerkin**

- For this implementation the balance law form is cast as

\[
\frac{\partial Q}{\partial t} + \frac{\partial \vec{F}}{\partial Q} \cdot \frac{\partial Q}{\partial x} = S + \kappa \nabla^2 Q_d
\]

- Lagrange polynomials are used for basis functions, \( \nu_r \)

\[
\int_\Omega \nu_r \frac{\partial Q}{\partial t} dV = \mathcal{L}_r(Q) = \int_\Omega \nu_r S dV - \int_\Omega \nu_r \frac{\partial \vec{F}}{\partial Q} \cdot \frac{\partial Q}{\partial x} dV + \kappa \int_\Omega \nu_r \nabla^2 Q_d dV
\]

- \( \theta \)-method time integration

\[
\mathcal{R}(Q^n) = \dot{\mathbf{M}} \frac{Q^{n+1} - Q^n}{dt} - \theta \mathcal{L}_r(Q^{n+1}) - (1 - \theta) \mathcal{L}_r(Q^n) = 0
\]

- \( \theta = 0.5 \) is used for 2\textsuperscript{nd} order accuracy

\[
\vec{J}(Q^n) = \frac{\partial \mathcal{R}(Q^n)}{\partial Q^n}, \quad \vec{J}(Q^n) \Delta Q = -\mathcal{R}(Q^n), \quad Q^{n+1} = Q^n + \Delta Q
\]

\[
\frac{\partial Q}{\partial t} + \frac{\partial \mathbf{F}}{\partial \mathbf{x}} = \mathbf{S}
\]

- Legendre polynomials are used for basis functions, \( v_p \).
- The hyperbolic equation is multiplied by the basis function,

\[
\int_{\Omega} v_p \frac{\partial Q}{\partial t} dV = \mathcal{L}_p(Q) = \int_{\Omega} v_p S dV - \int_{\partial \Omega} v_p \mathbf{F} \cdot d\mathbf{A} + \int_{\Omega} \mathbf{F} \cdot \nabla v_p dV
\]

- Explicit Runge-Kutta time integration
- \( CFL = c \Delta t / \Delta x \leq 1/(2p - 1) \), \( p \) is the polynomial order

\[
Q^{*} = Q^n + \Delta t \mathcal{L}_p(Q^n),
\]

\[
Q^{n+1} = \frac{1}{2} Q^* + \frac{1}{2} Q^n + \frac{1}{2} \Delta t \mathcal{L}_p(Q^*).
\]

Loverich et al, CCP 9 (2006)
\[
\frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} = 0, \quad Q(x, 0) = e^{-10(x-8)^2}, \quad ||\Delta Q||_2 = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\hat{Q} - Q_i)^2}
\]

**Spatial Convergence**

**Temporal Convergence**

- Simulations at fixed time-step
- Simulations at fixed CFL=1
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4. Application
1D soliton is a two-fluid plasma problem

The solution is smooth, therefore artificial dissipation can be small

The simulation uses 512 second-order elements

\[ B_z = 1.0, \ T_e = T_i = 0.01, \ u_i = u_e = 0 \]

\[ n_e = n_i = 1.0 + e^{-10(x-6)^2} \]

Baboolal, Math. and Comp. Sim. 55 (2001)
\[ \frac{m_i}{m_e} = 1836, \quad \frac{c}{c_{si}} = 1000\sqrt{2}, \quad \text{FV 5000 cells} \]

- DG Solution is very dispersive
- BFEM is less dissipative than the converged solution

Hakim et al, JCP 219 (2006)
### BFEM Computational Cost Savings

<table>
<thead>
<tr>
<th>case</th>
<th>$m_i/m_e$</th>
<th>$c/c_{si}$</th>
<th>DG time(s)</th>
<th>BFEM time (s)</th>
<th>BFEM cost over DG</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td>10/√2</td>
<td>0.32</td>
<td>37.7</td>
<td>+11681%</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>10/√2</td>
<td>1.28</td>
<td>37.7</td>
<td>+2845%</td>
</tr>
<tr>
<td>3</td>
<td>500</td>
<td>10/√2</td>
<td>6.82</td>
<td>37.7</td>
<td>+452.8%</td>
</tr>
<tr>
<td>4</td>
<td>1000</td>
<td>10/√2</td>
<td>12.4</td>
<td>38.2</td>
<td>+208.1%</td>
</tr>
<tr>
<td>5</td>
<td>1836</td>
<td>10/√2</td>
<td>23.5</td>
<td>40.4</td>
<td>+71.91%</td>
</tr>
<tr>
<td>6</td>
<td>3672</td>
<td>10/√2</td>
<td>47.2</td>
<td>39.2</td>
<td>-16.95%</td>
</tr>
<tr>
<td>7</td>
<td>3672</td>
<td>100/√2</td>
<td>520</td>
<td>265</td>
<td>-49.04%</td>
</tr>
<tr>
<td>8</td>
<td>3672</td>
<td>1000/√2</td>
<td>5274</td>
<td>2735</td>
<td>-48.14%</td>
</tr>
</tbody>
</table>

- Per time-step explicit DG is faster than BFEM, but it requires many more time-steps.
- BFEM is more efficient only when time-step are considerably larger than explicit DG.
$m_i/m_e = 1836$

- The BFEM seems to be less accurate than the DG implementation ($\sim 50\%$)
- When the mass ratio is one, the two methods have the same level of accuracy
- The discrepancy is due to the fact that the semi-implicit BFEM does not resolve the plasma frequency in this problem

$m_i/m_e = 1$
Fast rarefaction wave (FR), a slow compound wave (SC), a contact discontinuity (CD), a slow shock (SS), and another fast rarefaction wave (FR)

The problem exhibits limits of MHD and multi-fluid behavior by changing the Larmor radius, $r_L$

- MHD: $r_L \to 0$
- Multi-fluid: $r_L \sim L$

Brio and Wu, JCP 75 (1988)
SHOCK IN DENSITY BUT SMOOTH FIELDS.

- \( t = 0.05/\omega_{ci} \), \( c/c_{si} = 110 \), \( m_i/m_e = 1836 \)

The main features of the problem are captured by all three methods.

- BFEM does not properly resolve the fast electromagnetic waves which require accurately resolving the electron dynamics.

Mass density

Magnetic field (y-comp.)

Sousa (ERC/AFRL)
\[ \Delta t_{\text{max}} = \min \left( \frac{\Delta x}{c_{se}}, \frac{\Delta x}{c_{si}}, \frac{\Delta x}{c}, \frac{0.1}{\omega_{ce}}, \frac{0.1}{\omega_{ci}}, \frac{0.1}{\omega_{pe}}, \frac{0.1}{\omega_{pi}} \right) \]

- \( \Delta t_{\text{max}} \) corresponds to the maximum value allowed for explicit methods based on the CFL condition.
- \( \Delta t = 42.9 \Delta t_{\text{max}} \) is the maximum time step allowed by the BFEM due to ion dynamics.
Varying the artificial dissipation on the electron fluid, $\kappa_e$
Wave-like behavior of the problem is better resolved
Amplitude of the compound wave increases
Right fast rarefaction wave is not visible
Varying the artificial dissipation on the EM-field, $\kappa_{EM}$

- There is better agreement with the DG solution.
- This reinforces the point that the wave-like behavior arises from the interaction of the electron fluid with the electromagnetic fields.
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4 APPLICATION
- Deuterium and tritium are heated and compressed to fusion conditions.
- The compression is laser-driven.
- Deuterium can accelerate faster than the tritium.
- Low neutron yield measurements point towards separation.
- The phenomenon is not captured by single-fluid plasma models.
The ion species separation in both cases is the same although the solution behind the shock fronts differ.

Bellei et al., PoP 20 (2013) Sousa (ERC/AFRL)
The solution error increases as the computational time decreases.
The blended finite element method (BFEM) is presented

- DG spatial discretization with explicit Runge-Kutta time integration for Ions and neutrals
- CG spatial discretization with implicit Crank-Nicolson time integration for the electrons and EM fields
- DG captures shocks and discontinuities
- CG is efficient and robust for smooth solutions

Physics-based decomposition of the algorithm yields numerical solutions that resolve the desired timescales

DG method takes less computational time to advance the solution by one time-step, however $\Delta t$ is much smaller than that of the BFEM

Computational cost savings using the BFEM will only occur for relatively large implicit time-steps compared to explicit time-steps

Thank You.