The Blended Finite Element Method for Multi-fluid Plasma Modeling

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Viewgraph/Briefing Charts

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THE BLENDED FINITE ELEMENT METHOD FOR MULTI-FIUID PLASMA MODELING

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ICOSAHOM16, June 27th - July 1st, 2016
Rio de Janeiro, Brazil
THE MULTI-FLUID PLASMA MODEL

2  BLENDED FINITE ELEMENT METHOD
   • Blended Finite Element Method
   • Nodal Continuous Galerkin
   • Modal Discontinuous Galerkin
   • Model Verification

3  RESULTS
   • 1D Soliton problem
   • Electromagnetic Plasma Shock Problem

4  APPLICATION
There are multiple plasma models.

- 3-Dimensions + 3-Velocities
- Evolve the particles position and velocity
- e.g. Particle-In-Cell models

- Ensemble average of particles distribution, $f_s(x,v,t)$
- Evolve the distribution function
- e.g. Vlasov-Maxwell models
Modeling each particle velocity and position is not practical. Instead an average is performed to give a statistical description. Calculate the number of particles per unit volume having approximately the velocity $v$ near the position $x$ and at time $t$, distribution function $f(v, x, t)$

$$
\rho_s = m_s \int f_s(v) dv \\
\rho_s u_s = m_s \int v f_s(v) dv \\
\mathbb{P}_s = \mathbb{P}_s = m_s \int w w f_s(v) dv, \quad p_s = \frac{1}{3} m_s \int w^2 f_s(v) dv \\
H_s = m_s \int w w w f_s(v) dv, \quad h_s = \frac{1}{2} m_s \int w^2 w f_s(v) dv \\
w = v - u_s
$$
The Boltzmann eqn:

\[
\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \frac{\partial f_s}{\partial t} \bigg|_c
\]

Take the 0th, 1st, 2nd moments of the Boltzmann Eqn.

\[
m_s \int \mathbf{v}^n \frac{\partial f_s}{\partial t} d\mathbf{v} + m_s \int \mathbf{v}^{n+1} \cdot \frac{\partial f_s}{\partial \mathbf{x}} d\mathbf{v} + q_s \int \mathbf{v}^n (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} d\mathbf{v} = m_s \int \mathbf{v}^n \frac{\partial f_s}{\partial t} \bigg|_c d\mathbf{v}
\]

Each moment of the Boltzmann eqn gives an equation for the moment variable, and introduces the next higher moment variable.

This process can go on indefinitely.
Boltzmann equation evolves $f_s$. 

\[
\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{u}_s) = \left. \frac{\partial \rho_s}{\partial t} \right|_\Gamma
\]

\[
\frac{\partial \rho_s \mathbf{u}_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{u}_s \mathbf{u}_s + p_s \mathbf{I} + \Pi_s) = \frac{\rho_s q_s}{m_s} (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) - \sum_{s^*} R_{s,s^*} + \left. \frac{\partial \rho_s \mathbf{u}_s}{\partial t} \right|_\Gamma
\]

\[
\frac{\partial \varepsilon_s}{\partial t} + \nabla \cdot ((\varepsilon_s + p_s) \mathbf{I} + \Pi_s) \cdot \mathbf{u}_s + \mathbf{h}_s = \frac{\rho_s q_s}{m_s} \mathbf{u}_s \cdot \mathbf{E} + \sum_{s^*} Q_{s,s^*} + \left. \frac{\partial \varepsilon_s}{\partial t} \right|_\Gamma
\]

- System is truncated by relating higher moment variables to the lower ones.
- The fluids are coupled to each other and to the electromagnetic fields through Maxwell’s equations and interaction source terms.
**Advantages of the Model**

| Kinetic LTE, velocity moments | MFPM | $\varepsilon_0 \to 0$, $m_e \to 0$ | MHD |

**Ideal MHD Model is Valid When:**
- High collisionality, $\tau_{ii}/\tau \ll 1$
- Small Larmor radius, $r_{Li}/L \ll 1$
- Low Resistivity, \( \left( \frac{m_e}{m_i} \right)^{1/2} \left( \frac{r_{Li}}{L} \right)^2 \frac{\tau}{\tau_{ii}} \ll 1 \)

**Multi-Fluid Plasma Model**
- Less computationally expensive than kinetic models
- Multi-fluid effects become relevant at small spacial and temporal scales
- Finite electron mass and speed-of-light effects are included
- There is charge separation is modeled
- Displacement current effects are resolved in the MFPM
1 The Multi-fluid Plasma Model

2 Blended Finite Element Method
   - Blended Finite Element Method
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3 Results
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   - Electromagnetic Plasma Shock Problem

4 Application
The MFPM has dispersive sources.

\[ \frac{\partial Q}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = S \]

- The source Jacobian \( \frac{\partial S}{\partial Q} \) has imaginary eigenvalues.
- The equation system has dispersive sources.
- The dispersion is physical (may be difficult to distinguish from numerical dispersion).
- This dispersion is due to plasma waves that result from ion and electron plasma interactions with electromagnetic fields.
- An ideal numerical method for the MFPM should:
  - be high-order accurate
  - capture shocks
  - couple the flux and the sources
  - not impose strict time-step

Srinivasan et al, CCP 10 (2011)
Solution to the electron and EM fields is smooth and does not shock

- Continuous Galerkin
  - Electron fluid and EM fields
    \[ Q = \sum_{i} q_i v_i \]

- Discontinuous Galerkin
  - Multiple ion and neutral fluids
    \[ Q = \sum_{i} c_i v_i \]
**Implicit Continuous Galerkin**

- For this implementation the balance law form is cast as

\[
\frac{\partial Q}{\partial t} + \frac{\partial \mathbf{F}}{\partial Q} \cdot \frac{\partial Q}{\partial x} = \mathbf{S} + \kappa \nabla^2 Q_d
\]

- Lagrange polynomials are used for basis functions, \( v_r \)

\[
\int_{\Omega} v_r \frac{\partial Q}{\partial t} dV = \mathcal{L}_r(Q) = \int_{\Omega} v_r S dV - \int_{\Omega} v_r \frac{\partial \mathbf{F}}{\partial Q} \cdot \frac{\partial Q}{\partial x} dV + \kappa \int_{\Omega} v_r \nabla^2 Q_d dV
\]

- \( \theta \)-method time integration

\[
\mathcal{R}(Q^n) = \mathbf{M} \frac{Q^{n+1} - Q^n}{dt} - \theta \mathcal{L}_r(Q^{n+1}) - (1 - \theta) \mathcal{L}_r(Q^n) = 0
\]

- \( \theta = 0.5 \) is used for 2\(^{nd}\) order accuracy

\[
\llbracket J \rrbracket (Q^n) = \frac{\partial \mathcal{R}(Q^n)}{\partial Q^n}, \quad \llbracket J \rrbracket (Q^n) \Delta Q = -\mathcal{R}(Q^n), \quad Q^{n+1} = Q^n + \Delta Q
\]

Runge-Kutta Discontinuous Galerkin

\[
\begin{split}
\frac{\partial Q}{\partial t} + \frac{\partial \vec{F}}{\partial \vec{x}} = \vec{S}
\end{split}
\]

- Legendre polynomials are used for basis functions, \( \nu_p \)
- The hyperbolic equation is multiplied by the basis function,
  \[
  \int_{\Omega} v_p \frac{\partial Q}{\partial t} dV = \mathcal{L}_p(Q) = \int_{\Omega} v_p S dV - \oint_{\partial \Omega} v_p \vec{F} \cdot d\vec{A} + \int_{\Omega} \vec{F} \cdot \nabla v_p dV
  \]
- Explicit Runge-Kutta time integration
- \( CFL = c \Delta t / \Delta x \leq 1 / (2p - 1) \), \( p \) is the polynomial order

\[
Q^* = Q^n + \Delta t \mathcal{L}_p(Q^n),
\]

\[
Q^{n+1} = \frac{1}{2} Q^* + \frac{1}{2} Q^n + \frac{1}{2} \Delta t \mathcal{L}_p(Q^*).
\]

Loverich et al, CCP 9 (2006)
CONVERGENCE OF THE BFEM.

\[ \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} = 0, \quad Q(x, 0) = e^{-10(x-8)^2}, \quad ||\Delta Q||_2 = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\hat{Q} - Q_i)^2} \]

Spatial Convergence

Simulations at fixed time-step

Temporal Convergence

Simulations at fixed CFL=1
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4 APPLICATION
1D soliton is a two-fluid plasma problem

- The solution is smooth, therefore artificial dissipation can be small
- The simulation uses 512 second-order elements

- $B_z = 1.0, T_e = T_i = 0.01,$
- $u_i = u_e = 0$
- $n_e = n_i = 1.0 + e^{-10(x-6)^2}$

Baboolal, Math. and Comp. Sim. 55 (2001)
\[
\frac{m_i}{m_e} = 1836, \quad \frac{c}{c_{si}} = 1000\sqrt{2}, \text{ FV 5000 cells}
\]

- DG Solution is very dispersive
- BFEM is less dissipative than the converged solution

Hakim et al, JCP 219 (2006)
BFEM Computational Cost Savings

<table>
<thead>
<tr>
<th>case</th>
<th>$m_i/m_e$</th>
<th>$c/c_{si}$</th>
<th>DG time(s)</th>
<th>BFEM time (s)</th>
<th>BFEM cost over DG</th>
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<tr>
<td>2</td>
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<td>37.7</td>
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<tr>
<td>3</td>
<td>500</td>
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<td>37.7</td>
<td>+452.8%</td>
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<tr>
<td>4</td>
<td>1000</td>
<td>$10/\sqrt{2}$</td>
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<td>38.2</td>
<td>+208.1%</td>
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<tr>
<td>5</td>
<td>1836</td>
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<td>40.4</td>
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<tr>
<td>8</td>
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<td>5274</td>
<td>2735</td>
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- Per time-step explicit DG is faster than BFEM, but it requires many more time-steps
- BFEM is more efficient only when time-step are considerably larger than explicit DG
The BFEM seams to be less accurate than the DG implementation (∼ 50\%)

When the mass ratio is one, the two methods have the same level of accuracy

The discrepancy is due to the fact that the semi-implicit BFEM does not resolve the plasma frequency in this problem

\[ \frac{m_i}{m_e} = 1836 \]

\[ \frac{m_i}{m_e} = 1 \]
**Electromagnetic Plasma Shock Problem**

- Fast rarefaction wave (FR), a slow compound wave (SC), a contact discontinuity (CD), a slow shock (SS), and another fast rarefaction wave (FR)
- The problem exhibits limits of MHD and multi-fluid behavior by changing the Larmor radius, \( r_L \)
  - MHD: \( r_L \rightarrow 0 \)
  - Multi-fluid: \( r_L \sim L \)

Brio and Wu, JCP 75 (1988)
Shock in Density but Smooth Fields.

- $t=0.05/\omega_{ci}$, $c/c_{si}=110$, $m_i/m_e=1836$

- The main features of the problem are captured by all three methods.
- BFEM does not properly resolve the fast electromagnetic waves which require accurately resolving the electron dynamics.

**Diagrams:**
- Mass density
- Magnetic field (y-comp.)

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**Sousa (ERC/AFRL)**

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Maximum BFEM time-step

\[ \Delta t_{max} = \min \left( \frac{\Delta x}{c_{se}}, \frac{\Delta x}{c_{si}}, \frac{\Delta x}{c}, \frac{0.1}{\omega_{ce}}, \frac{0.1}{\omega_{ci}}, \frac{0.1}{\omega_{pe}}, \frac{0.1}{\omega_{pi}} \right) \]

- \( \Delta t_{max} \) corresponds to the maximum value allowed for explicit methods based on the CFL condition.

- \( \Delta t = 42.9 \Delta t_{max} \) is the maximum time step allowed by the BFEM due to ion dynamics.
Varying the artificial dissipation on the electron fluid, $\kappa_e$

Wave-like behavior of the problem is better resolved

Amplitude of the compound wave increases

Right fast rarefaction wave is not visible
Varying the artificial dissipation on the EM-field, $\kappa_{EM}$

There is better agreement with the DG solution

This reinforces the point that the wave-like behavior arises from the interaction of the electron fluid with the electromagnetic fields
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4 Application
- Deuterium and tritium are heated and compressed to fusion conditions
- The compression is laser-driven
- Deuterium can accelerate faster than the tritium
- Low neutron yield measurements point towards separation
- The phenomenon is not captured by single-fluid plasma models
The ion species separation in both cases is the same although the solution behind the shock fronts differ.

Bellei et al., PoP 20 (2013) SOUSA (ERC/AFRL)
Cost vs. Accuracy

- The solution error increases as the computational time decreases.
The blended finite element method (BFEM) is presented
- DG spatial discretization with explicit Runge-Kutta time integration for ions and neutrals
- CG spatial discretization with implicit Crank-Nicolson time integration for the electrons and EM fields
- DG captures shocks and discontinuities
- CG is efficient and robust for smooth solutions

Physics-based decomposition of the algorithm yields numerical solutions that resolve the desired timescales
- DG method takes less computational time to advance the solution by one time-step, however $\Delta t$ is much smaller than that of the BFEM
- Computational cost savings using the BFEM will only occur for relatively large implicit time-steps compared to explicit time-steps

Thank You.