Degrees of Freedom for Allan Deviation Estimates of Multiple Clocks

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Abstract—The three-cornered-hat (TCH) technique is used frequently in the timing community to estimate the stability of single frequency standards (clocks) when the only measurement data available are paired differences. In previous work, we presented an analytic method to estimate the reduced degrees of freedom produced when using the TCH estimation technique. In this paper, we extend that method to estimate the reduced degrees of freedom related to the more general case of the $M$-cornered-hat.

Index Terms—Allan deviation, degrees of freedom, Grubbs’ estimate, $M$-cornered-hat (MCH), three-cornered-hat (TCH).

I. INTRODUCTION

Evaluating the performance of a single frequency standard is very important in the production, analysis, and design of operational precise timing systems. Evaluating the stability of a single clock is complicated by the fact that many times a reference with an order of magnitude or better performance than the device under test does not exist. This means that the only available measurements may likely be paired differences against oscillators that have similar or nearly similar properties. The Allan deviation is the most widely used statistic to estimate the frequency stability of an oscillator. The three-cornered-hat (TCH) and $M$-cornered-hat (MCH) techniques used to estimate the performance of single oscillator from paired difference data.

In this paper, we propose a technique to estimate the reduced degrees of freedom that result from using MCH techniques as compared measurements against a perfect reference. Knowledge of the degrees of freedom then allows for the assignment of approximate confidence intervals to the MCH results using standard chi-squared procedures. This work is an extension of our previous work that addressed confidence intervals for TCHs [1].

The frequency standards discussed and simulated in this paper are assumed to be uncorrelated and that deterministic frequency characteristics, e.g., drift, have been removed from the data. We will be using the terms frequency standard, clock, and oscillator interchangeably throughout this paper.

II. ALLAN VARIANCE

The Allan variance is a widely used statistic in the timing community to evaluate the performance of frequency standards [2]. The Allan variance compares neighboring frequency averages and is defined as

$$\sigma_y^2(\tau) = \frac{1}{2} \left\langle (\hat{y}_{t+\tau} - \hat{y}_t)^2 \right\rangle$$

(1)

where $\hat{y}_t$ is the average frequency over an interval $\tau$ at time $t$ and the angle brackets correspond to the expectation value. Throughout this paper, we will now drop the subscript $y$ from the Allan deviation. Allan deviation will be represented by $\sigma$ and standard deviation will be represented by $\delta$.

In practice, when the Allan deviation of a clock is discussed or specified the underlying assumption is that characterization is of the “bare” clock against a mythical perfect reference.

III. DEGREES OF FREEDOM

The frequency stability of an oscillator can be estimated from phase or frequency data using (1). An important next step is to evaluate confidence levels for these stability estimates. Howe et al. [3] arrived at empirical estimates for the number of degrees of freedom associated with the Allan deviation of standard noise types. Once the number of degrees of freedom is known, an approximate confidence interval can be assigned by referring to the integrated chi-squared distribution for the number of degrees of freedom.

IV. GRUBBS’ ESTIMATE

The TCH technique is widely used in the timing community to estimate the frequency stability of each individual clock utilizing measurements between three clocks. It turns out that Grubbs worked on the paired difference problem in the 1940s [4]. It is interesting that Grubbs points out that some of the initial investigations of this technique were carried out in conjunction with electric clock timing measurements and also velocity measurements utilizing a chronograph.

The Grubbs’ estimate of the deviation for a single noise source obtained from paired difference measurements is given by

$$\hat{\sigma}_i^2 = \frac{1}{2} \left( \hat{\sigma}_i^2 + \hat{\sigma}_j^2 - \hat{\sigma}_{jk}^2 \right),$$

where $\hat{\sigma}_i^2 = \sigma_i^2, i, j,k = 1,2,3, k \neq j \neq i$. (2)
The hat (’) over the parameter denotes an estimated value. The single noise sources are assumed to be independent like in the TCH estimates. The associated variance of the variance is
\[
\text{var}(\hat{\delta}_i^2) = \left(\frac{2\delta_i^4 + \delta_i^2\delta_j^2 + \delta_i^2\delta_k^2 + \delta_j^2\delta_k^2}{\text{var}(\delta_i^2)}\right).
\]

The associated variance of the variance for \(M\) noise sources is then
\[
\text{var}(\hat{\delta}_i^2) = \left(2\delta_i^4 + \frac{4}{(M-1)^2} \sum_{k=1, k \neq i}^M \delta_i^2\delta_k^2 + \sum_{k=1, k \neq i}^M \sum_{j>k}^M \frac{\delta_i^2\delta_j^2}{(M-2)^2}\right).
\]

As in our previous work, we propose substituting the Allan deviation for the standard deviation used in the Grubbs’ estimate. The fractional loss in degrees of freedom using MCH compared to measurements against a perfect oscillator is then given by
\[
\Gamma_i = 2\sigma_i^4 \left[2\sigma_i^4 + \frac{4}{(M-1)^2} \sum_{k=1, k \neq i}^M \sigma_i^2\sigma_k^2 + \sum_{k=1, k \neq i}^M \sum_{j>k}^M \frac{\sigma_i^2\sigma_j^2}{(M-2)^2}\right]^{-1}.
\]

It should be noted that for \(M = 3\), (10) is identical to the TCH solution (7).

The special case of all \(M\) clocks having equivalent stability characteristics results in
\[
\Gamma = \left[1 + \frac{2}{M-1} + \frac{1}{(M-1)(M-2)}\right]^{-1}, \quad \sigma_1, 2, ..., M = \sigma.
\]

Another interesting example exists when the stability of one oscillator is a factor of \(\beta\) different from the other oscillators giving
\[
\Gamma_1 = \left[1 + \frac{2\beta^2}{M-1} + \frac{\beta^4}{(M-1)(M-2)}\right]^{-1}, \quad \sigma_1 = \sigma, \quad \sigma_2, ..., M = \beta\sigma.
\]

V. Simulations

In order to validate our use of the Allan deviation in Grubbs’ formulation, we simulated clock data for the standard noise types and compared the results with the predicted values.

A. Description

For each experiment, the data length for each synthesized clock was 1025 points with arbitrary constant sampling period. The Allan variance was then computed on 2000 separate instances of the simulated clock data with the characteristics under investigation. The standard deviation of the Allan variance estimates was then calculated for use in the estimate of the degrees of freedom. As expected, these results were consistent with the results of Howe et al.

The simulations were performed with white phase, white frequency, flicker frequency, and random walk frequency noise types. While all noise types produced results that validated
our hypotheses, we will present details for the white frequency noise simulations for the balance of this paper.

The individual simulated clock data were then differenced from other clocks to obtain the paired difference data. The MCH technique was then utilized to estimate the Allan variance for each individual member clock. Random processes along with poor confidence intervals associated with low degrees of freedom can result in negative variance estimates. In practice, limited amounts of data or unaccounted for correlations can cause negative variances [8]. In our analysis, negative variances are set to zero.

### B. Results for M Clocks With Equivalent Stability

Each simulation consisted of creating $M$ clocks with performance consistent with a given standard noise type (white phase, white frequency, flicker frequency, and random walk frequency modulation). All of the figures in this paper show results using simulated white frequency clock data. Fig. 1 shows the empirical estimates of the degrees of freedom resulting from the MCH simulations. Fig. 2 shows the ratio of the estimated number of degrees of freedom calculated using MCH compared to “bare” clocks, giving the respective estimates for $\Gamma$.

Fig. 3 plots the prediction of $\Gamma$ given in (11) along with the results of the simulations. The data represented by the open circles were obtained by taking the average of each respective MCH $\Gamma$ estimate (see Fig. 2) over an interval with relatively constant values corresponding to higher degrees of freedom.

### C. Results for M Clocks With Nonequivalent Stability

We also look to verify the expression given in (12) where a group of clocks are a factor of $\beta$ less stable than a single clock under test. Fig. 4 plots the prediction of $\Gamma$ for the single clock
and $\beta = 2$ with varying number of total clocks used in MCH calculation. Again the data represented by the open circles were obtained by taking the average of respective MCH $\Gamma$ estimates over an interval with higher degrees of freedom.

Fig. 5 is a mesh plot showing the dependence of the three variables in (12). Note as the stability ratio $\beta$ becomes greater than 2, $\Gamma$ remains small even with a fairly large number of clocks.

D. Negative Variance Estimates

As discussed earlier, it is possible to obtain nonphysical negative variance estimates from MCH calculations. This is most evident when the number of degrees of freedom gets small. The TCH technique shows the greatest reduction in the effective degrees of freedom. Therefore, we look at how the number of degrees of freedom affects the possibility of producing negative variance in TCH calculations (see Fig. 6). Based on the inspection in Fig. 6, we recommend keeping the degrees of freedom greater than 10 to reduce the possibility of negative variance estimates.

VI. Conclusion

We have provided analytic equations that can be utilized to estimate error bars for MCH Allan deviation estimates. The proposed equations have been validated by simulation for white phase, white frequency, flicker frequency, and random walk frequency modulation noise types.

REFERENCES


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