Experimental observation of dark solitons on water surface

A. Chabchoub1,∗, O. Kimmoun2, H. Branger3, N. Hoffmann1, D. Proment4, M. Onorato4,5, and N. Akhmediev6

1 Mechanics and Ocean Engineering, Hamburg University of Technology, 21079 Hamburg, Germany
2 École Centrale Marseille, 13013 Marseille, France
3 IRPHE, UMR 7342, CNRS, AMU Aix Marseille Université, 13013 Marseille, France
4 Dipartimento di Fisica, Università degli Studi di Torino, Torino 10125, Italy
5 Istituto Nazionale di Fisica Nucleare, INFN, Sezione di Torino, 10125 Torino, Italy
6 Optical Sciences Group, Research School of Physics and Engineering, Institute of Advanced Studies, The Australian National University, Canberra ACT 0200, Australia

We present first ever observation of dark solitons on the water surface. It takes the form of an amplitude drop of the carrier wave which does not change shape in propagation. The shape and width of the soliton depend on the water depth, carrier frequency and the amplitude of the background wave. The experimental data taken in a water tank show an excellent agreement with the theory. These results may improve our understanding of the nonlinear dynamics of water waves in finite depth.

There is a deep analogy between waves in optics and on the surface of water. Developing this analogy allows us to conduct research in one area and expand the ideas to another one. Such expansion has been particularly fruitful in the studies of rogue waves which first appeared from the seafarers gossips before finding solid grounds as objects of research in oceanography [1], later in optics [2] and now the new concept is widely used in many other fields of physics [3]. Unifying ideas [3] help to establish common grounds in this exciting area of research.

There is one particular type of nonlinear waves previously studied in optics and plasma physics which until now has not been observed in the case of water waves. As a result, we cannot estimate the importance of these waves in natural phenomena although they can surely be present among the variety of ocean waves destructively acting along the shores: tsunamis, seiches, bores, tidal waves etc. This special wave is commonly known as dark soliton. In optics, this wave can be described as a hole on a continuous wave background or on a constant amplitude plane wave. In case of water waves, the physics is similar but its observation requires special arrangements. Dark soliton can be classified as one of the fundamental waves in nonlinear dynamics in the sense that arbitrary wave configuration can be seen as nonlinear superposition of fundamental modes [4–7]. Clearly, studies in this area of research are important and must be started.

Generally speaking, dark solitons are localised reductions of the amplitude of the envelope field in nonlinear dispersive media [8]. There are a number of equations that admit dark soliton solution provided the dispersion and nonlinearity are related in specific way. In particular, the governing equation describing the dynamics of weakly nonlinear and quasi-monochromatic waves propagating on the surface of water with arbitrary depth is the nonlinear Schrödinger equation (NLS). Depending on the relative depth h of the water with respect to the wavenumber of the carrier wave κ, the water waves can be described by the NLS either of focussing or defocussing type. In deep-water and more precisely for κh > 1.363, the waves are governed by the NLS of focussing type which admits a family of stationary bright soliton solution and breathers. These waves have been investigated experimentally in [9, 10] and more recently, in [11]. For κh < 1.363, the sign of dispersion changes and wave propagation is described by the NLS of defocussing type which admits dark soliton solutions; they appear as envelope holes [12]. Here, we have to mention that dark solitons may also appear on waters of infinite depth, where the envelope is propagating in two spatial directions [13, 14]. Up to date, dark solitons have been observed only in fibre optics [15–17], in plasma [18, 19], in waveguide arrays [20] and Bose-Einstein condensates [21]. In the present work we report first observation of dark solitons generated in a water wave tank. We also discuss the shape and width of these localized structures which depend on the steepness parameter of the background, its frequency as well as on the relative water depth.

The NLS describes the space-time evolution of weakly nonlinear wave processes in various dispersive media [22–24]. In the case of water waves, it can be derived by applying the method of multiple scales expansion [25, 26]. For arbitrary depth, the equation can be written in the form:

\[ -i \left( \frac{\partial A}{\partial t} + c_g \frac{\partial A}{\partial x} \right) + \alpha \frac{\partial^2 A}{\partial x^2} + \beta |A|^2 A = 0, \]  

where:

\[ \alpha = -\frac{1}{2} \frac{\partial^2 \omega}{\partial k^2} \]

is the lowest order dispersion while

\[ \beta = \frac{\omega k^2}{16 \sinh^2 (\kappa h)} \left( \cosh (4\kappa h) + 8 - \tanh^2 (\kappa h) \right) - \frac{\omega}{2 \sinh^2 (2\kappa h)} \left( 2\omega \cosh^2 (\kappa h) + k c_g \right)^2 g h - c_g^2. \]
is the nonlinear coefficient expressed in terms of the depth $h$, frequency $\omega$ and wavenumber $k$ of the carrier wave and the group velocity $c_g = \frac{\partial \omega}{\partial k}$. Independent variables $x$ and $t$ are the space and time coordinates. The dispersion relation of the wave trains on the water surface with finite depth $h$ is

$$\omega = \sqrt{gk \tanh (kh)},$$

where $g$ denotes the gravitational acceleration.

The water surface elevation $\eta(x,t)$ is related to the amplitude $A(x,t)$ in the first order in steepness according to:

$$\eta(x,t) = \text{Re} \left( A(x,t) \exp \left[ i (kx - \omega t) \right] \right).$$

(2)

In the limit of infinite water depth, that is for the limiting case $kh \to \infty$, the expressions for $\alpha$ and $\beta$ can be simplified [24]:

$$\alpha = \frac{\omega}{8k^2}, \quad \beta = \frac{\omega k^2}{2}.$$  

For the arbitrary depth case, if $kh > 1.363$, then $\alpha \beta > 0$. In this case the plane wave solution may be unstable to long wave perturbations [27, 28]. This instability is usually referred to as the Benjamin-Feir-instability [24, 29, 30]. Exact breathing solutions describing this instability have been recently experimentally investigated in [11, 31, 32]. Such solutions may also appear naturally from random phase initial conditions provided that the wave spectrum is sufficiently energetic and narrow-banded [33, 34]. However, for $kh < 1.363$, the nonlinear coefficient $\beta$ becomes negative and the finite amplitude wave trains in this case are stable. In this work, we conducted experiments to deal with this case.

A scaled form of the NLS in finite depth for $kh < 1.363$ is the well-known defocusing NLS:

$$iq_T + q_{XX} - 2|q|^2q = 0,$$

(3)

which is obtained from (1) by introducing the scaled variables [1]:

$$X = x - c_g t, \quad T = \alpha t, \quad q = \sqrt{-\frac{\beta}{2\alpha}} A.$$  

(4)

Here, $X$ is the co-ordinate in a frame moving with the group velocity and $T$ is the scaled time. For a given carrier amplitude $a$, the defocusing NLS admits a one-parameter-family of localised soliton solutions, generally known as grey solitons [12, 35]. They are described by:

$$q_G = a \frac{\exp (im) + \exp (2aX \sin m)}{1 + \exp (2aX \sin m)} \exp (-2ia^2T).$$

(5)

where $m$ is the parameter of the family that controls the minimal amplitude at the centre of the soliton. For $m = \frac{\pi}{2}$, this minimal wave amplitude drops to zero. This limiting case is given by the simpler expression

$$q_D = a \tanh (aX) \exp (-2ia^2T).$$

(6)

It is called black soliton and it is illustrated in Fig. 1 with the value of $a = 1$.

![FIG. 2. (Color online) Wave channel used in the experiments.](image-url)

The experiments have been conducted in the wave tank of the Ecole Centrale Marseille/IRPHE. The tank is shown in Fig. 2. It is 17 m long and 0.65 m wide. A single flap-type wavemaker is installed at the far end of the tank. An efficient absorbing beach, made with submerged porous plate is installed at the other end. It is clearly visible at the right-hand side of Fig. 2 inside the water with fluorescent dye. The beginning of the beach is located at the distance of 13 m from the wavemaker. The vertical walls are made of transparent sections of glass supported by the metal frame. The water level of the free surface is measured with seven resistive wave gauges.
with a sampling frequency of 200 Hz. The location of the
gauges is given in the table I.

In order to generate dark solitons, we have to control
the flap displacement through the computerized equip-
ment and create initial conditions in dimensional units.
This means that Eqs.(6) and (2) have to be dimension-
alized with the use of inverted relations (4). Our ex-
periments have been conducted for two different water
depth values, \( h = 0.40 \) m and \( h = 0.25 \) m. In each case,
the condition of applicability of the defocusing NLS, i.e.
\( kh < 1.363 \), has been satisfied.

<table>
<thead>
<tr>
<th>The gauge number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Its position along the tank (m)</td>
<td>1.06</td>
<td>4.33</td>
<td>5.41</td>
<td>7.00</td>
<td>8.86</td>
<td>9.76</td>
<td>12.80</td>
</tr>
</tbody>
</table>

TABLE I. Wave gauge positions

Fig. 3 shows the evolution of a black soliton for the
carrier amplitude \( a = 0.04 \) m and the wavenumber \( k = 3 \)
m\(^{-1}\) while the water depth is of \( h = 0.40 \) m. Each time-
series has been shifted in time to position the zero of
the wave at the same location. As a result, all dia-
grams are aligned in time for convenience of compari-
son of their profiles. Moreover, all diagrams are aligned
in time by the theoretical value of the group velocity,
which is \( c_g = 1.18 \) m·s\(^{-1}\). Clearly, we observe that
the soliton does not change shape and propagates with
the corresponding group velocity in accordance with the-
ory since the stationary localizations are almost perfectly
aligned in all stages of propagation. In each panel, the
experimental curve is supplemented with the envelope
calculated using the first-order Fourier analysis. The
envelopes are consistent with the theoretical shape of the
dark soliton shown in Fig.1. Another interesting feature
of our data is that the difference between the group ve-
locity and the phase velocity leads to a continuous shift
of the dark soliton relative to the wave pattern of the car-
rier. Thus, the phase of the carrier in each of the seven
panels is also shifted relative to the previous one.

These data prove that we indeed observed dark soli-
tons. Fig. 4 shows similar set of experimental data for
the water depth \( h = 0.25 \) m. Here, the amplitude of
the carrier is \( a = 0.02 \) m while the wavenumber \( k = 4 \)
m\(^{-1}\), thus, the group velocity is \( c_g = 1.06 \) m·s\(^{-1}\). The
alignment of the bump, which is located at zero ampli-
tude level in both experiments, show that the theoretical
value of the group velocity is again in accordance with
the theoretical value and this is another proof that we
are dealing with the dark soliton although with the pa-
rameters of the experiment different from the previous
case.

A video showing the dynamics of surface elevation in
the flume can be found in the supplemental material. The
video demonstrates clearly the decrease of the amplitude
near zero point of the dark soliton profile.

Further verification that it is the dark soliton excited
on the surface of water can be obtained from confirm-
ing its effective width. In order to do that we calculated
the number of carrier waves within the soliton i.e. the
number of waves with modulated amplitude versus the
steepness of the background wave. We estimated this
dependance from our experimental data by defining the

![FIG. 3.](image-url) Evolution of the dark soliton along the
tank with the water depth \( h = 0.4 \) m. The carrier-amplitude
is \( a = 0.04 \) m, while \( kh = 1.2 \). Seven panels from top to
bottom correspond to experimental records of seven gauges
from 1 to 7 respectively shifted in time to keep zero amplitude
at the same position. The envelope over the experimental
curves computed using the first-order Fourier analysis is in
good agreement with theoretical dark soliton shape.

![FIG. 4.](image-url) Evolution of the dark soliton along the
tank with the water depth \( h = 0.25 \) m. The carrier-amplitude
is \( a = 0.02 \) m, while \( kh = 1.0 \). Seven panels from top to
bottom correspond to experimental records of seven gauges
from 1 to 7 respectively shifted in time to keep zero amplitude
at the same position. The envelope over the experimental
curves computed using the first-order Fourier analysis is in
good agreement with theoretical dark soliton shape.

Further verification that it is the dark soliton excited
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steepness of the background wave. We estimated this
dependance from our experimental data by defining the
modulated waves as those whose amplitude is less than 0.9 of the carrier amplitude. Then, we calculated the number of modulated waves defined this way within the soliton. Fractional values can be obtained if we best fit the envelope through the wave maxima. Fig. 5 shows these data for the first and the last gauges for the two \( h \)-values as a function of the steepness \( ak \). The data for all other gauges are very similar to these. Theoretical curves shown by the solid lines demonstrate that the number of modulated waves is inversely proportional to the steepness of the background. Comparison of experimental data with the theoretical curves proves once again that we do observe dark solitons.

The number of modulated waves within the dark soliton depends also on the wavenumber \( k \) for fixed \( h \). We calculated the number of modulated waves the same way as described above for several values of \( k \). Fig. 6 shows these data for the two values of the depth \( h \) along with the theoretical curves. The plot shows that our observations fit well the theoretical relationship between the number of modulated waves and the combined parameter \( kh \).

To conclude, our experimental study proves the existence of dark solitons in water waves. Our observations of these localized structures are in agreement with the theoretical prediction: the solitons preserve fixed shape during their evolution in the tank. Furthermore, the solitons propagate exactly with the group velocity for the corresponding wavenumber, wave frequency and water depth calculated theoretically. Generally, these results confirm that the NLS equation provides a good description of surface gravity waves even in the defocusing case. Thus, water waves have to be described in the frame of nonlinear dynamics rather than just a linear superposition of modes.

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FIG. 5. (Color online) The number of modulated waves within the dark soliton versus steepness \( ak \). The former is defined with the threshold amplitude of 0.9\( a \) for the corresponding depth. The triangles correspond the values obtained from the first gauge. The stars correspond the values obtained from the last gauge. Solid lines are obtained from theoretical calculations.

FIG. 6. (Color online) The number of modulated waves within the soliton versus \( kh \). The former is defined for the threshold amplitude 0.9\( a \) for the corresponding depth. The triangles correspond the values obtained from the first gauge. The stars correspond the values obtained from the last gauge. Solid lines are obtained from theoretical calculations.
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