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The monotone method extended to such systems is called the generalized monotone method. Here the generalized monotone method has been extended to the Caputo fractional differential equation of order q (where $0<q<1$) with an initial condition as well as the existence of coupled minimal and maximal solutions for such an equation and a numerical example is provided as an application of the theoretical results.

The linear sub-hyperbolic fractional partial differential equation in one dimensional space, that is the αth time order

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Final Report: Fractional Differential and Integral Inequalities with Applications

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The linear sub hyperbolic fractional partial differential equation in one dimensional space, that is the qth time order derivative is such that 1 < q < 2, and the linear super hyperbolic fractional partial differential equation in one dimensional space, that is the qth time order derivative is such that 2 < q < 3 are considered . In the special case when q = 1, it is the parabolic equation and when q = 2, it is the hyperbolic equation. The eigenfunction expansion method is used to obtain representation forms for the solution of the linear sub and the linear super hyperbolic fractional partial differential equations in terms of the nonhomogeneous terms, initial and boundary conditions. See [9,10, 11, 12, 14] and the references therein for qualitative study, applications and numerical study of fractional differential equations. The representation forms obtained are useful in computing numerical solutions to the sub and super hyperbolic partial differential equations.

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(a) Papers published in peer-reviewed journals (N/A for none)

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"A Representation Formula for the One Dimensional Caputo Fractional Reaction Diffusion Equation and a Numerical Example Using the Generalized Monotone Method." by Donna Stutson, Xavier University of Louisiana and Aghalaya S. Vatsala, ULL,(1083-35-163)at The Fall Southeastern section Meeting at Tulane University, New Orleans, LA, October 13-14, 2012, meeting # 1083.

"Numerical Methods for Fractional Differential Equations Via Generalized Monotone Method." by Aghalaya S. Vatsala, ULL,Sowmya Muniswamy, ULL and Donna Stutson, Xavier University of Louisiana (1086-34-1007) at the Joint Mathematics Meeting , January 9-12, 2013, San Diego Convention Center.

"A Numerical Example for the One Dimensional Caputo Fractional Wave Equation Using the Representative Solution and the Generalized Monotone Method." by Donna Stutson, Xavier University of Louisiana and Aghalaya S. Vatsala, ULL (1086-35-2418) at the Joint Mathematics Meeting , January 9-12, 2013, San Diego Convention Center.

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Superlinear Convergence via Iterative Methods for Scalar Caputo fractional differential equations with applications.
Sowmya Muniswamy*, University of Louisiana at Lafayette
Aghalaya S. Vatsala, University of Louisiana at Lafayette
(1096-34-1660)

Super Hyperbolic Linear Partial Fractional Differential Equations in One Dimensional Space. D S Stutson*, Xavier University of Louisiana
A S Vatsala, University of Louisiana at Lafayette
(1096-35-2439)

Sub and Super Hyperbolic Linear Partial Fractional Differential Equations with Numerical Results.
Aghalaya S. Vatsala*, University of Louisiana at Lafayette
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02/14/2016	10.00	Aghalaya S. Vatsala, Bhuvaneswari Sambandham. Laplace Transform Method for Sequential CaputoFractional Differential Equations / MESA, Nonlinear Studies (08 2015)
08/10/2012	2.00	Donna Stutson, A. S. Vatsala. GENERALIZED MONOTONE METHOD FORFRACTIONAL REACTION DIFFUSION EQUATIONS, Communications in Applied Analysis (09 2011)
08/10/2012	3.00	A.S. VATSALA, TH.T PHAM, J.D. RAM!!IREZ. GENERALIZED MONOTONE METHOD FOR CAPUTOFRACTIONAL DIFFERENTIAL EQUATION WITHAPPLICATIONS TO POPULATION MODELS, Neural, Parallel & Scientific Computations (04 2012)
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Fractional Differential and Integral Inequalities with Applications

Donna Stutson and A. S. Vatsala

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Introduction

It is well known that fractional differential equations both ordinary and fractional partial differential have played an important role in applications in theoretical modeling of scientific and engineering problems. The qualitative studies such as existence, uniqueness and computation of such dynamical models are useful in applications. Some approximate methods like the radial functions method and other methods are used in [3, 4, 7, 16] to compute exact or approximate solutions.

In solving nonlinear problems, the monotone method combined with the method of upper and lower solutions is a popular choice, because the existence of solutions by the monotone method is both theoretical and computational. The monotone method for various nonlinear problems has been developed in reference [8]. The monotone method (monotone iterative technique) combined with the method of lower and upper solutions yields monotone sequences, which converges to minimal and maximal solutions of the nonlinear differential equation. In many nonlinear problems (nonlinear dynamic systems), the nonlinear term is the sum of increasing and decreasing functions. The monotone method extended to such systems is called the generalized monotone method. Here the generalized monotone method has been extended to the Caputo fractional differential equation of order q (where $0 < q < 1$) with an initial condition as well as the existence of coupled minimal and maximal solutions for such an equation and a numerical example is provided as an application of the theoretical results.

The linear sub hyperbolic fractional partial differential equation in one dimensional space, that is the *qth* time order derivative is such that $1 < q < 2$, and the linear super hyperbolic fractional partial differential equation in one dimensional space, that is the *qth* time order derivative is such that $2 < q < 3$ are considered . In

the special case when $q = 1$, it is the parabolic equation and when $q = 2$, it is the hyperbolic equation. The eigenfunction expansion method is used to obtain representation forms for the solution of the linear sub and the linear super hyperbolic fractional partial differential equations in terms of the nonhomogeneous terms, initial and boundary conditions. See [9,10, 11, 12, 14] and the references therein for qualitative study, applications and numerical study of fractional differential equations. The representation forms obtained are useful in computing numerical solutions to the sub and super hyperbolic partial differential equations.

Preliminaries

In this section, we recall some known definitions and known results, which are useful to develop our main results. Initially, we recall some definitions:

Definition 2.1. *Caputo derivative of order q, when $N - 1 < q < N$ for $t \in [0,T]$, is defined as*

$$ {}^{c}D_{0^+}^{q}u(t) = \frac{1}{\Gamma(N-q)}\left(\int_0^t (t-s)^{N-q-1}u^N(s)ds\right). $$

In the special case when $N = 1$,

Definition 2.2. *Caputo fractional integral of order q, when $0 < q < 1$, is defined as*

$$ {}^{c}D_{0^+}^{q}u(t) = \frac{1}{\Gamma(1-q)}\left(\int_0^t (t-s)^{-q}u'(s)ds\right). $$

Definition 2.3. *Riemann Liouville fractional derivative of order q, when $0 < q < 1$ for $t \in [0,T]$, is defined as*

$$ D^q\, u(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{-q}u(s)ds. $$

Definition 2.4. *Riemann Liouville right-fractional integral of order q, when $0 < q < 1$ for $t \in [0,T]$, is defined as*

$$ {}_{t_0}D^{-q}u(t) = \frac{1}{\Gamma(q)}\frac{d}{dt}\int_0^t (t-s)^{-q}u(s)ds. $$

Definition 2.5. *The two parameter Mittag-Leffler function is defined as*

$$ E_{\alpha,\beta}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty}\frac{(t^{\alpha})^k}{\Gamma(\alpha k+\beta)} $$

and the single parameter Mittag-Leffler function is defined as $E_{\alpha}(\lambda t^{\alpha}) = E_{\alpha,1}(\lambda t^{\alpha})$.

Definition 2.6. *let $0 < q < 1$, $p = 1 - q$ a function $\phi \in C(J,R)$ is a C_p continuous function if $(t-t_0)^{1-q}\phi(t) \in C(J_0,R)$. The set of C_p functions is denoted by $C_p(J,R)$. Furthermore, given a function $\phi(t) \in C(J_0,R)$, the function $(t-t_0)^{1-q}\phi(t)$ is called the continuous extension of $\phi(t)$.*

Consider the linear Caputo fractional differential equation of order q when $N-1 < q < N$ with initial conditions of the form

$$\left({}^{c}D^{q}u\right)(t) - \lambda u(t) = f(t), u^{(k)}(0) = b_k \qquad (2.1)$$

where $b_k \in \mathbb{R}$ for $k = 0,1,2,\ldots N-1$ and $f \in C[J,\mathbb{R}]$ and $J = [0,T]$. Also, $u^{(k)}(0)$ denotes the k_{th} derivative of u with respect to t at $t = 0$. If $f(t)$ in (2.1) is in the space of $C^{k-1}[0,T]$. The problem (2.1) is equivalent to the Volterra integral equation

$$u(t) = \sum_{j=0}^{N-1} \frac{b_j}{j!} t^j + \frac{\lambda}{\Gamma(q)} \int_0^t \frac{u(s)ds}{(t-s)^{1-q}} + \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)ds}{(t-s)^{1-q}}.$$

The solution obtained is in the space of $C^N[0,T]$, see [7] for details.

Consider the equation
(2.5) $({}^{c}D^{q}u)(\mathrm{t}) = \mathrm{f}(t,u(t)) + g(t,u(t)),\ \ u(0) = u_0$, where $0 < q < 1$,
$f(t,u(t)), g(t,u(t)) \in C[J \times R, R]$, $f(t,u)$is increasing in u on J and $g(t,u)$ is decreasing in u on J.

Definition. Let $v_0, w_0 \in C^1[J,R]$. v_0, w_0 are said to be coupled lower and upper solutions of type I of (2.5) if
$({}^{c}D^{q}v)(t) \le f(t,v(t)) + g(t,w(t)),\ \ v(0) \le u_0$, and
$({}^{c}D^{q}w)(t) \ge f(t,w(t)) + g(t,v(t)),\ \ w(0) \ge u_0$.

Generalized Monotone Method for Caputo Fractional Differential Equation with an Initial Condition

The qualitative study of fractional differential equations has gained significant importance due to its applications in various branches of Science, Engineering and medicine, see [1,3,13,14,16,18]. In [14] it has been demonstrated that half order fractional differential equations give considerably better results for certain electro chemical problems than the classical approach. In addition [1,13,18] have studied linear fractional reaction diffusion equations which are applicable in random walks and nanotechnology.

Consider the equation
(3.1) $({}^{c}D^{q}u)(\mathrm{t}) = \mathrm{f}(t,u(t)) + g(t,u(t)),\ \ u(0) = u_0$,
where $0 < q < 1, f(t,u(t)), g(t,u(t)) \in C[J \times R, R]$, $f(t,u)$is increasing in u on J and $g(t,u)$ is decreasing in u on J.

Theorem 3.1 Assume that

I. $v_0, w_0 \in C^1[J,R]$. $v_0, w_0$are coupled lower and upper solutions of type I, with $v_0 \le w_0$on J.

II. $f(t,u(t)), g(t,u(t)) \in C[J \times R, R]]$, where $f(t,u)$is increasing in u on J and $g(t,u)$ is decreasing in u on J.

Then there exist monotone sequences, $v_n(t)$and $w_n(t)$, such that $v_n(t) \to \rho(t)$ and $w_n(t) \to r(t)$ uniformly and monotonically, where $\rho(t)$and $r(t)$ are coupled minimal and maximal solutions of equation (3.1) on J. That is, for any solution, $u(t)$, of (3.1) with $v_0(t) \le u_0(t) \le w_0(t)$on J, we get natural sequences, $\{v_n\}$ and $\{w_n\}$, satisfying the following,
$v_0(t) \le v_1(t) \le v_2(t) \le \cdots \le v_n(t) \le u(t) \le w_n(t) \le \cdots \le w_2(t) \le w_1(t) \le w_0(t)$,

for each $n \leq 1$ on J. Also$\rho(t) \leq u(t) \leq r(t)$ on J.

The next result uses intertwined sequences.

Theorem 3.2. Assume hypotheses, I. and II, of the above theorem hold. Then for any solution, $u(t)$, of equation (3.1) with $v_0(t) \leq u_0(t) \leq w_0(t)$on J, the alternating sequences, $\{v_{2n}, w_{2n+1}\}$ and $\{v_{2n+1}, w_{2n}\}$, satisfying intertwined sequences,

(3.7)$v_0(t) \leq w_1(t) \leq \cdots \leq v_{2n}(t) \leq w_{2n+1}(t) \leq u(t) \leq v_{2n+1}(t) \leq w_{2n}(t) \leq \cdots v_1(t) \leq w_0(t)$, for each $n \geq 1$ on J. This requires using type (iii) iterative schemes,

$({}^cD^q v_{n+1})(t) = f(t, w_n(t)) + g(t, v_n(t)), v_{n+1}(0) = u_0$ and
$({}^cD^q w_{n+1})(t) = f(t, v_n(t)) + g(t, w_n(t)), w_{n+1}(0) = u_0$.

Further, monotone sequences, $\{v_{2n}, w_{2n+1}\}$ and $\{w_{2n}, v_{2n+1}\}$, converge to $\rho(t)$ and $r(t)$, respectively, on J. $\rho(t)$ and $r(t)$ are coupled minimal and maximal solutions of (3.1), respectively. Also, $\rho(t) \leq u(t) \leq r(t)$ on J.

Theorem 3.4. If in addition to hypotheses I and II, suppose $f(t, u)$ satisfies left-hand-side Lipschitz condition, $f(t, u_2) - f(t, u_1) \leq L_1(u_2 - u_1), u_2 \geq u_1$, and g(t, u) satisfies right-hand-side Lipschitz condition, $g(t, u_2) - g(t, u_1) \geq -L_2(u_2 - u_1), u_2 \geq u_1$, where $v_0 \leq u_1 \leq u_2 \leq w_0$ implies that $\rho = r = u$, the unique solution of (3.1).

GENERALIZED MONOTONE METHOD FOR FRACTIONAL REACTION DIFFUSION EQUATIONS

Consider the fractional reaction diffusion equation of the form

$$\begin{gathered} {}^c\partial_t^q u - k\frac{\partial^2 u}{\partial x^2} = f(t,x,u) + g(t,x,u)\,, \quad (t,x) \in Q_T \\ u(t,0) = A(t), \quad u(t,L) = B(t) \quad (t,x) \in \Gamma_T \\ u(0,x) = h(x) \quad x \in \overline{\Omega} = \Omega, \end{gathered} \tag{4.1}$$

where $\Omega = [0, L], J = (0, T], Q_T = J \times \Omega$, $k > 0$ and $\Gamma_T = (0, T) \times \partial\Omega$.
Here ${}^c\partial_t^q u$ is the Caputo Partial Derivative with respect to t of order $q, 0 < q < 1$, which is defined as

$${}^c\partial_t^q u = \frac{1}{\Gamma(1-q)}\left(\int_0^t (t-s)^{-q}\frac{\partial u(s,x)}{\partial s}ds\right).$$

Assume that$f, g \in C^{1,2}[Q_T \times R, R]$, such that $f(t, x, u)$ is non-decreasing in u and $g(t, x, u)$ is non-increasing in u, $A(t), B(t) \in C^1[J, R]$ and $h(x) \in C^{2+\alpha}[\Omega, R]$, where $0 < \alpha < 1$.

In this section we develop the generalized monotone method for (4.1) combined with coupled lower and upper solutions of type I. The generalized monotone method yields monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of (4.1). Furthermore, assuming a uniqueness condition, we prove the existence of the unique solution of (4.1).

Theorem 4.1: Let (v_0, w_0) be coupled lower and upper solutions of (2.1) such that $v_0 \le w_0$ on Q_T. Furthermore, suppose $f(t,x,u)$ is nondecreasing in u on Q_T and $g(t,x,u)$is nonincreasing in Q_T and

$$f(t,x,u) - f(t,x,\bar{u}) \le L_1(u-\bar{u})$$
$$g(t,x,u) - g(t,x,\bar{u}) \ge -L_2(u-\bar{u}),$$

whenever $u \ge \bar{u}$ on Q_T. Then there exists monotone sequences $\{v_n(t,x)\}$ and $\{w_n(t,x)\}$ defined such that $v_n(t,x)$ and $w_n(t,x)$ are solutions to

$$\begin{gathered} {}^c\partial_t^q v_n(t,x) - k\frac{\partial^2 v_n(t,x)}{\partial x^2} = f(t,x,v_{n-1}) + g(t,x,w_{n-1}), \quad (t,x) \in Q_T \\ v_n(t,0) = A(t), \quad v_n(t,L) = B(t) \quad (t,x) \in \Gamma_T \\ v_n(0,x) = h(x) \quad x \in \bar{\Omega} = \Omega. \end{gathered} \tag{4.2}$$

and

$$\begin{gathered} {}^c\partial_t^q w_n(t,x) - k\frac{\partial^2 w_n(t,x)}{\partial x^2} = f(t,x,w_{n-1}) + g(t,x,v_{n-1}), \quad (t,x) \in Q_T \\ w_n(t,0) = A(t), \quad w_n(t,L) = B(t) \quad (t,x) \in \Gamma_T \\ w_n(0,x) = h(x) \quad x \in \bar{\Omega} = \Omega. \end{gathered} \tag{4.3}$$

respectively. Also, these sequences converge uniformly and monotonically to $\rho(t,x)$ and $r(t,x)$ such that $(\rho(t,x), r(t,x))$ are coupled minimal and maximal solutions of the nonlinear initial value problem (4.1), provided $v_0(t,x) \le u(t,x) \le w_0(t,x)$, where $u(t,x)$ is any solution of (2.1). Furthermore, $\rho(t,x) = r(t,x) = u(t,x)$, the unique solution of (2.1), on $\bar{Q}_T$.

A Representative Formula for the Linear Caputo Fractional Wave Equation

It is well known that fractional Brownian motion has been modeled as parabolic stochastic differential equation. Here a representation form for the solution of the deterministic one dimensional fractional wave equation with Caputo fractional derivative of order q, for $1 < q < 2$ has been developed. For $q = 1$ and $q = 2$, it reduces to the one dimensional parabolic equation and one dimensional wave equation, respectively.

Consider the linear Caputo fractional wave differential equation

$$\begin{gathered} {}^c\boldsymbol{\partial_t^q u} - \boldsymbol{c^2}\frac{\boldsymbol{\partial^2 u}}{\boldsymbol{\partial x^2}} = \boldsymbol{R(t,x)}, \quad \boldsymbol{(t,x) \in Q_T} \\ \boldsymbol{u(t,0) = A(t)}, \quad \boldsymbol{u(t,L) = B(t)} \quad \boldsymbol{(t,x) \in \Gamma_T} \\ \boldsymbol{u(0,x) = h(x), u_t(0,x) = g(x)} \quad \boldsymbol{x \in \bar{\Omega}}, \end{gathered} \tag{5.1}$$

Where $\boldsymbol{\Omega = [0,L], J = (0,T], Q_T = J \times \Omega, c^2 > 0}$ and $\boldsymbol{\Gamma_T = (0,T) \times \partial\Omega.}$

In order to obtain a representation formula we need to consider the corresponding homogeneous equation with homogeneous boundary condition. Using separation of variables method we can and the eigenfunctions corresponding to the eigenvalues λ_n. The eigenfunctions and the eigen- values vary depending on the type of

boundary conditions. In order to get a suitable final form of the solution we normalize the eigenfunctions. For the above type of boundary conditions, we get $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and the eigen- functions are $\phi_n(x) = sin\frac{n\pi x}{L}$.

$$u(t,x) = \int_0^L \left[\sum_{n=1}^{\infty} h(x_0)\phi_n(x_0)dx_0 E_{q,1}(-c^2\lambda_n t^q) + \int_0^L \left[\sum_{n=1}^{\infty} g(x_0)\phi_n(x_0)dx_0 t^q E_{q,2}(-c^2\lambda_n t^q)\right.\right.$$
$$+\int_0^t (t-s)^{q-1}E_{qq}(-c^2\lambda(t-s)^q)\int_0^L Q(s,x)\,ds + \frac{c^2 n\pi}{L}\int_0^t (t-s)^{q-1}E_{qq}(-c^2\lambda(t-s)^q)A(s)ds$$
$$\left.\left.+(-1)^{n+1}\frac{c^2 n\pi}{L}\int_0^t (t-s)^{q-1}E_{qq}(-c^2\lambda(t-s)^q)B(s)ds\right]\right].$$

Sub Hyperbolic Linear Partial Fractional Differential Equation in One Dimensional Space with Numerical Results

In this section a representation form for the sub hyperbolic linear fractional partial differential equation in one dimensional space is obtained. It is called sub hyperbolic if the time derivative is of order q when $1 < q < 2$. It is easy to observe that, if $q = 2$, it is a second order linear hyperbolic equation in one dimensional space. Consider the following linear Caputo fractional wave differential equation:

$$\begin{gathered} {}^c\partial_t^q u - c^2\frac{\partial^2 u}{\partial x^2} = F(x,t)\ on\ Q_T, \\ u(0,t) = A(t),\quad u(L,t) = B(t) \quad on\ \Gamma_T, \\ u(0,x) = h_0(x), \qquad u_t(0,x) = h_1(x)\quad on\ \overline{\Omega}\,, \end{gathered} \tag{6.1}$$

where $\Omega = [0,L]$, $J = (0,T]$, $Q_T = J \times \Omega$, $c_2 > 0$ and $\Gamma_T = (0,T) \times \partial\Omega$.

We develop a representation form for (6.1) which is useful in computing the solution of linear Caputo fractional partial differential equations of order $q\ when\ 1 < q < 2$, using the eigenfunction expansion method with eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and eigenfunctions $\phi_n(x) = sin\left(\frac{n\pi x}{L}\right)$. This will be useful in the development of the generalized monotone method to be developed in future work. We also assume that the initial boundary condition satisfy the compatability condition. Hence, we set

$$u(t,x) = \sum_{n=1}^{\infty} b_n(t) sin\frac{n\pi x}{L}. \tag{6.2}$$

We must then find $b_n(t)$ for each $n \geq 1$. Taking the Caputo derivative of each side, Green's formula and the boundary conditions yields the following fractional equation to be solved:

$${}^C D^q b_n(t) + c^2\left(\frac{n\pi}{L}\right)^2 b_n(t) = \frac{2}{L}\left[\frac{n\pi c^2}{L}\{A(t) + (-1)^n B(t)\} + \int_0^L F(t,x) sin\left(\frac{n\pi x}{L}\right)dx\right] \tag{6.3}$$

With initial conditions

$$b_n(0) = \frac{2}{L}\int_0^L h_0(x) sin\left(\frac{n\pi x}{L}\right)dx\,, \qquad b_n{}'(0) = \frac{2}{L}\int_0^L h_1(x) sin\left(\frac{n\pi x}{L}\right)dx.$$

Using Equation (2.1) to solve (6.3) substituting in for $b_n(t)$ in equation (6.2) and interchanging the order of integration and summation, we obtain the representation form:

$u(t,x) =$

$$\int_0^L \left[\sum_{n=1}^{\infty} h(x_0)\varphi_n(x_0)dx_0 E_{q,1}\left(-c^2\lambda_n t^q\right)\right]$$
$$+\int_0^L \left[\sum_{n=1}^{\infty} h_1(x_0)\varphi_n(x_0)\varphi_n(x)dx_0\, tE_{q,2}\left(-c^2\lambda_n t^q\right)\right]$$
$$+\frac{2}{L}\int_0^t\int_0^L \sum_{n=1}^{\infty}(t-s)^{(q-1)}E_{q,q}\left(-c^2\lambda_n(t-s)^q\right)[F(s,x_0)\varphi_n(x_0)\varphi_n(x)dx_0]ds$$
$$+\int_0^t \frac{2c^2n\pi}{L^2}(t-s)^{(q-1)}E_{q,q}(-c^2\lambda_n(t-s)^q)A(s)\varphi_n(x)ds$$
$$+(-1)^{n+1}\int_0^t \frac{2c^2n\pi}{L^2}(t-s)^{(q-1)}E_{q,q}(-c^2\lambda_n(t-s)^q)B(s)\varphi_n(x)ds.$$

Super Hyperbolic Linear Partial Fractional Differential Equation in One Dimensional Space with Numerical Results

Using the method of successive approximation by setting $u_0(t) = \sum_{j=0}^{N-1}\frac{b_j}{j!}t^j$ and taking the limit of $u_m(t)\ as\ \ m \to \infty$, the solution of (2.1) is given by

$$u(t) = \sum_{j=0}^{N-1} b_j t^j E_{q,j+1}(\lambda t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)f(s)ds.$$

Next we consider an eigenvalue problem on one dimensional space. For that purpose consider the eigenvalue problem

$$u'' + \lambda u = 0$$

with mixed homogeneous boundary conditions of the form:

$$B_u(a) = \alpha_i u(i) + (-1)^{(i+1)}\beta_i u'(i) = 0, for\ i = 0, 1.$$

The value of λ for which the nontrivial solution for the boundary value problem exists is called the eigenvalue denoted as λ_n and the corresponding nontrivial solution is called the eigenfunction denoted as $\varphi_n(x)$. The main result is developed using the eigenvalue problem with Direchlet boundary conditions,

$$u(0) = u(L) = 0.$$

In this section a representation form for the super hyperbolic linear fractional partial differential equation in one dimensional space is obtained. It is called super hyperbolic if the time derivative is of order q when $2 < q < 3$. It is easy to observe that, if $q = 2$, it is a second order linear hyperbolic equation in one dimensional space. Consider the following linear Caputo fractional wave differential equation:

$$\begin{gathered} {}^c\partial_t^q u - c^2\frac{\partial^2 u}{\partial x^2} = F(x,t)\ on\ Q_T, \\ u(0,t) = A(t),\quad u(L,t) = B(t) \quad on\ \Gamma_T, \\ u(0,x) = h_0(x), \qquad u_t(0,x) = h_1(x), \qquad u_{tt}(0,x) = h_2(x) \quad on\ \overline{\Omega}, \end{gathered} \tag{7.1}$$

where $\Omega = [0, L]$, $J = (0, T]$, $Q_T = J \times \Omega$, $c_2 > 0$ and $\Gamma_T = (0, T) \times \partial\Omega$. We also assume that $A(t), B(t) \in C^3[J, R]$, and $F(t, x) \in C^{3,2}[Q_T, R]$.

Assume the following compatability conditions relative to the initial and boundary conditions:

$$h_0(0) = A(0) = 0, \qquad h_0(L) = B(0) = 0$$

$$h_1(0) = A'(0) = 0, \qquad h_1(L) = B''(0) = 0$$

$$h_2(0) = A''(0) = 0, \qquad h_2(L) = B''(0) = 0.$$

This is needed when differentiating the integral representation of the terms related to the boundary conditions and when the problem is reduced to homogeneous boundary conditions. As in section 3, we set $u(t,x) = \sum_{n=1}^{\infty} b_n(t) sin \frac{n\pi x}{L}$. As before, the expression for $b_n(t)$ is found for each $n \geq 1$ by using Lagrange's identity along with Green's formula and the boundary conditions and substitute back into the expression for

$u(t,x)$. Changing the order of integration and summation, we obtain the solution :

$u(t,x) =$

$$\int_0^L \left[\sum_{n=1}^{\infty} \sum_{j=0}^{2} h_j(x_0)\, \varphi_n\,(x_0) dx_0 t^j E_{q,j+1}\left(-c^2\, \lambda_n t^q\right) \right]$$

$$+ \frac{2}{L} \int_0^t \int_0^L \sum_{n=1}^{\infty} (t-s)^{(q-1)}\, E_{q,q}\left(-c^2\, \lambda_n (t-s)^q\right) [F(s,x_0)\varphi_n(x_0)\varphi_n(x) dx_0] ds$$

$$+ \int_0^t \sum_{n=1}^{\infty} \frac{2c^2 n\pi}{L^2} (t-s)^{(q-1)} E_{q,q}(-c^2 \lambda_n (t-s)^q) A(s) \varphi_n(x) ds$$

$$+ (-1)^{n+1} \int_0^t \frac{2c^2 n\pi}{L^2} (t-s)^{(q-1)} E_{q,q}(-c^2 \lambda_n (t-s)^q) B(s) \varphi_n(x) ds.$$

This representation formula for the solution of (7.1) provides a methodology for computing the solution for different nonhomogeneous terms and different initial and boundary conditions. The convergence of the infinite series involves the Mittag-Leffler functions is yet to be established even when $\mathbf{0 < q < 1}$. However, if the boundary conditions are homogeneous and the initial and nonhomogeneous terms are chosen such that there are a finite number of Mittag-Leffler functions, then the solution can be numerically computed using MAPLE.

Riemann Liouville and Caputo Fractional Differential and Integral Inequalities

In this section, results for coupled fractional and ordinary integral inequalities where the nonlinear function is the sum of an increasing and a decreasing function have been developed. Also, the corresponding coupled fractional and ordinary differential inequality results have been developed without requiring the increasing or decreasing nature of the nonlinear function. In fact, results have been developed for coupled differential and integral inequalities for both ordinary and fractional equations. These results are useful in the qualitative study of ordinary and fractional dynamic systems of both Riemann Liouville and Caputo forms.

For the sake of simplicity, the obtained results will be on the interval $J = (t_0, T]$. Furthermore, $J_0 = (t_0, T] = \bar{J}$.

Theorem 8.1. *Let $v, w \in C_P[J, R], f, g, \in C[J_0 \times R, R]$ and satisfy the following coupled integral inequalities*
(i) $v(t) \leq \frac{v_0 (t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \left(f(s, v(s)) + g(s, w(s)) \right) ds,$

(ii) $w(t) \geq \frac{w_0(t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\ (f(s,w(s)) + g(s,v(s)))ds,$

(iii) $f(t,u)$ *is nondecreasing in* u *for each* $t \in J_0$, *and* $f(t,u)$ *satisfies the one sided Lipschitz condition of the form*

$$f(t,u_1) - f(t,u_2) \leq L(u_1 - u_2),$$

for some $L > 0$, *whenever* $u_1 \geq u_2$;

(iv) $g(t,u)$ *is non increasing in* u *for each* $t \in J_0$ *and* $g(t,u)$ *satisfies the one sided Lipschitz condition of the form*

$$g(t,u_1) - g(t,u_2) \geq -M(u_1 - u_2),$$

for some $M > 0$, *whenever* $u_1 \geq u_2$. *Then* $u_0 \leq w_0$ *implies* $v(t) \leq w(t)$ *on* J.

Consider the Riemann-Liouville fractional differential equation of the form:

$$D^q u(t) = f(t,u(t)) + g(t,u(t)), \quad \Gamma(q)(t-t_0)^{1-q}u(t)|_{t=t_0} = u^0, \qquad (8.1)$$

where $f, g \in C[J_0 \times R, R]$.

The next result is the differential inequality result relative to the Riemann-Liouville fractional differential equation (8.1).

Theorem 8.2. *Let* $v, w \in C_p[J,R], f, g, \in C[J_0 \times R, R]$ *and satisfy the following coupled differential inequalities*
(i) $D^q(v(t)) \leq f(t,v(t)) + g(t,w(t)), \qquad \Gamma(q)(t-t_0)^{1-q}v(t)|_{t=t_0} \leq u^0;$
(ii) $D^q(w(t)) \geq f(t,w(t)) + g(t,v(t)), \qquad \Gamma(q)(t-t_0)^{1-q}w(t)|_{t=t_0} \geq u^0;$
(iii) $f(t,u)$ *satisfies the one sided Lipschitz condition of the form*

$$f(t,u_1) - f(t,u_2) \leq L(u_1 - u_2),$$

for some $L > 0$, *whenever* $u_1 \geq u_2$;

(iv) $g(t,u)$ *satisfies the one sided Lipschitz condition of the form*

$$g(t,u_1) - g(t,u_2) \geq -M(u_1 - u_2),$$

for some $M > 0$, *whenever* $u_1 \geq u_2$. *Then* $v_0 \leq w_0$ *implies* $v(t) \leq w(t)$ *on* J.

Here the Caputo fractional differential and integral inequalities are developed.

Consider the Caputo fractional differential equation of order q where $0 < q < 1$, of the form :

$$^{C}D^q u(t) = \lambda\, u(t) + f(t), \quad u(t_0) = u_0, \qquad (8.2)$$

where $f, g \in C[J_0 \times R, R]$.

The integral representation of (8.2) is given by equation

$$u(t) = u_0 + E_{q,1}(\lambda(t-t_0)^q) + \int_{t_0}^{t}(t-s)^{q-1}\ E_{q,q}(\lambda(s-t_0)^q)f(s)ds, \quad (8.3)$$

where $\Gamma(q)$ is the Gamma function. Note, the above holds true if in place of equality we have less than or equal.

Now consider the Caputo fractional differential equation

$$ {}^{C}D^{q}u(t) = f(t,u) + g(t,u), \quad u(t_0) = u_0, \qquad (8.4) $$

Where $f, g \in C[J_0 \times R, R]$.

Definition 8.1. *The functions* $v, w \in C^1([t_0, T], R)$ *are called natural lower and upper solutions of (8.4) if :*

$$ {}^{C}D^{q}v(t) \le f(t,v) + g(t,v), \quad v(t_0) \le u_0, $$

and

$$ {}^{C}D^{q}w(t) \ge f(t,w) + g(t,w), \quad w(t_0) \ge u_0. $$

Definition 8.2. *The functions* $v, w \in C^1([t_0, T], R)$ *are called coupled lower and upper solutions of type I of (8.4) if :*

$$ {}^{C}D^{q}v(t) \le f(t,v) + g(t,w), \quad v(t_0) \le u_0, $$

and

$$ {}^{C}D^{q}w(t) \ge f(t,w) + g(t,v), \quad w(t_0) \ge u_0. $$

One can easily prove the existence of the solution of (8.4) on the interval $[t_0, T]$when natural lower and upper solution for (8.4) are known for which v(t) ≤ w(t). Coupled minimal and maximal solutions of (8.4) on the interval $[t_0, T]$ can be computed without any extra assumptions using the generalized monotone method when coupled lower and upper solutions of type I of (8.4) are known with $v(t) \le w(t)$.

Next we have the differential inequality result for coupled lower and upper solutions of type I:

Theorem 8.3. *Let* $v, w \in C^1(J, R)$, Where $f, g \in C[J \times R, R]$ *are coupled lower and upper solutions of type I, such that*

(i) $f(t,u)$ *satisfies the one sided Lipschitz condition of the form*

$$ f(t,u_1) - f(t,u_2) \le L(u_1 - u_2), $$

for some $L > 0$, *whenever* $u_1 \ge u_2$;

(ii) $g(t,u)$ *satisfies the one sided Lipschitz condition of the form*

$$ g(t,u_1) - g(t,u_2) \ge -M(u_1 - u_2), $$

for some $M > 0$, *whenever* $u_1 \ge u_2$. *Then* $v(t_0) \le w(t_0)$ *implies that* $v(t) \le w(t)$, *on J.*

Remark 5.1. Caputo integral inequalities results similar to Theorem 8.1 can also be developed. The integral inequality results need an extra assumption which is not required for the corresponding scalar differential inequality results. Thus, the differential inequality results are useful in iterative methods like the monotone quasilinearization methods. They are useful for proving uniqueness, continuous dependency on the initial condition and also in determining the order of convergence.

Numerical Graphs

Consider a special case of the Logistic model of the form

(4.1) ${}^{c}D^{\frac{1}{2}u(t)} = 0.99\mathrm{u(t)} - \mathrm{u}^2(\mathrm{t}), \mathrm{u}(0) = \frac{1}{2}$.

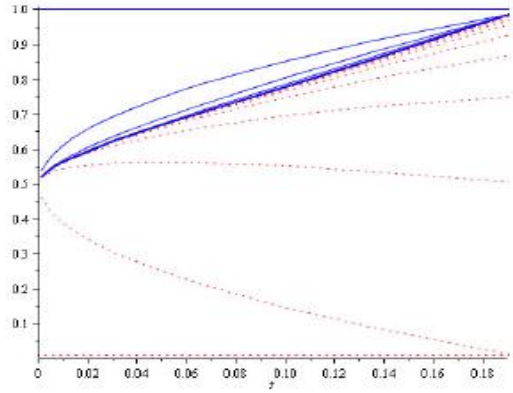

FIGURE 2. Dashed: $\{v_n\}$. Solid: $\{w_n\}$.

Conclusion

We have developed generalized monotone method for nonlinear fractional differential systems and fractional reaction diffusion equations via coupled lower and upper solutions of type 1. The advantage of the generalized monotone method is that each component of the iterates are scalar fractional differential equation whose solutions are easy to compute compared to computing the solution corresponding linear system.
Also, This representation formulas provide a methodology for computing solutions for different nonhomogeneous terms and different initial and boundary conditions.

References

[1] D. Baleanu, Z. B. Guvenc$_s$, J.A. T. Machado, New Trends in Nanotechnology and Fractional Calculus Applications, Springer, 2010.

[2] M. Caputo, Linear models of dissipation whose Q is almost independent, II, Geophysical Journal of the Royal Astronomical Society, **13** (1967), 529-539.

[3] Z. Denton and A.S.Vatsala, "Monotone Iterative Technique for Finite Systems of Nonlinear Fractional Differential Equations," *Opuscula Mathematica*, Vol. 31, No.3, 2011.

[4] Z. Denton and A.S.Vatsala, "Fractional Integral Inequalities and Applications," *Computers and Mathematics with Applications*, 59 (2010), 1087-1094.

[5] Z. Denton, P.W. Ng, and A.S. Vatsala, Quasilinearization Method via Lower and Upper Solutions for Riemann-Liouville Fractional Differential Equations, Nonlinear Dynamics and Systems Theory, 11 (3) (2011), 239-251.

[6] Kai Diethelm, *The Analysis of Fractional Differential Equations*, Springer, 2004.

[7] A. Kilbas, H. M. Srivastava, J. T. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematical studies, 204, Elsevier, 2006.

[8] G.S. Ladde, V. Lakshmikantham and A.S. Vatsala, Monotone Iterative Techniques for Nonlinear Differential Equations, Pitman Publishing, Inc., 1985.

[9] V. Lakshmikantham and A. S. Vatsala, *Generalized Quasilinearization for Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, Boston, London.

[10] V. Lakshmikantham, S. Leela and J. Vasundara Devi, Theory of Fractional Dynamic Systems, Cambridge Scientific Publishers, Ltd, 2009.

[11] G. S. Ladde, V. Lakshmikantham and A.S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman (1985).

[12] V. Lakshmikantham, S. Leela, *Differential and Integral Inequalities*, Volume I, II, Academic Press, New York, 1969.

[13] R. Metzler and J. Klafter,"The random walks's guide to anomalous diffusion: a fractional dynamics approach," Phys Rep 339(1):1-77.

[14] B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, New York, London, 1974.

[15] TH.T. Pham. J. D. Ramirez, A. S. Vatsala, Generalized Monotone Method for Caputo Fractional Differential Equations with Applications to Population Models, Neural, Parallel and Scientific Computations, 20(2012), 119-132.

[16] I. Podlubny, Fractional Differentials Equations, Mathematics in Science and Engineering, Volume 198, Academic Press, 1999.

[17] J. D. Ramirez, A.S. Vatsala, "Generalized Monotone Iterative Technique for Caputo Fractional Differential Equation with Periodic Boundary Condition via Initial Value Problem," submitted for publication to *Integral Inequalities and Applications*.

[18] M. Rivero, J. Trujillo and M. Pilar Velasco, "On Deterministic Fractional Models," New Trends in Nanotechnology and Fractional Calculus Applications, edited by D. Baleanu,et al, 123-150, Springer, 2010.

[19] M. Sokol and A.S. Vatsala, "A unified exhaustive study of monotone iterative method for initial value problems," Nonlinear Studies, 8 (4), 429-438,(2001).

[20] M. Sowmya and A. S. Vatsala, Numerical Approach via Generalized Monotone Method for Scalar Caputo Fractional Differential Equations, Neural, Parallel and Scientific Computations 21(2013), 19-30.

[21] M. Sowmya and A. S. Vatsala, Superlinear Convergence for Caputo Fractional Differential Equations with Applications, Dynamic Systems and Applications 22(2013), 479-492.

[22] D. Stutson and A. S. Vatsala, Generalized Monotone Method for Caputo Fractional Differential Systems via Coupled Lower and Upper Solutions, Dynamic Systems and Applications 20(2011), 495-504.

[23] A.S. Vatsala and I. H. West, "Generalized Monotone Iterative Method for Initial Value Problems," *Applied Mathematics Letters*, 17 (2004) 1231-1237.